Preference revelation games and strong cores of allocation problems with indivisibilities*

Koji TAKAMIYA†
Institute of Social and Economic Research,
Osaka University

This version, Aug 24 2005

Abstract

This paper studies the incentive compatibility of solutions to **generalized indivisible good allocation problems** introduced by Sönmez (1999), which contain the well-known **marriage problems** (Gale and Shapley, 1962) and the **housing markets** (Shapley and Scarf, 1974) as special cases. In particular, I consider the vulnerability to manipulation of solutions that are individually rational and Pareto optimal. By the results of Sönmez (1999) and Takamiya (2003), any individually rational and Pareto optimal solution is strategy-proof if and only if the strong core correspondence is essentially single-valued, and the solution is a strong core selection. Given this fact, this paper examines the equilibrium outcomes of the **preference revelation games** when the strong core correspondence is not necessarily essentially single-valued. I show that for the preference revelation games induced by any solution which is individually rational and Pareto optimal, the set of strict strong Nash equilibrium outcomes coincides with the strong core. This generalizes one of the results by Shin and Suh (1996) obtained in the context of the marriage probelms. Further, I examine the other preceding results proved for the marriage problems (Alcalde, 1996; Shin and Suh, 1996; Sönmez, 1997) to find that none of those results are generalized to the general model.

JEL Classification— C71, C72, C78, D71, D78.

Keywords— generalized indivisible good allocation problem, preference revelation game, strict strong Nash equilibrium, strong core.

---

*This paper is based on Chapter 4 of my dissertation submitted to Hokkaido University. I am grateful to Tayfun Sönmez for introducing some of his works related to this paper. I also thank Hiroshi Ono for encouragement, Tomoichi Shinotsuka for detailed comments and discussion and, Takuma Wakayama for comments. I also thank seminar participants at Otaru University of Commerce. All errors are my own responsibility.

†Institute of Social and Economic Research, Osaka University, 6-1 Mihogaoka Ibaraki Osaka 567-0047 JAPAN, E-mail: takamiya@iser.osaka-u.ac.jp.

0 Introduction

0.1 Motivation and method

This paper studies the vulnerability to manipulation of solutions to **generalized indivisible good allocation problems** introduced by Sönmez (1999). Specifically, I am interested in the manipulability of solutions which are **individually rational** and **Pareto optimal**. Generalized indivisible good allocation problems are a comprehensive economic model with indivisibilities. This class of allocation problems includes some well-known classes of problems as subclasses, such as the **marriage problems** (Gale and Shapley, 1962) and the **housing markets** (Shapley and Scarf, 1974).

When one examines the incentive compatibilty of solutions, it first comes to ask if they are **strategy-proof** or not.[1] Sönmez (1999) and Takamiya (2003) characterized strategy-proof solutions in this environment: Sönmez showed that whenever the strong core is nonempty for any admissible preference profile, if a solution is strategy-proof, individually rational and Pareto optimal, then the strong core correspondence must be essentially single-valued, and the solution is a selection from the strong core correspondence. (Here that the strong core correspondence is essentially single-valued means that for any preference profile any two strong core allocations are indifferent to all individuals.) Conversely, Takamiya showed that if the strong core correspondence is essentially single-valued, then any selection from the correspondence is strategy-proof (weakly coalition strategy-proof, in fact). These results shows that the possibility for reasonable strategy-proof solutions are very limited. This is equivalent to the essential single-valuedness of the strong core correspondence.

Given the above fact, this paper studies the incentive compatibility of solutions that are individually rational and Pareto optimal when the strong core correspondence is not necessarily essentially single-valued. In this case, such solutions need not be strategy-proof. This means that individuals are open to strategic behavior by misrepresenting their preferences. Thus it is expected that the resulting outcomes would be different from the ones arising from the true preferences. Now the relevant question of incentive compatibility is "where does the strategic behavior leads to?"

Given strategic behavior of individuals, the solution is regarded as a game in which each individual reports a preference relation as a strategy, and evaluates the outcome chosen by the solution fed with the reported preference profile. This game is called **preference revelation game**. (Thus, given a solution, one true preference profile defines one preference revelation game.) To study the above question, we analyze the equilibrium outcomes of the preference revelation games induced by the solution, i.e. the games.[2] Here the choice of equilibrium concept represents the way the individuals are assumed to behave.

To be more specific, in this paper two questions are asked. First, we have known that any selections from the strong core correspondence is strategy-

[1] Strategy-proofness is one of the most stringent criteria of nonmanipulability. An solution is called strategy-proof if no individual can be better off by misrepresenting his preference.

[2] This type of analysis was initiated by, among others, Hurwicz (1978), Otani and Sicilian (1982) and Thomson (1984, 1988) mainly for the Walrasian exchange economies.

proof when the correspondence is essentially single-valued. Then when it is not so, how much are these selections manipulable? Namely, in what equilibrium concept are the strong core outcomes played in the preference revelation games induced by strong core selections? Second, the analysis of preference revelation games has already been carried out by several authors for the **marriage problems** (Gale and Shapley, 1964), a special case of the present model. Then what part of this previous analysis can be extended to the present general model? Answers to the both questions (needless to say about the second, but also about the first) are to be found in generalizing the previous analysis to the present generalized model. I will review those previous results in Section 0.2.

The analysis in the sequel, I employ the concepts of **$\mathcal{G}$-proof Nash equilibirum** and **strict $\mathcal{G}$-proof equilibirum**.[3] Here $\mathcal{G}$-proof Nash equilibrium is a similar concept to strong Nash equilibrium except that the 'deviation power' is limited to the coalitions belonging to the prescribed class of coalitions $\mathcal{G}$. Thus depending on how $\mathcal{G}$ is set beforehand, it reduces to Nash equilibrium or strong Nash equilibrium. And strict $\mathcal{G}$-proof Nash equilibrium is a similar concept except that group deviations with weak improvement are also ruled out. It reduces to also Nash equilibrium or strict strong Nash equilibrium.

To describe the outcomes corresponding to these equilibrium concepts, I deal with the **$\mathcal{G}$-weak core** and the **$\mathcal{G}$-strong core**. Just like (strict) $\mathcal{G}$-proof Nash equilibrium, these core concepts are defined in the way that only the coalitions in $\mathcal{G}$ has the blocking power. The $\mathcal{G}$-weak core is defined by the blockings with strict improvement. This reduces to the **weak core** if $\mathcal{G}$ equals to the set of all coalitions. Also the $\mathcal{G}$-strict core is defined by the blockings with weak improvement. This reduces to the **strong core** similarly.

0.2 Previous and present results

The following looks back the analysis done in the context of the **marriage problems** (Gale and Shapley, 1964), a special case of the present model. Several authors have obtained "implementability" (i.e. "full conincidence") results.

Ma (1995) introduced the notion of **rematching-proof equilibrium**. A rematching-proof equilibrium is a Nash equilibrium in which no pair of 'man' and 'woman' can jointly deviate so that each of them be better off. He showed the following: Consider the preference revelation games induced by any **stable solution** (i.e. a selection from the stable matching correspondence).[4] Then the set of rematching-proof equilibrium outcomes coincides with the **stable matching correspondence**.

Shin and Suh (1996) pointed out that Ma's above result would still hold true if one uses the notions of **strong Nash equilibrium** or **strict strong Nash equilibrium** instead of rematching-proof equilibrium.

On the other hand, Alcalde (1996) studied **Nash equilibrium**. He showed that in the preference revelation games induced by any stable solution, the set

[3]The concept of $\mathcal{G}$-proof Nash equilibrium was introduced in Kalai, Postlewaite and Roberts (1979).

[4]For the case of the marriage problems, both the strong core and the weak core always coincide with the set of stable matchings. For a proof, refer to Roth and Sotomayor (1990).

of Nash equilibirum outcomes equals to the **individually rational matching correspondence** in

Finally, Sönmez (1997) adopted the concept of $\mathcal{G}$**-proof Nash equilibrium**. Sönmez proved the following: Let a solution be **individually rational** and **Pareto optimal**. (This is weaker than requiring the solution to be a stable solution.) Then the set of $\mathcal{G}$**-proof Nash equilibrium** coincides with the $\mathcal{G}$**-core correspondence** in the induced preference revelation games. The above-mentioned results concerning Nash equilibrium by Alcalde, rematching-proof equilibrium by Ma, and strong Nash equilibrium by Shin and Suh are all corollaries to Sönmez's result. However, note that the result concerning **strict** strong Nash equilibrium by Shin and Suh does not follow from Sönmez's result.

Given these existing results for the marriage problems, turning to the generalized model, this paper proves the following: For the preference revelation games induced by any solution which is individually rational and Pareto optimal, the set of strict $\mathcal{G}$-Nash equilibrium outcomes coincides with the $\mathcal{G}$-strong core for any "monotonic" class of coalitions $\mathcal{G}$. Here $\mathcal{G}$ is said to be monotonic if a coalition S belongs to $\mathcal{G}$, then any supercoalition of S also belongs to $\mathcal{G}$. By setting $\mathcal{G}$ equal to the set of all coalition, this result implies the coincidence between the set of **strict strong Nash equilibrium outcomes** and the **strong core**. This generalizes one of the two results by Shin and Suh (1996) in the marriage problems (since the strong core equals to the set of stable matchings in the marriage problems). On the other hand, by means of counter-examples, I show that neither the outcomes of Nash equilibrium nor strong Nash equilibrium are not even included in the strong core outcomes. These results anwser the first question I asked in Section 0.1.

In the above result, the monotonicity assumption of $\mathcal{G}$ is essential. In particular for the marriage problems, on the analogy of Sönmez's results, one may suspect that the above result still holds true without the monotonicity of $\mathcal{G}$. But there is a counter-example.

Next, to answer the second question I asked in Section 0.1. I examine what part of the rest of the previous results stated above are to be extended to the general model. I find that none of the other results are to be extended to the general model. Interestingly, although one does not obtain "full coincidence" any more, one direction of inclusion still preserves in general: For any class of coaltions $\mathcal{G}$, the set of $\mathcal{G}$-proof Nash equilibrium outcomes is included in the $\mathcal{G}$-weak core.

This paper is organized as follows: Section 1 introduces the basic concepts. Section 2 presents the main theorems which answer the first question of this paper. And Section 3 examines the generalizations of the preceding results. This answers the second question.

1 Definitions

1.1 Model

The following model is due to Sönmez (1999). This class of allocation problems is a comprehensive economic model with indivisibilities. It includes some well-known classes of problems, such as the **marriage problems** (Gale and Shapley, 1962) and the **housing markets** (Shapley and Scarf, 1974), and other models as subclasses.[5] A **generalized indivisible good allocation problem** is a list $(N, \omega, \mathcal{A}^f, R)$. Here N is the (nonempty) finite set of **individuals**. A **coalition** is a nonempty subset of N. For $i \in N$, $\omega(i)$ denotes the **initial endowment** of individual i. Assume that $\omega(i)$ is a finite set. For $S \subset N$, denote by $\omega(S)$ the set $\bigcup\{\omega(i)|i \in S\}$.[6] $\mathcal{A}^f$ is the set of **feasible allocations**. $\mathcal{A}^f$ is a nonempty subset of the set of all allocations $\mathcal{A}^0 := \{x : N \rightarrow\rightarrow \omega(N) \mid \forall a \in \omega(N) : \#\{i \in N | a \in x(i)\} = 1\}$. Assume $\omega \in \mathcal{A}^f$. $R = (R^i)_{i \in N}$ is a **preference profile**. Here for each $i \in N$, R^i is assumed to be a weak order on $\mathcal{A}^f$.[7]

xR^iy reads that to individual i, x is at least as good as y. As usual, I^i and P^i respectively denote the symmetric ('indifferent') and asymmetric ('strictly prefers') parts of R^i. For each individual i, $\mathcal{D}^i$ denotes the (nonempty) set of **admissible preferences** of i. For a coalition S, $\mathcal{D}^S$ denotes the Cartesian product $\prod_{i \in S} \mathcal{D}^i$. Then $\mathcal{D}^N$ represents the **domain of preferences**. One important example of preference domains is the domain $\mathcal{P}$ defined as follows: For each $i \in N$, let $\mathcal{P}^i$ denote the set of all preference relations in which individual i has strict preferences over his own assignments (i.e. $xI^iy \Rightarrow x(i) = y(i)$) and there is no consumption externality (i.e. $x(i) = y(i) \Rightarrow xI^iy$). Denote by $\mathcal{P}$ the Cartesian product $\prod_{i \in N} \mathcal{P}^i$. In this paper, whenever I refer to the **marriage problems** or the **housing markets**, the domain $\mathcal{P}$ is assumed.

Let a list $(N, \omega, \mathcal{A}^f)$ be given. Let $\mathcal{G}$ be a (nonempty) class of coalitions. Fix a preference profile R. Let $x, y \in \mathcal{A}^f$ and S be a coalition. Say that x **weakly dominates** y via S under R if

$$x(S) = \omega(S)\&[\forall i \in S : xR^iy]\&[\exists j \in S : xP^jy].$$

The **$\mathcal{G}$-strong core** is the set of allocations which are not weakly dominated by any other allocation via any coalition belonging to $\mathcal{G}$. The **$\mathcal{G}$-strong core correspondence** on $\mathcal{D}^N$ is the correspondence $\mathcal{C}^{\mathcal{G}} : \mathcal{D}^N \rightarrow\rightarrow \mathcal{A}^f$ such that for each $R \in \mathcal{D}^N$, $\mathcal{C}^{\mathcal{G}}(R)$ is the $\mathcal{G}$-strong core of the problem $(N, \omega, \mathcal{A}^f, R)$. Say that x **strongly dominates** y via S under R if

$$x(S) = \omega(S)\&[\forall i \in S : xP^iy].$$

The **$\mathcal{G}$-weak core** is the set of allocations which are not strongly dominated by any other allocation via any coalition belonging to $\mathcal{G}$. Denote the **$\mathcal{G}$-weak core correspondence** by $\mathcal{C}^{W\mathcal{G}}$.[8] If $\mathcal{G} = 2^N \setminus \{\emptyset\}$, then the $\mathcal{G}$-strong core ($\mathcal{G}$-weak core, respectively) is simply called the **strong core** (**weak core**, respectively). Denote the strong core correspondence by $\mathcal{C}$, and the weak core correspondence by $\mathcal{C}^W$.

An allocation x is **Pareto optimal** under R if there is no allocation which weakly dominates x via N. And an allocation x is **individually rational** under

[5] Sönmez (1999) discussed that six preceding models are included as subclasses.

[6] Throughout the paper, inclusion '$\subset$' is weak.

[7] A weak order is a complete and transitive binary relation.

[8] In some literature, weak core is simply called 'core.' And in others, strong core is referred to as 'core.' To avoid unnecessary confusion, here I do not use the term 'core.'

R if $\forall i \in N : xR^i\omega$. Denote by $\mathcal{I}$ the correspondence that chooses the set of individually rational allocations for every preference profile. Any allocation belonging to the strong core is both individually rational and Pareto optimal. Note that a weak core allocation is individually rational but not necessarily Pareto optimal.

A **solution** is a function $\varphi : \mathcal{D}^N \longrightarrow \mathcal{A}^f$. Say φ is **Pareto optimal** if for any $R \in \mathcal{D}^N$, the allocation $\varphi(R)$ is Pareto optimal under R. Call φ **individually rational** if for any $R \in \mathcal{D}^N$, the allocation $\varphi(R)$ is individually rational under R.

1.2 Preference revelation games

I introduce preference revelation games and some related concepts. Let a list $(N, \omega, \mathcal{A}^f, \mathcal{D}^N)$ be given. Let φ be a solution with the preference domain $\mathcal{D}^N$. A **direct mechanism** induced by φ is a pair $(\mathcal{D}^N, \varphi)$, where each $\mathcal{D}^i$ is interpreted as the **strategy space** of individual i, and $\varphi : \mathcal{D}^N \longrightarrow \mathcal{A}^f$ as the **outcome function**. Given a direct mechanism $(\mathcal{D}^N, \varphi)$, a preference profile $R \in \mathcal{D}^N$ defines the strategic game $(\mathcal{D}^N, \varphi, R)$. This game is called a **preference revelation game** induced by φ.

Let $(\mathcal{D}^N, \varphi, R)$ be a preference revelation game. Then a strategy profile $R^\star \in \mathcal{D}^N$ is a **$\mathcal{G}$-proof Nash equilibrium** if $\forall S \in \mathcal{G} : \forall R'^S \in \mathcal{D}^S$:

$$[\forall i \in S : \varphi(R^{\star -S}, R'^S)R^i\varphi(R^\star)] \Rightarrow [\exists j \in S : \varphi(R^{\star -S}, R'^S)I^j\varphi(R^\star)].$$

Denote by $N^{\mathcal{G}}(\mathcal{D}^N, \varphi, R)$ the set of $\mathcal{G}$-proof Nash equilibria. And call $R^\star \in \mathcal{D}^N$ a **strict $\mathcal{G}$-proof Nash equilibrium** if $\forall S \in \mathcal{G} : \forall R'^S \in \mathcal{D}^S$:

$$[\forall i \in S : \varphi(R^{\star -S}, R'^S)R^i\varphi(R^\star)] \Rightarrow [\forall i \in S : \varphi(R^{\star -S}, R'^S)I^i\varphi(R^\star)].$$

Denote by $sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R)$ the set of strict $\mathcal{G}$-proof Nash equilibria. If $\mathcal{G} = 2^N \setminus \{\emptyset\}$, then a $\mathcal{G}$-proof Nash equilibrium (strict $\mathcal{G}$-proof Nash equilibirum, respectively) is called a **strong Nash equilibrium** (**strict strong Nash equilibrium**, respectively). Denote the set of strong Nash equilibrium by $SN(\mathcal{D}^N, \varphi, R)$, and the set of strict strong Nash equilibrium by $SSN(\mathcal{D}^N, \varphi, R)$. If $\mathcal{G} = \{\{i\} \mid i \in N\}$, then a (strict) $\mathcal{G}$-proof Nash equilibirum is called a **Nash equilibrium**. Denote the set of Nash equilibrium by $Nash(\mathcal{D}^N, \varphi, R)$.

Let a direct mechanism $(\mathcal{D}^N, \varphi)$ be given. An **equilibrium concept** is a correspondence Eq which chooses a subset of the set of strategy profiles $\mathcal{D}^N$ for each strategic game $(\mathcal{D}^N, \varphi, R)$ with $R \in \mathcal{D}^N$, i.e. $Eq(\mathcal{D}^N, \varphi, R) \subset \mathcal{D}^N$. For example, Nash equilibrium is an equilibrium concept. Let $\Phi : \mathcal{D}^N \rightarrow\rightarrow \mathcal{A}^f$ be a nonempty-valued correspondence. Then say that the direct mechanism $(\mathcal{D}^N, \varphi)$ **weakly implements** the correspondence Φ in an equilibrium concept Eq if $\forall R \in \mathcal{D}^N : \varphi(Eq(\mathcal{D}^N, \varphi, R)) \subset \Phi(R)$ and $Eq(\mathcal{D}^N, \varphi, R) \neq \emptyset$. And say that the direct mechanism $(\mathcal{D}^N, \varphi)$ **fully implements** (or **implements**) the correspondence Φ in an equilibrium concept Eq if $\forall R \in \mathcal{D}^N : \Phi(R) = \varphi(Eq(\mathcal{D}^N, \varphi, R))$.

2 Main results

In the following, let the list $(N, \omega, \mathcal{A}^f, \mathcal{D}^N)$ be given. I require that in all the results which follow, Conditions A and B in the below are satisfied.

Let $A \subset \mathcal{A}^f$ and $R^i \in \mathcal{D}^i$. Let $\operatorname{top} R^i(A)$ denote the set $\{x \in A \mid \forall y \in A : xR^iy\}$.

Condition A: For each $i \in N$, $\mathcal{D}^i$ satisfies the following:

(i) There is no consumption externality, i.e. $\forall R^i \in \mathcal{D}^i : \forall x, y \in \mathcal{A}^f : x(i) = y(i) \Rightarrow xI^iy$.

(ii) For any $x \in \mathcal{A}^f$, there is some $R^i \in \mathcal{D}^i$ for which $\operatorname{top} R^i(\mathcal{A}^f) = \{z \mid z(i) = x(i)\}$, and $\omega(i) \neq x(i) \Rightarrow \operatorname{top} R^i(\mathcal{A}^f \setminus \operatorname{top} R^i(\mathcal{A}^f)) = \{z \mid z(i) = \omega(i)\}$.

Condition B: The set $\mathcal{A}^f$ satisfies the following: Let S be a coalition, and $x, y \in \mathcal{A}^f$. Then if $x(S) = \omega(S) = y(S)$ (which implies $x(N \setminus S) = \omega(N \setminus S) = y(N \setminus S)$), then $\mathcal{A}^f$ contains the allocation v such that $[\forall i \in S : v(i) = x(i)]$ and $[\forall i \in N \setminus S : v(i) = y(i)]$.

Condition A requires some "regularity" of the domain. And Condition B requires some "richness" of the feasible allocations. Note that the domain $\mathcal{P}$ defined in Section 1.1 satisfies Condition A. Also Condition B is satisfied in the **marriage problems** (Gale and Shapley, 1964) and the **housing markets** (Shapley and Scarf, 1974)

I present the main results. Firstly, I provide a lemma which asserts that, roughly speaking, the 'blocking power' of a coalition in the allocation problem implies their 'deviation power' in the induced preference revelation game.

Lemma 2.1: *Let φ be a solution that is individually rational and Pareto optimal. Let S be a coalition, $x \in \mathcal{A}^f$, and $R, R' \in \mathcal{D}^N$. Then if x weakly dominates the allocation $\varphi(R')$ via S under R, then there is some $R^{\star S} \in \mathcal{D}^S$ such that $[\forall i \in S : \varphi(R'^{-S}, R^{\star S})R^i\varphi(R')]\&[\exists j \in S : \varphi(R'^{-S}, R^{\star S})P^j\varphi(R')]$.*

Proof: Let S be a coalition, $x \in \mathcal{A}^f$, and $R, R' \in \mathcal{D}^N$. Assume x weakly dominates the allocation $\varphi(R')$ via S under R. Then choose $R^{\star S} \in \mathcal{D}^S$ satisfying for each $i \in S$, $\operatorname{top} R^{\star i}(\mathcal{A}^f) = \{z \mid z(i) = x(i)\}$, and $\omega(i) \neq x(i) \Rightarrow \operatorname{top} R^{\star i}(\mathcal{A}^f \setminus \operatorname{top} R^{\star i}(\mathcal{A}^f)) = \{z \mid z(i) = \omega(i)\}$. Such $R^{\star i}$ exists in $\mathcal{D}^i$ by Condition A. Since φ is individually rational, for each $i \in S$, $\varphi(R'^{-S}, R^{\star S})(i)$ equals $x(i)$ or $\omega(i)$. Denote $\varphi(R'^{-S}, R^{\star S})$ by y. Now I will show $\forall i \in S : x(i) = y(i)$. Suppose that for some $i \in S$, $x(i) \neq y(i)$ (which implies $y(i) = \omega(i) \neq x(i)$). Since x weakly dominates $\varphi(R')$ via S, it holds $x(S) = \omega(S)$. Then consider an allocation v which satisfies $[\forall i \in S : v(i) = x(i)]$ and $[\forall i \in N \setminus S : v(i) = y(i)]$. Condition B ensures $\mathcal{A}^f$ contains this allocation v. Then v Pareto-dominates y. But this contradicts the Pareto optimality of φ. Therefore, I have $\forall i \in S : x(i) = y(i)$. Recall that x weakly dominates $\varphi(R')$ via S under R. Then it follows $[\forall i \in S : yR^i\varphi(R')]\&[\exists j \in S : yP^j\varphi(R')]$. □

The following is immediate from Lemma 2.1.

Corollary 2.2: *Let φ be a solution that is individually rational and Pareto optimal. Then* $\forall R \in \mathcal{D}^N : \varphi(sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R)) \subset \mathcal{C}^{\mathcal{G}}(R)$.

Proof: Let $R \in \mathcal{D}^N$. Assume $x \notin \mathcal{C}^{\mathcal{G}}(R)$. Then there is some $y \in \mathcal{A}^f$ which weakly dominates x via some coalition $S \in \mathcal{G}$ under R. Pick up $R' \in \mathcal{D}^N$ be such that $\varphi(R') = x$. (If the image of φ does not contain x, then immediately $x \notin \varphi(sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R))$.) Then Lemma 2.1 implies that there is some $R^{\star S} \in \mathcal{D}^S$ such that $[\forall i \in S : \varphi(R'^{-S}, R^{\star S})R^i\varphi(R')]\&[\exists j \in S : \varphi(R'^{-S}, R^{\star S})P^j\varphi(R')]$. Since this is true for any $R' \in \mathcal{D}^N$ with $\varphi(R') = x$, I have $x \notin \varphi(sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R))$. □

Next I turn to the reverse direction of inclusion. Let us call $\mathcal{G}$ **monotonic** if $[S \in \mathcal{G}, S \subset T] \Rightarrow T \in \mathcal{G}$.

Lemma 2.3: *Let φ be a solution that is individually rational and Pareto optimal. Let $\mathcal{G}$ be a monotonic class of coalitions. Then* $\forall R \in \mathcal{D}^N : \mathcal{C}^{\mathcal{G}}(R) \subset \varphi(sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R))$.

Proof: Let $R \in \mathcal{D}^N$. And let $x \in \mathcal{C}^{\mathcal{G}}(R)$. Choose $R' \in \mathcal{D}^N$ such that for any $i \in N$, $\text{top}\, R'^i(\mathcal{A}^f) = \{z \mid z(i) = x(i)\}$, and $\omega(i) \neq x(i) \Rightarrow \text{top}\, R'^i(\mathcal{A}^f \setminus \mathit{top}\, R'^i(\mathcal{A}^f)) = \{z \mid z(i) = \omega(i)\}$. Then since φ is Pareto optimal, $\varphi(R') = x$. Now I will show $R' \in sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R))$ (thus $x \in \varphi(sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R)))$. Suppose not. Then there are some coalition $S \in \mathcal{G}$ and $R^{\star S} \in \mathcal{D}^S$ such that

$$[\forall i \in S : \varphi(R'^{-S}, R^{\star S})R^i\varphi(R')]\&[\exists j \in S : \varphi(R'^{-S}, R^{\star S})P^j\varphi(R')]. \tag{1}$$

Denote $\varphi(R'^{-S}, R^{\star S})$ by y. Then if $y(S) = \omega(S)$, then (1) implies y weakly dominates x via S under R, which says $x \notin \mathcal{C}^{\mathcal{G}}(R)$, a contradiction. Thus it must be $y(S) \neq \omega(S)$. Now let T denote an $\subset$-minimal element of the class of coalitions $\{G \subset N \mid S \subset G\&y(G) = \omega(G)\}$. Thus for any $i \in T \setminus S$, $y(i) \neq \omega(i)$. (Otherwise, T is not $\subset$-minimal.) Then since φ is individually rational, the construction of R' implies $\forall i \in T \setminus S : x(i) = y(i)$, which implies xI^iy. This and (1) together say $[\forall i \in T : yR^ix]$ and $[\exists j \in S \subset T : yP^jx]$, and by assumption $y(T) = \omega(T)$, i.e. y weakly dominates x via S under R. Since $\mathcal{G}$ is monotonic, $T \in \mathcal{G}$. Thus I have $x \notin \mathcal{C}^{\mathcal{G}}(R)$, a contradiction. This concludes $R' \in sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R)$. □

In Lemma 2.3 above, the monotonicity assumption of $\mathcal{G}$ is essential. When one considers the marriage problems **only**, on the analogy of the result of Sönmez (1997) (mentioned in Introdcution, also stated formally in Section 3), one may suspect that the above result still holds true without the monotonicity of $\mathcal{G}$. But this is not the case. One has the following counter-example.

Example 2.4: Let $M = \{1, 2\}$ and $W = \{3, 4\}$. Consider the case where $N = M \cup W$; $\forall i \in N, \omega(i) = \{i\}$; and

$$\mathcal{A}^f = \{x \in \mathcal{A}^0 \mid \forall i \in N, \sharp x(i) = 1;$$

$$\forall m \in M, x(m) \subset W \cup \{m\};$$
$$\forall w \in W; x(w) \subset M \cup \{w\};$$
$$\forall i, j \in N, x(i) = j \Leftrightarrow x(j) = i\}.$$

This is the **marriage problem** (Gale and Shapley, 1962) with two men and two women. Assume the domain $\mathcal{D}^N$ equals $\mathcal{P}$.[9] Let φ be a stable solution. Let $\mathcal{G}$ be $\{\{i\} \mid i \in N\} \cup \{\{1,2,3\}\}$. Note that $\mathcal{G}$ is not monotonic. The following gives one example of $R \in \mathcal{D}^N$ such that $\mathcal{C}^{\mathcal{G}}(R) \not\subset \varphi(sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R))$.

Let $R \in \mathcal{P}$ be the profile such that

$$3P^1 1P^1 4,$$
$$4P^2 2P^2 3,$$
$$1P^5 3P^3 2,$$
$$2P^4 4P^4 1.$$

Consider the matching $x \in \mathcal{A}^f$ such that

$$x(1) = 1,$$
$$x(2) = 4,$$
$$x(3) = 3.$$

It is easy to check $x \in \mathcal{C}^{\mathcal{G}}$. But $x \notin \varphi(sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R))$. The proof is as follows: Suppose that there is some $R' \in \mathcal{D}^N$ such that $x = \varphi(R')$ and $R' \in sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R)$. Consider the profile $(R^{\{1,2,3\}}, R'^4)$. Under this profile, there is the unique stable matching y such that

$$y(1) = 3,$$
$$y(2) = 4.$$

(This is because: (1) It is clear that 1 and 3 have to be matched each other under $(R^{\{1,2,3\}}, R'^4)$. (2) If 4 were not matched with 2 under $(R^{\{1,2,3\}}, R'^4)$, then 4 would have to be matched with himself. But this implies that 4 would have been matched with himself under R'.) Then $y = \varphi(R^{\{1,2,3\}}, R'^4)$. Note that yP^1x, yP^3x and yI^2x. Thus $R' \notin sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R)$. This completes the proof.

Now it follows $\mathcal{C}^{\mathcal{G}}(R) \not\subset \varphi(sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R))$. □

From Corollary 2.2 and Lemma 2.3, I conclude the following.

Theorem 2.5: *Let φ be a solution that is individually rational and Pareto optimal. Let $\mathcal{G}$ be a monotonic class of coalitions. Then $\forall R \in \mathcal{D}^N : \mathcal{C}^{\mathcal{G}}(R) = \varphi(sN^{\mathcal{G}}(\mathcal{D}^N, \varphi, R))$.*

Under the additional assumption of the nonempty-valuedness of $\mathcal{C}^{\mathcal{G}}$, this theorem says that the direct mechanism $(\mathcal{D}^N, \varphi)$ implements the correspondence $\mathcal{C}^{\mathcal{G}}$ in strict $\mathcal{G}$-proof Nash equilibrium.

[9] In this case, a preference relation on the feasible allocations (matchings) is identified with the corresponding relation on men (or women) plus herself (or himself).

In considering $\mathcal{G}$, in view of decentralized behavior of individuals, it should be assumed that $\forall i \in N, \{i\} \in \mathcal{G}$. This and the monotonicty of $\mathcal{G}$ together imply that $\mathcal{G}$ equals the set of all coaltions. Then Theorem 2.4 reduces to the following result.

Corollary 2.6: *Let φ be a solution that is individually rational and Pareto optimal. Then $\forall R \in \mathcal{D}^N : \mathcal{C}(R) = \varphi(SSN(\mathcal{D}^N, \varphi, R))$.*

Under the additional assumption of the nonempty-valuedness of $\mathcal{C}$, this corollary says that the direct mechanism $(\mathcal{D}^N, \varphi)$ implements the correspondence $\mathcal{C}$ in strict strong Nash equilibrium.

Shin and Suh (1996) proved that in the context of the marriage problems, the direct mechanism induced by any stable solution implements the strong core correspondence in strict strong Nash equilibrium. Corollary 2.6 strengthens this result by extending it to the general setting, and enlarging the set of solutions inducing the mechanism.

Shin and Suh (1996) also pointed out that in the above result of the marriage problems, strict strong Nash equilibrium can be replaced with strong Nash equilibrium. Does the same thing hold true in the present general model? The answer is negative. There is a counter-example: In the case of the "housing market" (Shapley and Scarf, 1974) with three traders, the direct mechanism induced by a strong core selection (that is trivially individually rational and Pareto optimal) does not even weakly implement the strong core correspondence in strong Nash equilibrium. Of course, neither does in Nash equilibirum since any strong Nash equilibrium is a Nash equilibrium.

Example 2.7: Consider the case where $N = \{1, 2, 3\}$; $\forall i \in N : \#\omega(i) = 1$; and $\mathcal{A}^f = \{z \in \mathcal{A}^0 \mid \forall i \in N : \#z(i) = 1\}$. This is the **housing market** (Shapley and Scarf, 1974) with three traders. Assume the domain $\mathcal{D}^N$ equals $\mathcal{P}$.[10] In this case, the strong core correspondence is single-valued on the domain $\mathcal{P}$ by the result of Roth and Postlewaite (1977). Denote $\omega(i)$ simply by i. Let φ be the strong core solution.[11] I give one example of $R \in \mathcal{P}$ such that $\varphi(SN(\mathcal{P}, \varphi, R)) \not\subset \mathcal{C}(R)$: Let $R \in \mathcal{P}$ be the profile such that

$$2P^1 1 P^1 3,$$
$$3P^2 1 P^2 2,$$
$$1P^3 3 P^3 2.$$

Note that $\varphi(R) = (2, 3, 1)$. Say $x = (2, 1, 3)(\neq \varphi(R))$. And choose another profile $R' \in \mathcal{P}$ such that

$$3P'^1 2 P'^1 1,$$
$$1P'^2 2 P'^2 3,$$
$$3P'^3 1 P'^3 2.$$

[10] In this case, a preference relation on $\mathcal{A}^f$ is identified with the corresponding relation defined on $\omega(N)$.

[11] Here and in the sequel, abusing notation, for an allocation x, x denotes the triple $(x(1), x(2), x(3))$.

Then $\varphi(R') = x$. And it is easy to check $R' \in SN(\mathcal{P}, \varphi, R)$. Thus I have $x \in \varphi(SN(\mathcal{P}, \varphi, R))$ and $x \notin \mathcal{C}(R)$, i.e. $\varphi(SN(\mathcal{P}, \varphi, R)) \not\subset \mathcal{C}(R)$. (Also, this implies $\varphi(Nash(\mathcal{P}, \varphi, R)) \not\subset \mathcal{C}(R)$.) $\square$

3 Additional results

Given the implementability results obtained in the previous section, this section examines which part of the preceding results in the context of the marriage problems are to be extended to the present model. Recall those results I have discussed in Introduction (Alcalde, 1996; Shin and Suh, 1996, Sönmez, 1997).[12]

(1) Alcalde: *If φ is a stable solution, then* $\forall R \in \mathcal{P} : \mathcal{I}(R) = \varphi(Nash(\mathcal{P}, \varphi, R))$.

(2) Shin and Suh: *If φ is a stable solution, then* $\forall R \in \mathcal{P} : \mathcal{C}^W(R) = \varphi(SN(\mathcal{P}, \varphi, R))$.

(3) Sönmez: *If φ is individually rational and Pareto optimal, then* $\forall R \in \mathcal{P} : \mathcal{C}^{W\mathcal{G}}(R) = \varphi(N^{\mathcal{G}}(\mathcal{P}, \varphi, R))$.

(4) Shin and Suh: *If φ is a stable solution, then* $\forall R \in \mathcal{P} : \mathcal{C}(R) = \varphi(SSN(\mathcal{P}, \varphi, R))$.

Note that (3) implies (1) and (2). However, (3) does not imply (4). As I have shown in the previous section, (4) is generalized to the general model. Then what about the other three results?

In the following, I show that only a 'half' of (4) (thus (1) and (2), too) is extended to the general model. That is, it holds true that $\forall R \in \mathcal{P} : \varphi(N^{\mathcal{G}}(\mathcal{P}, \varphi, R)) \subset \mathcal{C}^{W\mathcal{G}}(R)$. However, none of these results is generalized fully.

The following is a lemma analogous to Lemma 2.1 above.

Lemma 3.8: *Let φ be a solution that is individually rational and Pareto optimal. Let S be a coalition, $x \in \mathcal{A}^f$, and $R, R' \in \mathcal{D}^N$. Then if x strongly dominates the allocation $\varphi(R')$ via S under R, then there is some $R^{\star S} \in \mathcal{D}^S$ such that $\forall i \in S : \varphi(R'^{-S}, R^{\star S}) P^i \varphi(R')$.*

Lemma 3.8 will be proved along the same line to Lemma 2.1. So I omit the proof. Just like Corollary 2.2 follows from Lemma 2.1, one obtains the following Corollary 3.9 from Lemma 3.8

Corollary 3.9: *Let φ be a solution that is individually rational and Pareto optimal. Then* $\forall R \in \mathcal{D}^N : \varphi(N^{\mathcal{G}}(\mathcal{D}^N, \varphi, R)) \subset \mathcal{C}^{W\mathcal{G}}(R)$.

Turning to the reverse inclusion, I argue by the counter-example in the below (Example 3.10) that not only (3) but neither (1) nor (2) is to be fully

[12] Here I have dropped the result by Ma (1995). This employs rematching-proof equilibrium, which requires the "two-sided" structure that is specific to the marriage problems, and is not applicable to the general setting.

generalized. That is,

Example 3.10: Consider the housing market with three traders as defined in Example 2.6. Let φ be the strong core solution. I give one example of $R \in \mathcal{P}$ such that $\mathcal{C}^W(R) \not\subset \varphi(Nash(\mathcal{P}, \varphi, R))$: Choose $R \in \mathcal{P}$ such that

$$2P^1 1P^1 3,$$
$$1P^2 3P^2 2,$$
$$1P^3 3P^3 2.$$

Pick up $x = (2, 3, 1) \in \mathcal{C}^W(R)$. Let $R' \in \mathcal{P}$ be any profile such that $x = \varphi(R')$. Then it is easy to see that such R' satisfies $\forall i \in N : x(i)$ ranks top in R'^i. Now choose any $R^{\star 2} \in \mathcal{P}^2$ which ranks the good 1 as the best. Then $\varphi(R'^{-2}, R^{\star 2}) = (2, 1, 3)$. Now clearly, $\varphi(R'^{-2}, R^{\star 2})(2) P^2 \varphi(R')(2)$. This says that any such R' cannot be a Nash equilibrium of the game $(\mathcal{P}, \varphi, R)$. Thus I have $x \in \mathcal{C}^W(R)$ and $x \notin \varphi(Nash(\mathcal{P}, \varphi, R))$, i.e. $\mathcal{C}^W(R) \not\subset \varphi(Nash(\mathcal{P}, \varphi, R))$. This implies $\mathcal{I}(R) \not\subset \varphi(Nash(\mathcal{P}, \varphi, R))$ and $\mathcal{C}^W(R) \not\subset \varphi(SN(\mathcal{P}, \varphi, R))$. □

References

[1] Alcalde J (1996): Implementation of stable solutions to the marriage problem. *J Econ Theory* 69: pp1–18.

[2] Gale D and Shapley L(1962): College admissions and the stability of marriage. *Amer Math Monthly* 69: pp9–15

[3] Hurwicz L (1978): On the interaction between information and incentives in organizations In: Krippendorff K (ed): *Communication and Control in Society.* Scientific Publishers. New York. pp123–47.

[4] Kalai E, Postlewaite A and Roberts J (1979): A group incentive compatible mechanism yielding core allocations. *J Econ Theory* 20: pp13–22

[5] Ma J (1995): Stable matchings and rematching-proof equilibria in a two-sided matching market. *J Econ Theory* 66: pp252–69.

[6] Otani Y and Sicilian (1982): Equilibrium of Warlas preference games. *J Econ Theory* 27: pp47–68.

[7] Roth A and Sotomayor M (1990): *Two-Sided Matching: A Study in Game Theoretic Modeling and Analysis.* Cambridge University Press. London/New York.

[8] Roth A and Postlewaite A (1977): Weak versus strong domination in a market with indivisible goods. *J Math Econ* 4: pp131–37.

[9] Shapley L and Scarf H (1974): On cores and indivisibility. *J Math Econ* 1: pp23–37.

[10] Shin S and Suh S-C (1996): A Mechanism implementing the stable rule in marriage problems. *Econ Letters* 51: pp185–9.

[11] Sönmez T (1999): Strategy-proofness and essentially single-valued cores. *Econometrica* 67: pp677–89.

[12] Sönmez T (1997): Games of manipulation in marriage problems. *Games Econ Behavior* 20: pp169–76.

[13] Takamiya K (2003) On strategy-proofness and essentially single-valued cores: A converse result. *Soc Choice Welfare* 20: pp77–83.

[14] Thomson W (1984): The manipulability of resource allocation mechanisms. *Rev Econ Studies* 51: pp447–60.

[15] Thomson W (1988): The manipulability of the Shapley value. *Int J Game Theory* 17: pp101–27.