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A VIABILITY THEORY ANALYSIS OF A SIMPLE MACROECONOMIC MODEL

JACEK B. KRAWCZYK AND KUNHONG KIM

Abstract. Herbert A. Simon, 1978 Economics Nobel Prize laureate, talked about *satisficing* (his neologism) rather than *optimising* as being what the economists really need. We think that it is *viability theory*, which is a relatively young area of mathematics, that rigorously captures the essence of *satisficing*. We aim to use viability analysis to analyse a simple macroeconomic model and show how some adjustment rules can be endogenously obtained.

Keywords: Macroeconomic modelling; dynamic systems; viability theory
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1. Introduction

The aim of this paper is to explore the usefulness of viability theory for macroeconomic modelling.

Herbert A. Simon, 1978 Economics Nobel Prize laureate, talked about *satisficing* (his neologism) rather than *optimising* as being what the economists really need. We think that economic theory, which follows the Simon prescription, brings modelling closer to how people actually behave. We also think that it is *viability theory*, which is a relatively young area of continuous mathematics (see [1] and [2]), that rigorously captures the essence of *satisficing*. Therefore, viability theory appears to be an appropriate tool of achieving a *satisficing* solution to many economic problems. We aim to demonstrate this by solving a stylised Central Bank macroeconomic problem. The solution will enable us to analyse the system transition trajectory around a steady state (rather than towards the steady state). We believe that an evolutionary analysis enabled by viability analysis gives us a better insight into the system economics than just an equilibrium analysis. In particular, we hope to contribute to the discussion on how to avoid a *liquidity trap* (for an analysis of a liquidity trap problem performed through an established method see [6]).

In the next section, we provide a brief introduction to viability theory and, in Section 3, we apply it to a simple macroeconomic model[1]. The paper ends with concluding remarks.

2. What is viability theory?

Viability theory is an area of mathematics concerned with *viable evolution* of controlled dynamic systems. A system evolution is considered *viable* if the system trajectory remains within a prescribed region of the phase space, the so-called *viability domain*. The domain boundaries represent some normative constraints, which the system should satisfy for as long as the evolution is concerned. The basic problem that viability theory attempts to solve is whether a control strategy exists that prevents the system from leaving the viability domain. The *viability kernel* for such a problem is the set of all initial conditions, for which such a strategy exists.

Consider a dynamic economic system with several stock (or state) variables. At time $t \in \Theta$, where interval Θ can be finite or infinite, the state variables are

$$[x_1(t), x_2(t), \ldots x_N(t)]' \equiv x(t) \in X \equiv \mathcal{R}^N$$

and the instrument flows (or actions) are

$$[i_1(t), i_2(t), \ldots i_M(t)]' \equiv i(t) \in \mathcal{I} \subset \mathcal{R}^M.$$

Symbol $\mathcal{I}$ represents an instrument set (or action set) that could be split into two (or more) parts if there were two (or more) players who would decide upon the actions.

The state evolves according to the system dynamics $f(\cdot, \cdot)$ and instruments $i(t)$ as follows

$$\dot{x}(t) = f(x(t), i(t)) \qquad t \in \Theta, \quad i(t) \in \mathcal{I}, \quad x(t) \in X. \tag{1}$$

Evidently, at every state $x(t)$, the system *velocity* $\dot{x}(t)$ depends on action $i(t)$. We may say that the velocity is governed by the *set-valued* map (or correspondence)

$$F(x) \equiv \{f(x, i), i \in \mathcal{I}\} \quad \forall x \in X \tag{2}$$

where, for convenience, we dropped the time argument from notation. Combining the above formulae, the system dynamics can be rewritten in form of a *differential inclusion*:

$$\dot{x}(t) \in F(x(t)), \quad \text{for almost all} \quad t \in \Theta, \tag{3}$$

which determines the speed of the system variables given the instrument set $\mathcal{I}$.

In economic terms, the last relationship tells us that at time t, for a given composition of x (capital, labour, technology, *etc.*), the extent of growth (or decline), or steady state stability, are all dependent on the map $F : X \leadsto X$ whose values are limited by the scope of instruments contained in $\mathcal{I}$. Viability theory studies properties of differential inclusions (3) to say when the systems trajectories $x(t), t \in \Theta$ evolve *viably*[2] in the sense of the following definitions.

We will now say what we understand as *viable evolution*.

[1]For a viability theory application to environmental economics see [3]; also, see [8] for a viability analysis of an endogenous business cycle.

[2]This is possible to infer from the mathematical properties of F, see [1], [2].

Definition 2.1. *Let a viability domain be given in form of a closed and nonempty set of state constraints $K \subset X$. Evolution of dynamic system (3), (2) is* **viable in** *K if and only if, from any initial state x_0, at least one trajectory $x(t)$ starts such that $x(t) \in K, t \in \Theta$.*

So, this definition speaks of systems, which remain in K without any control. However, there are few real life dynamic systems that would be like that. A more realistic notion is that of *viability kernel* that specifically allows for an instrument set $\mathcal{I}$.

Definition 2.2. *The* **viability kernel** *of the constraint set K for the instrument set $\mathcal{I}$ is the set of initial conditions $x_0 \in K$ defined as the set V_F^K:*

(4) $$V_F^K \equiv \{x_0 \in K : \exists x(t) \in K, t \in \Theta \text{ while } x(t) \text{ solves } (3), (2) \text{ and } x(0) = x_0\}\,.$$

In other words, if a trajectory begins at the viability kernel then we know that it will forever remain in the constraint set.

Figure 1 illustrates the viability idea.

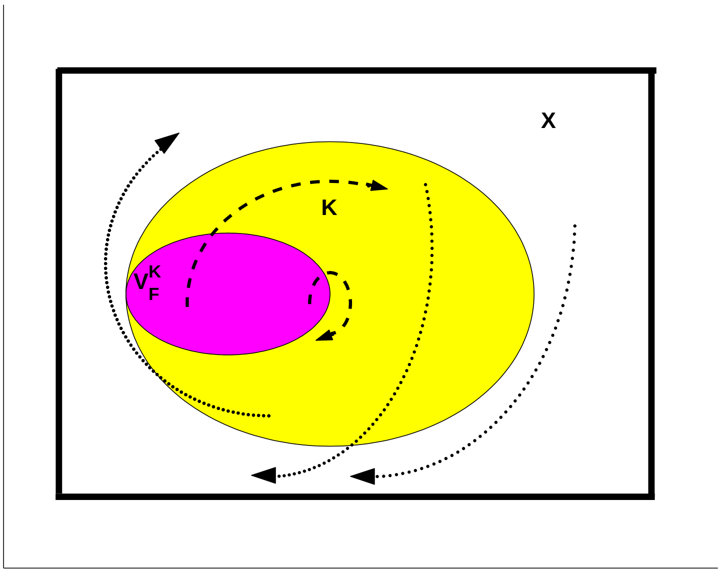

FIGURE 1. The viable and non viable trajectories.

The state constraints are represented by the yellow (or light shadowed) oval contour K contained in state space X. The dotted and dashed lines symbolise system evolution, which converges to where an arrows ends.

The viability kernel for the constraint set K, given instruments from a set $\mathcal{I}$, is the purple (darker) shadowed contour denoted V_I^K. The trajectories that start at the kernel (*i.e.*, the dashed lines) are *viable* in K *i.e.*, they remain in K. This is not the property of the other trajectories (dotted lines) that start outside the kernel. They may pass through K but do not remain there.

3. A MACROECONOMIC MODEL

3.1. **A viability theory problem.** Realistically, what a typical Central Bank wants to achieve is the maintenance of a few key macroeconomic variables within some bounds. Usually, the bank realises its multiple targets using *optimising* solutions that result from minimisation of the bank's loss function. Typically, the loss function includes penalties for violating an allowable inflation band and also for a non-smooth interest adjustment. The solution, which minimises the loss function, is unique for a given selection of parameter and the loss function values. In that, it does not allow for alternative strategies.

Our intention is to apply viability theory to a bank's problem, which is to keep variables of interest in a constrained set. This sounds very much like the viability theory problem illustrated in Figure 1. We will try to establish what the set of economy states is (*i.e.*, what the kernel V_F^K is) for a Central Bank to keep the economy's evolution viable (*i.e.*, such that the key variables remain within some prescribed bounds K) given available instruments.

In the next section we will use a stylised monetary rules model (inspired be [9]). We will then show that the solutions obtained through viability theory do not suffer from drawbacks typical of their optimising counterparts.

3.2. **A Central Bank problem.** Suppose a Central Bank is using nominal interest rate $i(t)$ as an instrumental variable to control inflation $\pi(t)$ and, to a lesser extent, output gap $y(t)$. A model that relates these variables may look like follows (see [9], p. 508 where the time step $h = 1$):

$$y_t = a_1 y_{t-h} + a_2 y_{t-2h} - a_3(i_{t-h} - E_{t-h}\pi_t) + u_t \tag{5}$$

$$\pi_t = \pi_{t-h} + \gamma y_t + \eta_t \tag{6}$$

where y_t is output gap, u_t, η_t are serially uncorrelated disturbances, with means equal to zero and a_1, a_2, a_3, γ are calibrated parameters.

Assume[3] $a_2 = 0$ in (5) and apply the expectation operator to both (5), (6). We could have used the full equation (5) in our study. However, this would have increased the state space dimensionality and made the viability analysis less transparent. As this paper's main purpose is to show how viability theory can be applied in macroeconomics we prefer to use a lower order system. We can therefore re-write (5), (6) as

$$y(t) = a\, y(t-h) - a_3(i(t-h) - \pi(t-h)) \tag{7}$$

$$\pi(t) = \pi(t-h) + \gamma y(t). \tag{8}$$

where we have defined $y(t) \equiv E_{t-h}y_t$ and $\pi(t) \equiv E_{t-h}\pi_t$ (obviously, $y(t-h) = E_{t-h}y_{t-h}$ and $\pi(t-h) = E_{t-h}\pi_{t-h}$). Then,

$$y(t) - y(t-h) = \alpha\, h\, y(t-h) - \xi\, h\, (i(t-h) - \pi(t-h)) \tag{9}$$

$$\pi(t) - \pi(t-h) = \zeta\, h y(t). \tag{10}$$

where $\alpha\, h = a-1, \quad \xi\, h = a_3, \quad \zeta\, h = \gamma$. Dividing each equation by h and requesting $h \to 0$ we get the *expected* inflation and output gap dynamics

$$\frac{dy}{dt} = \alpha\, y(t) - \xi\, (i(t) - \pi(t)) \tag{11}$$

$$\frac{d\pi}{dt} = \zeta\, y(t). \tag{12}$$

[3]Below, in Section 3.3, we explain how we have assigned a value to the one-lag term y_{t-h} coefficient a in (7).

This model tells us that the expected speed of inflation (12) changes proportionally to the expected output gap (however, for simplicity, we will drop "expected" from the subsequent text). Output gap, in turn, constitutes a "sticky" process driven by the difference between the interest and inflation rates.

3.3. **Parameter values.** We use the following parameter values (see [9]):

$$\xi = \frac{a_3}{h}\Big|_{h=1} = .35, \qquad \zeta = \frac{\gamma}{h}\Big|_{h=1} = .002\,.$$

Regarding α we know that $\alpha = \frac{a-1}{h}\big|_{h=1}$ but, we first need to say what the value of a is. In general, it is impossible to approximate a second order system (5) by a first order system (7). However, long term, both systems stalibilise if perturbed by a step function. We have chosen a such that, after some time, the responses of (5) and (7) are at the same level, see Figure 2.

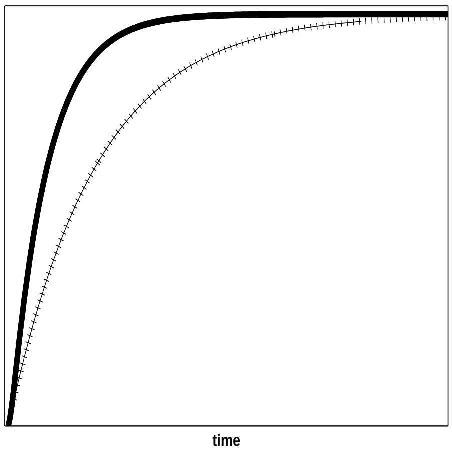

FIGURE 2. A second order and a first order system responses.

The solid line shows a step-function response of (5) with $a_1 = 1.53, a_2 = -.55$ (see [9][4]). The value of a for the dotted line is $a = .98$ [5] so, $\alpha = -.02$. Hence the macroeconomic model that we will analyse is

$$\frac{dy}{dt} = -0.02\,y(t) - 0.35\,(i(t) - \pi(t)) \tag{13}$$

$$\frac{d\pi}{dt} = 0.002\,y(t). \tag{14}$$

4. VIABLE SOLUTIONS

We will perform a viability theory analysis using model (13), (14) and for a given constraint set. This is *computational economics* and the results will be parameter specific; however, the procedure can be easily repeated for any plausible parameter selection.

[4]The parameter values come from a table published in [9], which quotes maximum likelihood estimates for a model originally studied by [4].

[5]Notice that 1.53-.55=.98 .

4.1. **The constraints.** Usually there is little doubt what the *politically* desired inflation bounds are. For example, in New Zealand, the inflation band has been legislated [.01, .03] . Less agreement is about what the desired output gap should be. We will assume a rather wide interval for output gap to reflect a lesser concern of the Central Bank for $y(t)$ (*e.g.*, $y(t) \in [-.04, .04]$). So, our viability domain (or the constraint set) K is

$$K \equiv \{(y(t), \pi(t)) \,:\, -.04 \leq y(t) \leq .04 \quad \text{and} \quad .01 \leq \pi \leq .03\}$$

see Figure 3.

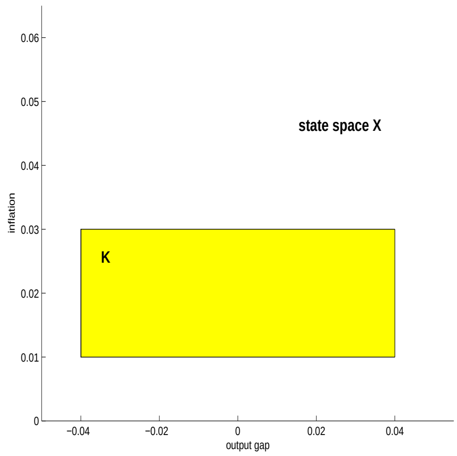

Figure 3. Constraint set K (viability domain).

Similarly to the desired size of the output gap, the instrument set composition also depends on political decisions. We will assume that $i(t) \in [0, .07]$. Independently of keeping the interest range constrained, many central banks are worried about the interest rate *smoothness*. That concern is usually modelled by adding $w(i_t - t_{t-h})^2$, $w > 0$ to the loss function. In continuous time, limiting the interest rate "velocity" $\dfrac{di}{dt}$ will produce smooth time profile $i(t)$. Bearing in mind that the central bank's announcements are usually made every quarter and that the typical change is a $\dfrac{1}{4}\%$, our instrument set $\mathcal{I}$ will be defined as

$$\mathcal{I} \equiv \left\{ i \,:\, i(t) \in [0, .07], \quad \text{and} \quad \frac{di}{dt} \in [-.005, .005] \right\} \tag{15}$$

i.e., the interest rate can drop, or increase, between 0 and .5% per quarter.

Hence, the dynamic system to analyse the relationship between the interest rate, inflation and output gap needs to be augmented by the interest rate velocity constraint and will now look as follows:

$$\frac{dy}{dt} = -0.02\, y(t) - 0.35\, (i(t) - \pi(t)) \tag{16}$$

$$\frac{d\pi}{dt} = 0.002\, y(t). \tag{17}$$

$$\frac{di}{dt} \in [-.005, .005]\,. \tag{18}$$

A 3D region, within which the system trajectory $[y(t), \pi(t), i(t)]$ will have to be contained, is shown in Figure 4.

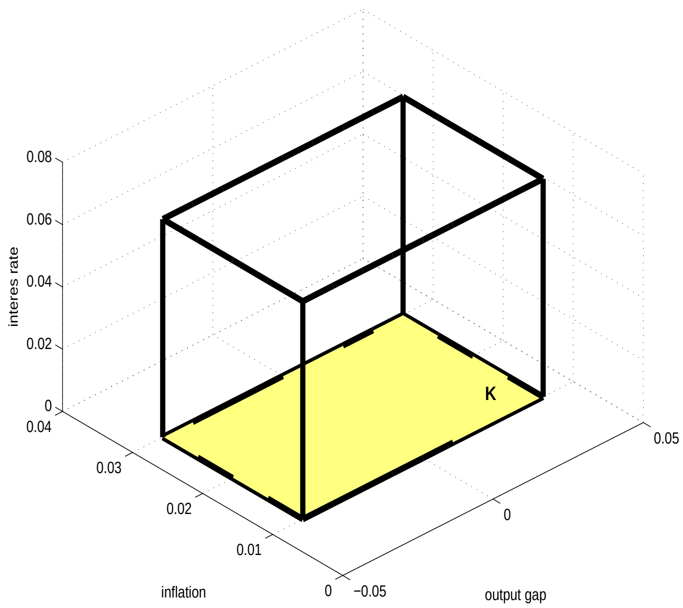

FIGURE 4. Constraint set K.

Let us briefly examine existence of steady states of (16), (17) and assess their stability. We add the plane $y = 0$ and another one $i = \pi$ to Figure 4. The steady states are at the intersection of those planes, see the dash-dotted line in Figure 5 top panel. The bottom panel shows the directions of system evolution on $y - \pi$ plane.

At any point where output gap is negative ($y < 0$) inflation decreases (arrow points toward "A"); reciprocally, inflation increases if output gap is positive (arrow points away from "A"). Above the plane $i = \pi$ interest rate dominates inflation, which decreases the output gap (above $i = \pi$, arrows point left). Below $i = \pi$ where inflation is higher than interest rate the arrows point right, which means that the output gap increases. (Basically, output gap diminishes where real interest rate is positive.)

Two interesting monetary control problems can be analysed with the help of Figure 5. The problems occur in the three dimensional space (upper panel) but can also be examined in two dimensions (bottom panel). First, the figure suggests that when output gap is positive, increasing i must not happen "too late", otherwise the inflation upper boundary will be violated.

A different problem may occur if output gap is negative. If the bank lowers the interest rate "too late", the economy might drift (with negative output gap) toward zero inflation where no instrument exists to lift the output. This means that the economy experiences a *liquidity trap i.e.*, remains in an area where output gap is negative and inflation is close to zero (positive or negative)[6].

[6]Again, we cite [6] for an analysis of a liquidity trap problem performed through an established method. Also notice that [7] is a recent publication where a liquidity trap problem is analysed in state space.

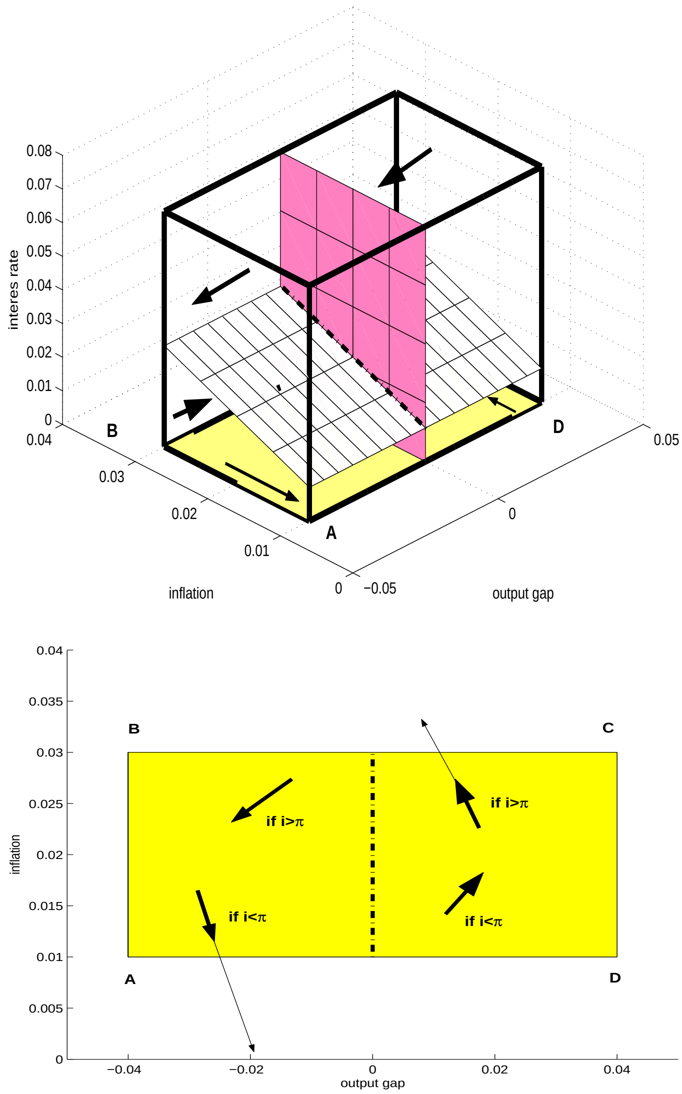

FIGURE 5. Constraint set K and steady states.

Situations like those ask for the determination of a collection of points from where the control from $\mathcal{I}$ (15) is sufficient to avoid leaving K. Such a collection is the viability kernel defined in Definition 2.2. In the next section we will determine V_F^K where the correspondence F is defined through (16)-(18).

4.2. **The viability kernel.** We will show the viability kernel's boundaries for two important economic situations that we call *overheating* and *liquidity trap.*

4.2.1. *Overheating.* Given model (16)-(18) and K we can determine V_F^K. We will determine the kernel's boundaries for the "north-east" corner (C) by running (16)-(18) backwards from $[y(T) = 0, \pi(T) = .03, i(T) = .03]$ (where T is some final time) with the maximal interest rate "velocity" (see (15). The results are shown in Figure 6 and also in the three dimensional Figure 8.

Figure 6 explains what will happen if the bank does not combat inflation early. The solid line shows a "booming" economy evolution that the bank started controlling when the output gap was maximal (4%) and inflation run[7] high but still below 3%. To "cool" the economy down, the interest rate applied at this moment must be close to the limit (which was 7%). The interest rate can gradually ease and, after more than 6 quarters, can reach 3%, the same value as inflation.

The solid line delimits the viability kernel (left, denoted V_F^K and marked by thin lines). Right from the line are states of the economy, from which violation of the 3% inflation limit is imminent, given the constraints on $i(t)$. The latter include $i_0 > 6.3\%$ (but less than 7%) and also a wish to achieve $i(t) \approx \pi(t)$ as quickly as possible to enable stabilisation of the output gap.

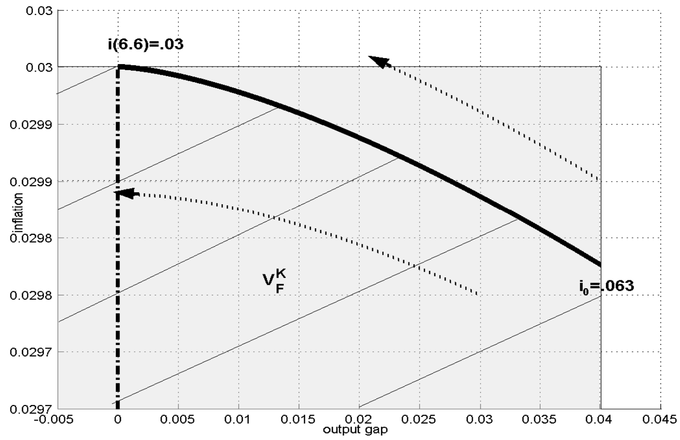

FIGURE 6. Viability kernel and trajectories at corner C.

[7]The numerical results suggest that a booming economy is *viable* in that a relatively high interest policy but such that remains in $\mathcal{I}$ can control the economy to $y = 0$ and not violate the inflation upper bound.

4.2.2. *Liquidity trap.* We will now determine the kernel's boundaries for the "south-west" corner (A) by running (16)-(18) backwards from $[0, .01, .0]$. The results are shown in Figure 7 and also in the three dimensional Figure 8.

Figure 7 explains what will happen if an economy had a negative output gap and inflation was close to the lower limit. The solid line in the left panel delimits a viability kernel (right, denoted V_F^K and marked by thin lines) from where the zero-interest rate policy guarantees an achievement of a positive output gap within 10 quarters. Any state to the left of the solid line does not have that property. So, an economy, which once fells outside the viability kernel V_F^K can remain for "long" at a negative output gap level and become deflationary. We can see that remaining in the viability kernel can prevent the liquidity trap.

A slightly different story tells the right panel. Here the viability kernel (again, right, denoted V_F^K and marked by thin lines) contains states of maximum-speed recovery from a negative output gap. The solid line is the trajectory, on which $\dfrac{di}{dt} = -.005$ and that leads to $y(t) > 0$ with inflation above its lower bound. Any point to the left from the line is non viable in that the economy, from that point, might slide to a liquidity trap.

A warning can be learned from this figure that even mildly negative output gaps can lead to a liquidity trap if inflation is very low and if the bank starts "late" to control the economy.

The two solid lines that define the kernels for different interest rates in Figure 7 are also visible in Figure 8. Here, we can appreciate the difference in the interest rate strategies between the situations described above.

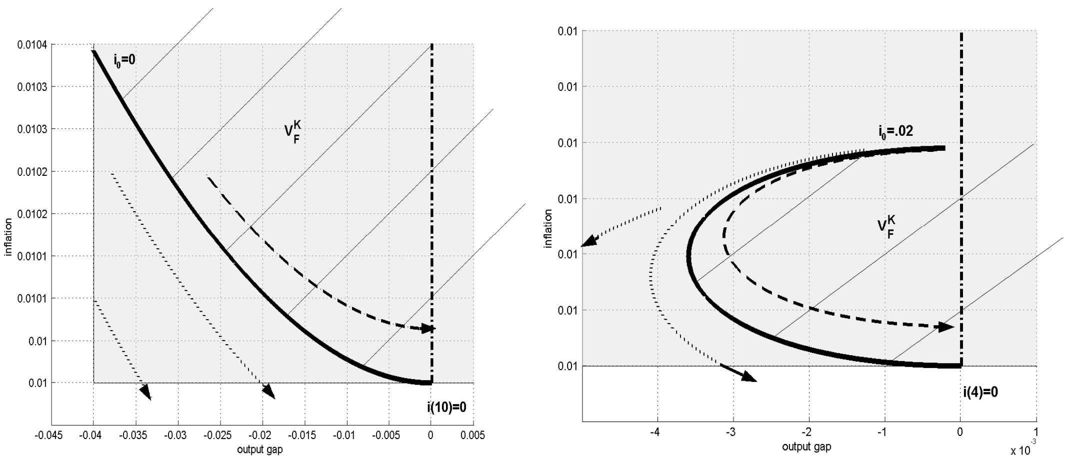

FIGURE 7. Viability kernel and trajectories at corner A.

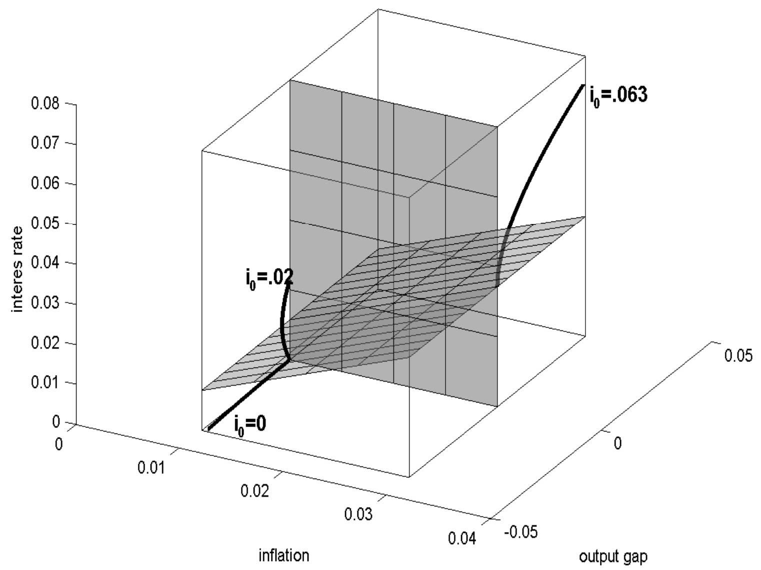

FIGURE 8. A 3D viability problem.

5. Concluding remarks

We have applied viability theory to a simple stylised model yet we were able to discuss many relevant issues related to macroeconomic modelling. In fact, we can also draw policy recommendations. At this stage we can say that the policy:

`every time interval` h `(could be a quarter) when` $y(t), \pi(t)$ `are assessed and are ''well inside"`[8] V_F^K`, set` $i(t) + h[-.005,\ .005]$

will be *satisficing*.

The *else* condition will be to apply $i(t) - .005\,h$ or $i(t) + .005\,h$ depending on where the economy is. However, the precise meaning of this rule will be our next research topic.

References

[1] Aubin, J.-P., (1997), Dynamical Economic Theory - A Viability Approach. Springer.
[2] Aubin, J.-P., G. Da Prato, H. Frankowska, (2000), Set-Valued Analysis. Springer.
[3] Bene, C., L. Doyen, D. Gabay, (2001), "A viability analysis for a bio-economic model", *Ecological Economics*, 36, pp. 385-396.
[4] Fuhrer, J.C, (1994), "Optimal monetary policy in a model of overlapping price contracts", Working Papers 94-2, Federal Reserve Bank of Boston.
[5] McCallum, B.T., (1999), "Issues in the design of monetary policy rules", in Handbook of Macroeconomics, Chapter 23. editors: J.B. Taylor & M. Woodford. North Holland.
[6] McCallum, B.T., (2004), "A monetary policy rule for automatic prevention of a liquidity trap", discussion paper, Carnegie Mellon University & NBER, May 2004.
[7] Nishiyama, S., (2003), "Inflation target as a buffer agains liquidity trap", manuscript, Institute for Monetary and Economic Studies, Bank of Japan, August 2003.

[8] This condition will depend on the governor's judgment.

[8] Saint-Pierre, P., (2001), "Controle viable et controle des crises de l'evolution salaire-emploi dans le modele de Goodwin", Centre de Recherche Viabilite, Jeux, Controle, Universite Paris IX - Dauphine.
[9] Walsh, C. E., (2003), Monetary Theory and Policy. MIT Press.

VICTORIA UNIVERSITY OF WELLINGTON, PO BOX 600, WELLINGTON, NEW ZEALAND. EMAIL: kunhong.kim@vuw.ac.nz.

VICTORIA UNIVERSITY OF WELLINGTON, PO BOX 600, WELLINGTON, NEW ZEALAND. EMAIL: jacek.krawczyk@vuw.ac.nz.