Chapter 1
Introduction

John Stachurski, Alain Venditti, and Makoto Yano

Kazuo Nishimura has made outstanding contributions to three main fields of economic theory: international trade, general equilibrium, and the theory of economic growth. With his characteristic mix of originality, technical clarity, elegance, and rigor, Kazuo's insights into nonlinear dynamics have transformed our understanding of economic growth, business cycles, and the relationship between them. For young scholars beginning work in these areas, Kazuo's papers are essential and inspiring.

Kazuo Nishimura obtained his Master's degree from Tokyo University in 1972. In 1973, he entered the graduate program in economics at the University of Rochester. His supervisor was Lionel McKenzie—one of the giants of twentieth-century neoclassical economics. After spending three years at Rochester, Kazuo joined the Department of Economics at Dalhousie, Canada, in 1976. He obtained his PhD in 1977. Subsequently, he returned to Japan with an appointment at the Tokyo Metropolitan University. He spent 10 years there, during which time he also taught at the State University of New York at Buffalo, and at the University of Southern California. In 1987, he joined the Institute of Economic Research at Kyoto University. He served as its director in 2006–2010.

Kazuo has had a highly distinguished career. He was elected fellow of the *Econometric Society* in 1992. He served as the vice president of the *Japanese Economic Association* in 1999–2000, and as its president in 2000–2001. He was co-editor of

J. Stachurski
Australian National University, Canberra, ACT, Australia
e-mail: john.stachurski@anu.edu.au

A. Venditti
National Center for Scientific Research, CNRS-GREQAM and EDHEC, Marseille, France
e-mail: alain.venditti@univmed.fr

M. Yano
Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Japan
e-mail: yano@kier.kyoto-u.ac.jp

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the *Japanese Economic Review* during 1995–1997, and its editor-in-chief from 1997 to 2000. In 2004, he established the *International Journal of Economic Theory*, a new journal for the profession that is already highly regarded within its field. He has served as a managing editor since its inception. He has also served on the editorial board of numerous leading journals in economics and applied mathematics, and he has edited a number of special issues for prestigious journals. He has organized successful conferences around the world.

Recently Kazuo has become an active and leading participant in the public debate in Japan about the science curriculum taught in high schools and universities. Through the popular press and scholarly journals on science and mathematics education, Kazuo has strongly advocated the need for improved training in science and mathematics for high school students and humanities students at universities. To further promote scientific and mathematical education, he founded the International Society for Education. He currently serves as president of the society.

To say that Kazuo is an extremely busy person is still an understatement. However, as his numerous co-authors, friends and colleagues all over the world will attest, he is always available to discuss research, provide insight, encouragement, and the benefit of his remarkable energy. A perfect day for Kazuo is to work through till evening with some co-authors on a difficult research problem, then to break for dinner and fine sake together in a Kyoto restaurant, all the while still discussing new ideas. If, late into night, a break for sleep appears to be necessary, there is no doubt that Kazuo will be back the next day with some hand-written notes providing the solution to the problem.

We, Kazuo's friends, colleagues and co-authors, have been privileged to benefit over many years from his loyalty, warmth and generosity. It is with great pleasure that we present this collection of selected papers, containing his most important and enduring contributions.

1.1 Overview of the Papers

In this section we provide an overview of the papers selected for this volume. While the optimal growth model provides a common framework linking these papers, they have been classified according to the particular qualitative property of the equilibrium that is analysed:

1. Optimal growth and endogenous cycles
2. Optimal growth and chaotic dynamics
3. Global dynamics in optimal growth models
4. Dynamic models with non-concave technologies
5. Stochastic optimal growth models
6. Indeterminacy in exogenous growth models
7. Indeterminacy in endogenous growth models

The complete list of papers can be found in Sect. 1.2.

1.1.1 Introductory Survey

As the title of the book suggests, all the selected papers involve nonlinear dynamics, a major field of research within economic theory. In collections such as this, it is usually desirable to provide the reader with a review of basic material and concepts common to the field. This task has already been accomplished by Kazuo, in a paper titled "Nonlinear dynamics in the infinite time horizon model," published with Gerhard Sorger in the *Journal of Economic Surveys* in 1999. Starting from the standard aggregate formulation of the optimal growth model, in which the unique optimal equilibrium converges toward a long-run balanced growth path, the paper then goes on to introduce the main features of nonlinear dynamics, including periodic cycles, chaotic dynamics, stochastic shocks, non-concave technologies, externalities, local indeterminacy, sunspot fluctuations and analysis of multisector models. It provides the perfect lead article of the present collection. It remains only for us to provide the reader with an overview of the selected papers, in order that he or she might quickly locate those papers most relevant to his or her field.

1.1.2 Optimal Growth and Endogenous Cycles

This section contains three important contributions on the existence of endogenous fluctuations in the perfect competitive framework of the multisector optimal growth model. The first and second belong to the set of papers written during a long and fruitful collaboration between Kazuo and Jess Benhabib (New York University). They deal with closed economies. The third paper is the first of many collaborations with Makoto Yano of Keio University. It studies a model of trade between two countries. These two collaborations have produced highly influential contributions on the analysis of non-linear dynamics under perfect competition.

The first paper, "The Hopf bifurcation and the existence and the stability of closed orbits in multi-sector models of economic growth," was published with Jess Benhabib in the *Journal of Economic Theory* in 1979. It shows that, under very general circumstances, the standard optimal growth model with two or more capital goods can give rise to optimal trajectories that are limit cycles. This was a striking result, as research on optimal growth in the 1960s and 1970s had typically found these models to be characterized by a strong form of stability, particularly when the discount rate is small (see, for instance, Brock and Scheinkman, 1976; Cass and Shell, 1976; McKenzie, 1976; and Scheinkman, 1976). In contrast, this paper, by treating the discount rate as a parameter, establishes via bifurcation theory that optimal paths can become closed orbits as the steady state loses stability.

Bifurcations for differential equation systems arise when, for some value of a parameter, the Jacobian of the function describing motion acquires eigenvalues with zero real parts at a stationary point. If a real eigenvalue becomes zero, this results in multiplicity of steady states. In this paper, the focus is on a bifurcation that results from pure imaginary roots: the Hopf bifurcation. In contrast to bifurcation from a real eigenvalue, no additional stationary points arise from a Hopf bifurcation. Instead, closed orbits emerge around the stationary point. Nonjoint production and a single, possibly composite, consumption good are assumed to assure the uniqueness of the steady state. Under some additional assumptions, it is shown that the Jacobian of the functions describing the motion of the system cannot vanish at a steady state, and, as a result, that only the Hopf bifurcation is possible. An example is given of a totally unstable steady state, giving rise to optimal paths that are closed orbits. The stability of orbits that arise from the optimal growth problem is discussed.

The second paper, "Competitive equilibrium cycles," was published in 1985 with Jess Benhabib in the *Journal of Economic Theory*, and provides a discrete-time extension of the previous contribution. Considering a two-sector optimal growth model with one pure consumption good and one investment good, it gives sufficient conditions for generating robust period-two cycles in outputs, stocks, and prices. These conditions are expressed in terms of the discount rate and differences in relative factor-intensities between the two industries.[1] Finally, some intuitive explanation for the existence of cyclical equilibria that arise in neoclassical technologies are also given. The paper is notable for the sophisticated techniques from differential topology used to prove the global existence of period-two cycles, corresponding locally to the existence of a flip bifurcation.

In the last paper of this section, "Interlinkage in the endogenous real business cycles of international economies," published in 1993 with Makoto Yano in *Economic Theory*, the transmission of business cycles between trading countries is analyzed. In a perfect foresight model with many consumers, it is well-known that competitive equilibrium paths behave like optimal growth paths.[2] Therefore, a perfect foresight equilibrium path may exhibit business cycles, even in a many-consumer model such as a large-country trade model. In this paper, a two-country optimal growth model in which the countries have independent technologies is considered. First, the determinants of each country's global accumulation pattern in autarkic equilibrium are analyzed using the results of Benhabib and Nishimura contained in the second paper of this section, providing conditions on the sectoral technologies that lead to existence of business cycles (in particular, period-two cycles). Global accumulation under free-trade is then addressed. Conditions for the international transmission of business cycles are given in terms of the parameters of each country's technologies.

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[1] In particular, the discount factor must be sufficiently large, and the consumption good sector must be more capital intensive than the investment good sector.

[2] See, for instance, Becker (1980), Bewley (1982), Epstein (1987), and Yano (1983, 1984).

1.1.3 Optimal Growth and Chaotic Dynamics

The two papers by Jess Benhabib and Kazuo Nishimura contained in the previous section had a strong impact on the group of researchers studying macroeconomic dynamics and business cycles, initiating a common program of research that aimed to provide detailed analysis of the qualitative properties of capital accumulation paths in optimal growth models. For example, the landmark paper of Boldrin and Montrucchio (1986), which provides an anything-goes result for optimal paths in standard multisector optimal growth models, was initiated at the beginning of the 1980s after Luigi Montrucchio attended a presentation of the "Competitive equilibrium cycles" paper.

The findings of Boldrin and Montrucchio indicated that equilibrium time paths in deterministic equilibrium economies may exhibit complex—even chaotic—dynamics. Building on the results of McKenzie (1976) and Scheinkman (1976), they showed specifically that almost any capital accumulation path can be realized as a solution of an optimal growth model, provided that the discount rate is large enough. However, this leaves open the question as to whether or not chaotic optimal paths can appear under more reasonable rates of discounting. The two papers with Makoto Yano in this section, titled "Nonlinear dynamics and chaos in optimal growth: an example," and "Chaotic solutions in dynamic linear programming," published in *Econometrica* in 1995 and in *Chaos, Solitons and Fractals* in 1996 respectively, showed that they can.

The first paper demonstrates the possibility of ergodically chaotic optimal accumulation in a two-sector model with linear preferences and Leontief production. A condition is provided under which the optimal transition function is unimodal and expansive. It is then shown that the set of parameter values satisfying this condition is nonempty, no matter how weakly future utilities are discounted. The second paper is a complement to the first, as it provides a simpler and more intuitive proof of the same result.

While these two papers provide strong conclusions on the compatibility between chaotic optimal paths and low discounting of future utilities, their main limitation concerns the fact that preferences and technologies are specified. This extra structure inhibits the formation of a general picture. In consequence, the paper "On the least upper bound of discount factors that are compatible with period-three cycles," published in the *Journal of Economic Theory* in 1996 with Makoto Yano, studies a class of general, reduced form optimal growth models. In these models time is discrete, and an indirect utility function summarizes both preferences and the social production function. The authors derive the least upper bound of the set of discount factors such that a cyclical optimal period-three path emerges in a standard model of optimal growth.[3] For this purpose, they characterize a cyclical optimal period-three

[3]The existence of a cyclical path of period three is a fundamental criterion for the emergence of complex nonlinear dynamics, in particular of cyclical paths of any other periodicity (see Sarkovskii (1964) and Li and Yorke (1975)).

path in terms of the dual path of prices. They then demonstrate that a condition for the existence of a cyclical optimal period-three path can be expressed by the existence of a solution to a system of linear equations. By solving this system, they finally demonstrate that the least upper bound mentioned above is equal to $(3-\sqrt{5})/2$.

1.1.4 Global Dynamics in Optimal Growth Models

Although they focus on the particular case of chaotic optimal paths, the contributions contained in the previous section are concerned more generally with the following question: "How is long-run optimal behavior affected by changes in the rate at which the future is discounted?" The paper contained in the present section, "Intertemporal complementarity and optimality: a study of a two-dimensional dynamical system," published in *International Economic Review* in 2005 with Tapan Mitra, provides additional insights on this point. It studies the underlying structure of the two-dimensional dynamical system generated by a class of optimal growth models that allow for intertemporal complementarity between adjacent periods, but preserves the time-additively separable framework of standard Ramsey models. The purpose of the article is to complete the program sketched in the contribution of Samuelson (1971), providing both a complete local and global analysis of the model under which the standard results of the Ramsey model continue to hold, even with dependence of tastes between periods. Global convergence results are established and related to the local analysis using the mathematical theory of two-dimensional dynamical systems. The local stability property of the stationary optimal stock is also related to the differentiability of the optimal policy function near the stationary optimal stock, using the stable manifold theorem.

1.1.5 Dynamic Models with Non-Concave Technologies

Traditionally, optimal growth theory is concerned with the derivation of an optimal consumption plan of a representative agent when the production function in the economy is concave and preferences of the consumer exhibit decreasing marginal utility. In these settings, there exists a unique long-run steady state capital stock that is globally stable, in the sense that the optimal capital stock converges to the steady state level independent of the initial condition. The paper contained in this section, "A complete characterization of optimal growth paths in an aggregated model with a non-concave production function," published in the *Journal of Economic Theory* in 1983 with W. Davis Deckert, provides some extensions of this framework by considering a non-concave production function and thus the local existence of increasing returns. It appeared as a path-breaking study of optimal one-sector growth under non-convex technology. When the technology is convex, the Euler equation is sufficient to prove the monotonicity property of the equilibrium path. This is no longer the case when non-convexities are introduced. Deckert and Nishimura show that the Euler equation in conjunction with the Principle of Optimality is sufficient to prove monotonicity for non-concave production functions. For certain interest rates, they prove that the optimal path converges to a steady state only if the initial capital stock is above a critical level, otherwise it converges to zero. They also demonstrate that the set of points for which the value function is differentiable is precisely the set of initial capital stocks from which there is a unique optimal path.

1.1.6 Stochastic Optimal Growth Models

In the first paper in this section, "Stochastic optimal growth with nonconvexities," published with Ryszard Rudnicki and John Stachurski in the *Journal of Mathematical Economics* in 2006, the problem of optimal growth with non-concave production (i.e., the problem studied in the paper with W. Davis Deckert discussed in the last section) is revisited, but now with random shocks to productivity. A number of key results from the deterministic case are replicated, such as almost everywhere differentiability of the value function, and monotonicity of optimal policies. In addition, the authors establish a basic dichotomy in growth dynamics when sufficient mixing is present: Either global stability is observed, or the economy is globally asymptotically collapsing to the origin.

While the dichotomy result of this paper is useful in its own right, it does not provide a set of conditions on the primitives under which global stability is shown to hold. For the concave case, this problem was essentially solved as early as the 1970s (see, e.g., Brock and Mirman (1972)), but the non-concave case proved far less tractable. The third paper in this section, "Stability of stochastic optimal growth models: a new approach," published with John Stachurski in the *Journal of Economic Theory* in 2005, provided a breakthrough. The paper gave a set of stability conditions on the model primitives that could be viewed as natural analogues of the classical deterministic conditions. The breakthrough resulted from a new method developed by the authors for studying stability of optimal dynamic models, based on viewing marginal utility of consumption as a Lyapunov function.

1.1.7 Indeterminacy in Exogenous Growth Models

A wide-spread perception among economists is that macroeconomic business-cycle fluctuations can be driven by changes in expectations of fundamentals. A major strand of the literature focusing on fluctuations derived from agents' beliefs is based on the concept of sunspot equilibria, dating back to the early works of Shell (1977), Azariadis (1981), and Cass and Shell (1983). As shown by Woodford (1986), the existence of sunspot equilibria is closely related to the indeterminacy of perfect foresight equilibrium (i.e., the existence of a continuum of equilibrium paths converging toward one steady state from the same initial value of the state variable).

Following the seminal contribution of Benhabib and Farmer (1994), infinite horizon Ramsey-type models augmented to include external effects in production have been shown to exhibit multiple equilibria and local indeterminacy. Much of the research in this area has been concerned with the empirical plausibility of indeterminacy in markets with external effects, which exhibit some degree of increasing returns. While the early results on indeterminacy relied on relatively large increasing returns, Benhabib and Farmer (1996) showed that indeterminacy can also occur in two-sector models with small sector-specific external effects and mild increasing returns. Nevertheless, a number of empirical researchers on disaggregated US data (see for instance Basu and Fernald (1997)) have concluded that returns to scale seem to be roughly constant.

The three papers contained in this section show how indeterminacy can occur in standard growth models with sector-specific externalities, constant social returns, decreasing private returns and small or negligible external effects. The first one, "Indeterminacy and sunspots with constant returns," published with Jess Benhabib in the *Journal of Economic Theory* in 1998, is an influential contribution that has initiated many extensions and developments. Considering a two-sector economy with Cobb-Douglas technology and a utility function that is linear in consumption, Benhabib and Nishimura show that local indeterminacy arises if and only if the sector-specific externalities generate a capital intensity reversal between the private and the social levels. Specifically, indeterminacy requires that the consumption good sector is capital intensive at the private level but labor intensive at the social level. This conclusion is fundamentally based on the breakdown of the duality between the Rybczynski and Stolper-Samuelson effects, caused by introduction of market imperfections.

The second paper, "Trade and indeterminacy in a dynamic general equilibrium model," published with Koji Shimomura in the *Journal of Economic Theory* in 2002, introduces sector-specific externalities in a Heckscher-Ohlin, two-country, dynamic general equilibrium model. In their model, factors are internationally immobile, and the countries both use identical Cobb-Douglas technologies. Under the same capital intensity reversal condition as in the previous paper, Nishimura and Shimomura show that there are multiple equilibrium paths from the same initial distribution of capital in the world market, and that the distribution of capital in the limit differs among equilibrium paths. One equilibrium path converges to a long-run equilibrium in which the international ranking of factor endowment ratios differs from the initial ranking, while another equilibrium path maintains the initial ranking and converges to a different long-run equilibrium. Since the path realized is indeterminate, so is the long-run trade pattern. Therefore, the long-run Heckscher–Ohlin prediction, that each country exports those goods such that the abundantly endowed factor of production is intensively used for producing it, is vulnerable to the introduction of externality.

In most papers focusing on the existence of indeterminacy within two-sector models with constant social returns, the utility function is assumed to be linear, which implies an infinite elasticity of intertemporal substitution in consumption. However, as shown recently by Mulligan (2002) and Vissing-Jorgensen (2002) for instance, this elasticity is estimated around unity or even below. The last paper of this section, "Indeterminacy in discrete-time infinite-horizon models with nonlinear utility and endogenous labor," published with Alain Venditti in the *Journal of Mathematical Economics* in 2007, considers a two-sector economy in which preferences are given by a additively separable CES utility function, defined over consumption and leisure. Three particular cases are considered:

1. Labor supply is infinitely elastic,
2. The elasticity of intertemporal substitution in consumption is infinite, and
3. Labor supply is inelastic.

In the first case, the steady state is a saddle-point for any value of elasticity of intertemporal substitution in consumption, and any level of external effects. In the second case, local stability of the steady state is independent from elasticity of labor supply, and local indeterminacy relies only on the loss of duality between the Rybczynski and Stolper-Samuelson effects. In the third case, a geometrical method of Grandmont et al. (1998) is used to show that the steady state is locally indeterminate if and only if the elasticity of intertemporal substitution in consumption is large enough. On this basis, the main result of the paper is established: the steady state is locally indeterminate if and only if the elasticity of intertemporal substitution in consumption is sufficiently large, and that of labor supply is sufficiently low. Moreover, period-two cycles always occur as the elasticity of labor supply is increased, and the steady state becomes saddle-point stable.

1.1.8 Indeterminacy in Endogenous Growth Models

The fundamental contributions of Romer (1986) and Lucas (1988) showed that if preferences are homothetic and external effects in production generate constant returns to scale for reproducible factors, then endogenous growth can occur in infinite-horizon models. This means that the growth process is endogenously determined, perpetual, and can be characterized by the dynamics of growth rates for output and consumption. This fact, combined with some of the previous results provided in this collection, suggests the possible existence of growth rate fluctuations. The three contributions contained in this section are concerned precisely with this point.

The first, "Indeterminacy under constant returns to scale in multisector economies," published with Jess Benhabib and Qinglai Meng in *Econometrica* in 2000, is an extension to the endogenous growth framework of the first paper discussed in Sect. 1.1.7. The authors consider a multisector economy with one consumable capital good and n pure capital goods produced without fixed factors from Cobb-Douglas technologies. The economy is characterized by sector-specific externalities and constant social returns. Instantaneous utility is homogeneous with a constant intertemporal elasticity of substitution. Under some conditions on the matrices of private and social input coefficients, local indeterminacy exists. In the particular case of a two-sector endogenous growth model, these conditions imply that the consumable capital good is intensive in the pure capital good from the private perspective, but it is intensive in itself from the social perspective. Contrary to the case of exogenous growth, this result holds for any value of the elasticity of intertemporal substitution in consumption.

Most of the papers in this collection considering multisector economies with productive external effects assume sector-specific externalities, and show that a capital intensive consumption good sector at the private level is a necessary condition for the occurrence of local indeterminacy. However, such a formulation is quite different from the one initially considered by Romer (1986) and Lucas (1988), where global externalities are assumed. In this case, some additional inter-relationships between the sectors are introduced, and these new mechanisms may provide different channels through which local indeterminacy can arise. The second paper of this section, "Global externalities, endogenous growth and sunspot fluctuations," published with Harutaka Takahashi and Alain Venditti in *Advanced Studies in Pure Mathematics* in 2009, considers a two-sector economy with Cobb-Douglas technologies, labor-augmenting global external effects, and increasing social returns. The paper focuses on the local indeterminacy of endogenous growth paths. After proving the existence of a balanced growth path, existence of sunspot fluctuations is shown to be compatible with a labor intensive consumption good sector at the private level, provided that elasticity of intertemporal substitution in consumption admits intermediary values. In particular, the existence of period-two cycles requires precisely such a capital intensity configuration.

The last paper of this collection, "A homoclinic bifurcation and global indeterminacy of equilibrium in a two-sector endogenous growth model," published with Paolo Mattana and Tadashi Shigoka in the *International Journal of Economic Theory* in 2009, also focuses on the consequences of global externalities, discussed in the context of an endogenous growth model similar to that of Lucas (1988). In the Lucas model, an external effect appears in the physical-goods sector. In this paper, it appears in the educational sector. Contrary to the Lucas formulation, this external effect yields multiple balanced growth paths. Moreover, the model undergoes a global homoclinic bifurcation, and exhibits global indeterminacy of equilibrium.

1.1.9 Summary

This selection of Kazuo Nishimura's papers, since the contribution published in the *Journal of Economic Theory* in 1979, provides a precise description of the usefulness of nonlinear dynamics for the analysis of infinite-horizon intertemporal optimization models. As co-authors and friends, it is our pleasure to show how Kazuo's works contributed to the development of nonlinear dynamics in economic theory. We look forward to seeing even further progress along the lines of these selected papers in the coming years.

1.2 List of Papers

The selected papers of Kazuo Nishimura chosen for this volume are as follows:

Introductory Survey:

1. "Nonlinear Dynamics in the Infinite Time Horizon Model," (with Gerhard Sorger), *Journal of Economic Surveys,* Vol. 13, 619–652, 1999

Optimal Growth and Endogenous Cycles:

2. "The Hopf Bifurcation and the Existence and the Stability of Closed Orbits in Multi-Sector Models of Economic Growth," (with Jess Benhabib), *Journal of Economic Theory,* Vol. 21, pp. 421–444, 1979
3. "Competitive Equilibrium Cycles," (with Jess Benhabib), *Journal of Economic Theory,* Vol. 35, pp. 284–306, 1985
4. "Interlinkage in the Endogenous Real Business Cycles of International Economies," (with Makoto Yano), *Economic Theory,* Vol. 3, pp. 151–168, 1993

Optimal Growth and Chaotic Dynamics:

5. "Nonlinear Dynamics and Chaos in Optimal Growth: An Example," (with Makoto Yano) *Econometrica,* 63, 981–1001, 1995
6. "Chaotic Solutions in Dynamic Linear Programming," (with M. Yano), *Chaos, Solitons and Fractals,* 7, 1191–1953, 1996
7. "On the Least Upper Bound of Discount Factors that are Compatible with Optimal Period-Three Cycles," (with Makoto Yano), *Journal of Economic Theory,* 66, 306–333, 1996

Global Dynamics in Optimal Growth Models:

8. "Intertemporal Complementarity and Optimality: A Study of a Two-Dimensional Dynamical System," (with Tapan Mitra), *International Economic Review,* Vol. 46, No. 1, pp. 93–131, 2005

Dynamic Models with Non-concave Technologies:

9. "A Complete Characterization of Optimal Growth Paths in an Aggregated Model with a Non-concave Production Function," (with W. Davis Dechert), *Journal of Economic Theory,* Vol. 31, pp. 332–354, 1983

Stochastic Optimal Growth Models:

10. "Stochastic Optimal Growth with Nonconvexities," (with Ryszard Rudnicki and John Stachurski), *Journal of Mathematical Economics,* Vol. 42(1), pp. 74–96, 2006
11. "Stability of Stochastic Optimal Growth Models: a New Approach," (with John Stachurski), *Journal of Economic Theory,* Vol.22, pp. 100–118, 2005

Indeterminacy and Externalities in Exogenous Growth Models:

12. "Indeterminacy and Sunspots with Constant Returns," (with Jess Benhabib), *Journal of Economic Theory,* Vol. 81, pp. 58–96, 1998
13. "Trade and Indeterminacy in a Dynamic General Equilibrium Model," (with Koji Shimomura), *Journal of Economic Theory,* Vol. 105, pp. 244–259, 2002
14. "Indeterminacy in discrete-time infinite- horizon models with nonlinear utility and endogenous labor," (with Alain Venditti), *Journal of Mathematical Economics,* Vol. 43(3–4), pp. 446–476, 2007

Indeterminacy and Externalities in Endogenous Growth Models:

15. "Indeterminacy Under Constant Returns to Scale in Multisector Economies," (with Jess Benhabib and Qinglai Meng), *Econometrica,* Vol. 68, pp. 1541–48, 2000
16. "Global Externalities, Endogenous Growth and Sunspot Fluctuations," (with Harutaka Takahashi and Alain Venditti), *Advanced Studies in Pure Mathematics,* Vol. 53, pp. 227–238, 2009
17. "A Homoclinic Bifurcation and Global Indeterminacy of Equilibrium in a Two-Sector Endogenous Growth Model," (with Paolo Mattana and Tadashi Shigoka), *International Journal of Economic Theory* Vol. 5, pp. 25–60, 2009