

# Dynamics of Monopolistic Competition in the Short-run and in the Long-run

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## Abstract

We study stability and instability of short-run and long-run equilibria of monopolistic competition with product differentiation. Since monopolistic competition is not either a game of complete information or of incomplete information, there should exist dynamic adjustment process of prices in markets given the set of incumbent firms. Moreover, entry and exit of firms occur depending on the levels of individual profits. In the economies under investigation, the price index and total income level of consumers determine whether the differentiated commodities are substitutes or complements. We show that there may exist multiple short-run equilibria including stable and unstable ones. The cause is characterized by decreasing marginal cost and endogenous complementarities between differentiated commodities.

**Keywords:** *monopolistic competition, short-run equilibrium, long-run equilibrium, decreasing marginal cost, complementarities*

## 1 Introduction

The theory of monopolistic competition, pioneered by Chamberlin (1933), has been extensively applied in economic analysis. Our interests are in dynamic adjustment processes to lead an economy to equilibria in the "short-run" and "long-run" for monopolistically competitive markets with product differentiation. In such markets, each single firm is not only a monopolist in the sense that its product is not a perfect substitute for others, but also a non-strategic competitor in the sense that it regards its effect of its own action on the whole markets negligible. We mean by "short-run" that the set of firms which participate in markets is fixed, and by "long-run" that all potential firms are free

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to entry and exit. Then a short-run market equilibrium represents product prices, levels of individual profits, and income level given the set of firms, all of which sell their products without taking into account effects of their own prices on the "price index" of market. On the other hand, a long-run market equilibrium characterizes the set of incumbent firms endogenously determined: those profits are nonnegative and no new entrants can earn positive profits as long as the incumbent firms exist. This distinction is standard.

So far the two types of equilibria have been studied with separate interests in the literature: Short-run equilibrium is discussed about conditions for its existence to explain possibilities of consistent decision makings by firms with market powers (Negishi (1961), Roberts and Sonnenschein (1977), Hart (1985a, 1985b, 1985c)). Long-run Equilibrium is examined to analyze economies of scales and product differentiations in various contexts such as industrial organization, international trade, and economic geography (Spence (1976), Dixit and Stiglitz (1977), Krugman (1979, 1991), Fujita, Krugman and Venables (1991), Ottaviano, Tabuchi and Thisse (2002)).

Although there are vast range of articles that discuss existence, welfare analysis, and comparative statics ("comparative dynamics" in the sense that long-run equilibrium is investigated) of monopolistically competitive equilibrium, the number of works concerning dynamic stability of short-run and long-run equilibria is extremely limited. Chao and Takayama (1990) emphasize the importance of stability analysis of long-run equilibrium of monopolistic competition. They point out that a number of useful results have been obtained by the zero-profit situations supposed to be attained through the free entry-exit market mechanism, which is necessary but not sufficient to attain equilibrium. They present a necessary and sufficient condition for long-run stability of monopolistically competitive equilibrium with a utility function of Dixit and Stiglitz (1977), and a neoclassical constant-returns production functions in terms of average and marginal factor intensities in the production side.

The purpose of this paper is to study what kind of conditions make short-run and long-run equilibria dynamically stable or not, and to characterize such conditions in terms of complementarities of differentiated products on the consumption side and propensities of marginal cost on the production side. We employ a utility function that exhibits "love of product diversity" as in Dixit and Stiglitz (1977) and a constant marginal utility of income as in Spence (1976). The demand functions derived from this type of utility functions have constant marginal propensities of income when the income level is low than a "critical level," and zero income effects when the level is higher than the critical one. They also show that the differentiated goods are always substitutes when the income level is lower than the critical one, and they are substitute or complements depending on a primitive condition when the income level is higher than the critical level.

In the short-run analysis, we present that there may be two equilibria one of which is unstable when products are complements and the marginal cost is highly decreasing. In the long-run analysis, we show that the dynamics of switching from substitutes to complements and its converse work through changes in the total income. Namely, the total profit of an industry starts to decrease when the size of industry is large enough and individual profits are quite small, and then the total income is so small that income effect arises. We also give examples of economies where profits per firm increase when entry occurs. We characterize those phenomena by primitive conditions of parameters

of economies. We show that in the presence of income effect there may exist stable and unstable short-run equilibria and a long-run equilibrium is always stable. We prove that in the absence of income effect complementarities between differentiated products and decreasing marginal cost are necessary conditions for short-run instability, and the long-run instability is implied by the complementarities and the short-run stability.

## 2 Consumer

First we focus on short-run performances of market with product differentiation, and ignore the entry process and endogenous selections of product variety. We tentatively assume that the economy produces a fixed set of differentiated products, each of which is supplied by a sole monopolist. We also suppose that there is a perfectly competitive market in which all the firms produce a homogenous good with identical linear cost functions whose values are measured in terms of numéraire. We thus always set the price of the good to be  $\gamma$ , the constant marginal cost of the competitive firms.

Let  $M$  to be a positive real number which denotes a *size* of the industry under investigation. Then we represent by the interval  $[0, M]$  the set of indices of differentiated products sold on market. A *product diversity* on  $[0, M]$  is a function from  $[0, M]$  to the set of nonnegative real numbers.

We suppose that there exists a representative agent in economy, whose behavior coincides with the aggregation over the whole group of the existing consumers. We then assume that the agent is endowed with  $N$  units of numéraire, holds ownership shares of all firms, and has a preference relation represented by the following utility function:

$$\frac{\beta}{\eta} \left( \int_0^M q(i)^\rho di \right)^{\eta/\rho} x^\alpha + z,$$

$$\alpha + \eta < 1, \quad 0 < \eta < 1, \quad 0 < \alpha < 1, \quad \beta > 0, \quad \text{and} \quad 0 < \rho < 1$$

where  $q(\cdot)$  is a product diversity on  $[0, M]$ ,  $x$  is a consumption of the competitive good and  $z$  is that of numéraire. The representative agent solves the following problem:

$$\begin{aligned} \text{Maximize} \quad & \frac{\beta}{\eta} \left( \int_0^M q(i)^\rho di \right)^{\eta/\rho} x^\alpha + z, \\ \text{subject to} \quad & \int_0^M p(i)q(i)di + \gamma x + z \leq Y \end{aligned}$$

where  $p(i)$  is the price of differentiated product  $i$  and  $Y$  is income.

We take two steps to compute the solution. The first step is to consider

$$\text{Minimize} \quad \int_0^M p(i)q(i)di, \quad \text{subject to} \quad \left[ \int_0^M q(i)^\rho di \right]^{1/\rho} = Q.$$

The first order condition for interior maximum is:

$$p(i) = \mu \rho q(i)^{\rho-1} \quad \text{for all } i \in [0, M], \quad \text{and}$$

$$\int_0^M q(i)^\rho di = Q^\rho$$

where  $\mu$  is the Lagrangian multiplier. This gives equality of marginal rate of substitution to price ratios,

$$\frac{p(i)}{p(j)} = \frac{q(i)^{\rho-1}}{q(j)^{\rho-1}}$$

for any pair  $i, j \in [0, M]$ . We then set

$$\frac{q(i)}{p(i)^{1/(\rho-1)}} = \frac{q(j)}{p(j)^{1/(\rho-1)}} = R, \quad \text{i.e., } q(i) = Rp(i)^{1/(\rho-1)}.$$

We introduce the *price index* for differentiated goods by

$$P \equiv \left[ \int_0^M p(i)^{\rho/(\rho-1)} di \right]^{(\rho-1)/\rho}. \quad (1)$$

In general, the price index  $P$  is the minimal expenditure for one unit of  $Q$ , aggregate utility from differentiated goods (See Appendix A). Then we obtain  $Q$  and  $q(i)$ :

$$Q = R \left( \int_0^M p(i)^{\rho/(\rho-1)} di \right)^{1/\rho} = RP^{1/(\rho-1)} \quad \left( \text{i.e., } R = \frac{Q}{P^{1/(\rho-1)}} \right),$$

$$q(i) = Rp(i)^{1/(\rho-1)} = Q \left( \frac{p(i)}{P} \right)^{1/(\rho-1)} \quad \text{for all } i \in [0, M]. \quad (2)$$

The second step is to substitute (1) and (2) into the original maximization problem. Then we have:

$$\text{Maximize } \frac{\beta}{\eta} Q^\eta x^\alpha + z, \quad \text{subject to } PQ + \gamma x + z \leq Y.$$

We discuss the case that both  $Q$  and  $x$  are positive, but take into account the fact that  $z = 0$  when  $Y$  is low enough. By the Kuhn-Tucker theorem, the first order condition is:

$$\beta Q^{\eta-1} x^\alpha = \lambda P, \quad (3)$$

$$\frac{\alpha\beta}{\eta} Q^\eta x^{\alpha-1} = \lambda \gamma, \quad (4)$$

$$z(1 - \lambda) = 0, \quad z \geq 0, \quad 1 - \lambda \leq 0; \quad \text{and} \quad (5)$$

$$Y - PQ - \gamma x - z = 0. \quad (6)$$

where  $\lambda$  is the Lagrangian multiplier.

First, we investigate the case of  $z > 0$ . Then (5) reduces to  $\lambda = 1$ . Substituting  $\lambda = 1$  into (3) and (4), we obtain:

$$\beta Q^{\eta-1} x^\alpha = P; \quad \text{and} \quad (7)$$

$$\frac{\alpha\beta}{\eta} Q^\eta x^{\alpha-1} = \gamma. \quad (8)$$

From (7) and (8),

$$Q = \frac{\eta\gamma}{\alpha} \frac{x}{P}. \quad (9)$$

Substituting (9) into (7), we obtain:

$$x = \left\{ \beta \left( \frac{\alpha}{\eta\gamma} \right)^{1-\eta} P^{-\eta} \right\}^{1/(1-\eta-\alpha)}. \quad (10)$$

Substituting (10) into (9),

$$Q = \frac{\eta\gamma}{\alpha} \frac{1}{P} \left\{ \beta \left( \frac{\alpha}{\eta\gamma} \right)^{1-\eta} P^{-\eta} \right\}^{1/(1-\eta-\alpha)} = \left\{ \left( \frac{\alpha}{\eta\gamma} \right)^\alpha \beta \right\}^{1/(1-\eta-\alpha)} P^{\frac{\alpha-1}{1-\eta-\alpha}}.$$

Let  $K = \left( \frac{\alpha}{\eta\gamma} \right)^\alpha \beta$ . From (6), (9) and (10), we have:

$$z = Y - PQ - \gamma x = Y - \frac{\eta + \alpha}{\eta} K^{1/(1-\eta-\alpha)} P^{-\eta/(1-\eta-\alpha)}. \quad (11)$$

Hence,

$$\begin{aligned} q(i) &= Q \left( \frac{p(i)}{P} \right)^{1/(\rho-1)} = \left( \frac{p(i)}{P} \right)^{1/(\rho-1)} K^{1/(1-\eta-\alpha)} P^{\frac{\alpha-1}{1-\eta-\alpha}}, \\ &= p(i)^{1/(\rho-1)} K^{1/(1-\eta-\alpha)} P^{\frac{\rho(1-\alpha)-\eta}{(1-\rho)(1-\eta-\alpha)}} \equiv d(p(i), P). \end{aligned} \quad (12)$$

This is the formula of the demand function for product  $i$  when  $z > 0$ , i.e.,

$$P^{\eta/(1-\eta-\alpha)} Y > \frac{\eta + \alpha}{\eta} K^{1/(1-\eta-\alpha)}.$$

**Remark 1** From (12), if  $z > 0$ , we say that differentiated goods are substitutes (i.e.,  $\partial d(p(i), P)/\partial P > 0$ ) if  $\rho(1-\alpha) - \eta > 0$ , and they are complements (i.e.,  $\partial d(p(i), P)/\partial P < 0$ ) if  $\rho(1-\alpha) - \eta < 0$ . Notice that these relations are symmetric in our model although they are generally asymmetric.

Next we examine the case of  $z = 0$ . Even demand functions derived from quasi-linear utility functions exhibits income effects when the income level is low enough (See Fig.1). Income effects exist only if  $z = 0$ .

On the consumer side, the first order condition for utility maximization is:

$$\beta Q^{\eta-1} x^\alpha = \lambda P, \quad (13)$$

$$\frac{\alpha\beta}{\eta} Q^\eta x^{\alpha-1} = \lambda \gamma, \quad (14)$$

$$z = 0; \quad \text{and}$$

$$Y - PQ - \gamma x = 0. \quad (15)$$

Then obtain

$$\begin{aligned} Q &= \left( \int_0^M q(i)^\rho di \right)^{1/\rho} = R P^{1/(\rho-1)} \left( \text{i.e., } R = \frac{Q}{P^{1/(\rho-1)}} \right), \\ q(i) &= R p(i)^{1/(\rho-1)} = Q \left( \frac{p(i)}{P} \right)^{1/(\rho-1)} \quad \text{for all } i \in [0, M]. \end{aligned}$$

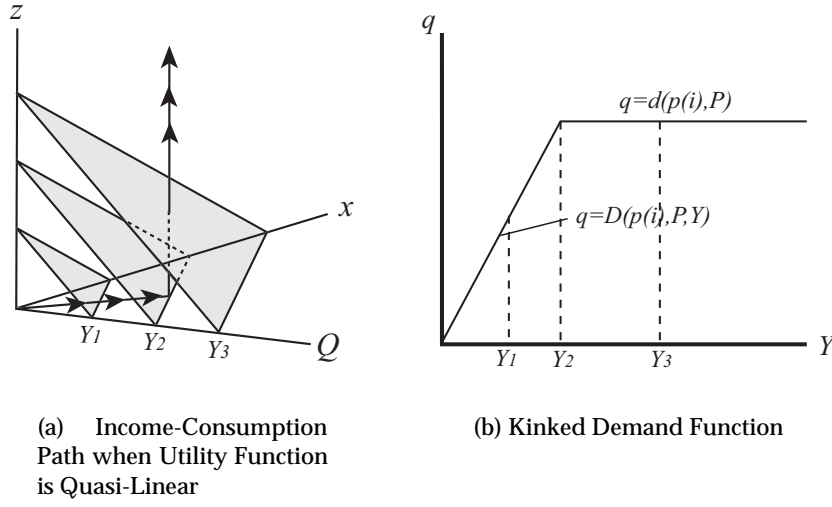


Figure 1: Income-Consumption Path and the Graph of Demand Function

From (13) and (14), we obtain:

$$PQ = \frac{\eta\gamma x}{\alpha}. \quad (16)$$

Substituting (16) into (15), we have:

$$x = \frac{\alpha}{\gamma(\eta + \alpha)}Y.$$

Then we obtain  $Q$  and  $q(i)$ :

$$Q = (Y - \gamma x)P^{-1} = \frac{\eta}{\eta + \alpha}YP^{-1}, \quad (17)$$

$$q(i) = \frac{\eta}{\eta + \alpha}Yp(i)^{1/(\rho-1)}P^{\rho/(1-\rho)} \equiv D(p(i), P, Y). \quad (18)$$

**Remark 2** From (18), if  $z = 0$ , the differentiated goods are always substitutes (i.e.,  $\partial D(p(i), P, Y)/\partial P > 0$ ). Namely, complements change to substitutes and vice versa according to the price index and the income level.

### 3 Monopolistically Competitive Firm

#### 3.1 The Case without Income Effect

Consider the market of differentiated goods in which all firms are monopolistically competitive. We assume that each firm  $i$  has the identical cost function

$$cq(i)^\theta + f, \quad \theta > \rho$$

where  $cq(i)^\theta$  is the variable cost and  $f$  is the fixed cost. The cost is measured in terms of numéraire. Note that the marginal cost is decreasing if  $0 < \theta < 1$  and is increasing  $\theta > 1$ .

A monopolistically competitive firm maximizes its profit subject to the demand function for its own product given the market price index. Hence, firm  $i$  chooses  $q(i)$  and  $p(i)$  given  $P$ :

$$\begin{aligned} \text{Maximize} \quad & p(i)q(i) - (cq(i)^\theta + f) \\ \text{subject to} \quad & q(i) = d(p(i), P). \end{aligned}$$

Since  $d(p(i), P) = p(i)^{1/(\rho-1)} K^{1/(1-\eta-\alpha)} P^{\frac{\rho(1-\alpha)-\eta}{(1-\rho)(1-\eta-\alpha)}}$ , the inverse demand is:

$$p(i) = q(i)^{\rho-1} K^{\frac{\rho-1}{1-\eta-\alpha}} P^{\frac{\eta-\rho(1-\alpha)}{1-\eta-\alpha}},$$

so that the associated profit is:

$$\begin{aligned} \pi(i) &\equiv p(i)q(i) - (cq(i)^\theta + f) = q(i)^\rho K^{\frac{\rho-1}{1-\eta-\alpha}} P^{\frac{\eta-\rho(1-\alpha)}{1-\eta-\alpha}} - (cq(i)^\theta + f) \\ &= cq(i)^\rho \left\{ c^{-1} K^{\frac{\rho-1}{1-\eta-\alpha}} P^{\frac{\eta-\rho(1-\alpha)}{1-\eta-\alpha}} - q(i)^{\theta-\rho} \right\} - f. \end{aligned}$$

Since  $\theta > \rho$ ,  $\pi(i)$  attains a single peak. From (12), the first order condition is:

$$\rho - c\theta q(i)^{\theta-\rho} K^{\frac{\rho-1}{1-\eta-\alpha}} P^{\frac{\eta-\rho(1-\alpha)}{1-\eta-\alpha}} = 0.$$

We thus obtain the profit-maximizing price level and output level independently of  $i$ :

$$p(P) \equiv \left( \frac{\rho}{c\theta} \right)^{\frac{\rho-1}{\theta-\rho}} K^{\frac{(1-\rho)(\theta-1)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{(\theta-1)\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}}, \quad \text{and} \quad (19)$$

$$q(P) \equiv \left( \frac{\rho}{c\theta} \right)^{1/(\theta-\rho)} K^{\frac{1-\rho}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\rho(1-\alpha)-\eta}{(\theta-\rho)(1-\eta-\alpha)}}. \quad (20)$$

### 3.2 The Case with Income Effect

The profit maximization problem of firm  $i$  is:

$$\begin{aligned} \text{Maximize} \quad & p(i)q(i) - (cq(i)^\theta + f) \\ \text{subject to} \quad & q(i) = D(p(i), P, Y). \end{aligned}$$

From (18), the first order condition for profit maximization is rewritten as

$$\rho - c\theta \left( \frac{\eta}{\eta + \alpha} Y \right)^{\rho-1} q(i)^{\theta-\rho} P^{-\rho} = 0.$$

Hence we have  $p(P, Y)$  and  $q(P, Y)$  independently of  $i$ :

$$p(P, Y) = \left( \frac{\rho}{c\theta} \right)^{\frac{\rho-1}{\theta-\rho}} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{(\theta-1)(\rho-1)}{\rho-\theta}} P^{\frac{\rho(\theta-1)}{\theta-\rho}}, \quad (21)$$

$$\text{and } q(P, Y) = \left( \frac{\rho}{c\theta} \right)^{1/(\theta-\rho)} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{1-\rho}{\theta-\rho}} P^{\rho/(\theta-\rho)}. \quad (22)$$

## 4 Short-run Equilibrium and Dynamics

We mean by "short-run" that the set of incumbent firms is fixed. We formulate a *dynamics á la Cournot*, which is a price adjustment process of monopolistic competition.

In this dynamical process, each firm produces and supplies its own product given price index of the industry and income level of consumers. The price is determined by market demand, and agents make trades. The earned profit becomes part of the new income of consumers. Then, the price index and income level are updated (See Fig.2).

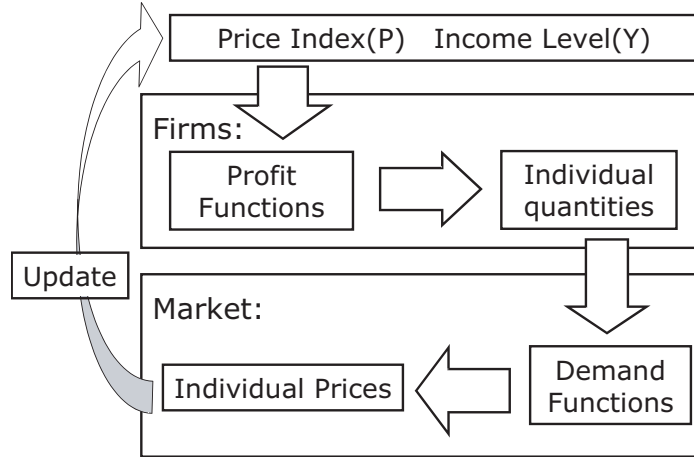


Figure 2: Price Adjustment Process of Monopolistic Competition in the Short-run

This dynamic process continues until neither the price index nor income level change. The new price system  $(p^*, P^*)$  and the income  $Y^*$  to clear the markets solves the following simultaneous equations:

$$g\left(\frac{p^*}{P^*k}\right)Q(P^*, Y^*) = q(P, Y), \quad (23)$$

$$P^* = Mp^*g\left(\frac{p^*}{P^*k}\right), \quad \text{and} \quad (24)$$

$$Y^* = N + M(p^*q(P, Y) - C(q(P, Y))), \quad (25)$$

where  $g$  is the inverse function of  $v'$ . Equation (23) means that supply equals demand, and (24) and (25) respectively follows from the definitions of price index and income. See Appendix A for these general settings and derivations.

We suppose that individual prices are determined by the demand functions and output levels. Hence,  $p^* = p^*(P, Y, M)$ ,  $P^* = P^*(P, Y, M)$ , and  $Y^* = Y^*(P, Y, M)$ . It means that given  $M$ ,  $(P, Y)$  is updated by  $P^*(\cdot)$  and  $Y^*(\cdot)$ . Thus, the short-run dynamic adjustment system is:

$$\begin{aligned} \dot{P} &= P^*(P, Y, M) - P; \quad \text{and} \\ \dot{Y} &= Y^*(P, Y, M) - Y. \end{aligned}$$

### 4.1 Two Kinds of Demand Functions

We consider two kinds of demand functions categorized by income effect.



There exists no income effect when either  $Y$  or  $P$  is high enough, but it emerges with increase in  $Y$  or  $P$ . In this section, we investigate the boundary dividing the "Spence case" and the "Dixit-Stiglitz case", i.e., the border between existence and non-existence of income effects. From (11), we define the function  $\zeta$  by:

$$\zeta(P, Y) = Y - \frac{\eta + \alpha}{\eta} K^{1/(1-\eta-\alpha)} P^{-\eta/(1-\eta-\alpha)}. \quad (26)$$

Equation (26) means that  $\zeta(P, Y)$  may be negative. Therefore, we need to investigate two cases of output levels according to the signs of  $z$ . We refer to the case of  $\zeta(P, Y) > 0$  as the *SP (Spence) case*. In this case, we deal with the profit-maximization output level  $q(P)$  as (20). On the other hand, when  $\zeta(P, Y) \leq 0$ , we deal with a output level  $q(P, Y)$  as (22). We call this the *DS (Dixit-Stiglitz) case*.

Furthermore, there are two types of demand function according to the signs of  $\zeta(P, Y)$ . Namely,

$$g\left(\frac{p^*}{P^*k}\right) Q(P^*, Y^*) \equiv d(p^*, P^*) = p^{*1/(\rho-1)} K^{1/(1-\eta-\alpha)} P^{*\frac{\rho(1-\alpha)-\eta}{(1-\rho)(1-\eta-\alpha)}} \quad \text{if} \quad \zeta(P^*, Y^*) > 0, \quad \text{and} \quad (27)$$

$$\equiv D(p^*, P^*, Y^*) = \frac{\eta}{\eta + \alpha} Y^* p^{*1/(\rho-1)} P^{*\rho/(1-\rho)}. \quad \text{if} \quad \zeta(P^*, Y^*) \leq 0. \quad (28)$$

Hence, we have to consider four kinds of dynamics.

## 4.2 Range of $P$ and $Y$

### 4.2.1 SP Case

First, we deal with the SP case, in which output level  $q(P)$  is (20). The SP case can be classified by two kinds of demand functions. One is the *SP-SP case*. In this case, firms face the demand function of the SP case, and the representative consumer reveals that of the same case. It is because  $\zeta(P, Y)$  and  $\zeta(P^*(P, Y, M), Y^*(P, Y, M))$  are both positive. Then we solve the following simultaneous equations:

$$p^{*1/(\rho-1)} K^{1/(1-\eta-\alpha)} P^{*\frac{\rho(1-\alpha)-\eta}{(1-\rho)(1-\eta-\alpha)}} = \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} K^{\frac{1-\rho}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\rho(1-\alpha)-\eta}{(\theta-\rho)(1-\eta-\alpha)}}, \quad (29)$$

$$P^* = \left[ \int_0^M p^{*\rho/(\rho-1)} di \right]^{(\rho-1)/\rho} = p^* M^{(\rho-1)/\rho}; \quad \text{and} \quad (30)$$

$$Y^* = N + M \left\{ p^* q(P) - c q(P)^\theta - f \right\}. \quad (31)$$

The other is the *DS-SP case*. In this case, firms are subject to the demand function of the DS case, and the market demand is of the SP case. It means that  $\zeta(P^*(P, Y, M), Y^*(P, Y, M)) \leq$

0 and  $\zeta(P, Y) > 0$ . Then we solve the following simultaneous equations:

$$\begin{aligned} \frac{\eta}{\eta + \alpha} Y^* p^{*1/(\rho-1)} P^{*\rho/(1-\rho)} \\ = \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} K^{\frac{1-\rho}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\rho(1-\alpha)-\eta}{(\theta-\rho)(1-\eta-\alpha)}}, \end{aligned} \quad (32)$$

$$P^* = \left[ \int_0^M p^{*\rho/(\rho-1)} di \right]^{(\rho-1)/\rho} = p^* M^{(\rho-1)/\rho}; \quad \text{and} \quad (33)$$

$$Y^* = N + M \left\{ p^* q(P) - cq(P)^\theta - f \right\}. \quad (34)$$

Now we investigate the boundary of the SP-SP case and the DS-SP case. Consider  $(P^*, Y^*)$  as variables. Substituting (30) into (29), we have:

$$P^{*\frac{\alpha-1}{1-\eta-\alpha}} = \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} K^{\frac{1-\rho}{(\theta-\rho)(1-\eta-\alpha)}} M^{1/\rho} P^{\frac{\rho(1-\alpha)-\eta}{(\theta-\rho)(1-\eta-\alpha)}}. \quad (35)$$

Substituting (33) into (32), we have:

$$\frac{\eta}{\eta + \alpha} Y^* = P^* \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} K^{\frac{1-\rho}{(\theta-\rho)(1-\eta-\alpha)}} M^{1/\rho} P^{\frac{\rho(1-\alpha)-\eta}{(\theta-\rho)(1-\eta-\alpha)}}. \quad (36)$$

From (11), the equation that satisfies  $\zeta(P^*, Y^*) = 0$  is:

$$\frac{\eta}{\eta + \alpha} Y^* = K^{1/(1-\eta-\alpha)} P^{*\eta/(1-\eta-\alpha)}. \quad (37)$$

The curves (35), (36) and (37) intersect at the point  $(Y_1^*, P_1^*)$ :

$$\begin{aligned} Y_1^* &= \frac{\eta + \alpha}{\eta} \left(\frac{\rho}{c\theta}\right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{\frac{\eta(1-\theta)-(\theta-\rho)(\alpha-1)}{(\theta-\rho)(1-\alpha)(1-\eta-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} P^{\frac{\eta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\alpha)(1-\eta-\alpha)}}, \\ P_1^* &= \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{(\theta-\rho)(\alpha-1)}} K^{\frac{1-\rho}{(\theta-\rho)(\alpha-1)}} M^{\frac{1-\eta-\alpha}{\rho(\alpha-1)}} P^{\frac{\rho(1-\alpha)-\eta}{(\theta-\rho)(\alpha-1)}}. \end{aligned}$$

From (30) and (31), we obtain:

$$\begin{aligned} Y^* &= N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \\ &\quad + P^* M^{1/\rho} \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} K^{\frac{1-\rho}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\rho(1-\alpha)-\eta}{(\theta-\rho)(1-\eta-\alpha)}}. \end{aligned} \quad (38)$$

In the SP-SP case, (35) intersects (38) in the domain of  $z > 0$ . Namely,

$$\begin{aligned} F(P, M) &\equiv N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \\ &\quad - \frac{\alpha}{\eta} \left(\frac{\rho}{c\theta}\right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{\frac{\eta(1-\theta)-(\theta-\rho)(\alpha-1)}{(\theta-\rho)(1-\alpha)(1-\eta-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} P^{\frac{\eta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\alpha)(1-\eta-\alpha)}} > 0. \end{aligned} \quad (39)$$

On the other hand, in the DS-SP case, (36) intersects (38) in the domain of  $z < 0$ , that is:

$$\begin{aligned} F(P, M) &= N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \\ &\quad - \frac{\alpha}{\eta} \left(\frac{\rho}{c\theta}\right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{\frac{\eta(1-\theta)-(\theta-\rho)(\alpha-1)}{(\theta-\rho)(1-\alpha)(1-\eta-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} P^{\frac{\eta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\alpha)(1-\eta-\alpha)}} \leq 0. \end{aligned} \quad (40)$$

Namely,  $F(P, M) = 0$  is the boundary line between the SP-SP case and the DS-SP case.

In the SP-SP case, we obtain  $P^*(P, M)$  and  $Y^*(P, M)$  from (35) and (38):

$$P^*(P, M) = \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{(\alpha-1)(\theta-\rho)}} K^{\frac{1-\theta}{(\alpha-1)(\theta-\rho)}} P^{\frac{\rho(1-\alpha)-\eta}{(\alpha-1)(\theta-\rho)}} M^{\frac{1-\eta-\alpha}{\rho(\alpha-1)}},$$

$$Y^*(P, M) = N - Mf + \left(\frac{\rho}{c\theta}\right)^{\frac{\eta}{(1-\alpha)(\theta-\rho)}} K^{\frac{(\alpha-1)(\theta-\rho)+\eta(\theta-1)}{(\alpha-1)(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\eta\{\eta-\rho(1-\alpha)\}}{(\alpha-1)(\theta-\rho)(1-\eta-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}}.$$

For regularity, we set a restriction that the overall input cannot exceed the total amount of numéraire and the competitive good is produced however less it is. That is,

$$N > c \int_0^M q(i)^\theta di + Mf = M(cq(P)^\theta + f)$$

$$= cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} + Mf. \quad (41)$$

Then  $Y^*(P, M)$  is always positive. We thus define the time-derivative of  $P$  and  $Y$  in the SP-SP case by

$$\dot{P} = P^*(P, M) - P$$

$$= \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{(\alpha-1)(\theta-\rho)}} K^{\frac{1-\theta}{(\alpha-1)(\theta-\rho)}} P^{\frac{\rho(1-\alpha)-\eta}{(\alpha-1)(\theta-\rho)}} M^{\frac{1-\eta-\alpha}{\rho(\alpha-1)}} - P, \quad (42)$$

$$\dot{Y} = Y^*(P, M) - Y$$

$$= N - Mf + \left(\frac{\rho}{c\theta}\right)^{\frac{\eta}{(1-\alpha)(\theta-\rho)}} K^{\frac{(\alpha-1)(\theta-\rho)+\eta(\theta-1)}{(\alpha-1)(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\eta\{\eta-\rho(1-\alpha)\}}{(\alpha-1)(\theta-\rho)(1-\eta-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} - c \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} M - Y. \quad (43)$$

Rearranging (36) and (38), we also obtain  $P^*(P, M)$  and  $Y^*(P, M)$  in the DS-SP case:

$$P^*(P, M) = \frac{\eta}{\alpha} \left(\frac{\rho}{c\theta}\right)^{1/(\rho-\theta)} K^{\frac{\rho-1}{(\theta-\rho)(1-\eta-\alpha)}} M^{-1/\rho} P^{\frac{\eta-\rho(1-\alpha)}{(\theta-\rho)(1-\eta-\alpha)}}$$

$$\left[ N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \right],$$

$$Y^*(P, M) = \frac{\eta + \alpha}{\alpha} \left[ N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \right].$$

From (41), both  $P^*(P, M)$  and  $Y^*(P, M)$  are always positive. We thus define the time-derivative of  $P$  and  $Y$  in the DS-SP case by

$$\dot{P} = P^*(P, M) - P$$

$$= \frac{\eta}{\alpha} \left(\frac{\rho}{c\theta}\right)^{1/(\rho-\theta)} K^{\frac{\rho-1}{(\theta-\rho)(1-\eta-\alpha)}} M^{-1/\rho} P^{\frac{\eta-\rho(1-\alpha)}{(\theta-\rho)(1-\eta-\alpha)}}$$

$$\left[ N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \right] - P, \quad (44)$$

$$\dot{Y} = Y^*(P, M) - Y$$

$$= \frac{\eta + \alpha}{\alpha} \left[ N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \right] - Y. \quad (45)$$

#### 4.2.2 DS Case

Next, we deal with the DS case in which output level  $q(P, Y)$  is (22). The DS case can also be classified two cases by two types of demand functions. One is the *SP-DS* case, in which the demand function is  $d(p^*, P^*)$ . Then we solve the following simultaneous equations:

$$\begin{aligned} p^{*1/(\rho-1)} K^{1/(1-\eta-\alpha)} P^{*\frac{\rho(1-\alpha)-\eta}{(1-\rho)(1-\eta-\alpha)}} \\ = \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{1-\rho}{\theta-\rho}} P^{\rho/(\theta-\rho)}, \end{aligned} \quad (46)$$

$$P^* = \left[ \int_0^M p^{*\rho/(\rho-1)} di \right]^{(\rho-1)/\rho} = p^* M^{(\rho-1)/\rho}; \quad \text{and} \quad (47)$$

$$Y^* = N + M \left\{ p^* q(P, Y) - cq(P, Y)^\theta - f \right\}. \quad (48)$$

The other is the *DS-DS* case. In this case, we employ  $D(p^*, P^*, Y^*)$  as the demand function. Then we solve the following simultaneous equations:

$$\frac{\eta}{\eta+\alpha} Y^* p^{*1/(\rho-1)} P^{*\rho/(1-\rho)} = \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{1-\rho}{\theta-\rho}} P^{\rho/(\theta-\rho)}, \quad (49)$$

$$P^* = \left[ \int_0^M p^{*\rho/(\rho-1)} di \right]^{(\rho-1)/\rho} = p^* M^{(\rho-1)/\rho}; \quad \text{and} \quad (50)$$

$$Y^* = N + M \left\{ p^* q(P, Y) - cq(P, Y)^\theta - f \right\}. \quad (51)$$

Now we investigate the boundary of the *SP-DS* case and the *DS-DS* case. Consider  $(P^*, Y^*)$  as variables. Substituting (47) into (46), we have:

$$P^{*\frac{\alpha-1}{1-\eta-\alpha}} = \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} K^{-1/(1-\eta-\alpha)} M^{1/\rho} P^{\rho/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{1-\rho}{\theta-\rho}}. \quad (52)$$

Substituting (50) into (49), we have:

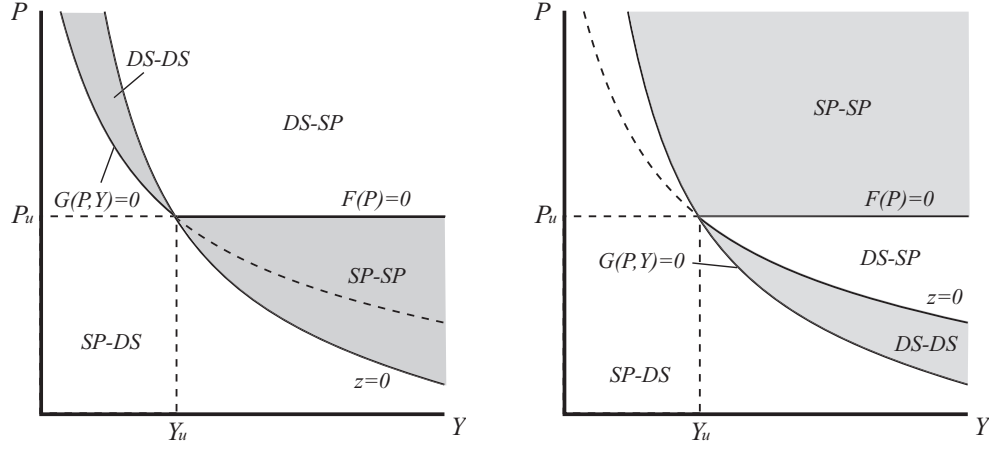
$$\frac{\eta}{\eta+\alpha} Y^* = P^* \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} M^{1/\rho} P^{\rho/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{1-\rho}{\theta-\rho}}. \quad (53)$$

The curves (52), (53) and (37) intersect at a point  $(Y_2^*, P_2^*)$ :

$$\begin{aligned} Y_2^* &= \frac{\eta+\alpha}{\eta} \left(\frac{\rho}{c\theta}\right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{1/(1-\alpha)} M^{\frac{\eta}{\rho(1-\alpha)}} P^{\frac{\rho\eta}{(\theta-\rho)(1-\alpha)}} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\eta(1-\rho)}{(\theta-\rho)(1-\alpha)}}, \\ P_2^* &= \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{(\theta-\rho)(\alpha-1)}} K^{1/(1-\alpha)} M^{\frac{1-\eta-\alpha}{\rho(\alpha-1)}} P^{\frac{\rho(1-\eta-\alpha)}{(\theta-\rho)(\alpha-1)}} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{(1-\rho)(1-\eta-\alpha)}{(\theta-\rho)(\alpha-1)}}. \end{aligned}$$

From (47) and (48), we obtain:

$$\begin{aligned} Y^* &= N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\rho\theta/(\theta-\rho)} \\ &\quad + P^* M^{1/\rho} \left(\frac{\rho}{c\theta}\right)^{1/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{1-\rho}{\theta-\rho}} P^{\rho/(\theta-\rho)}. \end{aligned} \quad (54)$$



(a)  $\rho(1 - \alpha) > \eta$

(b)  $\rho(1 - \alpha) < \eta$

Figure 3: The Range of Four Dynamics Cases

In the SP-DS case, (52) intersects (54) in the domain of  $z > 0$ . Namely,

$$\begin{aligned}
 G(P, Y, M) = & N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \\
 & - \frac{\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{1/(1-\alpha)} M^{\frac{\eta}{\rho(1-\alpha)}} P^{\frac{\rho\eta}{(\theta-\rho)(1-\alpha)}} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{\eta(1-\rho)}{(\theta-\rho)(1-\alpha)}} > 0.
 \end{aligned} \tag{55}$$

On the other hand, in the DS-DS case, (53) intersects (54) in the domain of  $z < 0$ . That is:

$$\begin{aligned}
 G(P, Y, M) = & N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \\
 & - \frac{\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{1/(1-\alpha)} M^{\frac{\eta}{\rho(1-\alpha)}} P^{\frac{\rho\eta}{(\theta-\rho)(1-\alpha)}} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{\eta(1-\rho)}{(\theta-\rho)(1-\alpha)}} \leq 0.
 \end{aligned} \tag{56}$$

Namely,  $G(P, Y, M) = 0$  is the boundary line between the DS-DS case and the SP-DS case.

In the SP-DS case, we obtain  $P^*(P, Y, M)$  and  $Y^*(P, Y, M)$  from (52) and (54):

$$\begin{aligned}
 P^*(P, Y, M) = & \left( \frac{\rho}{c\theta} \right)^{\frac{1-\eta-\alpha}{(\theta-\rho)(\alpha-1)}} K^{1/(1-\alpha)} M^{\frac{1-\eta-\alpha}{\rho(\alpha-1)}} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{(1-\rho)(1-\eta-\alpha)}{(\alpha-1)(\theta-\rho)}} P^{\frac{\rho(1-\eta-\alpha)}{(\theta-\rho)(\alpha-1)}}, \\
 Y^*(P, Y, M) = & N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \\
 & + \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{1/(1-\alpha)} M^{\frac{\eta}{\rho(1-\alpha)}} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{\eta(1-\rho)}{(\theta-\rho)(1-\alpha)}} P^{\frac{\rho\eta}{(\theta-\rho)(1-\alpha)}}.
 \end{aligned}$$

For regularity, we set a restriction that the overall input cannot exceed the total amount of numéraire and the competitive good is produced however less it is. That is,

$$\begin{aligned} N &> M(cq(P, Y)^\theta + f) \\ &= cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\theta(\rho-1)}{\rho-\theta}} P^{\frac{\rho\theta}{\theta-\rho}} + Mf. \end{aligned} \quad (57)$$

Then  $Y^*(P, Y, M)$  is always positive. We thus define the time-derivative of  $P$  and  $Y$  in the SP-DS case by

$$\begin{aligned} \dot{P} &= P^*(P, Y, M) - P \\ &= \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{(\theta-\rho)(\alpha-1)}} K^{1/(1-\alpha)} M^{\frac{1-\eta-\alpha}{\rho(\alpha-1)}} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{(1-\rho)(1-\eta-\alpha)}{(\alpha-1)(\theta-\rho)}} P^{\frac{\rho(1-\eta-\alpha)}{(\theta-\rho)(\alpha-1)}} - P, \end{aligned} \quad (58)$$

$$\begin{aligned} \dot{Y} &= Y^*(P, Y, M) - Y \\ &= N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \\ &\quad + \left(\frac{\rho}{c\theta}\right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{1/(1-\alpha)} M^{\frac{\eta}{\rho(1-\alpha)}} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\eta(1-\rho)}{(\theta-\rho)(1-\alpha)}} P^{\frac{\rho\eta}{(\theta-\rho)(1-\alpha)}} - Y. \end{aligned} \quad (59)$$

Rearranging (53) and (54), we also obtain  $P^*(P, Y, M)$  and  $Y^*(P, Y, M)$  in the DS-DS case:

$$\begin{aligned} P^*(P, Y, M) &= \frac{\eta}{\alpha} \left(\frac{\rho}{c\theta}\right)^{1/(\rho-\theta)} M^{-1/\rho} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\rho-1}{\theta-\rho}} P^{\rho/(\rho-\theta)} \\ &\quad \left[ N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \right], \\ Y^*(P, Y, M) &= \frac{\eta+\alpha}{\alpha} \left[ N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \right]. \end{aligned}$$

From (57),  $P^*(P, Y, M)$  and  $Y^*(P, Y, M)$  are always positive. We thus define the time-derivative of  $P$  and  $Y$  in the DS-DS case by

$$\begin{aligned} \dot{P} &= P^*(P, Y, M) - P \\ &= \frac{\eta}{\alpha} \left(\frac{\rho}{c\theta}\right)^{1/(\rho-\theta)} M^{-1/\rho} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\rho-1}{\theta-\rho}} P^{\rho/(\rho-\theta)} \\ &\quad \left[ N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \right] - P. \end{aligned} \quad (60)$$

$$\begin{aligned} \dot{Y} &= Y^*(P, Y, M) - Y \\ &= \frac{\eta+\alpha}{\alpha} \left[ N - Mf - cM \left(\frac{\rho}{c\theta}\right)^{\theta/(\theta-\rho)} \left(\frac{\eta}{\eta+\alpha} Y\right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \right] - Y. \end{aligned} \quad (61)$$

Then we separate the  $Y$ - $P$  plane into these four cases (See Fig.3). In the next section, we investigate the locus of  $\dot{P} = 0$  and  $\dot{Y} = 0$  in each range.

### 4.3 The loci of $\dot{P} = 0$ and $\dot{Y} = 0$

First, we consider the locus of  $\dot{P} = 0$ .

We draw  $\dot{P} = 0$  in Figs.4 and 5 according to the signs of  $\rho(1-\alpha) - \eta$  and  $\theta(1-\alpha) - \eta$ . Fig.4 shows the case that  $\dot{P} = 0$  of the SP-SP case exists in the SP-SP case and Fig.5 is not. See Appendix B for  $\dot{P} = 0$  of each range and coordinate value of the intersections  $A$ ,  $B$  and  $C$ . Define the function  $Z$  by

$$Z(M) \equiv N - Mf - \frac{\alpha\theta + \rho\eta}{\eta\theta} \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}}. \quad (62)$$

Then the condition that  $\dot{P} = 0$  of the SP-SP case exists in the SP-SP case is  $Z(M) > 0$ .

Next, we consider the locus of  $\dot{Y} = 0$ . We draw  $\dot{Y} = 0$  as Figs.6 and 7. See Appendix C for  $\dot{Y} = 0$  of each range and coordinate values of the intersections  $D$ ,  $E$ ,  $F$  and  $G$ .

### 4.4 The Short-run Equilibria

#### 4.4.1 The locus of $z(M)$ and the Range of $M$

To preparing for the dynamic analysis of  $\dot{P} = 0$  and  $\dot{Y} = 0$ , we investigate the boundary of  $M$  according to the signs of  $z$ . We combine Figs.5, 6, 7 and 8 and see whether short-run equilibria exist or not in each case.

From (109) and (116) in Appendix B, there exists a unique equilibrium in the SP-SP case:

$$Y_s^*(M) = N - Mf + M^{\frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}} \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} \left(1 - \frac{\rho}{\theta}\right), \quad \text{and} \quad (63)$$

$$P_s^*(M) = \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{\eta-\theta(1-\alpha)}} K^{\frac{\theta-1}{\theta(1-\alpha)-\eta}} M^{\frac{(\rho-\theta)(1-\eta-\alpha)}{\rho\{\theta(1-\alpha)-\eta\}}}. \quad (64)$$

We call  $(Y_s^*(M), P_s^*(M))$  the *SP short-run equilibrium*.

From (112) and (119), we also have a unique equilibrium  $(Y_d^*(M), P_d^*(M))$  in the DS-DS case,

$$Y_d^*(M) = \frac{\theta(\eta + \alpha)}{\alpha\theta + \rho\eta} (N - Mf), \quad \text{and} \quad (65)$$

$$P_d^*(M) = \left(\frac{\rho}{c\theta}\right)^{-1/\theta} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} (N - Mf) \right\}^{(\theta-1)/\theta} M^{\frac{\rho-\theta}{\rho\theta}}. \quad (66)$$

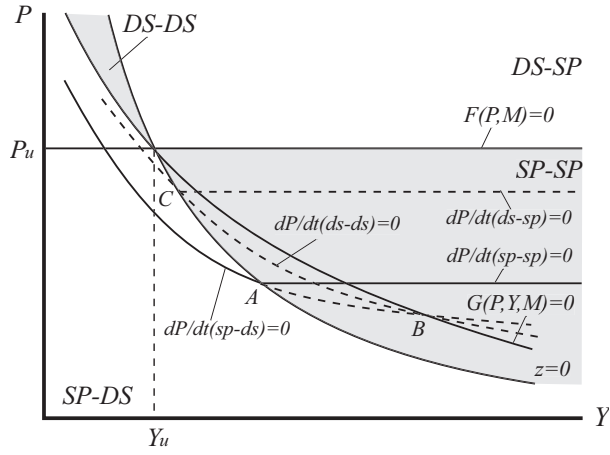
We call  $(Y_d^*(M), P_d^*(M))$  the *DS short-run equilibrium*.

We investigate the boundary of  $M$  of the SP case. Substituting (63) and (64) into (11), we define the function  $z_s$  by:

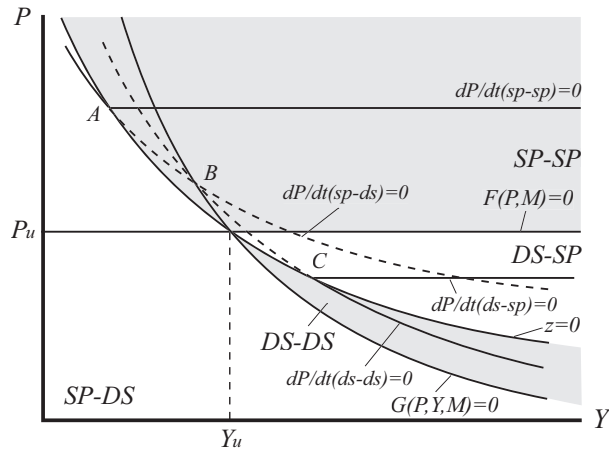
$$\begin{aligned} z_s(M) &\equiv \zeta(P_s^*(M), Y_s^*(M)) \\ &= Y_s^*(M)^* - \frac{\eta + \alpha}{\eta} K^{1/(1-\eta-\alpha)} P_s^*(M)^{*-\eta/(1-\eta-\alpha)} \\ &= N - Mf - \frac{\alpha\theta + \rho\eta}{\eta\theta} \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}}. \end{aligned} \quad (67)$$

Note that  $N > Mf$ . Then

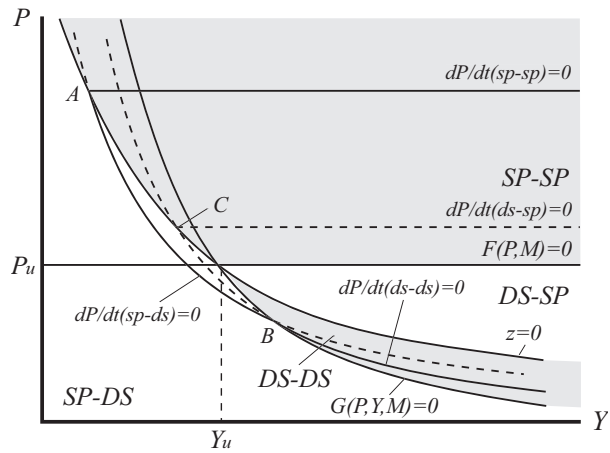
$$\frac{\partial z_s^*(M)}{\partial M} = -f - \frac{(\theta - \rho)(\alpha\theta + \rho\eta)}{\rho\theta\{\theta(1-\alpha) - \eta\}} \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\theta\{\eta-\rho(1-\alpha)\}}{\rho\{\theta(1-\alpha)-\eta\}}}. \quad (68)$$



(a)  $\rho(1 - \alpha) > \eta$



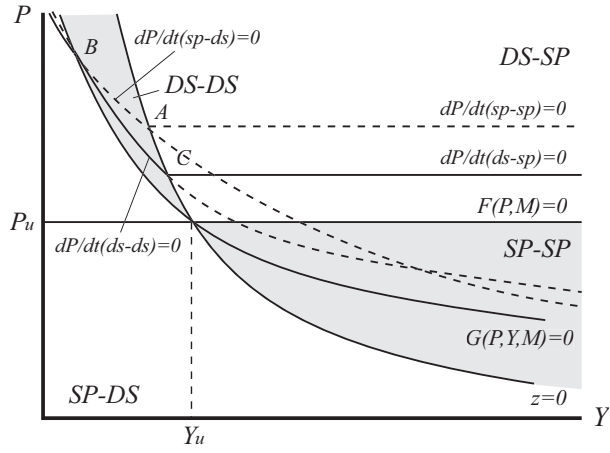
(b)  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$



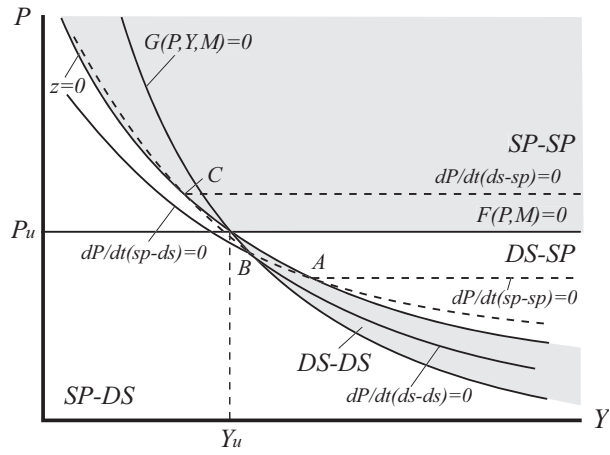
(c)  $\rho(1 - \alpha) < \eta$  and  $\theta(1 - \alpha) < \eta$

Figure 4: The Locus of  $\dot{P} = 0$  when  $Z(M) > 0$

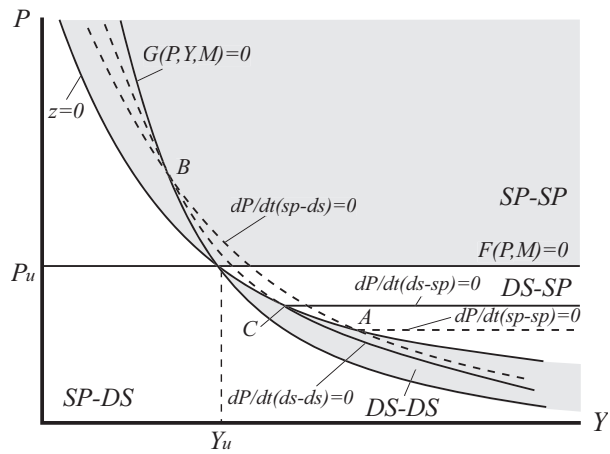




(a)  $\rho(1 - \alpha) > \eta$

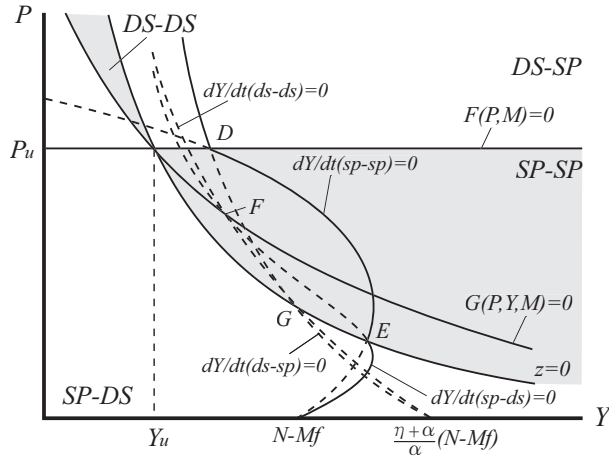


(b)  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$

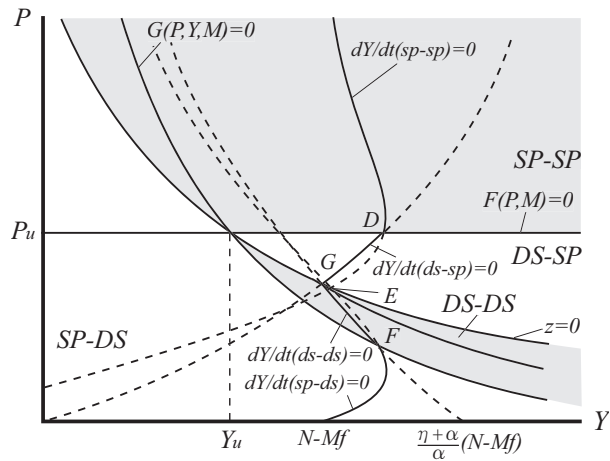


(c)  $\rho(1 - \alpha) < \eta$  and  $\theta(1 - \alpha) < \eta$

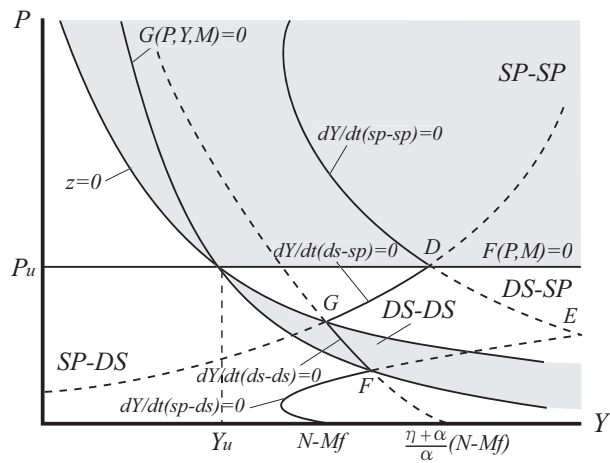
Figure 5: The Locus of  $\dot{P} = 0$  when  $Z(M) \leq 0$



(a)  $\rho(1 - \alpha) > \eta$

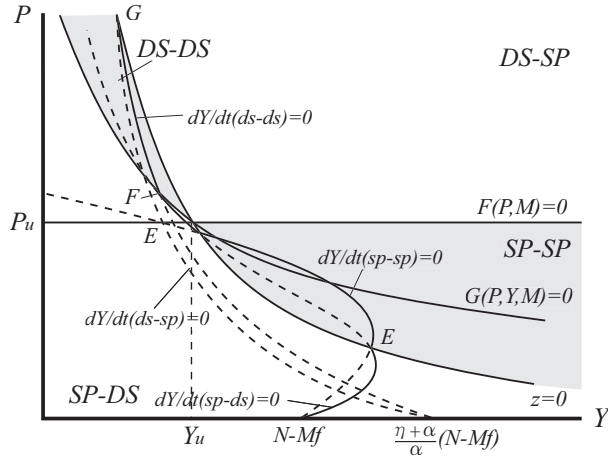


(b)  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$

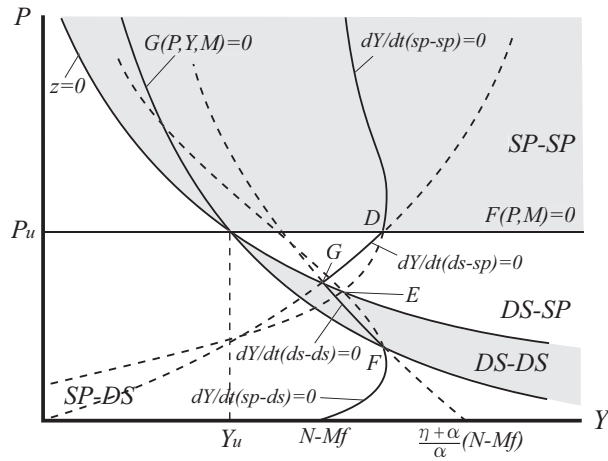


(c)  $\rho(1 - \alpha) < \eta$  and  $\theta(1 - \alpha) < \eta$

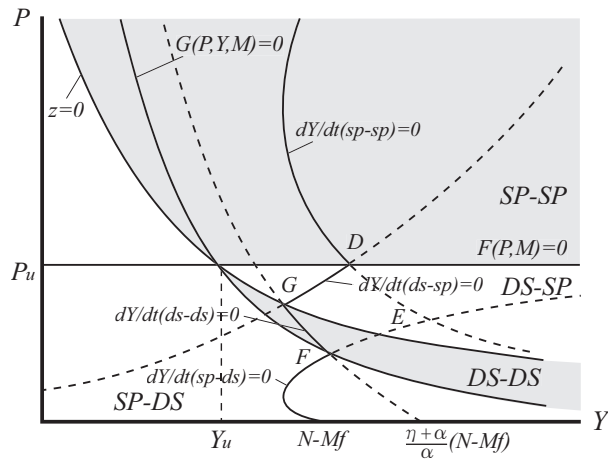
Figure 6: The Locus of  $\dot{Y} = 0$  when  $Z(M) > 0$



(a)  $\rho(1 - \alpha) > \eta$



(b)  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$



(c)  $\rho(1 - \alpha) < \eta$  and  $\theta(1 - \alpha) < \eta$

Figure 7: The Locus of  $\dot{Y} = 0$  when  $Z(M) \leq 0$

When  $\theta(1 - \alpha) > \eta$ , (68) is always negative. This means that  $z_s(\cdot)$  is monotonically decreasing for  $M$ . Since  $z_s(\cdot)$  becomes negative if  $M$  goes to infinity, there exists a unique point  $M^b$  such that  $z_s(M^b) = 0$  (See Fig.8(a)). Namely, the boundary level of the SP case is  $M^b$  defined below:

$$N - M^b f = \frac{\alpha\theta + \rho\eta}{\eta\theta} \left( \frac{\rho}{c\theta} \right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^b \frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}. \quad (69)$$

From (67),  $M$  satisfy the following inequality when  $Z(M) > 0$ :

$$N - Mf > \frac{\alpha\theta + \rho\eta}{\eta\theta} \left( \frac{\rho}{c\theta} \right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M \frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}. \quad (70)$$

Recall (62). When  $M$  is low enough,  $z_s(M)$  is positive. We thus define  $0 < M < M^b$  as the domain in the SP case (See Fig.8(a)).

When  $\theta(1 - \alpha) < \eta$ , there exists  $\hat{M}$  such that (68) is zero. Then we have:

$$\hat{M} = \left\{ \frac{\alpha\theta + \rho\eta}{\rho\{\eta - \theta(1 - \alpha)\}} \right\}^{\frac{\rho\{\eta - \theta(1 - \alpha)\}}{\theta\{\eta - \rho(1 - \alpha)\}}} \left\{ \frac{1}{f} \left( 1 - \frac{\rho}{\theta} \right) \right\}^{\frac{\rho\{\eta - \theta(1 - \alpha)\}}{\theta\{\eta - \rho(1 - \alpha)\}}} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1 - \alpha) - \eta\}}} K^{\frac{\rho}{\rho(1 - \alpha) - \eta}}. \quad (71)$$

From equation (68), we obtain the slope of the function  $z(\cdot)$  as:

$$\begin{aligned} \frac{\partial z_s(M)}{\partial M} &> 0, & \text{if } 0 < M < \hat{M}; \\ \frac{\partial z_s(M)}{\partial M} &= 0, & \text{if } M = \hat{M}; \text{ and} \\ \frac{\partial z_s(M)}{\partial M} &< 0, & \text{if } \hat{M} < M < \frac{N}{f}. \end{aligned}$$

Thus the function  $z_s(\cdot)$  has the maximum value at  $\hat{M}$ . From (67), we also have:

$$\begin{aligned} z_s(M) &\rightarrow -\infty & \text{as } M \rightarrow 0+; \text{ and} \\ z_s(M) &< 0 & \text{as } M \rightarrow \frac{N}{f} -. \end{aligned}$$

Let  $T = \frac{1}{f} \left( 1 - \frac{\rho}{\theta} \right)$ . Substituting (71) into (67) and rearranging,

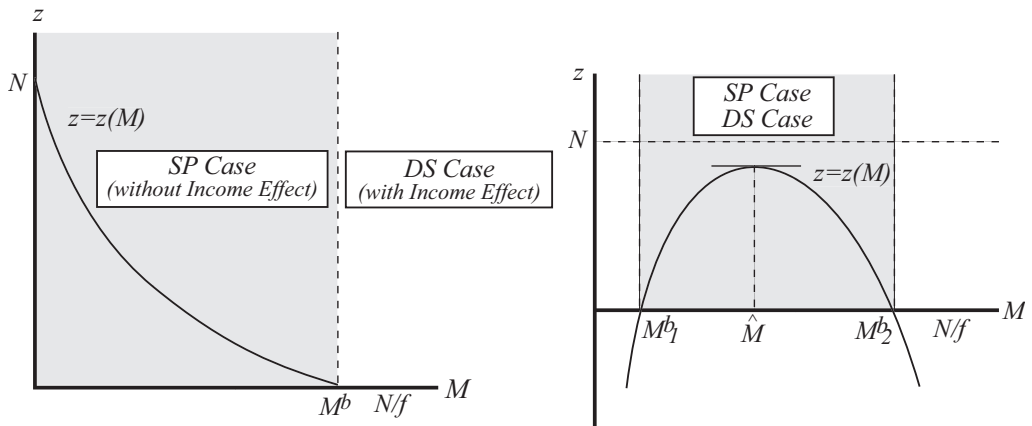
$$z_s(\hat{M}) = N - \frac{\eta - \rho(1 - \alpha)}{\eta} \left\{ \frac{\alpha\theta + \rho\eta}{\rho\{\eta - \theta(1 - \alpha)\}} \right\}^{\frac{\rho\{\eta - \theta(1 - \alpha)\}}{\theta\{\eta - \rho(1 - \alpha)\}}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\frac{\rho}{\rho(1-\alpha)-\eta}}.$$

Thus, there exist two boundaries  $M_1^b$  and  $M_2^b$ ,  $M_1^b < M_2^b$ , which satisfy (69) if

$$N > \frac{\eta - \rho(1 - \alpha)}{\eta} \left\{ \frac{\alpha\theta + \rho\eta}{\rho\{\eta - \theta(1 - \alpha)\}} \right\}^{\frac{\rho\{\eta - \theta(1 - \alpha)\}}{\theta\{\eta - \rho(1 - \alpha)\}}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\frac{\rho}{\rho(1-\alpha)-\eta}}.$$

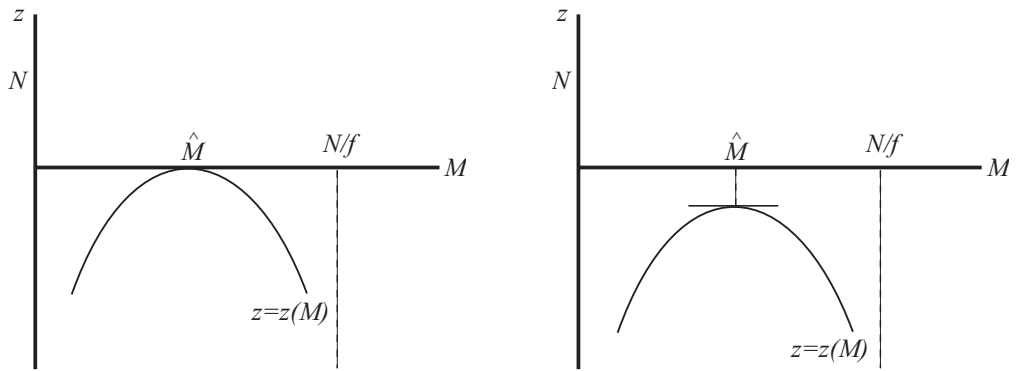
On the other hand,  $z_s(\cdot)$  is always negative when

$$N < \frac{\eta - \rho(1 - \alpha)}{\eta} \left\{ \frac{\alpha\theta + \rho\eta}{\rho\{\eta - \theta(1 - \alpha)\}} \right\}^{\frac{\rho\{\eta - \theta(1 - \alpha)\}}{\theta\{\eta - \rho(1 - \alpha)\}}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\frac{\rho}{\rho(1-\alpha)-\eta}}.$$



(a)  $\theta(1 - \alpha) > \eta$

(b)  $\theta(1 - \alpha) < \eta$



(c)  $\theta(1 - \alpha) < \eta$

(d)  $\theta(1 - \alpha) < \eta$

**Figure 8: Domains of the SP case and the DS case**

If there exist two boundaries, the domain of the SP case is denoted by  $M_1^b < M < M_2^b$  (See Fig.8(b)). On the other hand, if  $z_s(\cdot)$  is always negative, there exists no  $M$  which is of the SP case (See Fig.8(c)).

Next, we investigate the boundary of  $M$  of the DS case. Substituting (65) and (66) into (11), we have:

$$\begin{aligned}
z_d(M) &\equiv \zeta(P_d^*(M), Y_d^*(M)) \\
&= Y_d^*(M) - \frac{\eta + \alpha}{\eta} K^{1/(1-\eta-\alpha)} P_d^*(M)^{-\eta/(1-\eta-\alpha)} \\
&= \frac{\theta(\eta + \alpha)}{\alpha\theta + \rho\eta} (N - Mf) \\
&\quad - \frac{\eta + \alpha}{\eta} \left(\frac{\rho}{c\theta}\right)^{\eta/(1-\eta-\alpha)} K^{1/(1-\eta-\alpha)} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} (N - Mf) \right\}^{\frac{\eta(1-\theta)}{\theta(1-\eta-\alpha)}} M^{\frac{\eta(\theta-\rho)}{\rho\theta(1-\eta-\alpha)}}.
\end{aligned} \tag{72}$$

The domain in the DS case is defined  $z_d(M) \leq 0$ . Rearranging (72), we obtain the following inequality, which indicates  $z_d(M) \leq 0$ :

$$\left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} (N - Mf) \right\}^{\theta(1-\alpha)-\eta} \leq \left(\frac{\rho}{c\theta}\right)^\eta K^\theta M^{\frac{\eta(\theta-\rho)}{\rho}}. \tag{73}$$

If  $\theta(1 - \alpha) > \eta$ , we obtain from (73) the domain in the DS case:

$$N - Mf \leq \frac{\alpha\theta + \rho\eta}{\eta\theta} \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}}. \tag{74}$$

We also obtain the domain of  $M$  in the DS case when  $\theta(1 - \alpha) < \eta$ :

$$N - Mf \geq \frac{\alpha\theta + \rho\eta}{\eta\theta} \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}}. \tag{75}$$

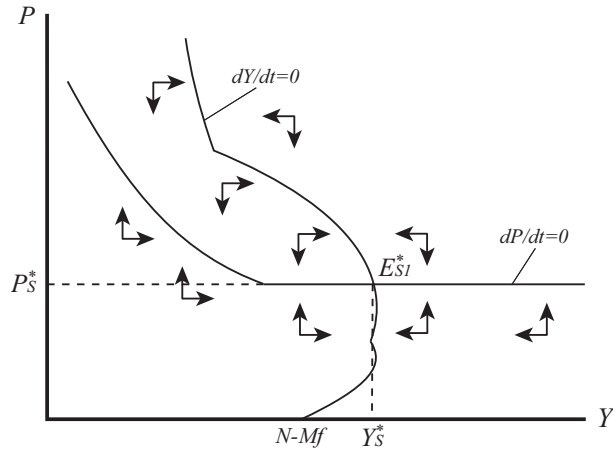
Compare (70), (74) and (75). When  $\theta(1 - \alpha) > \eta$ , we obtain the domain of  $M$  in the DS case as the complement of that in the SP case (See Fig.8(a)). On the other hand, if  $\theta(1 - \alpha) < \eta$ , the domain in the DS case coincides with that in the SP case (See Fig.8(b)). Furthermore, there could exist  $\hat{M}$  as a unique point of the DS case (See Fig.8(c)), and there is no  $M$  that satisfies either of two cases. (See Fig.8(d)).

Now we draw three patterns of the dynamics of  $\dot{P} = 0$  and  $\dot{Y} = 0$  according to the signs of  $Z(M)$ .

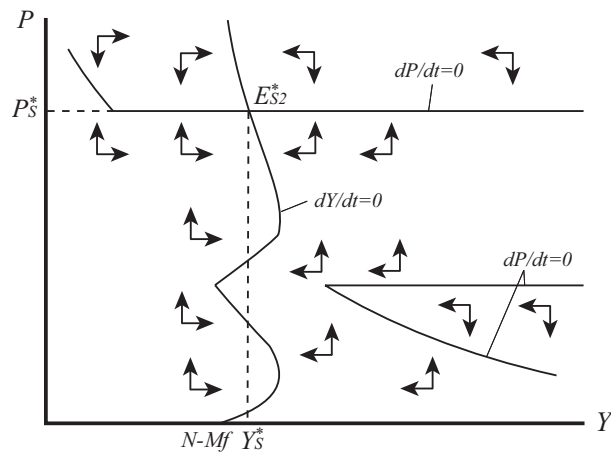
Under the condition of  $Z(M) > 0$ , there exists a unique equilibrium of the SP case when  $\theta(1 - \alpha) > \eta$  (See Fig.9(a) and Fig.9(b)). On the other hand, if  $\theta(1 - \alpha) < \eta$ , there exist two equilibria  $E_{s3}^*$ , and  $E_{d1}^*$  in the SP and the DS case, respectively (See Fig.9(c)).

Compare  $E_{s3}^*(P_{s3}^*(M), Y_{s3}^*(M))$  and  $E_{d1}^*(P_{d1}^*(M), Y_{d1}^*(M))$ . From (64) and (66), we obtain:

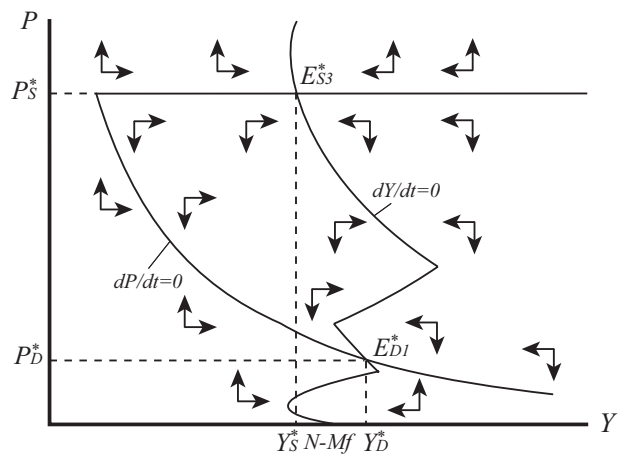
$$\begin{aligned}
&P_{s3}^*(M) - P_{d1}^*(M) \\
&= \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{\eta-\theta(1-\alpha)}} K^{\frac{\theta-1}{\theta(1-\alpha)-\eta}} M^{\frac{(\rho-\theta)(1-\eta-\alpha)}{\rho\{\theta(1-\alpha)-\eta\}}} - \left(\frac{\rho}{c\theta}\right)^{-1/\theta} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} (N - Mf) \right\}^{(\theta-1)/\theta} M^{\frac{\rho-\theta}{\rho\theta}} \\
&= \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{\eta-\theta(1-\alpha)}} K^{\frac{\theta-1}{\theta(1-\alpha)-\eta}} M^{\frac{(\rho-\theta)(1-\eta-\alpha)}{\rho\{\theta(1-\alpha)-\eta\}}} \\
&\quad \left[ 1 - \left(\frac{\rho}{c\theta}\right)^{\frac{\eta(\theta-1)}{\theta\{\eta-\theta(1-\alpha)\}}} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} (N - Mf) \right\}^{(\theta-1)/\theta} M^{\frac{\eta(\rho-\theta)(\theta-1)}{\rho\theta\{\theta(1-\alpha)-\eta\}}} \right].
\end{aligned}$$



(a)  $\rho(1 - \alpha) > \eta$

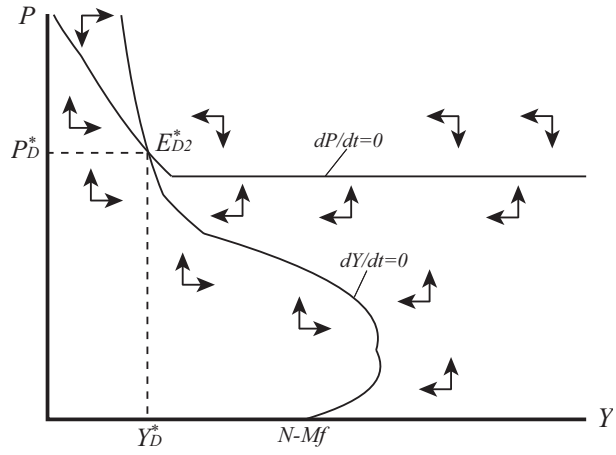


(b)  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$

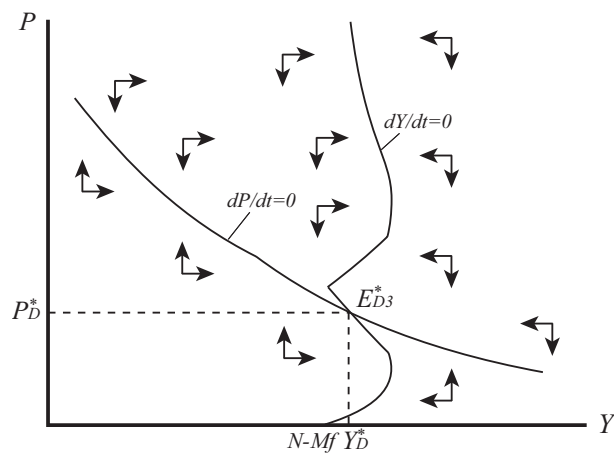


(c)  $\rho(1 - \alpha) < \eta$  and  $\theta(1 - \alpha) < \eta$

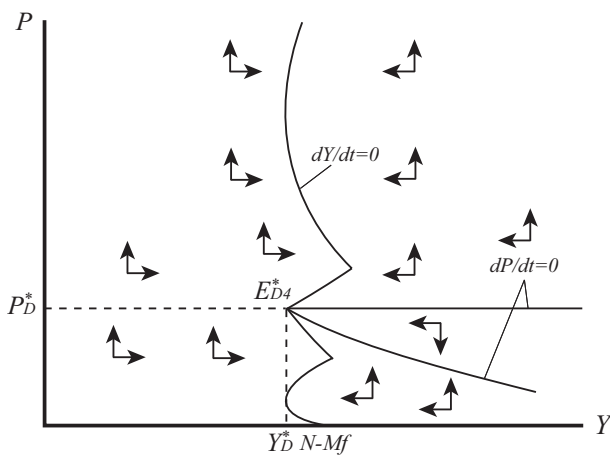
Figure 9: Dynamics of Short-run Equilibria when  $Z(M) > 0$



(a)  $\rho(1 - \alpha) > \eta$



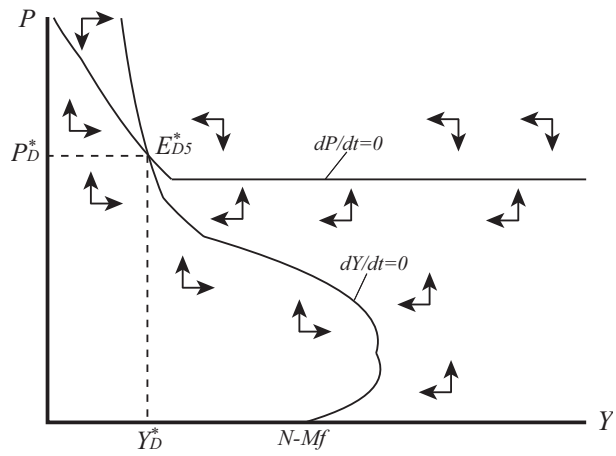
(b)  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$



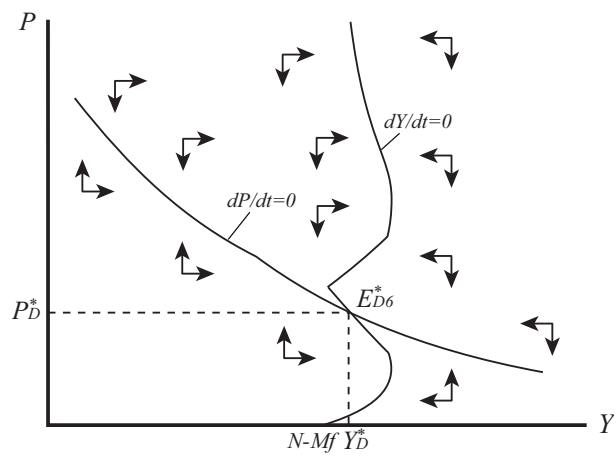
(c)  $\rho(1 - \alpha) < \eta$  and  $\theta(1 - \alpha) < \eta$

Figure 10: Dynamics of Short-run Equilibria when  $Z(M) = 0$

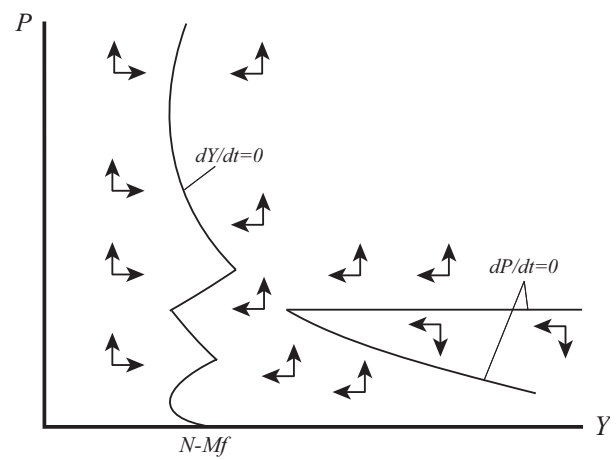




(a)  $\rho(1 - \alpha) > \eta$



(b)  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$



(c)  $\rho(1 - \alpha) < \eta$  and  $\theta(1 - \alpha) < \eta$

Figure 11: Dynamics of Short-run Equilibria when  $Z(M) < 0$

Thus the condition of  $P_{s3}^*(M) > P_{d1}^*(M)$  is:

$$1 > \left(\frac{\rho}{c\theta}\right)^{\frac{\eta(\theta-1)}{\theta\{\eta-\theta(1-\alpha)\}}} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} (N - Mf) \right\}^{(\theta-1)/\theta} M^{\frac{\eta(\rho-\theta)(\theta-1)}{\rho\theta\{\theta(1-\alpha)-\eta\}}}.$$

We consider when  $\theta(1 - \alpha) < \eta$ . Rearranging this inequality, we have:

$$N - Mf > \frac{\alpha\theta + \rho\eta}{\eta\theta} \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}}.$$

It means that  $P_{s3}^*(M)$  is always larger than  $P_{d1}^*(M)$  when  $\theta(1 - \alpha) < \eta$  and  $Z(M) > 0$ .

When  $Z(M) = 0$ , there exists a unique equilibrium in the DS case (See Fig.10(a), Fig.10(b) and Fig.10(c)).

Under the condition of  $Z(M) < 0$ , there exists a unique equilibrium in the DS case when  $\theta(1 - \alpha) > \eta$  (See Fig.11(a) and Fig.11(b)). On the other hand, there exists no equilibrium if  $\theta(1 - \alpha) < \eta$  (See Fig.11(c)).

As is seen above, the key thing is the sign of  $\theta(1 - \alpha) - \eta$ . In a general the SP case, namely when there is no income effect, it follows from equations (23) and (24) that

$$\frac{\partial \dot{P}}{\partial P} = \frac{\partial P^*(P, Y, M)}{\partial P} - 1 = \frac{g_P - g_{Q_P}}{g_{Q_P}}.$$

Suppose that  $Q_P(P^*(P, Y), Y^*(P, Y)) < 0$ , then the sign of  $\frac{\partial \dot{P}}{\partial P}$  coincides with  $\frac{PQ_P}{Q} - \frac{Pq_P}{q}$ , which is the difference between elasticities of  $Q(P, Y)$  and  $q(P, Y)$ . It can be shown that the sign of the difference is that of  $\theta(1 - \alpha) - \eta$ .

## 4.5 Stability and Instability of Short-run Equilibria

### 4.5.1 SP-SP Case

There exist short-run equilibria in the cases of the "SP-SP" and "DS-DS". In this section, we investigate whether these equilibria are dynamically stable or not.

We start with the SP-SP case. Let  $H_1(P)$  and  $H_2(P, Y)$  be the LHS's of (42) and (43), respectively. By (63) and (64), we have Jacobian matrix of the dynamics system at  $(P_s^*(M), Y_s^*(M))$  is:

$$A_1^*(P_s^*, Y_s^*) \equiv \begin{bmatrix} \frac{\partial H_1(P_s^*)}{\partial P} & \frac{\partial H_1(P_s^*)}{\partial Y} \\ \frac{\partial H_2(P_s^*, Y_s^*)}{\partial P} & \frac{\partial H_2(P_s^*, Y_s^*)}{\partial Y} \end{bmatrix}$$

where

$$\begin{aligned} \frac{\partial H_1(P_s^*)}{\partial P} &= \frac{\theta(1 - \alpha) - \eta}{(\alpha - 1)(\theta - \rho)}, & \frac{\partial H_1(P_s^*)}{\partial Y} &= 0, \\ \frac{\partial H_2(P_s^*, Y_s^*)}{\partial P} &= \frac{\{\rho(1 - \alpha) - \eta\}^2}{(\alpha - 1)(\theta - \rho)(1 - \eta - \alpha)} \left(\frac{\rho}{c\theta}\right)^{\frac{1-\alpha}{\theta(1-\alpha)-\eta}} K^{1/\{\theta(1-\alpha)-\eta\}} M^{\frac{(\theta-\rho)(1-\alpha)}{\rho\{\theta(1-\alpha)-\eta\}}}, & \text{and} \\ \frac{\partial H_2(P_s^*, Y_s^*)}{\partial Y} &= -1. \end{aligned}$$

Then the trace and determinant are:

$$\begin{aligned} tr A_1(P_s^*, Y_s^*) &= \frac{\partial H_1(P_s^*)}{\partial P} + \frac{\partial H_2(P_s^*, Y_s^*)}{\partial Y} = \frac{\theta(1-\alpha) - \eta}{(\alpha-1)(\theta-\rho)} - 1, \quad \text{and} \\ det A_1(P_s^*, Y_s^*) &= \frac{\partial H_1(P_s^*)}{\partial P} \frac{\partial H_2(P_s^*, Y_s^*)}{\partial Y} - \frac{\partial H_1(P_s^*)}{\partial Y} \frac{\partial H_2(P_s^*, Y_s^*)}{\partial P} = \frac{\theta(1-\alpha) - \eta}{(1-\alpha)(\theta-\rho)}. \end{aligned}$$

When  $\theta(1-\alpha) > \eta$ ,  $tr A_1(P^*, Y^*) < 0$  and  $det A_1(P^*, Y^*) > 0$ . It means that the SP short-run equilibrium is locally stable if  $\theta(1-\alpha) > \eta$ . On the other hand,  $tr A_1(P^*, Y^*) < 0$  and  $det A_1(P^*, Y^*) < 0$  if  $\theta(1-\alpha) < \eta$ . That is, the short-run SP equilibrium is locally unstable. We make the point that complementarity and decreasing marginal cost are necessary conditions for short-run instability.

#### 4.5.2 DS-DS Case

Next, we move to the DS-DS case. Let  $H_3(P, Y)$  and  $H_4(P, Y)$  be the LHS's of (60) and (61), respectively. By (65) and (66), we have Jacobian matrix at  $(P_d^*(M), Y_d^*(M))$  is:

We obtain the following Jacobian matrix from (112) and (119):

$$A_2^*(P_d^*, Y_d^*) \equiv \begin{bmatrix} \frac{\partial H_3(P_d^*, Y_d^*)}{\partial P} & \frac{\partial H_3(P_d^*, Y_d^*)}{\partial Y} \\ \frac{\partial H_4(P_d^*, Y_d^*)}{\partial P} & \frac{\partial H_4(P_d^*, Y_d^*)}{\partial Y} \end{bmatrix},$$

where

$$\begin{aligned} \frac{\partial H_3(P_d^*, Y_d^*)}{\partial P} &= \frac{\alpha\theta + \eta\rho^2}{\alpha(\rho - \theta)}, \\ \frac{\partial H_3(P_d^*, Y_d^*)}{\partial Y} &= \frac{\eta(1-\rho)(\alpha + \rho\eta)}{\alpha(\eta + \alpha)(\rho - \theta)} \left(\frac{\rho}{c\theta}\right)^{-1/\theta} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} (N - Mf) \right\}^{-1/\theta} M^{\frac{\rho-\theta}{\rho\theta}}, \\ \frac{\partial H_4(P_d^*, Y_d^*)}{\partial P} &= \frac{\rho^2(\eta + \alpha)}{\alpha(\rho - \theta)} \left(\frac{\rho}{c\theta}\right)^{1/\theta} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} (N - Mf) \right\}^{1/\theta} M^{\frac{\theta-\rho}{\rho\theta}}, \quad \text{and} \\ \frac{\partial H_4(P_d^*, Y_d^*)}{\partial Y} &= \frac{\alpha(\rho - \theta) - \rho\eta(1-\rho)}{\alpha(\theta - \rho)}. \end{aligned}$$

Then the trace and determinant are:

$$\begin{aligned} tr A_2(P_d^*, Y_d^*) &= \frac{\partial H_3(P_d^*, Y_d^*)}{\partial P} + \frac{\partial H_4(P_d^*, Y_d^*)}{\partial Y} \\ &= \frac{\alpha\theta + \eta\rho^2}{\alpha(\rho - \theta)} + \frac{\alpha(\rho - \theta) - \rho\eta(1-\rho)}{\alpha(\theta - \rho)} = \frac{\alpha(\theta - \rho) + \alpha\theta + \rho\eta}{\alpha(\rho - \theta)} < 0, \quad \text{and} \\ det A_2(P_d^*, Y_d^*) &= \frac{\partial H_3(P_d^*, Y_d^*)}{\partial P} \frac{\partial H_4(P_d^*, Y_d^*)}{\partial Y} - \frac{\partial H_3(P_d^*, Y_d^*)}{\partial Y} \frac{\partial H_4(P_d^*, Y_d^*)}{\partial P} \\ &= \frac{(\alpha\theta + \eta\rho^2)\{\alpha(\rho - \theta) - \rho\eta(1-\rho)\}}{\alpha^2(\rho - \theta)^2} - \frac{\rho^2\eta(1-\rho)(\alpha + \rho\eta)}{\alpha^2(\rho - \theta)^2} = \frac{\alpha\theta + \eta\rho}{\alpha(\theta - \rho)} > 0. \end{aligned}$$

Hence, the short-run DS equilibrium is locally stable.

Under the condition of  $Z(M) > 0$  and  $\theta(1-\alpha) > \eta$ , the short-run equilibria of the SP case  $E_{S_1}^*$  (Fig.9(a)) and  $E_{S_2}^*$  (Fig.9(b)) are stable.

When  $\theta(1-\alpha) < \eta$ , the short-run equilibrium of the SP case  $E_{S_3}^*$  (Fig.9(c)) is unstable, while the short-run equilibrium of the DS case  $E_{D_1}^*$  is stable.

When  $Z(M) = 0$ , there exists stable equilibria  $E_{S2}^*$  and  $E_{S3}^*$  of the DS case if  $\theta(1 - \alpha) > \eta$  (See Figs.10(a) and 10(b)). Furthermore, when  $Z(M) = 0$  and  $\theta(1 - \alpha) < \eta$ ,  $E_{D4}^*$  is "semi-stable" in the sense that it is unstable but there is a continuum of stable paths (See Fig.10(c)).

Under the condition of  $Z(M) < 0$ , there exists a stable equilibrium of the DS case when  $\theta(1 - \alpha) > \eta$  (See Figs.11(a) and 11(b)).

Then we obtain the following theorems:

**Theorem 1** *If the differentiated goods are substitutes, i.e.,  $\rho(1 - \alpha) > \eta$ , there exists a unique and stable short-run equilibrium.*

**Theorem 2** *If the differentiated goods are complements and marginal cost is not highly decreasing, i.e.,  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$ , there exists a unique and stable short-run equilibrium.*

**Theorem 3** *If the differentiated goods are complements and marginal cost is highly decreasing, i.e.,  $\theta(1 - \alpha) < \eta$ , there exist no, one or two short-run equilibria according to the size of  $M$ . If two equilibria exist, one is stable and the other is unstable.*

#### 4.6 Stability of Long-run Equilibrium: The Case without Income Effect

From (64), the individual price, quality, and the aggregate utility of the short-run SP equilibrium are:

$$\begin{aligned} p_s^*(M) &= \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{\eta-\theta(1-\alpha)}} K^{\frac{\theta-1}{\theta(1-\alpha)-\eta}} M^{\frac{(1-\theta)\{\rho(1-\alpha)-\eta\}}{\rho\{\theta(1-\alpha)-\eta\}}}, \\ q_s^*(M) &= \left(\frac{\rho}{c\theta}\right)^{\frac{1-\alpha}{\theta(1-\alpha)-\eta}} K^{1/\{\theta(1-\alpha)-\eta\}} M^{\frac{\eta-\rho(1-\alpha)}{\rho\{\theta(1-\alpha)-\eta\}}}, \\ Q_s^*(M) &= \left(\frac{\rho}{c\theta}\right)^{\frac{1-\alpha}{\theta(1-\alpha)-\eta}} K^{1/\{\theta(1-\alpha)-\eta\}} M^{\frac{(\theta-\rho)(1-\alpha)}{\rho\{\theta(1-\alpha)-\eta\}}}. \end{aligned}$$

Thus the short-run SP equilibrium profit of firm  $i$  is:

$$\pi_s^*(M) = \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\theta\{\eta-\rho(1-\alpha)\}}{\rho\{\theta(1-\alpha)-\eta\}}} \left(1 - \frac{\rho}{\theta}\right) - f. \quad (76)$$

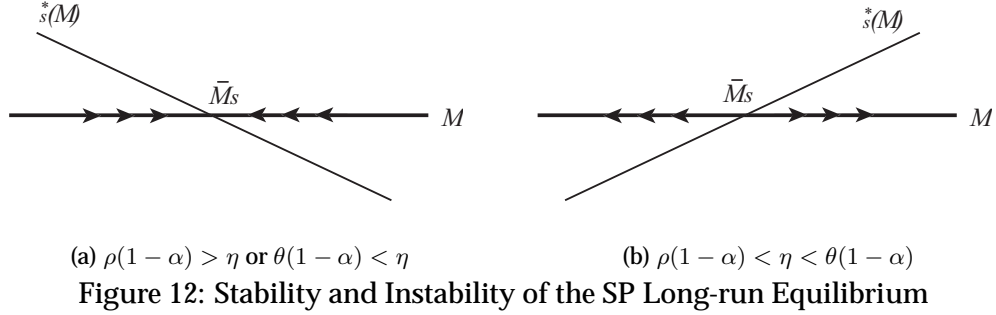
The size of  $M$  of industry expands when the individual profit level is positive and shrinks when the level is negative. Note that the industry never shrink from  $M = 0$ . Hence, we call  $M \geq 0$  a *long-run SP equilibrium* if  $\pi_s^*(M) = 0$  or  $M = 0$  and  $\lim_{M \rightarrow 0} \pi_s^*(M) < 0$ .

When  $\pi_s^*(M) = 0$ , we obtain the long-run SP equilibrium size of industry:

$$\bar{M}_s = \left(\frac{\rho}{c\theta}\right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\rho\{\theta(1-\alpha)-\eta\}}{\theta\{\rho(1-\alpha)-\eta\}}}. \quad (77)$$

Substituting (77) into (63) and (64), we obtain:

$$\begin{aligned} P_s^*(\bar{M}_s) &= \left(\frac{\rho}{c\theta}\right)^{\frac{\rho(1-\eta-\alpha)}{\theta\{\eta-\rho(1-\alpha)\}}} K^{\frac{1-\rho}{\eta-\rho(1-\alpha)}} T^{\frac{(\theta-\rho)(1-\eta-\alpha)}{\theta\{\eta-\rho(1-\alpha)\}}}, \\ Y_s^*(\bar{M}_s) &= N. \end{aligned}$$



We also have:

$$p_s^*(\bar{M}_s) = \left(\frac{\rho}{c\theta}\right)^{-1/\theta} T^{(1-\theta)/\theta},$$

$$q_s^*(\bar{M}_s) = \left(\frac{\rho}{c\theta}\right)^{1/\theta} T^{-1/\theta},$$

$$Q_s^*(\bar{M}_s) = \left(\frac{\rho}{c\theta}\right)^{\frac{\rho(1-\alpha)}{\theta\{\rho(1-\alpha)-\eta\}}} K^{1/\{\rho(1-\alpha)-\eta\}} T^{\frac{(\theta-\rho)(1-\alpha)}{\theta\{\rho(1-\alpha)-\eta\}}}.$$

From equation (77), we have:

$$\frac{\partial \pi^*(M)}{\partial M} = \frac{(\theta - \rho)\{\eta - \rho(1 - \alpha)\}}{\rho\{\theta(1 - \alpha) - \eta\}} \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\frac{\theta}{\theta(1-\alpha)-\eta}} M^{\frac{\theta\{\eta-\rho(1-\alpha)\}}{\rho\{\theta(1-\alpha)-\eta\}}-1}.$$

Recall  $\theta > \rho$ . The above formula tells us:

$$\frac{\partial \pi(M)}{\partial M} < 0 \quad \text{if} \quad \frac{\eta}{1 - \alpha} < \rho \quad \text{or} \quad \theta < \frac{\eta}{1 - \alpha},$$

$$\frac{\partial \pi(M)}{\partial M} > 0 \quad \text{if} \quad \rho < \frac{\eta}{1 - \alpha} < \theta.$$

This implies that the individual profit increases as the size of industry expands only if the products are complements, and that it may decrease as the size expands even if these are complements (See Fig.12). Notice that there exists a "reswitching" phenomenon of stability (See the fourth column of Table.1)

	$\rho(1 - \alpha) > \eta$	$\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$	$\theta(1 - \alpha) < \eta$
	substitutes	complements	complements
Short-run Equilibrium	locally stable $E_{S1}^*$	locally stable $E_{S2}^*$	locally unstable (saddle point) $E_{S3}^*$
Long-run Equilibrium	globally stable	locally unstable	globally stable

Table 1: Stability and Instability in the Short-run and in the Long-run (The Case without Income Effect)

#### 4.7 Stability of Long-run Equilibrium: The Case with Income Effect

From (65) and (66):

$$\begin{aligned} p_d^*(M) &= \left(\frac{\rho}{c\theta}\right)^{-1/\theta} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta}(N - Mf) \right\}^{(\theta-1)/\theta} M^{(1-\theta)/\theta}, \\ q_d^*(M) &= \left(\frac{\rho}{c\theta}\right)^{1/\theta} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta}(N - Mf) \right\}^{1/\theta} M^{-1/\theta}, \\ Q_d^*(M) &= \left(\frac{\rho}{c\theta}\right)^{1/\theta} \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta}(N - Mf) \right\}^{1/\theta} M^{\frac{\theta-\rho}{\rho\theta}}. \end{aligned}$$

Thus the profit of each firm is:

$$\pi_d^*(M) = \frac{\eta(\theta - \rho)}{\alpha\theta + \rho\eta} \frac{N - Mf}{M} - f. \quad (78)$$

We call  $M \geq 0$  a *long-run DS equilibrium* if  $\pi_d^*(M) = 0$  or  $M = 0$  and  $\lim_{M \rightarrow 0} \pi_d^*(M) < 0$ .

We obtain:

$$\frac{\partial \pi(M)}{\partial M} = \frac{\eta(\rho - \theta)}{\alpha\theta + \rho\eta} \frac{N}{M^2} < 0.$$

It means that the long-run equilibrium is always stable (See Fig.13). The long-run equilibrium size of industry is:

$$\bar{M}_d = \frac{\eta}{\eta + \alpha} NT. \quad (79)$$

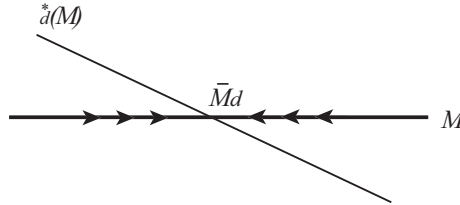


Figure 13: Stability of the DS Long-run Equilibrium

Substituting (78) into (65) and (66), we obtain:

$$\begin{aligned} P_d^*(\bar{M}_d) &= \left(\frac{\rho}{c\theta}\right)^{-1/\theta} \left(\frac{\eta}{\eta + \alpha} N\right)^{(\rho-1)/\rho} T^{\frac{\rho-\theta}{\rho\theta}}, \\ Y_d^*(\bar{M}_d) &= N. \end{aligned}$$

We also have:

$$\begin{aligned} p_d^*(\bar{M}_d) &= \left(\frac{\rho}{c\theta}\right)^{-1/\theta} T^{(1-\theta)/\theta}, \\ q_d^*(\bar{M}_d) &= \left(\frac{\rho}{c\theta}\right)^{1/\theta} T^{-1/\theta}, \\ Q_d^*(\bar{M}_d) &= \left(\frac{\rho}{c\theta}\right)^{1/\theta} \left(\frac{\eta}{\eta + \alpha} N\right)^{1/\rho} T^{\frac{\theta-\rho}{\rho\theta}}. \end{aligned}$$

	$\rho(1 - \alpha) > \eta$ substitutes	$\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$ complements	$\theta(1 - \alpha) < \eta$ complements
Short-run Equilibrium when $Z(M) > 0$	-	-	locally stable $E_{D1}^*$
Short-run Equilibrium when $Z(M) = 0$	locally stable $E_{D2}^*$	locally stable $E_{D3}^*$	semi-stable $E_{D4}^*$
Short-run Equilibrium when $Z(M) < 0$	locally stable $E_{D5}^*$	locally stable $E_{D6}^*$	-
Long-run Equilibrium	globally stable	globally stable	globally stable

Table 2: Stability and Instability in the Short-run and in the Long-run (The Case with Income Effect)

## 5 Existence and Non-Existence of Income Effects

### 5.1 Two Kinds of Demand Functions

There exists no income effect when either  $Y$  or  $P$  is high enough, but it emerges with increase in  $Y$  or  $P$ . In this section, we investigate the boundary dividing the SP case and the DS case, i.e., the border between existence and non-existence of income effects.

### 5.2 Conversion of Profit Function

We also consider the connecting point of profit functions. In the SP case, the individual profit is:

$$\pi_s(M) = \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\frac{\theta}{\theta(1-\alpha)-\eta}} M^{\frac{\theta\{\eta-\rho(1-\alpha)\}}{\rho\{\theta(1-\alpha)-\eta\}}} \left(1 - \frac{\rho}{\theta}\right) - f. \quad (80)$$

In the the DS case, the profit of each firm is:

$$\pi_d(M) = \frac{\eta(\theta - \rho)}{\alpha\theta + \rho\eta} \left(\frac{N}{M} - f\right) - f. \quad (81)$$

From (69) and (80), we obtain the profit at the boundary value  $M^b$ :

$$\begin{aligned} \pi_s(M^b) &= \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\frac{\theta}{\theta(1-\alpha)-\eta}} M^{b\frac{\theta\{\eta-\rho(1-\alpha)\}}{\rho\{\theta(1-\alpha)-\eta\}}} \left(1 - \frac{\rho}{\theta}\right) - f \\ &= \frac{\eta(\theta - \rho)}{\alpha\theta + \eta\rho} \left(\frac{N}{M^b} - f\right) - f. \end{aligned} \quad (82)$$

It is clear from (81) and (82) that  $\pi_d(M^b) = \pi_s(M^b)$ . From (82),

$$\begin{aligned} \pi_s(M^b) > 0 &\quad \text{if } M^b < \frac{\eta}{\eta + \alpha} NT, \\ \pi_s(M^b) < 0 &\quad \text{if } M^b > \frac{\eta}{\eta + \alpha} NT. \end{aligned}$$

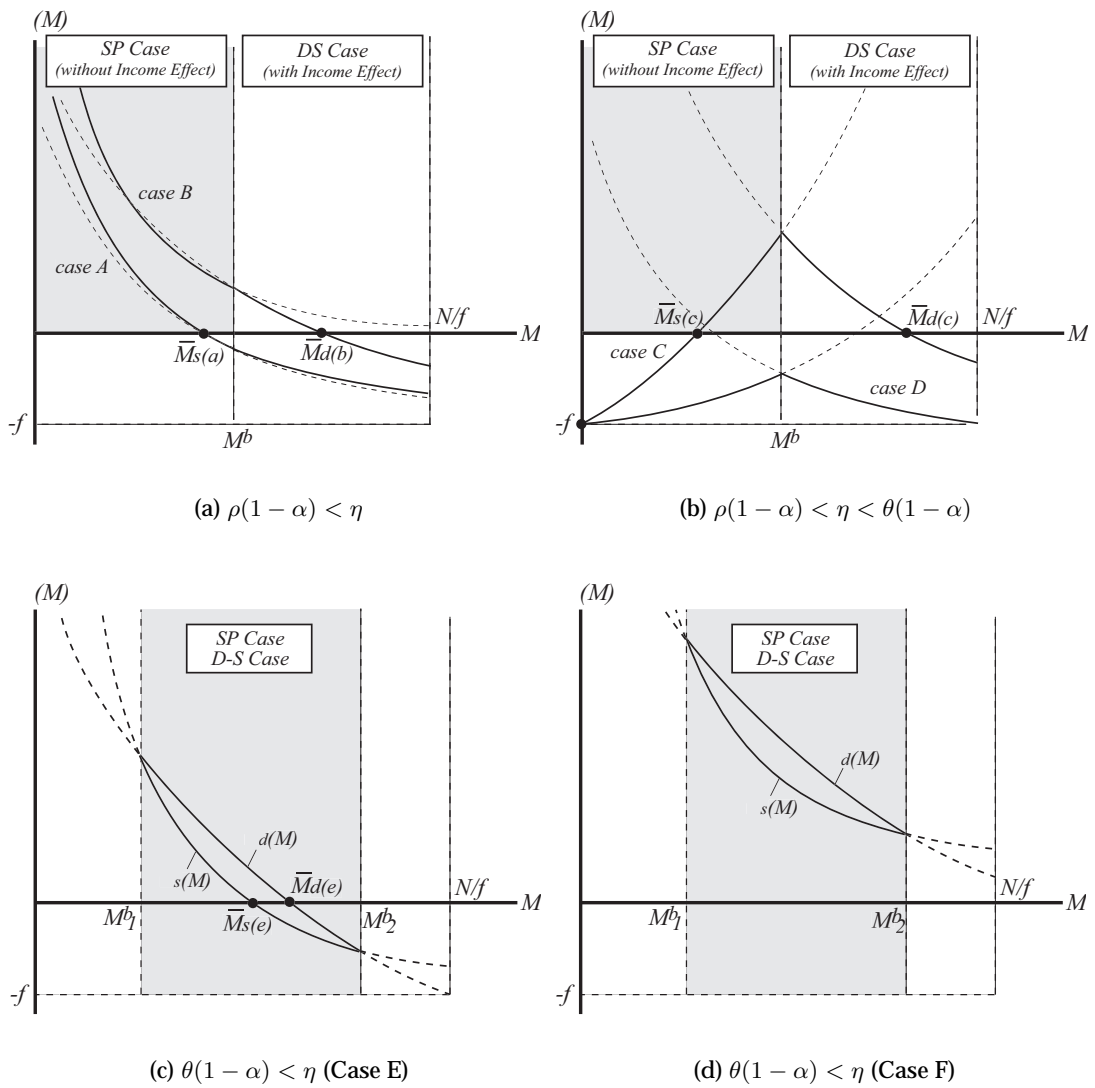


Figure 14: The Conversion of Profit Function



We can prove that  $M^b$  is in proportion to the size of  $N/f$ . In other words,  $M^b$  is relatively large if  $N$  is large enough or  $f$  is small enough.

There are two types of graphs of  $\pi_s(\cdot)$  depending on the sign of  $\theta(1 - \alpha) - \eta$  (See Fig.14). First, we consider the case of  $\theta(1 - \alpha) > \eta$ . From (80), we have:

$$\frac{\partial \pi_s(M)}{\partial M} = \frac{\theta\{\eta - \rho(1 - \alpha)\}}{\rho\{\theta(1 - \alpha) - \eta\}} \left(\frac{\rho}{c\theta}\right)^{\eta/\{\theta(1 - \alpha) - \eta\}} K^{\frac{\theta}{\theta(1 - \alpha) - \eta}} M^{\frac{\theta\{\eta - \rho(1 - \alpha)\}}{\rho\{\theta(1 - \alpha) - \eta\}} - 1} \left(1 - \frac{\rho}{\theta}\right) - f. \quad (83)$$

From (83), we obtain:

$$\begin{aligned} \frac{\partial \pi_s(M)}{\partial M} &> 0 && \text{if } \rho(1 - \alpha) < \eta < \theta(1 - \alpha), \\ \frac{\partial \pi_s(M)}{\partial M} &< 0 && \text{if } \theta(1 - \alpha) < \eta \text{ and } \rho(1 - \alpha) > \eta. \end{aligned}$$

Hence,  $\pi_s(\cdot)$  is monotonically increasing if  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$ , and monotonically decreasing if  $\theta(1 - \alpha) < \eta$  and  $\rho(1 - \alpha) > \eta$ . From (81),  $\pi_d(\cdot)$  is always monotonically decreasing.

First, we consider  $\pi_s(\cdot)$ . When  $\theta(1 - \alpha) > \eta$ , the domain of the SP case is defined  $0 < M < M^b$  (See Fig.8(a)). If  $\rho(1 - \alpha) > \eta$ , i.e.,  $\theta$  is relatively high and the differentiated goods are substitutes,  $\pi_s(\cdot)$  is monotonically decreasing (See Fig.14(a)). In addition, when  $M^b < \frac{\eta}{\eta + \alpha} NT$ , there exists a unique stable equilibrium ( $\bar{M}_s(a)$ ) on the domain of the SP case (Case A, See Fig.14(a)). On the other hand, there is no equilibrium of the SP case if  $M^b > \frac{\eta}{\eta + \alpha} NT$ , (Case B, See Fig.14(a)).

When  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$ , namely, the differentiated goods are complements and  $\theta$  is relatively high,  $\pi_s(\cdot)$  is monotonically increasing. Furthermore, under the condition of  $M^b < \frac{\eta}{\eta + \alpha} NT$ , there exists a unique unstable equilibrium ( $\bar{M}_s(c)$ ) of the SP case (Case C, See Fig.14(a)). On the other hand, when  $M^b > \frac{\eta}{\eta + \alpha} NT$ , there is no equilibrium of the SP case (Case D, See Fig.14(a)).

Next, we consider  $\pi_d(\cdot)$  on the domain of  $M^b < M < N/f$ , that is, in the DS case. Since  $\pi_d(\cdot)$  is monotonically decreasing and goes to  $-f$  when  $M$  approaches to  $N/f$ , there exists a stable equilibrium in Case B ( $\bar{M}_d(b)$ ) and Case C ( $\bar{M}_d(c)$ ) (See Fig.14(a)).

Namely, under the condition of  $\theta(1 - \alpha) > \eta$ , we obtain four cases of conversion of profit function. In Case A, there exists a unique stable equilibrium ( $\bar{M}_s(a)$ ) in the SP case. In Case B, there exists a unique stable equilibrium ( $\bar{M}_d(b)$ ) in the DS case.

We also show that in Case C there exist equilibria  $\bar{M}_s(c)$  and  $\bar{M}_d(c)$  on the domains. The equilibrium of the SP case is stable and that of the DS case is unstable. Furthermore, we can prove that there exists no equilibrium on any domain (Case D).

Next, we consider the condition of  $\theta(1 - \alpha) < \eta$ . In this case, the domain of the SP case is  $M_1^b < M < M_2^b$  and that of the DS case is  $M_1^b \leq M \leq M_2^b$  (See Fig.8(b)).

Under this condition, both  $\pi_s(\cdot)$  and  $\pi_d(\cdot)$  is monotonically decreasing. We consider the condition that  $\bar{M}_s$  exists on the domain of the SP case.

Substituting (77) into (67), we have:

$$z_s(\bar{M}_s) = N - \frac{\eta + \alpha}{\eta} \left(\frac{\rho}{c\theta}\right)^{\frac{\rho\eta}{\theta\{\rho(1 - \alpha) - \eta\}}} K^{\rho/\{\theta(1 - \alpha) - \eta\}} T^{\frac{\eta(\theta - \rho)}{\theta\{\rho(1 - \alpha) - \eta\}}}. \quad (84)$$

When  $z_s(\bar{M}_s) > 0$ ,  $\bar{M}_s$  exists in the SP case, that is,

$$\frac{\eta}{\eta + \alpha} N > \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}}. \quad (85)$$

Substituting (79) into (72), we have:

$$z_d(\bar{M}_d) = N - \frac{\eta + \alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\eta/(1-\eta-\alpha)} K^{1/(1-\eta-\alpha)} \left( \frac{\eta}{\eta + \alpha} N \right)^{\frac{\eta(1-\rho)}{\rho(1-\eta-\alpha)}} T^{\frac{\eta(\theta-\rho)}{\rho\theta(1-\eta-\alpha)}}. \quad (86)$$

When  $z_d(\bar{M}_d) \leq 0$ ,  $\bar{M}_d$  exists in the DS case. If  $\theta(1 - \alpha) < \eta$ , we obtain inequality which satisfy  $z_d(\bar{M}_d) \leq 0$  as follows:

$$\frac{\eta}{\eta + \alpha} N \geq \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}}. \quad (87)$$

Note that (87) coincides with (85) except equality. We draw graphs in Fig.14(b), Fig.14(c) and Fig.14(d). When either (85) or (87) are satisfied, there exist two equilibria  $\bar{M}_s(e)$ ,  $\bar{M}_d(e)$  and both of them are stable (Case E, See Fig.14(b)). Furthermore, rearranging (87), we obtain:

$$\frac{\eta}{\eta + \alpha} NT \geq \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\rho\{\theta(1-\alpha)-\eta\}}{\theta\{\rho(1-\alpha)-\eta\}}}.$$

From (77) into (79), we obtain the inequality when both  $\bar{M}_s(e)$  and  $\bar{M}_d(e)$  exist:

$$\bar{M}_d(e) > \bar{M}_s(e). \quad (88)$$

When only (87) is satisfied, that is,

$$\frac{\eta}{\eta + \alpha} N = \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}},$$

there exists a unique stable equilibrium  $\bar{M}_d(f)$  (Case F, See Fig.14(c)). When neither (85) nor (87) are satisfied, there exists no equilibrium (Case G, See Fig.14(d)).

Then we obtain the following theorems:

**Theorem 4** *If the differentiated goods are substitutes, i.e.,  $\rho(1 - \alpha) > \eta$ , there exists a unique and stable long-run equilibrium.*

**Theorem 5** *If the differentiated goods are complements and marginal cost is not highly decreasing, i.e.,  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$ , there exist three long-run equilibria which are stable one with no firms, unstable one and stable one with firms.*

**Theorem 6** *If the differentiated goods are complements and marginal cost is highly decreasing, i.e.,  $\theta(1 - \alpha) < \eta$ , two kinds of equilibria exist.*

### 5.3 Social Welfare

In this section, we consider social welfare in the SP case and the DS case. We investigate the optimal size of industry that maximize social welfare subject to short-run equilibrium, and compare it with long-run equilibrium size of industry. Furthermore, we compare social welfare in the SP case with that in the DS case when  $\theta(1 - \alpha) < \eta$ , that is, when  $\bar{M}_s$  and  $\bar{M}_d$  exist on the same domain.

The social welfare  $W$  is calculated by the following formula:

$$\begin{aligned}
W &= \frac{\beta}{\eta} \left( \int_0^M q(i)^\rho di \right)^{\eta/\rho} x^\alpha + z \\
&= \frac{\beta}{\eta} Q^\eta \left( \frac{\alpha}{\eta\gamma} PQ \right)^\alpha + N + \int_0^M (p(i)q(i) - cq(i)^\theta - f) di - PQ - \gamma x \\
&= \frac{\beta}{\eta} Q^\eta \left( \frac{\alpha}{\eta\gamma} PQ \right)^\alpha + N - Mf - cM^{(\rho-\theta)/\rho} Q^\theta - \frac{\alpha}{\eta} PQ. \tag{89}
\end{aligned}$$

#### 5.3.1 SP case

When  $0 < M < M^b$ , we obtain social welfare  $W_s(M)$  in the the SP case as:

$$W_s(M) = N - Mf + \frac{\theta(1 - \alpha) - \eta\rho}{\eta\theta} \left( \frac{\rho}{c\theta} \right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}}. \tag{90}$$

Then we have:

$$\frac{\partial W_s(M)}{\partial M} = \frac{(\theta - \rho)\{\theta(1 - \alpha) - \eta\rho\}}{\rho\theta\{\theta(1 - \alpha) - \eta\}} \left( \frac{\rho}{c\theta} \right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\theta\{\eta-\rho(1-\alpha)\}}{\rho\{\theta(1-\alpha)-\eta\}}} - f. \tag{91}$$

When  $\theta(1 - \alpha) < \eta$ , (91) is always negative, that is, (90) is monotonically decreasing for  $M$ . When  $\theta(1 - \alpha) > \eta$ , there exists  $M_s^\circ$  that satisfy  $\frac{\partial W_s(M)}{\partial M} = 0$ . Rearranging (91), we obtain the inequality when  $\frac{\partial W_s(M)}{\partial M} > 0$ :

$$M^{\frac{\theta\{\eta-\rho(1-\alpha)\}}{\rho\{\theta(1-\alpha)-\eta\}}} > \frac{\rho\theta\{\theta(1 - \alpha) - \eta\}}{(\theta - \rho)\{\theta(1 - \alpha) - \eta\rho\}} \left( \frac{\rho}{c\theta} \right)^{\eta/\{\eta-\theta(1-\alpha)\}} K^{\theta/\{\eta-\theta(1-\alpha)\}} f. \tag{92}$$

From (92),  $M_s^\circ$  is the minimum value when  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$ , and the maximum value when  $\eta < \rho(1 - \alpha) < \theta(1 - \alpha)$ . Suppose that  $W_s^\circ$  denotes  $W_s(M_s^\circ)$ . Then  $M_s^\circ$  and  $W_s^\circ$  are given by the following formula:

$$M_s^\circ = \left[ \frac{\theta(1 - \alpha) - \eta\rho}{\rho\{\theta(1 - \alpha) - \eta\}} \right]^{\frac{\rho\{\theta(1-\alpha)-\eta\}}{\theta\{\rho(1-\alpha)-\eta\}}} T^{\frac{\rho\{\theta(1-\alpha)-\eta\}}{\theta\{\rho(1-\alpha)-\eta\}}} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}}, \tag{93}$$

$$\begin{aligned}
W_s^\circ &= N + \frac{\rho(1 - \alpha) - \eta}{\eta} \left[ \frac{\theta(1 - \alpha) - \eta\rho}{\rho\{\theta(1 - \alpha) - \eta\}} \right]^{\frac{\rho\{\theta(1-\alpha)-\eta\}}{\theta\{\rho(1-\alpha)-\eta\}}} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} \\
&\quad T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}}. \tag{94}
\end{aligned}$$

Compare  $M_s^\circ$  and  $\bar{M}_s$ :

$$\bar{M}_s - M_s^\circ = \left(\frac{\rho}{c\theta}\right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K \rho^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\rho\{\theta(1-\alpha)-\eta\}}{\theta\{\rho(1-\alpha)-\eta\}}} \left[ 1 - \left(\frac{\theta(1-\alpha)-\eta\rho}{\rho\{\theta(1-\alpha)-\eta\}}\right)^{\frac{\rho\{\theta(1-\alpha)-\eta\}}{\theta\{\rho(1-\alpha)-\eta\}}} \right]. \quad (95)$$

Hence,  $M_s^\circ > \bar{M}_s$  when  $\rho(1-\alpha) > \eta$ . It means that  $\bar{W}_s$ , social welfare at  $\bar{M}_s$ , is smaller than  $W_s^\circ$ , that is, under-entry of firms occurs when goods are substitutes. When  $\rho(1-\alpha) < \eta < \theta(1-\alpha)$ ,  $M_s^\circ < \bar{M}_s$ . It means that under-entry of firms occurs. When  $\theta(1-\alpha) < \eta$ , (90) is monotonically decreasing, namely, excess-entry of firms occurs. Substituting (77) into (90), we obtain:

$$\bar{W}_s \equiv W_s(\bar{M}_s) = N + \frac{1-\eta-\alpha}{\eta} \left(\frac{\rho}{c\theta}\right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K \rho^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}}.$$

Then it is clear that  $\bar{W}_s$  is always larger than  $N$ .

We show the graphs of social welfare as Figs.15 and 16 according to the signs of  $\theta(1-\alpha) - \eta$ . When  $\theta(1-\alpha) > \eta$ , the domain in the SP case is  $0 < M < M_b$  and we can draw two kinds of graphs according to the signs of  $\rho(1-\alpha) - \eta$ .

In the range of the SP case, (90) has the maximum value at  $M_s^\circ$  when  $\rho(1-\alpha) > \eta$  (See Figs.15(a), 15(b), 15(c) and 15(d)). Furthermore, we draw the graphs according to the profit functions (See Fig.14). When  $\bar{M}_s$  exists in the SP case (See Case A in Fig.14(a)), it is clear that under-entry of firms occurs. We show some graphs by the size of  $M_s^\circ$  (See Figs.15(a), 15(b), 15(c) and Appendix D for more information). When  $\rho(1-\alpha) < \eta < \theta(1-\alpha)$  and there exists  $\bar{M}_s$  in the SP case (See Case C in Fig.14(b)), we see that  $M_s^\circ$  is smaller than  $\bar{M}_s$ .

When  $\theta(1-\alpha) < \eta$ , the domain in the SP case is  $M_{b1} < M < M_{b2}$  and  $\bar{W}_s$  is monotonically decreasing (See Fig.16).

### 5.3.2 DS case

When  $M^b \leq M < N/f$ , we obtain the social welfare  $W_d(M)$  in the DS case as follows:

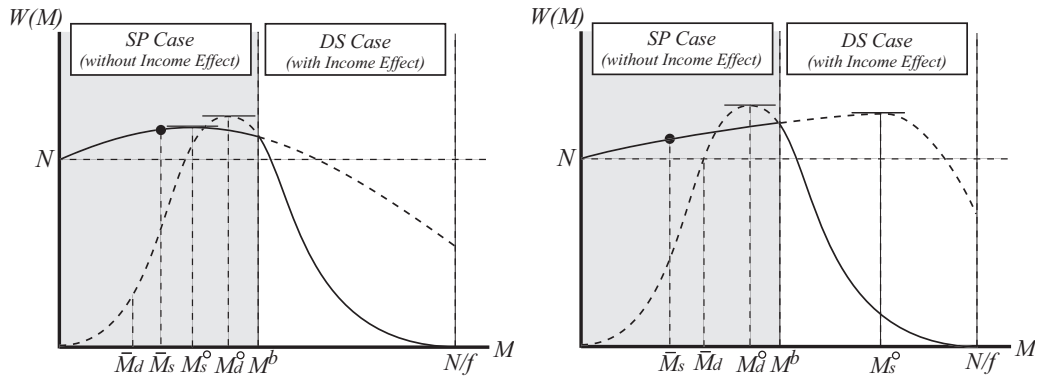
$$W_d(M) = \frac{1}{\eta} \left(\frac{\rho}{c\theta}\right)^{\eta/\theta} K \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} (N - Mf) \right\}^{(\eta+\alpha\theta)/\theta} M^{\frac{\eta(\theta-\rho)}{\rho\theta}}. \quad (96)$$

Then we have:

$$\frac{\partial W_d(M)}{\partial M} = \frac{1}{\eta} \left(\frac{\rho}{c\theta}\right)^{\eta/\theta} K \left\{ \frac{\eta\theta}{\alpha\theta + \rho\eta} \right\}^{(\eta+\alpha\theta)/\theta} (N - Mf)^{\{\eta-\theta(1-\alpha)\}/\theta} M^{\frac{\eta(\theta-\rho)-\rho\theta}{\rho\theta}} \left\{ \frac{\eta(\theta-\rho)}{\rho\theta} N - \frac{\alpha\rho + \eta}{\rho} Mf \right\}. \quad (97)$$

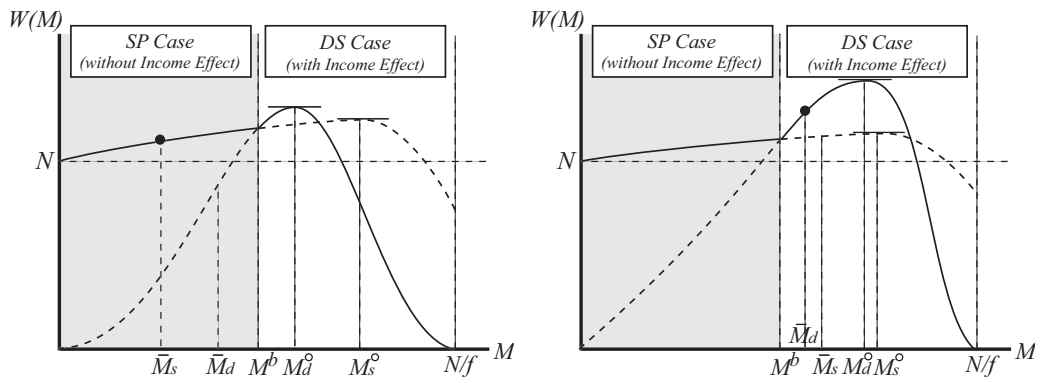
Thus there exists  $M_d^\circ$  such that (97) is zero:

$$M_d^\circ = \frac{\eta}{\alpha\rho + \eta} NT. \quad (98)$$



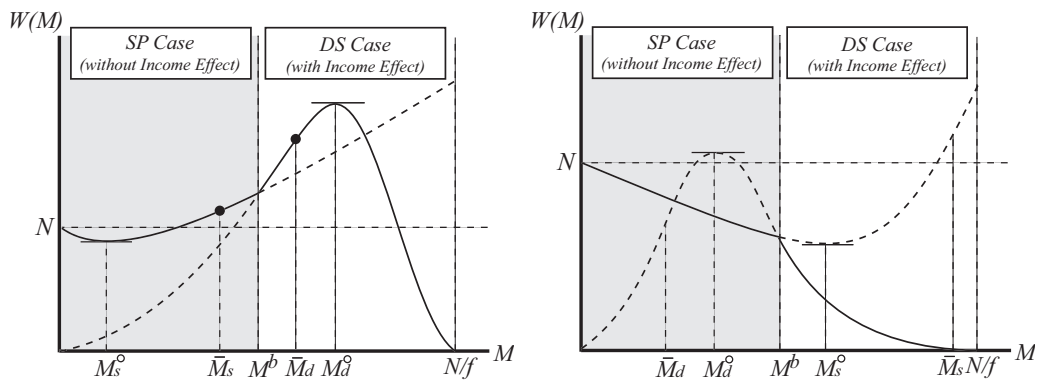
(a)  $\rho(1 - \alpha) > \eta$  when  $\bar{M}_s \in SP$

(b)  $\rho(1 - \alpha) > \eta$  when  $\bar{M}_s \in SP$



(c)  $\rho(1 - \alpha) > \eta$  when  $\bar{M}_s \in SP$

(d)  $\rho(1 - \alpha) > \eta$  when  $\bar{M}_d \in SP$



(e)  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$

(f)  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$

Figure 15: Social Welfare when  $\eta < \theta(1 - \alpha)$

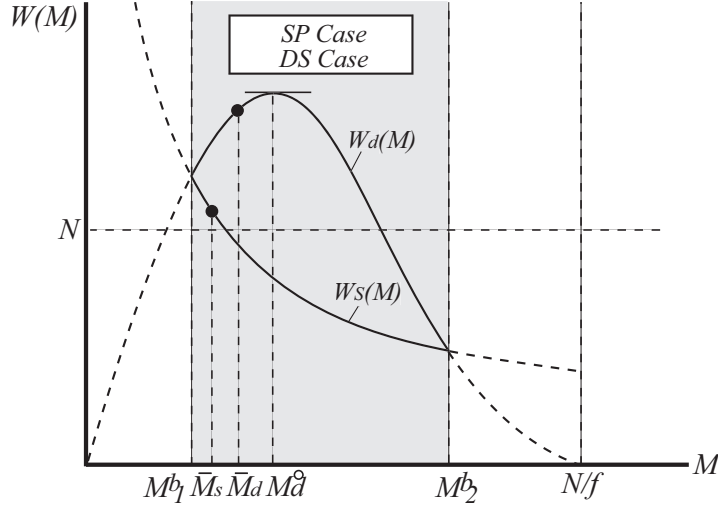


Figure 16: Social Welfare when  $\theta(1 - \alpha) < \eta$

Since  $\frac{\partial W_d(M)}{\partial M} > 0$  if  $M < M_d^\circ$  and  $\frac{\partial W_d(M)}{\partial M} < 0$  if  $M > M_d^\circ$ , (96) has the maximum value at  $M_d^\circ$  (See Fig.15). Substituting (98) into (96), we have:

$$W_d(M_d^\circ) = \frac{1}{\eta} \left( \frac{\rho}{c\theta} \right)^{\eta/\theta} K \left\{ \frac{\rho(\alpha\rho + \eta)}{\alpha\theta + \rho\eta} \right\}^{(\eta+\alpha\theta)/\theta} \left( \frac{\eta}{\alpha\rho + \eta} N \right)^{(\alpha\rho+\eta)/\rho} T^{\frac{\eta(\theta-\rho)}{\rho\theta}}.$$

From (77) and (96), we also obtain:

$$\bar{W}_d \equiv W_d(\bar{M}_d) = \frac{1}{\eta} \left( \frac{\rho}{c\theta} \right)^{\eta/\theta} K \left( \frac{\eta}{\alpha\rho + \eta} N \right)^{(\alpha\rho+\eta)/\rho} T^{\frac{\eta(\theta-\rho)}{\rho\theta}}.$$

Compare  $M_d^\circ$  and  $\bar{M}_d$ . From (79) and (98),

$$\bar{M}_d - M_d^\circ = \frac{\eta}{\eta + \alpha} NT - \frac{\eta}{\alpha\rho + \eta} NT = \frac{\eta\alpha(\rho - 1)}{(\eta + \alpha)(\alpha\rho + \eta)} NT < 0. \quad (99)$$

It means that the number of firms of the long run equilibrium is less than the optimum level  $M_d^\circ$ .

When  $\theta(1 - \alpha) > \eta$ , the range of the DS case is  $M_b < M < N/f$ . When  $\rho(1 - \alpha) > \eta$ , there are two cases whether  $\bar{M}_d$  exists in the DS case (Fig. 15(d)) or not (Figs.15(a), 15(b), 15(c)). If  $\bar{M}_d$  exists in the DS case, we draw social welfare as Fig. 15(d) (See Appendix D for more information). It is clear that under-entry of firms occurs.

When  $\rho(1 - \alpha) < \eta < \theta(1 - \alpha)$ , there may exist  $\bar{M}_s$  in the SP case and  $\bar{M}_d$  in the DS case simultaneously (See Case C in Fig.14(b) and Fig.15(e)).

In this case,  $W_s^\circ$  is the minimum value at  $M_s^\circ$  and  $M_s^\circ < \bar{M}_s < M_b$ . It means that  $W_s^\circ < \bar{W}_s < W_b$ . We also have the condition of  $W_b < \bar{W}_d < W_d^\circ$  because  $W_d^\circ$  is the maximum value at  $M_d^\circ$  and  $M_b < \bar{M}_d < M_d^\circ$ . Recall that  $\bar{W}_s$  is always larger than  $N$ . Hence, we have the condition of  $N < \bar{W}_s < \bar{W}_d$ .

In the case that there exists no equilibrium on any domain (See Case D in Fig.14(c)), social welfare is monotonically decreasing for  $M$  (See Fig.15(f)).

When  $\theta(1 - \alpha) < \eta$ , the domain in the DS case coincides with that in the SP case (See Fig.16).

Compare  $W_s(M)$  and  $W_d(M)$  when  $\theta(1 - \alpha) < \eta$ . Then we know that  $W_d(M)$  has the maximum value  $W_d(M_d^\circ)$  and the values of M-intercept are 0 and  $N/f$ . Since  $W_s(M)$  and  $W_d(M)$  intersect at two values  $M_{b1}$  and  $M_{b2}$ , and  $W_s(M)$  is monotonically decreasing for  $M$ , it is clear that  $M_d^\circ < M_{b2}$ .

Furthermore, from (88) and (99),  $\bar{M}_s < \bar{M}_d < M_d^\circ$  is the condition which both  $\bar{M}_s$  and  $\bar{M}_d$  exist on the domain of  $M_{b1} < M < M_{b2}$ .

Then it can be seen that  $W_s(M)$  is below  $W_d(M)$  on the domain of  $M_{b1} < M < M_{b2}$  (See Fig.16). We thus obtain the following theorems:

**Theorem 7** *Social welfare level of the DS case is always higher than that of the SP case.*

**Theorem 8** *When two equilibria exist (See Fig.10(c)), the social welfare level of the stable equilibrium  $E_{D1}^*$  is higher than that of the unstable equilibrium  $E_{S3}^*$ .*

## Appendix A

This appendix is to explain from more general settings the maximization behaviors of the representative consumer and monopolistically competitive firms shown in the paper.

The representative consumer has the utility function  $U(Q, x, z)$  for consumption of the three types of goods:  $Q$  denotes a composite index of differentiated goods,  $x$  is consumption of the agricultural good, and  $z$  is the numéraire. The quantity index  $Q$  is given by subutility functions over the interval  $[0, M]$  of varieties of manufactured goods where  $M$  is the range of varieties produced.

The composite index  $Q$  of differentiated goods is defined by subutility functions  $V$  and  $v$  as

$$Q = V \left( \int_0^M v(q(i)) di \right).$$

We impose one-homogeneity on the composite function of  $V$  and the integral of  $v$ . Namely,

$$V \left( \int_0^M v(tq(i)) di \right) = tV \left( \int_0^M v(q(i)) di \right)$$

for all  $t > 0$  and all  $M > 0$ .

We assume that the preference is *doubly separable*, that is, the utility of  $q(\cdot)$ ,  $x$  and  $z$  is measured by a composite function of  $U$ ,  $V$  and  $v$  as

$$U \left( V \left( \int_0^M v(q(i)) di \right), x, z \right)$$

Given income  $Y$  and a system of prices,  $p(i)$  for each manufactured goods  $i$  and  $r$  for the homogeneous product, the consumer maximizes the utility subject to the budget constraint:

$$\text{Maximize } U \left( V \left( \int_0^M v(q(i)) di \right), x, z \right), \quad \text{subject to } \int_0^M p(i)q(i) + rx + z \leq Y.$$

This maximization problem is solved in two steps. First, however much the value of the manufacturing composite  $Q$  is, each  $q(i)$  needs to be chosen so as to minimize the expenditure of attaining  $M$ . This means:

$$\text{Minimize } \int_0^M p(i)q(i) di \quad \text{subject to } V \left( \int_0^M v(q(i)) di \right) = Q.$$

For each  $i \in [0, M]$ , the compensated demand function  $h(i, p, Q, M)$  for manufactured good indexed  $i$  is defined by

$$\begin{aligned} V \left( \int_0^M v(h(i, p, Q, M)) di \right) &= Q, \quad \text{and} \\ \int_0^M p(i)h(i, p, Q, M) di &\leq \int_0^M p(i)q(i) di \\ \text{for all } q \in (\mathbb{R}_+)^{[0, M]} \quad \text{such that } &V \left( \int_0^M v(q(i)) di \right) = Q. \end{aligned}$$



By the one-homogeneity of  $V \left( \int_0^M v(\cdot) di \right)$ , we can show that  $h(i, p, Q, M) = h(i, p, 1, M)Q$ . Define  $\bar{q}(i, p, M) \equiv h(i, p, 1, M)$ , then

$$1 = V^{-1} \left( \int_0^M v(\bar{q}(i, p, 1, M)) di \right) \quad \text{and} \quad h(i, p, Q, M) = Q\bar{q}(i, p, M).$$

Hence, the expenditure function  $e$  is defined by

$$e(p, Q, M) \equiv \int_0^M p(i)h(i, p, Q, M) di = Q \int_0^M p(i)\bar{q}(i, p, M) di$$

The price index is therefore defined by

$$P \equiv \frac{e(p, Q, M)}{Q} = \int_0^M p(i)\bar{q}(i, p, M) di = e(p, 1, M).$$

Next consider the maximization problem as follows:

$$\text{Maximize} \quad U(Q, x, z), \quad \text{subject to} \quad PQ + \gamma x + z \leq Y.$$

Since  $\gamma$  is the marginal cost of the competitive sector, which is constant, the maximizer is denoted by  $(Q(P, Y), x(P, Y), z(P, Y))$ . Let  $Q = 1$ , and consider

$$\text{Minimize} \quad \int_0^M p(i)q(i) di, \quad \text{subject to} \quad V \left( \int_0^M v(q(i)) di \right) = 1.$$

The first-order condition is:

$$p(i) = \mu V' \left( \int_0^M v(q(i)) di \right) v'(q(i)), \quad (100)$$

$$1 = V \left( \int_0^M v(q(i)) di \right), \quad (101)$$

where  $\mu$  is a Lagrangian multiplier. From equation (101),

$$V^{-1}(1) = \int_0^M v(q(i)) di. \quad (102)$$

Substituting (102) into (100), we have:

$$p(i) = \mu V' (V^{-1}(1)) v'(q(i)).$$

Let  $k = V' (V^{-1}(1))$ , then

$$v'(q(i)) = \frac{p(i)}{\mu k} \quad (103)$$

Let  $\bar{q}$  be the expenditure minimizer. By homogeneity,

$$V \left( \int_0^M v(t\bar{q}(i)) di \right) = t.$$

Then we obtain

$$1 = V' \left( \int_0^M v'(t\bar{q}(i)) di \right) \int_0^M v'(t\bar{q}(i)) \bar{q}(i) di. \quad (104)$$

Substituting  $t = 1$  into equation (104),

$$1 = V' (V^{-1}(1)) \int_0^M v'(\bar{q}(i)) \bar{q}(i) di = k \int_0^M v'(\bar{q}(i)) \bar{q}(i) di. \quad (105)$$

Substituting (103) into (105) and rearranging,

$$\mu = \int_0^M p(i) \bar{q}(i) di.$$

Then  $P = \mu$ , and we obtain from (103),

$$\bar{q}(i) = (v')^{-1} \left[ \frac{p(i)}{Pk} \right]$$

Hence,

$$q(i) = \bar{q}(i) Q(P, Y) = (v')^{-1} \left( \frac{p(i)}{Pk} \right) Q(P, Y)$$

Define  $g \equiv (v')^{-1}$ , then

$$q(i) = g \left( \frac{p(i)}{Pk} \right) Q(P, Y).$$

Recall  $P = e(p, 1, M)$ , then we have  $(q(\cdot), x, z)$  that solves the original maximization problem.

Next consider behavior of firms and performances of markets of monopolistic competition. Let  $C$  be the identical total cost function of firms in the industry. We present some dynamics of monopolistic competition in the short-run. We mean by 'short-run' that the set of incumbent firms is fixed. Let  $M$  be the size of the industry. We propose three versions of formulations of price adjustment process of monopolistic competition. Each of them is based on its own characteristics of the commodity.

The first version is á la Cournot. Each firm produces and supplies its own product given price index of the industry and income level of consumers. The price is determined by market demand, and agents make trades. The earned profit becomes part of the new income of consumers. Then, the price index and income level are updated. This dynamic process continues until neither the price index nor the income level change. This is formulated as follows: Note that  $p(i) = Pkv'(q(i))$ . Given  $(P, Y)$ , each firm  $i$  selects quantity  $q(i)$  to maximize its profit

$$Pkv'(q(i))q(i) - C(q(i)).$$

By functional form, the maximizer is written as  $q(i) = q(P, Y)$ , which is independent of  $i$ .

Suppose that  $q(P, Y)$  is actually produced and supplied to market  $i$ . Then the new price system  $(p^*, P^*)$  and the income  $Y^*$  to clear the markets solves the following simultaneous equations

$$g\left(\frac{p^*}{P^*k}\right)Q(P^*, Y^*) = q(P, Y), \quad (106)$$

$$P^* = Mp^*g\left(\frac{p^*}{P^*k}\right), \quad \text{and} \quad (107)$$

$$Y^* = N + M(p^*q(P, Y) - C(q(P, Y))). \quad (108)$$

Hence,  $p^* = p^*(P, Y, M)$ ,  $P^* = P^*(P, Y, M)$ , and  $Y^* = Y^*(P, Y, M)$ . Thus, the short-run dynamic adjustment system is:

$$\begin{aligned} \dot{P} &= P^*(P, Y, M) - P; \quad \text{and} \\ \dot{Y} &= Y^*(P, Y, M) - Y. \end{aligned}$$

The second version is á la Bertrand. Each firm sets the price of its own product given price index of the industry and income level of consumers. The quantity is determined by market demand, and agents make trades. The earned profit becomes part of the new income of consumers. Then, the price index and income level are updated. This dynamic process continues until the neither the price index nor income level change: Recall that the market demand is expressed as  $g\left(\frac{p(i)}{Pk}\right)Q(P, Y)$ . Given  $(P, Y)$ , each firm  $i$  selects  $p(i)$  to maximize its profit

$$p(i)g\left(\frac{p(i)}{Pk}\right)Q(P, Y) - C\left(g\left(\frac{p(i)}{Pk}\right)Q(P, Y)\right).$$

By functional form, the solution is independent of  $i$ , then it is written as  $p(i) = p(P, Y)$ . Suppose that  $p(P, Y)$  is set and fixed in market  $i$ , then the new value of the price index  $P^*$  and the income level  $Y^*$  solve the simultaneous equations:

$$\begin{aligned} P^* &= Mp(P, Y)g\left(\frac{p(P, Y)}{P^*k}\right); \quad \text{and} \\ Y^* &= N + M\left(p(P, Y)g\left(\frac{p(P, Y)}{P^*k}\right)Q(P^*, Y^*) - C\left(g\left(\frac{p(P, Y)}{P^*k}\right)Q(P^*, Y^*)\right)\right). \end{aligned}$$

Hence,  $P^* = P^*(P, Y, M)$ , and  $Y^* = Y^*(P, Y, M)$ . Thus, the short-run dynamic adjustment system is:

$$\begin{aligned} \dot{P} &= P^*(P, Y, M) - P; \quad \text{and} \\ \dot{Y} &= Y^*(P, Y, M) - Y. \end{aligned}$$

The last version is á la Walras. Each firm tentatively sets the price of its own product given price index of the industry and income level of consumers. Its anticipated profit is informed to consumers as part of their new income, and the information of price index and income level is updated. This adjustment process continues until the anticipated

values coincides with the actual ones. Agents trade only after the adjustment process terminates: Given  $(P, Y)$ , each firm  $i$  selects  $p(i)$  to maximize its profit

$$p(i)g\left(\frac{p(i)}{Pk}\right)Q(P, Y) - C\left(g\left(\frac{p(i)}{Pk}\right)Q(P, Y)\right).$$

By functional form, the solution is independent of  $i$ , then it is written as  $p(i) = p(P, Y)$ . Then the anticipated profit of firm  $i$  is:

$$\pi(P, Y) = p(P, Y)g\left(\frac{p(P, Y)}{Pk}\right)Q(P, Y) - C\left(g\left(\frac{p(P, Y)}{Pk}\right)Q(P, Y)\right),$$

and the new value of the price index  $P^*$  and the income level  $Y^*$  are determined by:

$$P^* = Mp(P, Y)g\left(\frac{p(P, Y)}{P^*k}\right); \quad \text{and}$$

$$Y^* = N + M\pi(P, Y).$$

Hence,  $P^* = P^*(P, Y, M)$ , and  $Y^* = Y^*(P, Y, M)$ . Thus, the short-run dynamic adjustment system is:

$$\dot{P} = P^*(P, Y, M) - P; \quad \text{and}$$

$$\dot{Y} = Y^*(P, Y, M) - Y.$$

We can see that the short-run equilibrium  $(P(M), Y(M))$ , which is defined by

$$P^*(P(M), Y(M), M) = P(M); \quad \text{and}$$

$$Y^*(P(M), Y(M), M) = Y(M)$$

is the same regardless of adjustment processes.

Finally, we give the standard formulation of dynamics of monopolistic competition in the long-run. The size of industry expands when the individual profit level is positive and shrinks when the level is negative, so that

$$\dot{M} = \phi(\pi^*(M)),$$

where  $\phi'(\pi) > 0$  for all  $\pi \in \mathfrak{R}$  and  $\phi(0) = 0$ . Then the long-run equilibrium size of industry is  $\bar{M}$  such that  $\pi^*(\bar{M}) = 0$ .

## Appendix B

This appendix is to indicate  $\dot{P} = 0$  of four cases: the SP-SP, DS-SP, SP-DS and DS-DS cases and coordinate values of the intersections  $A$ ,  $B$  and  $C$ .

From (42), (44), (58) and (60), we have the following four equations:  
 $\dot{P} = 0$  of the SP-SP case:

$$P = \left(\frac{\rho}{c\theta}\right)^{\frac{1-\eta-\alpha}{\eta-\theta(1-\alpha)}} K^{\frac{1-\theta}{\eta-\theta(1-\alpha)}} M^{\frac{(\theta-\rho)(1-\eta-\alpha)}{\rho\{\eta-\theta(1-\alpha)\}}}. \quad (109)$$

$\dot{P} = 0$  of the DS-SP case:

$$P^{\frac{\theta(1-\eta-\alpha)-\eta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} = \frac{\eta}{\alpha} \left( \frac{\rho}{c\theta} \right)^{1/(\rho-\theta)} K^{\frac{\rho-1}{(\theta-\rho)(1-\eta-\alpha)}} M^{-1/\rho} \left[ N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \right]. \quad (110)$$

$\dot{P} = 0$  of the SP-DS case:

$$P = \left( \frac{\rho}{c\theta} \right)^{\frac{1-\eta-\alpha}{\theta(\alpha-1)+\rho\eta}} K^{\frac{\rho-\theta}{\theta(\alpha-1)+\rho\eta}} M^{\frac{(\theta-\rho)(1-\eta-\alpha)}{\rho\{\theta(\alpha-1)+\rho\eta\}}} \left( \frac{\eta}{\eta+\alpha} Y \right)^{\frac{(1-\rho)(1-\eta-\alpha)}{\theta(\alpha-1)+\rho\eta}}. \quad (111)$$

$\dot{P} = 0$  of the DS-DS case:

$$P^{\theta/(\theta-\rho)} = \frac{\eta}{\alpha} \left( \frac{\rho}{c\theta} \right)^{1/(\rho-\theta)} M^{-1/\rho} \left( \frac{\eta}{\eta+\alpha} Y \right)^{\frac{\rho-1}{\theta-\rho}} \left[ N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} \left( \frac{\eta}{\eta+\alpha} Y \right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \right]. \quad (112)$$

From (26), (40) and (56), we also obtain the boundaries between these cases.

The boundary between the SP case and the DS case is  $z = 0$ . That is:

$$\frac{\eta}{\eta+\alpha} Y = K^{1/(1-\eta-\alpha)} P^{-\eta/(1-\eta-\alpha)}. \quad (113)$$

The boundary between the SP-SP case and the DS-SP case is  $F(P, M) = 0$ . That is:

$$\begin{aligned} N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \\ = \frac{\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{\frac{\eta(1-\theta)-(\theta-\rho)(\alpha-1)}{(\theta-\rho)(1-\alpha)(1-\eta-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} P^{\frac{\eta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\alpha)(1-\eta-\alpha)}}. \end{aligned} \quad (114)$$

The boundary between the SP-DS case and the DS-DS case  $G(P, Y, M) = 0$ . That is:

$$\begin{aligned} N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} \left( \frac{\eta}{\eta+\alpha} Y \right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \\ = \frac{\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{1/(1-\alpha)} M^{\frac{\eta}{\rho(1-\alpha)}} P^{\frac{\rho\eta}{(\theta-\rho)(1-\alpha)}} \left( \frac{\eta}{\eta+\alpha} Y \right)^{\frac{\eta(1-\rho)}{(\theta-\rho)(1-\alpha)}}. \end{aligned} \quad (115)$$

The curves (109), (111) and (113) intersect at the point  $A(Y_A, P_A)$ :

$$\begin{aligned} Y_A &= \frac{\eta+\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{\theta(1-\alpha)-\eta}} K^{\frac{\theta}{\theta(1-\alpha)-\eta}} M^{\frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}}, \\ P_A &= \left( \frac{\rho}{c\theta} \right)^{\frac{1-\eta-\alpha}{\eta-\theta(1-\alpha)}} K^{\frac{1-\theta}{\eta-\theta(1-\alpha)}} M^{\frac{(\theta-\rho)(1-\eta-\alpha)}{\rho\{\eta-\theta(1-\alpha)\}}} \end{aligned}$$

The curves (111), (112) and (115) intersect at the point  $B(Y_B, P_B)$ , which satisfies the following equations:

$$\begin{aligned}
N - Mf - c \left( \frac{\rho}{c\theta} \right)^{\frac{\theta(\alpha-1)}{\theta(\alpha-1)+\rho\eta}} K^{\frac{-\rho\theta}{\theta(\alpha-1)+\rho\eta}} M^{\frac{\eta(\rho-\theta)}{\theta(\alpha-1)+\rho\eta}} \left( \frac{\eta}{\eta+\alpha} Y_B \right)^{\frac{\theta(1-\rho)(\alpha-1)}{\theta(\alpha-1)+\rho\eta}} \\
= \frac{\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{-\eta}{\theta(\alpha-1)+\rho\eta}} K^{\frac{-\theta}{\theta(\alpha-1)+\rho\eta}} M^{\frac{\eta(\rho-\theta)}{\rho\{\theta(\alpha-1)+\rho\eta\}}} \left( \frac{\eta}{\eta+\alpha} Y_B \right)^{\frac{\eta(\rho-1)}{\theta(\alpha-1)+\rho\eta}}, \\
N - Mf - cK^{\theta/(1-\eta-\alpha)} M^{(\rho-\theta)/\rho} P_B^{\frac{\theta(\alpha-1)}{1-\eta-\alpha}} \\
= \frac{\alpha}{\eta} K^{1/(1-\eta-\alpha)} P_B^{-\eta/(1-\eta-\alpha)}.
\end{aligned}$$

The curves (110), (112) and (113) intersect at the point  $C(Y_C, P_C)$ , which satisfies the following equations:

$$\begin{aligned}
N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} K^{\frac{\rho\theta}{\eta(\theta-\rho)}} \left( \frac{\eta}{\eta+\alpha} Y_C \right)^{\frac{\theta\{\eta-\rho(1-\alpha)\}}{\eta(\theta-\rho)}} \\
= \frac{\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{1/(\theta-\rho)} K^{\frac{\theta}{\eta(\theta-\rho)}} M^{1/\rho} \left( \frac{\eta}{\eta+\alpha} Y_C \right)^{\frac{\theta(1-\eta-\alpha)-\eta(1-\rho)}{\eta(\rho-\theta)}}, \\
N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P_C^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \\
= \frac{\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{1/(\theta-\rho)} K^{\frac{1-\rho}{(\theta-\rho)(1-\eta-\alpha)}} M^{1/\rho} P_C^{\frac{\theta(1-\eta-\alpha)-\eta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}}.
\end{aligned}$$

## Appendix C

This appendix is to indicate  $\dot{P} = 0$  of four cases: the SP-SP, DS-SP, SP-DS and DS-DS cases and coordinate values of the intersections  $D$ ,  $E$ ,  $F$  and  $G$ .

From (42), (44), (58) and (60), we have the following four equations:

$\dot{Y} = 0$  of the SP-SP case:

$$\begin{aligned}
Y = N - Mf + \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(1-\alpha)(\theta-\rho)}} K^{\frac{(\alpha-1)(\theta-\rho)+\eta(\theta-1)}{(\alpha-1)(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\eta\{\eta-\rho(1-\alpha)\}}{(\alpha-1)(\theta-\rho)(1-\eta-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} \\
- c \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} M. \tag{116}
\end{aligned}$$

$\dot{Y} = 0$  of the DS-SP case:

$$Y = \frac{\eta+\alpha}{\alpha} \left[ N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \right]. \tag{117}$$

$\dot{Y} = 0$  of the SP-DS case:

$$\begin{aligned}
Y = N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} \left( \frac{\eta}{\eta+\alpha} Y \right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \\
+ \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{1/(1-\alpha)} M^{\frac{\eta}{\rho(1-\alpha)}} \left( \frac{\eta}{\eta+\alpha} Y \right)^{\frac{\eta(1-\rho)}{(\theta-\rho)(1-\alpha)}} P^{\frac{\rho\eta}{(\theta-\rho)(1-\alpha)}}. \tag{118}
\end{aligned}$$

$\dot{Y} = 0$  of the DS-DS case:

$$Y = \frac{\eta + \alpha}{\alpha} \left[ N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} \left( \frac{\eta}{\eta + \alpha} Y \right)^{\frac{\theta(1-\rho)}{\theta-\rho}} P^{\frac{\rho\theta}{\theta-\rho}} \right]. \quad (119)$$

The curves (114), (116) and (117) intersect at the point  $D(Y_D, P_D)$ , which satisfies the following equations:

$$\begin{aligned} N - Mf - \frac{\alpha}{\eta + \alpha} Y_D &= cK^{-\theta/\eta} M^{(\rho-\theta)/\rho} \left( \frac{\eta}{\eta + \alpha} Y_D \right)^{\frac{\theta(1-\alpha)}{\eta}}, \\ N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\frac{\theta}{\theta-\rho}} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P_D^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \\ &= \frac{\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{\frac{(\theta-\rho)(\alpha-1)-\eta(1-\theta)}{(\theta-\rho)(\alpha-1)(1-\eta-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} P_D^{\frac{\eta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\alpha)(1-\eta-\alpha)}}. \end{aligned}$$

The curves (113), (116) and (118) intersect at the point  $E(Y_E, P_E)$ , which satisfies the following equations:

$$\begin{aligned} Y_E &= N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} K^{\frac{\rho\theta}{\eta(\theta-\rho)}} \left( \frac{\eta}{\eta + \alpha} Y_E \right)^{\frac{\theta\{\eta-\rho(1-\alpha)\}}{\eta(\theta-\rho)}} \\ &\quad + \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{\frac{\theta}{(\theta-\rho)(1-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} \left( \frac{\eta}{\eta + \alpha} Y_E \right)^{\frac{\eta-\rho(1-\alpha)}{(\theta-\rho)(1-\alpha)}}, \\ \frac{\eta + \alpha}{\eta} K^{1/(1-\eta-\alpha)} P_E^{-\eta/(1-\eta-\alpha)} \\ &= N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P_E^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}} \\ &\quad + \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)}} K^{\frac{(\theta-\rho)(\alpha-1)-\eta(1-\theta)}{(\theta-\rho)(\alpha-1)(1-\eta-\alpha)}} M^{\frac{\eta}{\rho(1-\alpha)}} P_E^{\frac{\eta\{\eta-\rho(1-\alpha)\}}{(\theta-\rho)(\alpha-1)(1-\eta-\alpha)}}. \end{aligned}$$

The curves (115), (118) and (119) intersect at the point  $F(Y_F, P_F)$ , which satisfies the following equations:

$$\begin{aligned} N - Mf - \frac{\alpha}{\eta + \alpha} Y_F &= cK^{-\theta/\eta} M^{(\rho-\theta)/\rho} \left( \frac{\eta}{\eta + \alpha} Y_F \right)^{\frac{\theta(1-\alpha)}{\eta}}, \\ \frac{\alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\eta}{(\theta-\rho)(1-\alpha)-\eta(1-\rho)}} K^{\frac{\theta-\rho}{(\theta-\rho)(1-\alpha)-\eta(1-\rho)}} M^{\frac{\eta(\theta-\rho)}{\rho\{(\theta-\rho)(1-\alpha)-\eta(1-\rho)\}}} P_F^{\frac{\rho\eta}{(\theta-\rho)(1-\alpha)-\eta(1-\rho)}} \\ &= N - Mf - c \left( \frac{\rho}{c\theta} \right)^{\frac{\theta(1-\alpha)}{(\theta-\rho)(1-\alpha)-\eta(1-\rho)}} K^{\frac{\eta(\theta-\rho)(1-\rho)}{(\theta-\rho)(1-\alpha)-\eta(1-\rho)}} M^{\frac{(\theta-\rho)\{\rho(1-\eta-\alpha)+\eta\}}{\rho\{(\theta-\rho)(1-\alpha)-\eta(1-\rho)\}}} \\ &\quad P_F^{\frac{\rho\theta(1-\alpha)}{(\theta-\rho)(1-\alpha)-\eta(1-\rho)}}. \end{aligned}$$

The curves (113), (117) and (119) intersect at the point  $G(Y_G, P_G)$ , which satisfies the following equations:

$$\begin{aligned} N - Mf - \frac{\alpha}{\eta + \alpha} Y_G &= cM \left( \frac{\rho}{c\theta} \right)^{\frac{\theta}{\theta-\rho}} K^{\frac{\rho\theta}{\eta(\theta-\rho)}} \left( \frac{\eta}{\eta + \alpha} Y_G \right)^{\frac{\theta\{\eta-\rho(1-\alpha)\}}{\eta(\theta-\rho)}}, \\ \frac{\alpha}{\eta} K^{1/(1-\eta-\alpha)} P_G^{-\eta/(1-\eta-\alpha)} \\ &= N - Mf - cM \left( \frac{\rho}{c\theta} \right)^{\theta/(\theta-\rho)} K^{\frac{\theta(1-\rho)}{(\theta-\rho)(1-\eta-\alpha)}} P_G^{\frac{\theta\{\rho(1-\alpha)-\eta\}}{(\theta-\rho)(1-\eta-\alpha)}}. \end{aligned}$$

We obtain  $E_{D4}^*(Y_{E4}^*, P_{E4}^*)$  as the point that  $C$  coincides with  $G$ , that is:

$$Y_{E4}^* = \frac{\eta + \alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\eta/\{\theta(1-\alpha)-\eta\}} K^{\theta/\{\theta(1-\alpha)-\eta\}} M^{\frac{\eta(\theta-\rho)}{\rho\{\theta(1-\alpha)-\eta\}}},$$

$$P_{E4}^* = \left( \frac{\rho}{c\theta} \right)^{\frac{1-\eta-\alpha}{\eta-\theta(1-\alpha)}} K^{\frac{1-\theta}{\eta-\theta(1-\alpha)}} M^{\frac{(\theta-\rho)(1-\eta-\alpha)}{\rho\{\eta-\theta(1-\alpha)\}}}$$

## Appendix D

This appendix is to prepare for drawing Fig.15. We consider the condition of the existence of  $\bar{M}_s$ ,  $\bar{M}_d$ ,  $M_s^\circ$  and  $M_d^\circ$ . From (77) and (70), the condition that  $\bar{M}_s$  exists in the SP case is:

$$N > \frac{\eta + \alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}} \equiv \bar{N}_s.$$

We thus see that  $\bar{M}_s$  exists in the DS case when  $N < \bar{N}_s$ . From (70) and (94), the condition of  $M_s^\circ$  in the SP case is:

$$N > \left[ \frac{\alpha\theta + \rho\eta}{\eta\theta} + \frac{(\theta - \rho)\{\theta(1 - \alpha) - \rho\eta\}}{\rho\theta\{\theta(1 - \alpha) - \eta\}} \right] \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}}$$

$$\left[ \frac{\theta(1 - \alpha) - \rho\eta}{\rho\{\theta(1 - \alpha) - \eta\}} \right]^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}} \equiv N_s^\circ.$$

We thus see that  $M_s^\circ$  exists in the DS case when  $N < M_s^\circ$ . From (79) and (74), the condition of  $\bar{M}_d$  in the DS case is:

$$N < \frac{\eta + \alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}} \equiv \bar{N}_d \quad \text{if } \rho(1 - \alpha) > \eta,$$

$$N > \frac{\eta + \alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}}. \quad \text{if } \rho(1 - \alpha) < \eta.$$

Note that  $\bar{N}_s$  equals to  $\bar{N}_d$ .

If we define  $N_d^\circ$  as follows, the condition of  $M_d^\circ$  in the DS case is:

$$N_d^\circ \equiv \frac{(\alpha\rho + \eta)(\alpha\theta + \rho\eta)}{\eta\rho(\alpha\theta + \eta)} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}} \left[ \frac{\alpha\theta + \rho\eta}{\rho(\alpha\theta + \eta)} \right]^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}}.$$

$$N < N_d^\circ \quad \text{if } \rho(1 - \alpha) > \eta,$$

$$N > N_d^\circ \quad \text{if } \rho(1 - \alpha) < \eta.$$

From (95),  $\bar{N}_s < N_s^\circ$  is a necessary and sufficient condition for  $\bar{M}_s > M_s^\circ$  when  $\rho(1 - \alpha) > \eta$  and  $\bar{N}_s > N_s^\circ$  when  $\rho(1 - \alpha) < \eta$ . We also obtain  $\bar{N}_d < N_d^\circ$  from (99). Assume that  $\bar{N}_s = \bar{N}_d \equiv \bar{N}$ . Then we prove that  $\bar{N} < N_d^\circ < N_s^\circ$  when  $\rho(1 - \alpha) > \eta$  and  $N_s^\circ < \bar{N} < N_d^\circ$  when  $\rho(1 - \alpha) < \eta$ .

Compare  $\bar{M}_s$  and  $\bar{M}_d$ . From (77) and (79),

$$\bar{M}_d - \bar{M}_s = \frac{\eta}{\eta + \alpha} \left\{ N - \frac{\eta + \alpha}{\eta} \left( \frac{\rho}{c\theta} \right)^{\frac{\rho\eta}{\theta\{\rho(1-\alpha)-\eta\}}} K^{\rho/\{\rho(1-\alpha)-\eta\}} T^{\frac{\eta(\theta-\rho)}{\theta\{\rho(1-\alpha)-\eta\}}} \right\}.$$



Then we obtain the following inequalities:

$$\begin{aligned}\bar{M}_d &> \bar{M}_s & \text{if } N > \bar{N}, \\ \bar{M}_d &< \bar{M}_s & \text{if } N < \bar{N}.\end{aligned}$$

First, we consider social welfare when  $\rho(1 - \alpha) > \eta$ .

When  $N > N_s^\circ$ ,  $\bar{M}_s$ ,  $M_s^\circ$ ,  $\bar{M}_d$  and  $M_d^\circ$  are in the SP Case. We thus obtain the inequality  $\bar{M}_s < M_s^\circ < \bar{M}_d < M_d^\circ < M^b$  and draw Fig.15(a). When  $N_d^\circ < N < N_s^\circ$ ,  $\bar{M}_s$ ,  $\bar{M}_d$  and  $M_d^\circ$  are in the SP Case and  $M_s^\circ$  is in the DS Case. We thus obtain the inequality  $\bar{M}_s < \bar{M}_d < M_d^\circ < M^b < M_s^\circ < N/f$  and draw Fig.15(b). When  $\bar{N} < N < N_d^\circ$ ,  $\bar{M}_s$  and  $\bar{M}_d$  are in the SP Case and  $M_s^\circ$  and  $M_d^\circ$  are in the DS Case. We thus obtain the inequality  $\bar{M}_s < \bar{M}_d < M^b < M_d^\circ < M_s^\circ < N/f$  and draw Fig.15(c). When  $N < \bar{N}$ ,  $\bar{M}_s$ ,  $M_s^\circ$ ,  $\bar{M}_d$  and  $M_d^\circ$  are in the DS Case. We thus obtain the inequality  $M^b < \bar{M}_d < \bar{M}_s < M_s^\circ < M_d^\circ < N/f$  and draw Fig.15(d).

Next, we consider social welfare when  $\rho(1 - \alpha) < \eta$ .

When  $\bar{N} < N$ ,  $\bar{M}_s$  and  $M_s^\circ$  are in the SP Case and  $\bar{M}_d$  and  $M_d^\circ$  are in the DS Case. We thus obtain the inequality  $M_s^\circ < \bar{M}_s < M^b < \bar{M}_d < M_d^\circ < N/f$  and draw Fig.15(e). When  $\bar{N} > N$ ,  $\bar{M}_d$  and  $M_d^\circ$  are in the SP Case and  $\bar{M}_s$  and  $M_s^\circ$  are in the DS Case. We thus obtain the inequality  $\bar{M}_d < M_d^\circ < M^b < M_s^\circ < \bar{M}_s < N/f$  and draw Fig.15(f).

## Appendix E

We consider the loci of  $P(M)$  and  $Y(M)$ . First we deal with the SP case. The equilibrium of the the SP case is denoted by (63) and (64). Then we express  $Y$  as a function of  $P$ :

$$Y_s(P) = N - P^{\frac{\rho\{\eta-\theta(1-\alpha)\}}{(\theta-\rho)(1-\eta-\alpha)}} \left(\frac{\rho}{c\theta}\right)^{\rho/(\rho-\theta)} K^{\frac{\rho(\theta-1)}{(\theta-\rho)(1-\eta-\alpha)}} f + P^{\frac{-\eta}{1-\eta-\alpha}} K^{1/(1-\eta-\alpha)} \left(1 - \frac{\rho}{\theta}\right). \quad (120)$$

Then we have:

$$\begin{aligned}\frac{\partial Y_s(P)}{\partial P} &= \frac{\rho\{\theta(1-\alpha) - \eta\}}{(\theta-\rho)(1-\eta-\alpha)} P^{\frac{\rho\{\eta-\theta(1-\alpha)\}}{(\theta-\rho)(1-\eta-\alpha)} - 1} \left(\frac{\rho}{c\theta}\right)^{\rho/(\rho-\theta)} K^{\frac{\rho(\theta-1)}{(\theta-\rho)(1-\eta-\alpha)}} f \\ &\quad - \frac{\eta}{1-\eta-\alpha} P^{\frac{-\eta}{1-\eta-\alpha}} K^{1/(1-\eta-\alpha)} \left(1 - \frac{\rho}{\theta}\right).\end{aligned} \quad (121)$$

When  $\theta(1 - \alpha) < \eta$ , (121) is always negative, i.e.,  $Y_s(P)$  is monotonically decreasing function as Case A (See Fig.17(a)).

On the other hand, there exists  $P_s^m$  that satisfy (121) is zero, when  $\theta(1 - \alpha) > \eta$ . Namely,

$$P_s^m = \left\{ \frac{\rho\{\theta(1-\alpha) - \eta\}}{\eta(\theta-\rho)} \right\}^{\frac{(\theta-\rho)(1-\eta-\alpha)}{\theta\{\rho(1-\alpha) - \eta\}}} T^{\frac{(\theta-\rho)(1-\eta-\alpha)}{\eta-\theta\{\rho(1-\alpha)\}}} \left(\frac{\rho}{c\theta}\right)^{\frac{\rho(1-\eta-\alpha)}{\theta\{\eta-\rho(1-\alpha)\}}} K^{\frac{\rho-1}{\rho(1-\alpha)-\eta}}.$$

When  $\rho(1 - \alpha) < \eta$ ,  $Y \rightarrow -\infty$  as  $P \rightarrow 0$  (Case B, See Fig.17(a)), and  $Y \rightarrow \infty$  as  $P \rightarrow 0$  when  $\rho(1 - \alpha) > \eta$  (Case C, See Fig.17(a)). Then it is clear that the function  $Y_s(P)$  has the maximum value  $(P_s^m, Y_s^m)$  in Case B, and has the minimum value  $(P_s^m, Y_s^m)$  in Case C.

From (113), we obtain the following function  $Y_{z=0}(P)$  such as:

$$Y_{z=0}(P) = \frac{\eta + \alpha}{\eta} K^{1/(1-\eta-\alpha)} P^{-\eta/(1-\eta-\alpha)}. \quad (122)$$

Recall that  $z > 0$  is the domain of the SP case, that is, we consider  $Y_s(P)$  when  $Y_{z=0}(P) > 0$ . And we have to take into account the coverage limitation that  $N - Mf$  is always positive. Namely,

$$Y_s(P) > \left(1 - \frac{\rho}{\theta}\right) K^{1/(1-\eta-\alpha)} P^{-1/(1-\eta-\alpha)}.$$

There could exist two intersections ( $(P_{s1}^b, Y_{s1}^b)$  and  $(P_{s2}^b, Y_{s2}^b)$ ,  $P_{s1}^b < P_{s2}^b$ ) of (120) and (122). These intersections satisfy the following equation.

$$N = \left(\frac{\rho}{c\theta}\right)^{\rho/(\rho-\theta)} K^{\frac{\rho(\theta-1)}{(\theta-\rho)(1-\eta-\alpha)}} P^{\frac{\rho\{\eta-\theta(1-\alpha)\}}{(\theta-\rho)(1-\eta-\alpha)}} f + \frac{\alpha\theta + \rho\eta}{\eta\theta} K^{1/(1-\eta-\alpha)} P^{\frac{-\eta}{1-\eta-\alpha}}. \quad (123)$$

On the other hand, there exists a unique intersection  $(P_s^b, Y_s^b)$  that satisfy (123) when  $\theta(1-\alpha) > \eta$ .

Next we deal with the DS case. From (65) and (66), we obtain  $P_d(Y)$ :

$$P_d(Y) = \left(\frac{\rho}{c\theta}\right)^{-1/\theta} \left(\frac{\eta}{\eta + \alpha} Y\right)^{(\theta-1)/\theta} \left\{ N - \frac{\alpha\theta + \rho\eta}{\theta(\eta + \alpha)} Y \right\}^{\frac{\rho-\theta}{\rho\theta}} f^{\frac{\theta-\rho}{\rho\theta}}. \quad (124)$$

Recall  $N - Mf > 0$ . Then (65) gives the domain  $Y \in \left[0, \frac{\theta(\eta + \alpha)N}{\alpha\theta + \rho\eta} f\right]$ . Furthermore, from the equation (122), the range of  $P_d(Y)$  is

$$P_d(Y) \leq K^{1/\eta} \left(\frac{\eta}{\eta + \alpha} Y\right)^{-(1-\eta-\alpha)/\eta}.$$

From (124), we have:

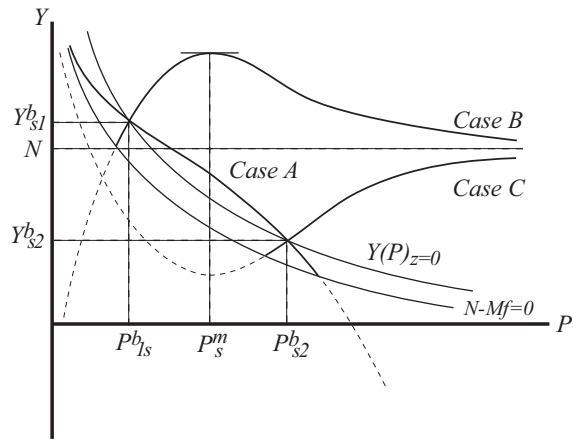
$$\begin{aligned} \frac{\partial P_d(Y)}{\partial Y} &= \left(\frac{\rho}{c\theta}\right)^{-1/\theta} \left(\frac{\eta}{\eta + \alpha}\right)^{(\theta-1)/\theta} Y^{-1/\theta} \left\{ N - \frac{\alpha\theta + \rho\eta}{\theta(\eta + \alpha)} Y \right\}^{\frac{\rho-\theta-\rho\theta}{\rho\theta}} f^{\frac{\theta-\rho}{\rho\theta}} \\ &\quad \left\{ \frac{\theta-1}{\theta} N + \frac{(1-\rho)(\alpha\theta + \rho\eta)}{\rho\theta(\eta + \alpha)} Y \right\}. \end{aligned} \quad (125)$$

When  $\theta > 1$ , (125) is always positive, i.e.,  $P_d(Y)$  is monotonically increasing function (Case D, See Fig.17(b)). And there exists a unique intersection of (122) and (124) ( $Y_{d1}^b, P_{d1}^b$ ) that satisfy the following equation.

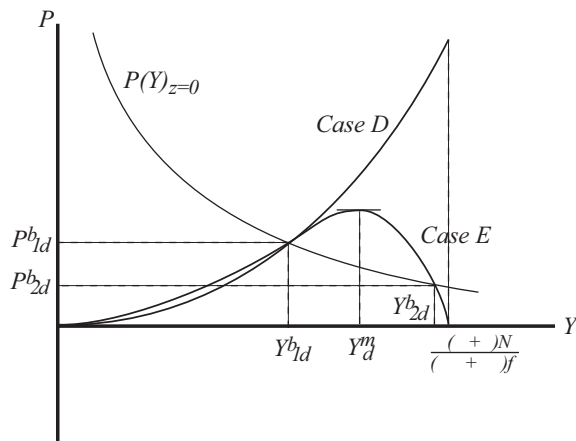
$$\left\{ N - \frac{\alpha\theta + \rho\eta}{\theta(\eta + \alpha)} Y \right\}^{\frac{\rho-\theta}{\rho\theta}} = \left(\frac{\rho}{c\theta}\right)^{1/\theta} K^{1/\eta} \left(\frac{\eta}{\eta + \alpha} Y\right)^{\frac{\eta-\theta(1-\alpha)}{\eta\theta}} f^{\frac{\theta-\rho}{\rho\theta}}. \quad (126)$$

When  $0 < \theta < 1$ , there exists  $Y_d^m$  that satisfy (125) is zero where

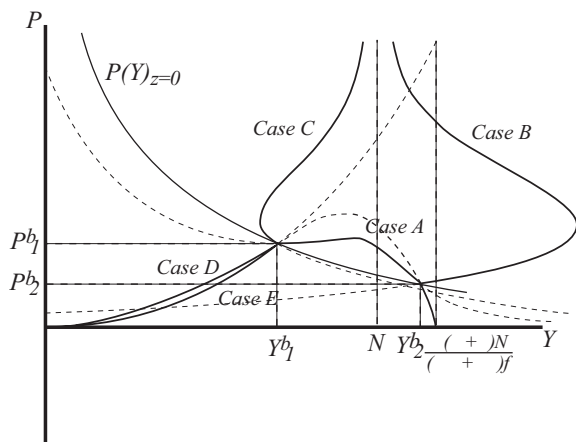
$$Y_d^m = \frac{\rho(1-\theta)(\eta + \alpha)}{(1-\rho)(\alpha\theta + \rho\eta)} N.$$



(a) the SP case



(b) the DS case



(c) Conversion of the Model

Figure 17: The loci of  $P(M)$  and  $Y(M)$

And if  $Y$  satisfy  $0 < Y < Y_d^m$ , (125) is positive, and is negative when  $Y_d^m < Y < \frac{\theta(\eta + \alpha)}{\alpha\theta + \rho\eta}N$ . It means that  $(Y_d^m, P_d^m)$  is maximum value. Then, there could exist two intersections  $((Y_{d1}^b, P_{d1}^b)$  and  $(Y_{d2}^b, P_{d2}^b)$ ,  $Y_{d1}^b < Y_{d2}^b$ ) may exist (Case E, See Fig.17(b)). These intersections also fill the equation (126).

We can show the conversion between the SP case and the DS case (See Fig.17(c)). We prove that  $(Y_{s1}^b, P_{s1}^b) = (Y_{d1}^b, P_{d1}^b)$  and  $(Y_{s2}^b, P_{s2}^b) = (Y_{d2}^b, P_{d2}^b)$ .

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