Interdependent Utility Functions in an Intergenerational Context

Tomoichi Shinotsuka [1]

Department of Economics
Otaru University of Commerce
3-5-21, Midori, Otaru 047-8501 Japan
June 2001
This Version, November 2002

Running Title: Interdependent Utility Functions
Correspondence: Tomoichi Shinotsuka
Department of Economics
Otaru University of Commerce
3-5-21, Midori, Otaru 047-8501
Hokkaido, Japan
e-mail: shino@res.otaru-uc.ac.jp
Phone:0134-27-530
Fax: 0134-27-5320

[1] I gratefully acknowledge the useful comments of seminar and conference participants in Hitotsubashi, Kansai, Waseda, and Caltech (Social Choice and Welfare Meeting).

We investigate the question of representing nonpaternalistic functions (aggregators) in paternalistic form, which was posed by Ray [1987], in an intergenerational setting. As in Hori [2001], the aggregators in this paper may differ across generations and depend possibly on the utility levels of all other generations. We discuss two approaches to deal with an infinite horizon. The first one explores monotonicity structures inherent in nonpaternalistic altruism. By means of lattice-theoretic arguments, we establish the existence of representations of nonpaternalistic functions in paternalistic form. The second approach uses the requirement of small degree of altruism. Keywords: Nonpaternalistic intergenerational altruism, Paternalistic representation, Aggregator. JEL Classification Numbers: D11, D64

1. Introduction

To analyze intertemporal economic problems, the notion of intergenerational altruism has been playing an important role. [2] Researchers in this field identified two models of intergenerational altruism. To borrow terminologies from Ray [1987], the paternalistic model, on the one hand, incorporates intergenerational altruism into the utility function of each generation as a function of consumption allocations among all generations. On the other hand, the nonpaternalistic model captures intergenerational altruism by means of aggregators which relate the utility level of each generation to the utility levels of other generations as well as one's own consumption. The idea of nonpaternalistic altruism was formulated by Becker [1974] in the context of altruism among family members. In intergenerational contexts, the same idea was employed by Barro [1974], Kimball [1987], Ray [1987], Hori and Kanaya [1989], Hori [1992], and Hori [2000].

Ray [1987, 113-114] addressed the following question concerning the two approaches:

> The representation of nonpaternalistic functions in paternalistic form has also been the object of limited attention;.... But a systematic analysis of the relationship between these two frameworks is yet to be written, and appears to be quite a challenge, especially for models with an infinite horizon.

Several interesting results on this question have been delivered recently. Bergstrom [1999] identified the relevance of an infinite version of McKenzie´s [1960] dominant diagonal condition for a given list of linear aggregators to possess a unique representation. Hori [2001] considered the representation problem for the case of a finite number of agents with possibly nonlinear aggregators. Hori [2001] showed that McKenzie´s [1960] dominant diagonal matrix can be useful in the case of non-linear aggregators as well. The model in this paper is an extension of Hori´s [2001] to the case of countably many generations. As in Hori [2001], the aggregators in this paper may differ across generations and depend possibly on the utility levels of all other generations.

We discuss two approaches to deal with an infinite horizon. The first one explores monotonicity structures inherent in nonpaternalistic altruism. By means of lattice-theoretic arguments alone, we establish the existence of representations of nonpaternalistic functions (aggregators) in paternalistic form. Becker [1974] discussed the problem of infinite regress to require that the degree of altruism be small. The second approach uses the requirement of the same spirit expressed in terms of uniformly small Fréche derivative (with respect to the utility level of other generations). We regard this approach as a natural extension of Hori´s [2001] . We also discuss the case of linear aggregators. As

[2] For prominent examples, the reader is referred to the references in Ray [1987] and Hori and Kanaya [1989].

Bergstrom [1999] showed, a certain infinite matrix with a dominant diagonal expresses the idea of small degree of altruism in this case and it offers a powerful tool to represent nonpaternalistic functions in paternalistic form. Our treatment is a little different from Bergstrom [1999] in that we view the infinite matrix as a representation of a continuous linear operator on l_∞ (the set of bounded utility allocations of all generations) into itself while Bergstrom viewed it as a certain limit of finite dimensional square matrices.

The rest of this paper is organized as follows. In the next section, we present the model. In section 3, we discuss the lattice-theoretic approach to the representation problem. In section 4, we discuss the approach based on the contraction mapping theorem. In section 5, we consider the case of linear aggregators. In the last section, we show that the contraction approach is an extension of Hori´s [2001] to the case of an infinite horizon.

2. The Model

For simplicity, we assume that there is one consumer for each generation. The integers t=1, 2, ... denote generations. For each t, $X_t = \mathbf{R}^l_+$ denotes the consumption set of generation t.

Let $X=\prod_{t=1}^{\infty} X_t$. For each t, let U_t be a nonempty subset of $\mathbf{R}^\infty$. A generic element $u_{-t} = (u_1,..., u_{t-1}, u_{t+1}, ...) \in U_t$ signifies a profile of utilities other than generation t.

We employ the following terminologies. Let f be a real-valued function on an ordered set Y. We say that f is non-decreasing (resp. strictly increasing) if $f(x) \geq f(y)$ (resp. $f(x) > f(y)$) for all x, y $\in$Y with $x \geq y$ and $x \neq y$.

For each t, a real-valued function G_t on $X_t \times U_t$ is given. We call it the **aggregator for generation t**. Let $G=(G_1, G_2, ...)$ be the profile of the aggregators.

Representation Problem (RP): Given the profile G of aggregators, find a profile $u=(u_1, u_2, ...)$ of real-valued functions on X such that for each x $\in$X and t, $u_t(x) = G_t(x_t, u_{-t}(x))$, where $u_{-t}(\cdot)$ denotes the profile with the t-th component $u_t(\cdot)$ deleted, $u_t(x)$ is strictly increasing in x_t and non-decreasing in $x_{-t} = (x_1,..., x_{t-1}, x_{t+1}, ...)$.

If RP has a solution $u = (u_1, u_2,...)$, we call it **a paternalistic representation of $\mathbf{G = (G_1, G_2,...)}$**.We call the t-th component $u_t(\cdot)$ of the representation u the **utility function of generation t.** Two questions immediately arise.

Question 1: Does G have a paternalistic representation?

Question 2: Is the representation unique?

3. The Lattice-Theoretic Approach

In this section, we assume the following on the aggregators.

Pointwise Boundedness (PB): For each t and $x_t \in X_t$, $\{G_t(x_t, u_{-t})$: $u_{-t} \in U_t\}$ is bounded.

Monotonicity(MON): For each t, $G_t(x_t, u_{-t})$ is strictly increasing in x_t and non-decreasing in u_{-t}.

Now, we present the first main result.

Theorem 1: Under PB and MON, there exists a paternalistic representation of a given profile of aggregators.

Proof. By PB, we can define the following real-valued functions. For each t and x = $(x_1, x_2, ...)\in X$, let $\alpha_t(x) = \inf\{G_t(x_t, u_{-t})$: $u_{-t} \in U_t\}$, $\beta_t(x) = \sup\{G_t(x_t, u_{-t})$: $u_{-t} \in U_t\}$. We consider the following function spaces. $\mathcal{U}_t = \{u_t(\cdot)|\ u_t(\cdot)$ is non-decreasing and for each $x\in X$, $\alpha_t(x)\leq u(x)\leq \beta_t(x)\}$. The set $\mathcal{U}_t$ is non-empty since $\alpha_t(\cdot)$ and $\beta_t(\cdot)$ belong to it. Let $\mathcal{U} = \prod_{t=1}^{\infty}\mathcal{U}_t$. We equip $\mathcal{U}$ with the natural order $\geq$, i. e. $u \geq v$ if $u_t(x) \geq v_t(x)$ for every x and t. For $u = (u_1, u_2, ...)$, $v = (v_1, v_2, ...)\in \mathcal{U}$, let $u\wedge v = \inf\{u, v\}$ and $u \vee v = \sup\{u, v\}$. Then, for each $x\in X$, $(u \wedge v)(x) = (\min\{u_1(x), v_1(x)\}, \min\{u_2(x), v_2(x)\},...)$ and $(u\vee v)(x) = (\max\{u_1(x), v_1(x)\}, \max\{u_2(x), v_2(x)\},...)$. These operations, $\wedge$ and $\vee$, make $\mathcal{U}$ a complete lattice, i. e. for every non-empty subset $\mathcal{T}$ of $\mathcal{U}$, $\inf\mathcal{T}$ and $\sup\mathcal{T}$ exist and belong to $\mathcal{U}$. Indeed, $\inf\mathcal{T}\ (x)= (\inf\{u_1(x)\ |u\in \mathcal{T}\}, \inf\{u_2(x)\ |u\in \mathcal{T}\},...)$ and $\sup\mathcal{T}\ (x)= (\sup\{u_1(x)\ |u\in \mathcal{T}\}, \sup\{u_2(x)\ |u\in \mathcal{T}\},...)$ are non-decreasing in x and belong to $\mathcal{U}$.

For each $u(\cdot)=(u_1(\cdot), u_2(\cdot),...)\in \mathcal{U}$ and t, let $F(u(\cdot))_t(x)= G_t(x_t, u_{-t}(x))$. Clearly, $F(u(\cdot))_t(x)$ is strictly increasing in x_t and non-decreasing in x_{-t}. It is also trivial that $F(u(\cdot))_t(\cdot) \in \mathcal{U}_t$. Hence, the operator F maps $\mathcal{U}$ into itself. Clearly, F(u) is non-decreasing in u. Hence, by Tarski´s fixed point theorem [1955], there exists $u(\cdot)=(u_1(\cdot), u_2(\cdot),...)\in \mathcal{U}$ such that for every x and t, $u_t(x) = G_t(x, u_{-t}(x))$. By MON, u_t satisfies the desired monotonicity properties. ■

Example 1: To see how crucial PB is in Theorem 1, let us consider the following profile of aggregators $G = (G_1, G_2, G_3, ...)$: $G_1(x_1, u_{-1}) = p\cdot x_1 + \alpha u_2$, $G_2(x_2, u_{-2}) = p\cdot x_2 +\beta\ u_1$, $G_t(x_t, u_{-t}) = p\cdot x_t$ (t = 3, 4, ...), where p is an l-dimensional vector with strictly positive components, and α and β are positive constants satisfying $\alpha\beta>1$. Clearly G satisfies MON but violates PB. Suppose G possesses a system of utility functions $u = (u_1, u_2, u_3, ...)$. Then, $u_1(x) = p\cdot x_1 + \alpha u_2(x)$ and $u_2(x) = p\cdot x_2 +\beta\ u_1(x)$ for all x. Hence, $u_1(x) = \frac{p\cdot x_1+\alpha p\cdot x_2}{1-\alpha\beta}$. Since $1 - \alpha\beta < 0$, $u_1(x)$ cannot be strictly increasing in own consumption x_1 (or non-decreasing in x_2 for that matter). A contradiction obtains. Therefore, there is no paternalistic representation.

4. The Contraction Approach

In this section, we obtain a unique paternalistic representation of a given profile of aggregators.

To this end, we add a few more assumptions on the aggregators. For simplicity, we put a restriction on the domains of the aggregators: For each t, U_t is equal to $\mathbf{l}_\infty$. For $u\in \mathbf{l}_\infty$, $||u||_\infty$ denotes the sup norm of u. $\mathbf{1}$ denotes the constant sequence (1, 1,...). Note that the domain of the aggregator, $X_t \times U_t$ is a subset of $\mathbf{R}^\infty$. We equip $X_t \times U_t$ with the relative product topology. From now on, we refer it as the product topology.

Continuity(CONT): For each t, the aggregator G_t is product continuous.

Uniform Boundedness(UB): For every $\alpha \in \mathbf{R}$, $\sup_t \sup_{x_t \in X_t} |G_t(x_t, \alpha\mathbf{1})| < \infty$.

Lipschitz Condition(LC): There exists $\delta \in (0, 1)$ such that for every t, x_t, u_{-t} and v_{-t}, $|G_t(x_t, u_{-t}) - G_t(x_t, v_{-t})| \leq \delta ||u_{-t} - v_{-t}||_\infty$.

CONT is standard. UB may be weakened at the cost of elaborating the choice of relevant function spaces (Boyd [1990]), which we do not pursue in this paper. LC expresses the idea that the utility level of each generation does not depend too much on those of other generations.

Theorem 2: Under CONT, UB, and LC, there uniquely exists a paternalistic representation of a given profile of aggregators.

Proof. We set up different function spaces from those in the previous section. Let $\mathcal{U} = \{u = (u_1, u_2, ...)|$ For each t, u_t is a product continuous, real-valued function on X, and $\sup_{x \in X} \sup_t |u_t(x)| < \infty\}$. For $u = (u_1, u_2, ...) \in \mathcal{U}$, let $||u||_\infty = \sup_{x \in X} \sup_t |u_t(x)|$. By the standard argument, $\mathcal{U}$ is a Banach space under the norm $||u||_\infty$.

Let $\mathcal{U}^{inc} = \{u = (u_1, u_2, ...) \in \mathcal{U}|$ For each t, u_t is non-decreasing.$\}$. Clearly, $\mathcal{U}^{inc}$ is a closed subset of U so that it is a complete metric space.

Now, we define a operator T on $\mathcal{U}^{inc}$. For $u = (u_1, u_2, ...) \in \mathcal{U}^{inc}$ and $x \in X$, let $T(u)(x) = (G_1(x_1, u_{-1}(x)), G_2(x_2, u_{-2}(x)), ...)$, where $u_{-t}(x) = (u_1(x), u_2(x), ..., u_{t-1}(x), u_{t+1}(x), ...)$ for every t. To see that T maps $\mathcal{U}^{inc}$ into itself, for every $x \in X$, $u = (u_1, u_2, ...) \in \mathcal{U}^{inc}$, and t, $G_t(x_t, -||u||\mathbf{1}) \leq G_t(x_t, u_{-t}(x)) \leq G_t(x_t, ||u||\mathbf{1})$ by MON.

Thus, for every t,

$|G_t(x_t, u_{-t}(x))| \leq \max\{\sup_{x \in X} \sup_\tau |G_\tau(x_\tau, ||u||\mathbf{1})|, \sup_{x \in X} \sup_\tau |G_\tau(x_\tau, -||u||\mathbf{1})|\}$.

Thus, by UB, $\sup_{x \in X} \sup_t |G_t(x_t, u_{-t}(x))| < \infty$. Clearly, for every t, $G_t(x_t, u_{-t}(x))$ is non-decreasing in x and product continuous in x. Hence, T maps $\mathcal{U}^{inc}$ into itself.

By LC, T is a contraction. Hence, by the contraction mapping theorem, there exists a unique $u^* = (u_1^*, u_2^*, ...) \in \mathcal{U}^{inc}$ such that $u^* = T(u^*)$, i.e. for every $x \in X$ and t, $u_t(x) = G_t(x_t, u_{-t}(x))$. By MON, $u_t(x)$ is strictly increasing in x_t.

5. Linear Representation Problem

We call a real-valued, increasing function ν_t on X_t a **felicity function of generation t**. $\nu = (\nu_1, \nu_2, ...)$ denotes a profile of felicity functions. Let $\nu = (\nu_1, \nu_2, ...)$ be a profile of felicity functions and let $\{a_{tj}\}_{t=1 j=1}^{\infty \infty}$ be a double sequence such that for each t and j, $a_{tj} \geq 0$ and $a_{tt}=0$, and $\{a_{tj}\}_{j=1}^{\infty}$ is summable. We say that the aggregator $G_t(\cdot, \cdot)$ is **linear** if it is of the form $G_t(x_t, U_{-t}) = \nu_t(x_t)+\sum_{j=1}^{\infty} a_{tj} U_j$.

Linear Representation Problem (LRP): Given a profile of linear aggregators, find a paternalistic representation.

Two immediate questions arise.

Question 3: Does LRP possess a solution?

Question 4: Is a solution to LRP unique?

To give a positive answer to each question, we propose a condition which generalizes Hori´s [2001]. To this end, let B be the infinite matrix defined by

$$\begin{bmatrix} 1 & -a_{12} & -a_{13} & \cdot & \cdot \\ -a_{21} & 1 & -a_{23} & \cdot & \cdot \\ -a_{31} & -a_{32} & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \end{bmatrix}$$

Let b_{ij} be the (i, j)-element of the matrix B, i. e. $b_{ij} = 1$ if i = j, and $b_{ij} = -a_{ij}$ otherwise. Let n be a positive integer and let $I_1 = \{1, 2, ..., n\}$,..., $I_k = \{n(k-1)+1, n(k-1)+2, ..., n(k-1)+n\}$ (k = 2, 3, ...). Then, the set $\{I_k\}_{k=1}^{\infty}$ partitions the set $\mathbf{N}$ of all positive integers. For every i and j $\in \mathbf{N}$, let B_{ij} be the sub-matrix $[b_{lm}]_{l\in I_i, m\in I_j}$.

Dominant Diagonal Blocks (DDB): The matrix B has a dominant diagonal blocks, i.e. there exists $n \in \mathbb{N}$ such that for all i, B_{ii} satisfies the Hawkins-Simon condition, and there exists a norm $||\cdot||$ on $\mathbf{R}^n$ such that $\sup_i \sup_{x\in \mathbf{R}^n:||x||=1} ||B_{ii}^{-1}x|| < \infty$ and $\sup_i \sup_{x\in \mathbf{R}^n:||x||=1} \Sigma_{j\neq i}^{\infty} ||B_{ii}^{-1} B_{ij} x|| < 1$.[3]

DDB means that off-diagonal blocks are small in terms of some norm. This intuition may easily be seen in a special case n = 1. In this case,all the diagonal blocks B_{ii} degenerate into 1×1 matrix 1.

Dominant Diagonal (DD): $\sup_t \sum_{j\neq t}^{\infty} a_{tj} < 1$.

The series $\Sigma_{j\neq t}^{\infty} a_{tj}$ may be regarded as the degree of intergenerational altruism. Then, DD clearly expresses the idea that the degree of intergenerational altruism is small.

To see the relevance of DDB, let us look at the system of simultaneous equations:

$U_t = G_t(x_t, U_{-t}) = \nu_t(x_t)+\sum_{j=1}^{\infty} a_{tj} U_j$ (t=1,2,...).

[3] Araujo and Scheinkman [1979] applied this version of diagonal dominance assumption to deliver comparative dynamics results in infinite horizon optimization problems.

We search for a bounded sequence U = (U_1, U_2,...) that solves the simultaneous equation. This immediately raises a question of invertibility of the continuous linear operator T:$\mathbf{l}_\infty \to \mathbf{l}_\infty$ represented by the infinite matrix B.

Let I:$\mathbf{l}_\infty \to \mathbf{l}_\infty$ be the identity operator. By DDB, ||T - I|| <1. Hence, T is invertible and $T^{-1} = \sum_{j=0}^{\infty}(I - T)^j$. See Lang[1969, Ch.5], for example. The last formula shows the inverse operator T is represented by a non-negative infinite matrix. Thus, by DDB, the system has the unique solution:

$$U(x) = T^{-1}\nu(x) = \nu(x) + \sum_{j=1}^{\infty}(I - T)^j \nu(x).$$

Let U(x) =($U_1(x_1, x_{-1})$, $U_2(x_2, x_{-2})$,...). Since each $\nu_t(x_t)$ is strictly increasing in x_t, each $U_t(x_t, x_{-t})$ is strictly increasing in x_t. Since $\sum_{j=1}^{\infty}(I - T)^j$ is non-negative, $\sum_{j=1}^{\infty}(I - T)^j\nu(x)$ is non-decreasing in x. Hence, U(x) gives the unique solution to LRP.

Now, we discuss diagonal dominance introduced by Bergstrom [1999].

Bergstrom Dominant Diagonal (BDD): There exists a bounded sequence d = (d_1, d_2,...) such that for all t, $d_t > 0$, and $\inf_t$ (d_t - $\sum_{j=1}^{\infty} a_{t_j} d_j$) > 0.

Suppose that the infinite matrix B satisfies BDD. Then, the continuous linear operator T: $l_\infty \to l_\infty$ represented by the infinite matrix B is invertible. The infinite matrix representing the inverse operator of T is of the following form: $DC^{-1}D^{-1}$, where D = diag(d_1, d_2, ...),C = (c_{t_j}), c_{t_j} = ($a_t d_j)/d_t$. Note that the existence of the inverse matrix of C follows from ||C - I|| <1, where I denotes the identity matrix and $\|\cdot\|$ denotes the sup-norm. Since $C^{-1} = \sum_{j=0}^{\infty}(I - C)^j$, C^{-1} is nonnegative. Hence, $DC^{-1}D^{-1}$ is nonnegative also. Hence, under BDD , LRP has a unique solution.

6. Link between the Contraction Approach and DDB

In this section, we consider the logical implications of differentiable aggregators. To be more specific, we extend Hori´s result [2001] by means of the contraction approach.

Smoothness(S):For each t and x_t, $G_t(x_t, u_{-t})$ is continuously Fréchet differentiable with respect to u_{-t}.

Let $D_{u_{-t}}G_t(x_t, u_{-t})$ be the derivative of $D_{u_{-t}}G_t(x_t, u_{-t})$ with respect to u_{-t}. Note that $D_{u_{-t}}G_t(x_t, u_{-t})$ is a sup norm continuous, linear functional on l_∞. By MON, it is nonnegative. By definition of the dual norm, $||D_{u_{-t}}G_t(x_t, u_{-t})|| = \sup_{h\in l_\infty:||h||_\infty=1}|D_{u_{-t}}G_t(x_t, u_{-t})(h)|$. Since $D_{u_{-t}}G_t(x_t, u_{-t})$ is nonnegative, $||D_{u_{-t}}G_t(x_t, u_{-t})||$ can be written as $D_{u_{-t}}G_t(x_t, u_{-t})(\mathbf{1})$. To see the link between the contraction approach in the previous section and the condition developed by Hori [2001], it is useful to consider the following condition.

Limited Utility Dependence (LUD): $\sup_{u\in\mathcal{U}^{inc}}\sup_t\sup_{x_t\in X_t}||D_{u_{-t}}G_t(x_t, u_{-t}(x))||<1$.

By the mean value theorem (see Lang [1969, Corollary 1, Ch.5] for example), for every t, x, u_{-t} and v_{-t},
$|G_t(x_t, u_{-t}) - G_t(x_t, v_{-t})| \leq \sup_{w_{-t}} ||D_{u_{-t}}G_t(x_t, w_{-t})|| \, ||u_{-t} - v_{-t}||_\infty$, where

the $\sup_{w_{-t}}$ is taken over any w_{-t} on the line segment between u_{-t} and v_{-t}. Let $\delta = \sup_{u\in\mathcal{U}^{inc}}\sup_t\sup_{x_t\in X_t}||D_{u_{-t}}G_t(x_t, u_{-t}(x))||$. Then, by LUD, $\delta<1$. Since $\sup_{w_{-t}} ||D_{u_{-t}}G_t(x_t, w_{-t})|| \leq \delta$, we have $|G_t(x_t, u_{-t}) - G_t(x_t, v_{-t})| \leq \delta \, ||u_{-t} - v_{-t}||_\infty$. Thus, LUD implies LC.

In order to see the link between our results and Hori´s[2001], we need to invoke the Yosida-Hewitt decomposition theorem (see Yosida and Hewitt [1952]): $D_{u_{-t}}G_t(x_t, u_{-t})$ can be expressed as

$D_{u_{-t}}G_t(x_t, u_{-t})(h)= \sum_{j\neq t}^{\infty} p_{tj}(x_t, u_{-t})h_j + \lambda_t(x_t, u_{-t})(h)$ for every $h\in l_\infty$, where $\{p_{tj}(x_t, u_{-t})\}_{j\neq t}^{\infty}$ is an absolutely summable, nonnegative sequence and $\lambda_t(x_t, u_{-t})$ is a purely finitely additive, nonnegative linear functional on l_∞.

Let $j_0 \neq t$, and let $e^{j_0} = \{e_j^{j_0}\}_{j\neq t}^{\infty}$ be the sequence defined by $e_{j_0}^{j_0} =1$ and $e_j^{j_0}=0$ for $j\neq t, j_0$. Then, $D_{u_{-t}}G_t(x_t, u_{-t})(e^{j_0})= p_{tj_0}(x_t, u_{-t})$. Since $D_{u_{-t}}G_t(x_t, u_{-t})(e^{j_0})$ is the partial derivative of $G_t(x_t, u_{-t})$ with respect to u_{j_0},denoted by $G_{tj_0}(x_t, u_{-t})$, $\{G_{tj}(x_t, u_{-t})\}_{j\neq t}^{\infty}$ is absolutely summable and nonnegative.

Let $a_{tj} = \sup_{u\in\mathcal{U}^{inc}}\sup_{x\in X}G_{tj}(x_t, u_{-t})(t \neq j)$. Clearly, $\sup_{u\in\mathcal{U}^{inc}}\sup_t\sup_{x\in X}\{\sum_{j\neq t}^{\infty} G_{tj}(x_t, u_{-t})\} \leq \sup_t \sum_{j\neq t}^{\infty} a_{tj}$. Now, let us consider the following two conditions.

Exclusion(EX): For each t, $x\in X$, and u_{-t}, the purely finitely addtive part $\lambda_t(x_t, u_{-t})$ of the Fréchet derivative $D_{u_{-t}}G_t(x_t, u_{-t})$ vanishes.

Uniformly Dominant Diagonal Blocks(UDDB): There exists a nonnegative infinite matrix $A =[a_{tj}]_{t=1}^{\infty}{}_{j=1}^{\infty}$ such that for each t, j, and (x_t, u_{-t}), $a_{tt} = 0$, $a_{tj} \geq \partial G_t(x_t, u_{-t})/\partial u_j$ and that the infinite matrix I-A satisfies DDB.

It follows from the above discussions that UDDB, along with EX, imply LUD. This explains why UDDB, the analogue of Hori´s condition ((4.1) in Hori[2001]), is useful in obtaining the unique solution to RP.

References

Araujo, A. and Scheinkman, J. A., Notes on Comparative Dynamics, in *General equilibrium, Growth, and Trade:Essays in Honor of Lionel Mckenzie* (J. Green and J. A. Scheinkman eds.), 1979.

Barro, R., Are Government Bond Net Wealth?, *Journal of Political Economy* **82**, 1095-1117.

Becker, G. S., A Theory of Social Interactions, *Journal of Political Economy* **82**, 1063-1093.

Bergstrom, T. C., Systems of Benevolent Utility Functions, *Journal of Public Economic Theory*, **1**, 71-100.

Boyd, J., Recursive Utility and the Ramsey Problem, *J. Econ. Theory* **50**, 326-346, 1990.

Hori, H., Utility Functionals with Nonpaternalistic Intergenerational Altruism: The Case Where Altruism Extends to Many Generations, *Journal of Economic Theory* **56**, 451-467, 1992.

Hori, H., Non-stationary Intergenerational Altruism, mimeo. Tohoku University, 2000.

Hori, H., Non-paternalistic Altruism and Utility Interdependence *Japanese Economic Review* **52**, No.2, 137-155, 2001.

Hori, H. and Kanaya S., Utility Functionals with Nonpaternalistic Intergenerational Altruism, *Journal of Economic Theory* **49**, 241-265, 1989.

Kimball, M. S., Making Sense of Two-sided Altruism, *Journal of Monetary Economics* **20**, 301-326, 1987.

Lang, S, *Real Analysis*, Addison-Wesley, 1969.

McKenzie, L. W., Matrices with Dominant Diagonals and Economic Theory, in K.J. Arrow, S. Karlin, and P. Suppes, eds., *Mathematical Methods in the Social Sciences*, 1959, Stanford: Stanford University Press, pp. 47-62.

Ray, D., Nonpaternalistic Intergenerational Altruism, *Journal of Economic Theory* **41**, 112-132, 1987.

Tarski, A., A Lattice-Theoretical Fixed Point Theorem and Its Applications, *Pacific J. Math.* **5**, 285-309, 1955.

Yosida, K. and Hewitt E., Finitely Additive Measures, *Trans. Amer. Math. Soc.* **72**, 46-66.