Non-Manipulable Division Rules in Claim Problems and Generalizations*

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Abstract

This paper studies an abstract class of problems of dividing an amount when each recipient is characterized by a number or a vector. The class can deal with problems of bankruptcy, cost sharing, income redistribution, social choice with transferable utilities, probability updating, and probability aggregation. We give a full characterization of the family of division rules for which no group of recipients can increase the total amount of their awards by transferring their characteristic vectors within the group. Any rule that satisfies the non-manipulability condition and a mild boundedness condition consists of a "priority part," which assigns fixed (possibly asymmetric) awards, and a "proportional part," which assigns awards in proportion to characteristic vectors. This family of rules includes the proportional rule, equal division, and weighted versions of "equal-distance" type rules, and is closed under convex combinations. A number of existing and new results in specific contexts are obtained as corollaries.

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1 Introduction

Consider a situation where a certain amount of a good has to be divided among a set of agents. Each agent is characterized by a number or a vector, which is taken into consideration when a division is determined. There exist a variety of real-life allocation problems that take this abstract form. For example, when the liquidation value of a bankrupt firm is divided among creditors, the amount of credits that the creditors hold are taken into account; when the surplus of a project is divided among investors, the contribution levels of the investors are taken into account. We search for systematic methods, or *rules*, of determining a division. We are interested in rules for which no group of agents can increase their total awards by reallocating their characteristic vectors within the group. This condition, introduced by Moulin (1985a), prevents distortion of adjudication processes from tactical maneuvers and is referred to as *reallocation-proofness*.

We consider a general class of allocation problems to accommodate a number of specific classes of problems studied in the literature. We consider a situation where there is a set of recipients, or entities, N.[1] Each entity is identified by a characteristic vector, denoted by $c_i \equiv (c_{ik})_{k \in K} \in \mathbb{R}_+^K$ where K is a finite set of issues. A *problem* is described by (c, E), where $c \equiv (c_i)_{i \in N} \in \mathbb{R}_+^{N \times K}$ is a profile of characteristic vectors and E is an amount to be divided. A *rule* is a function that specifies a vector of awards for each admissible problem. Our results hold for any class of problems, provided that the class is *rich* in the sense that it is closed under transfers of characteristic vectors within any group of entities.

Our framework can deal with a variety of specific classes of problems studied in the literature. An example is *bankruptcy problems* (O'Neill, 1982), in which E is a liquidation value of a bankrupt firm and this value has to be divided among creditors. Each creditor is characterized by a number $c_i \in \mathbb{R}_+$, which is the amount that the creditor can claim against the bankrupt firm. It is assumed that the liquidation value of the firm is not sufficient to satisfy all claims, i.e., $\sum_{i \in N} c_i \geq E$. A "dual" of bankruptcy problems is *surplus sharing problems* (Moulin, 1987), in which E is total surplus from a cooperative venture, $c_i \in \mathbb{R}_+$ denotes agent i's contribution to the venture, and $\sum_{i \in N} c_i \leq E$.[2]

In bankruptcy and surplus-sharing problems, the amount to be divided is exogenous, but our framework enables us to consider problems in which the amount to be divided depends on characteristics. For example, in *cost allocation problems* with a cost function $g \colon \mathbb{R}_+ \to \mathbb{R}_+$, the total cost to be allocated is given by $g(\sum_{i \in N} c_i)$, which depends on the total usage of a service. Also, in *income redistribution problems*, the total income to be redistributed is given by $\sum_{i \in N} c_i$ where c_i is individual i's income.

In all the previous examples, K is a singleton. An example where K is not a singleton is *social choice problems with transferable utility* (Moulin, 1985a), where K is the set of alternatives and c_i is the vector of agent i's valuations for the alternatives. The amount

[1] We use this rather neutral term, "entity," because N has various meanings in applications.

[2] For surveys of studies of these problems, see Moulin (2002) and Thomson (2003a,b).

to be divided is given by the maximum total valuation, i.e., $E = \max_{k \in K} \sum_{i \in N} c_{ik}$.

Other examples of specific division problems with somewhat different interpretations are *probability updating problems* (Stalnaker, 1968; Lewis, 1973) and *probability aggregation problems* (McConway, 1981; Rubinstein and Fishburn, 1986). In these problems, N is the set of states of the world, K is the set of agents, c_{ik} is the probability that agent k assigns to state i, and $E = 1$ is the total probability to be allocated to the states.

In probability aggregation problems, rules specify how to aggregate $|K|$ probability distributions into a single distribution. To see what reallocation-proofness means in this context, consider two situations that are identical except that agents may put different probabilities to the states in an event $S \subseteq N$. Thus, in the two situations, each agent puts the same probability on the event S as a whole but may distribute the probability over S differently. Then, reallocation-proofness says that the aggregated (social) distribution puts the same probability on the event S in the two situations (but may distribute the probability over S differently).

This suggests that, in the context of probability aggregation, reallocation-proofness can be interpreted as a requirement of informational efficiency. In reality, it is often the case that an event consists of a large number of states. Under a reallocation-proof aggregation rule, if a society is interested in an event but not in the individual states that constitute it, then the society can treat the event as a single composite state without any loss and does not have to collect information about agents' beliefs over those individual states.

A simplest well-known probability aggregation rule is to compute a weighted average of the individual distributions (what is called a *linear opinion pool* by McConway, 1981), which is reallocation-proof.

In probability updating problems, an agent (or a society) has a prior probability distribution over the state space N^* (here $|K| = 1$). If event $N \subseteq N^*$ occurs, then the prior probability distribution needs to be updated. A rule specifies how to update the prior distribution according to this new information. A commonly used updating rule is Bayes' rule, which is also reallocation-proof since for Bayes' rule it is immaterial whether the event is treated as a single composite state.

Our main result characterizes the family of reallocation-proof rules (Theorem 1). Every rule in this family can be written as the sum of two parts: a "priority part," which may treat entities asymmetrically on the basis of their identities but ignores differences among their characteristic vectors, and an "additive part," which treats entities symmetrically and depends on their characteristic vectors in an additive way. If a rule in this family satisfies a mild boundedness condition, its additive part is proportional to characteristic vectors (Theorem 2). This family includes the proportional rule (when characteristic vectors are one-dimensional), equal division, and weighted versions of "equal-distance" type rules, and is closed under convex combinations. This characterization holds for any class of problems as long as the richness condition is satisfied and there exist three or more entities.

Several existing and new results in specialized contexts are obtained as corollaries. In particular, our theorem generates the characterizations of the proportional rule in O'Neill (1982), Chun (1988), de Frutos (1999), Ching and Kakker (2001), Chambers and Thomson (2002), and Moulin (2002), some families of rules studied by Chun (1988) and Moulin (1985a, 1987), and "linear opinion pools" studied by McConway (1981). We also show that, for the characterization of the proportional rule, reallocation-proofness can be weakened to its pairwise version; i.e., it suffices to require that no pair of entities can manipulate.

The remaining part of the paper is organized as follows. Section 2 introduces definitions and notation. Section 3 presents our main results. Section 4 examines applications of our results in the setting where the set of entities is fixed. Section 5 examines applications in the setting where the set of entities is variable, where we discuss, among others, an axiom called *merging-splitting-proofness*, which is closely related to reallocation-proofness. The appendix gives proofs omitted in the text.

2 Definitions

2.1 Division Problems

There is a finite set $N = \{1, 2, \ldots, |N|\}$ of *entities*. Each entity $i \in N$ is characterized by a finite dimensional vector $c_i \equiv (c_{ik})_{k\in K} \in \mathbb{R}^K_+$ where $K = \{1, 2, \ldots, |K|\}$ is a finite set of issues. We refer to c_i as i's *characteristic vector*. A profile of characteristic vectors is denoted by $c \equiv (c_i)_{i\in N} \in \mathbb{R}^{N\times K}_+$ and the sum of these vectors is denoted by

$$\bar{c} \equiv (\bar{c}_k)_{k\in K} \equiv (\sum_{i\in N} c_{ik})_{k\in K} \in \mathbb{R}^K_+.$$

A *problem* is a pair $(c, E) \in \mathbb{R}^{N\times K}_+ \times \mathbb{R}_{++}$, where $c \in \mathbb{R}^{N\times K}_+$ is a profile of characteristic vectors and $E \in \mathbb{R}_{++}$ is an amount to be divided. For simplicity, we only consider problems such that for each $k \in K$, $\bar{c}_k > 0$.

A *domain* is a non-empty set of problems and is denoted by $\mathcal{D}$. A division rule, or briefly, a *rule* over a domain $\mathcal{D}$ is a function f associating with each problem $(c, E) \in \mathcal{D}$ a vector of awards $f(c, E) \in \mathbb{R}^N$. A domain $\mathcal{D}$ is *rich* if, for each problem $(c, E) \in \mathcal{D}$ and each profile $c' \in \mathbb{R}^{N\times K}_+$ such that $\bar{c}' = \bar{c}$, we have $(c', E) \in \mathcal{D}$. That is, $\mathcal{D}$ is rich if it is closed under reallocations of characteristic vectors. We restrict our attention to rich domains. For each problem $(c, E) \in \mathcal{D}$, let

$$\mathcal{D}(\bar{c}, E) \equiv \{(c', E) \in \mathbb{R}^{N\times K}_+ \times \mathbb{R}_{++} : \bar{c}' = \bar{c}\}.$$

Then richness says that, for each $(c, E) \in \mathcal{D}$, we have $\mathcal{D}(\bar{c}, E) \subseteq \mathcal{D}$.

The notion of richness enables us to investigate various classes of problems in a unified way. Here are well-known examples of classes that satisfy richness:

Bankruptcy. As mentioned in the introduction, a bankruptcy problem deals with the problem of how to divide the liquidation value E of a bankrupt firm among the set of creditors N (O'Neill, 1982). In this problem, $|K| = 1$, and $c_i \in \mathbb{R}_+$ is the claim that creditor i has against the bankrupt firm. The liquidation value is not sufficient to satisfy all claims. Thus $\mathcal{D} = \{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++} : \sum_{i \in N} c_i \geq E\}$.

A bankruptcy problem can also be interpreted as a problem of collecting income tax. In this problem, c_i is individual i's income level and E is the total amount of tax to be collected (Young, 1987).

In practice, a firm issues a variety of financial assets and bankruptcy laws distinguish types of financial assets that the creditors hold. This motivates us to consider the following multi-dimensional generalization of bankruptcy problems:

Multi-Dimensional Bankruptcy. As in the single-dimensional case, E is the liquidation value of a bankrupt firm and N is the set of creditors. Let K denote the set of types of financial assets and c_{ik} the claim that creditor i holds in the form of asset k. Thus, the class of multi-dimensional bankruptcy problems is given by $\mathcal{D} = \{(c, E) \in \mathbb{R}^{N \times K}_+ \times \mathbb{R}_{++} : \sum_{k \in K} \bar{c}_k \geq E,\ \text{and } \bar{c}_k > 0 \text{ for all } k \in K\}$.

Surplus Sharing. The problem is how to divide the profit from a project among the contributors (Young, 1987). Here, $|K| = 1$, c_i is the amount of the opportunity cost for contributor i, and $E \geq \sum c_i$ is the profit that the project generates. Thus $\mathcal{D} = \{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++} : 0 < \sum_{i \in N} c_i \leq E\}$.

Claim Problems. This class is simply the union of the classes of (single-dimensional) bankruptcy and surplus-sharing problems (Moulin, 1987; Chun, 1988).[3] That is, no inequality between E and $\sum_{i \in N} c_i$ is imposed. Thus $\mathcal{D} = \{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++} : \sum_{i \in N} c_i > 0\}$.

Social Choice with Transferable Utility. In this class, N is the set of agents and K is the set of alternatives. Agents have quasi-linear preferences and c_{ik} denotes agent i's valuation of alternative k. Given the feasibility of monetary transfers, the total surplus to be divided is given by $E = \max_{k \in K} \bar{c}_k$, i.e., the total valuation of the efficient alternatives. Thus $\mathcal{D} = \{(c, E) \in \mathbb{R}^{N \times K}_+ \times \mathbb{R}_{++} : E = \max_{k \in K} \bar{c}_k,\ \text{and } \bar{c}_k > 0 \text{ for each } k \in K\}$. This class of problems differs from the previous ones since the amount to be divided depends on c. This problem is studied by Moulin (1985a).

Cost Sharing. Let N be the set of agents and $|K| = 1$. Each agent $i \in N$ has a demand $c_i \geq 0$ for a good. For each profile of demands $c \in \mathbb{R}^N_+$, the aggregate cost to

[3] Moulin (1987) interprets this problem as pure-surplus sharing after all opportunity costs are returned to contributors.

be shared among the agents is given by $C(\bar{c})$ where $C\colon \mathbb{R}_+ \to \mathbb{R}_+$ is a cost function. Then $\mathcal{D} = \{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++} : E = C(\bar{c})\}$. This problem is studied by (Moulin and Shenker, 1992).

Income Redistribution. Let N be the set of individuals in society and $|K| = 1$. Each individual $i \in N$ has an income $c_i \geq 0$. The problem is how to redistribute the total income in the society, $\bar{c}$, among the individuals. Then $\mathcal{D} = \{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_{++} : E = \bar{c}\}$.

Probability Updating. Let N^* be the set of all states of the world. A person initially has a probability distribution over N^*. We then consider a situation in which the person is informed that event $N \subseteq N^*$ has occurred. For each state $i \in N$, $c_i \in \mathbb{R}_+$ denotes the probability that the person initially assigns to state i (thus $|K| = 1$). Since $N \subseteq N^*$, we have $\sum_{i \in N} c_i \leq 1$. The problem is how to update the person's probability distribution according to the information. Since the total probability to be allocated to N is 1, we always have $E = 1$. Thus $\mathcal{D} = \{(c, 1) \in \mathbb{R}^N_+ \times \{1\} : 0 < \sum_{i \in N} c_i \leq 1\}$. This problem is similar to the belief updating problem studied in Stalnaker (1968) and Lewis (1973).

Probability Aggregation. Here N is the set of states of the world and K is the set of agents. Each agent k has a probability distribution over N, denoted by $(c_{ik})_{i \in N} \in \Delta^{|N|-1}$. The problem is how to aggregate $|K|$ individual distributions into a single distribution. Since the total probability to be allocated is 1, we have $E = 1$. Thus $\mathcal{D} = \{(c, 1) \in \mathbb{R}^{N \times K}_+ \times \{1\} : \bar{c}_k = 1 \text{ for each } k \in K\}$. This problem is studied by McConway (1981) and Rubinstein and Fishburn (1986).

We use the following notation for vector inequalities: given $x, y \in \mathbb{R}^M$, $x \geqq y$ means that $x_m \geq y_m$ for each m; $x \geq y$ means that $x \geqq y$ and $x \neq y$; and $x > y$ means that $x_m > y_m$ for each m. When $x \geqq y$, the multi-dimensional interval between x and y is denoted by $[y, x] \equiv \{z \in \mathbb{R}^M : x \geqq z \geqq y\}$.

2.2 Axioms

This subsection defines a number of properties that might be satisfied by rules. We start with the main axiom in this paper.

The main axiom states that no group of entities can change the total amount of their awards by reallocating characteristic vectors among themselves.

Reallocation-Proofness. For each $(c, E) \in \mathcal{D}$, each $S \subseteq N$, and each $c' \in \mathbb{R}^{N \times K}_+$, if $\sum_{i \in S} c'_i = \sum_{i \in S} c_i$, then

$$\sum_{i \in S} f_i(c'_S, c_{N \setminus S}, E) = \sum_{i \in S} f_i(c, E).$$

In the contexts of claim problems and their variants, if the left-hand side of the equation exceeds the right-hand side, then group S with claim profile $(c_i)_{i \in S}$ can increase their total awards by reallocating the members' claims into $(c'_i)_{i\in S}$. If the reverse inequality holds, then group S with the claim profile $(c'_i)_{i\in S}$ can gain from the reverse arrangement. This axiom was introduced by Moulin (1985a) in the context of social choice with transferable utilities.[4] We refer readers to his paper for more discussion on this axiom.

In the context of probability aggregation, *reallocation-proofness* has a rather different meaning. Given a set of states $S \subseteq N$, consider two profiles of beliefs $(c_k)_{k\in K} \equiv ((c_{ik})_{i\in N})_{k\in K}$ and $(c'_k)_{k\in K} \equiv ((c_{ik})_{i\in N})_{k\in K}$ such that, for each agent $k \in K$, c_k and c'_k differ only in probabilities that are put on the states in S. Thus the probability of the event S itself is the same under c_k and c'_k. Then *reallocation-proofness* states that the aggregate probability of the event S is the same under c and c'. That is, aggregate probability of an event S depends on the agents' beliefs over the states in S only through the probabilities that agents assign to the event S as a whole. Similarly, in the context of probability updating, *reallocation-proofness* states that the updated probability of a given event S depends on the initial belief over the states in S only through the total probability that the initial belief puts on S as a whole.

We also consider a pairwise version of *reallocation-proofness*, which deals only with the reallocation of characteristic vectors between two entities:

Pairwise Reallocation-Proofness. For each $(c, E) \in \mathcal{D}$, each $i, j \in N$ with $i \neq j$, and each $c' \in \mathbb{R}^{N\times K}_+$, if $c'_i + c'_j = c_i + c_j$ and $c'_k = c_k$ for all $k \neq i, j$, then

$$f_i(c', E) + f_j(c', E) = f_i(c, E) + f_j(c, E).$$

The pairwise version is particularly relevant for problems in which N is the set of agents (e.g., claim problems), since it is reasonable to believe that strategic reallocations of characteristic vectors are easier to implement for smaller groups of agents (because of smaller "transaction costs").

In the remainder of this subsection, we define a number of basic axioms.

The following axiom requires that awards add up to the amount to divide:

Efficiency. For each $(c, E) \in \mathcal{D}$, $\sum_{i\in N} f_i(c, E) = E$.

The following axiom excludes rules whose image of the compact set $\mathcal{D}(\bar{c}, E)$ is unbounded below and above:

One-Sided Boundedness. For each $(c, E) \in \mathcal{D}$, there exists $i \in N$ such that $f_i(\,\cdot\,, E)$ is bounded from either above or below over $\mathcal{D}(\bar{c}, E)$.

This axiom is implied by each of the following two axioms. The first one requires awards to be non-negative:

Non-Negativity. For each $(c, E) \in \mathcal{D}$ and each $i \in N$, $f_i(c, E) \geq 0$.

[4] Moulin calls this axiom "no advantageous reallocation."

Another axiom that implies *one-sided boundedness* is *no transfer paradox* (Moulin, 1985a), which states that no entity can increase its award by transferring part of its characteristic vector to other entities:

No Transfer Paradox. For each $(c, E) \in \mathcal{D}$, each $c' \in \mathbb{R}_+^{N\times K}$, each $i, j \in N$ with $i \neq j$, and each $t \in [0, c_i]$,

$$f_i(c_i - t, c_j + t, c_{-\{i,j\}}, E) \leq f_i(c_i, c_j, c_{-\{i,j\}}, E).$$

This axiom implies *one-sided boundedness* since, for each $(c, E) \in \mathcal{D}$ and each $i \in N$, $f_i(\,\cdot\,, E)$ on $\mathcal{D}(\bar{c}, E)$ is bounded above by $f_i(c', E)$ where $c'_i = \bar{c}$ and $c'_j = 0$ for each $j \neq i$.

The next axiom states that no amount is awarded to entities whose characteristic vectors are zero:

No Award for Null. For each $(c, E) \in \mathcal{D}$ and each $i \in N$, if $c_i = 0$, then $f_i(c, E) = 0$.

For example, in the context of probability updating, *no award for null* means that, if a state initially receives no probability, so does it after updating.

The next axiom states that two entities with the same characteristic vector receive the same amount:

Equal Treatment of Equals. For each $(c, E) \in \mathcal{D}$ and each $i, j \in N$, if $c_i = c_j$, then $f_i(c, E) = f_j(c, E)$.

The next symmetry axiom is stronger than *equal treatment of equals*.

Anonymity. For each permutation $\tau \colon N \to N$, each $(c, E) \in \mathcal{D}$, and each $i \in N$, $f_{\tau(i)}(c^\tau, E) = f_i(c, E)$ where c^τ is defined by $c^\tau \equiv (c_{\tau(i)})_{i \in N}$.

The next axiom is also stronger than *equal treatment of equals* and says that, if i's characteristic vector weakly dominates j's in every dimension, then i receives at least as much as j receives:

Order Preservation in Gains. For each $(c, E) \in \mathcal{D}$ and each $i, j \in N$, if $c_i \geqq c_j$, then $f_i(c, E) \geq f_j(c, E)$.

2.3 Generalized Proportional Rules

For the case when characteristic vectors are single-dimensional (i.e., $|K| = 1$), one of the simplest and best-known rules is the proportional rule, which divides the total amount proportionally to characteristic vectors.

Definition 1 (Proportional Rule, $|K| = 1$). For each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = \frac{c_i}{\bar{c}} E.$$

The right-hand side is well-defined since we rule out problems for which $\bar{c} = 0$. In the context of probability updating, the proportional rule is *Bayes' rule*. In the context of cost sharing, it is the *average-cost rule*.

We now extend the definition of the proportional rule to the case when characteristic vectors are multi-dimensional. Let us define a *weight function* as a function $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \Delta^{|K|-1}$, which assigns a weight vector $W(\bar{c}, E)$ on K as a function of $(\bar{c}, E)$. With this definition, we define proportional rules in the multi-dimensional case as follows:

Definition 2 (Proportional Rule). There exists a weight function W such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = \sum_{k \in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) E.$$

We let P^W denote the proportional rule associated with W.

This rule P^W first applies the proportional rule to each single-dimensional sub-problem (c^k, E) where $c^k \equiv (c_{ik})_{i \in N}$ and then takes the weighted average of the solutions to the sub-problems using the vector of weights $W(\bar{c}, E)$. The weights depend on the problem being considered but depend only on $(\bar{c}, E)$. Proportional rules are *efficient* since $\sum_{k \in K} W_k(\bar{c}, E) = 1$. Proportional rules also satisfy all other axioms defined in Section 2.2. It is evident that, if $|K| = 1$, Definition 2 reduces to Definition 1.

In the context of probability aggregation, we have $E = 1$ and $\bar{c}_k = 1$ for each $k \in K$. This means that a weight function reduces to a single weight vector $w \equiv W((1, \ldots, 1), 1)$. A proportional rule then simply takes a weighted average of individual probability distributions using a fixed vector of weights, and is called a *linear opinion pool* (McConway, 1981).

We now introduce what we call *generalized proportional rules*. These rules are characterized by two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$, and the award to i is given by the sum of the following two terms. The first term is $A_i(\bar{c}, E)$, which is independent of i's characteristic vector but may treat i differently from others based on i's identity. The second term is proportional to i's characteristic vector and treats entities symmetrically. On the other hand, the second term may treat issues asymmetrically, and the degree of importance attached to each issue $k \in K$ is given by $W_k(\bar{c}, E)$. Formally,

Definition 3 (Generalized Proportional Rule). There exist two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = A_i(\bar{c}, E) + \sum_{k \in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) E. \tag{1}$$

Note that W is not required to be a weight function, i.e., neither $W_k(\bar{c}, E) \geq 0$ nor $\sum_{k \in K} W_k(\bar{c}, E) = 1$ is required. Proportional rules are special cases where $A_i = 0$ and W is a weight function.

Since, given $(\bar{c}, E)$, the second term on the right-hand side of (1) is linear in c_{ik}, generalized proportional rules satisfy *reallocation-proofness* and *one-sided boundedness*. On the other hand, these rules do not necessarily satisfy other axioms in Section 2.2. We will specify necessary and sufficient conditions for (A, W) to satisfy each of those axioms.

The following are two examples of generalized proportional rules that are not proportional rules:

Example 1 (Equal Division). For each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = \frac{E}{|N|},$$

i.e., $A_i(\bar{c}, E) = \frac{E}{|N|}$ and $W(\bar{c}, E) = 0$.

Example 2 (Weighted Rights Egalitarian Rule for $|K| = 1$). There is a weight vector $\lambda \in \mathrm{int}(\Delta^{|N|-1})$ such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = c_i + \lambda_i(E - \bar{c}),$$

i.e., $A_i(\bar{c}, E) = \lambda_i(E - \bar{c})$ and $W(\bar{c}, E) = 1$. When $\lambda = (\frac{1}{|N|}, \dots, \frac{1}{|N|})$, the rule is what is called the *rights egalitarian rule* in Herrero, Maschler, and Villar (2000).

3 Main Results

Our first main result is a characterization of the family of *reallocation-proof* rules.

Theorem 1. Assume $|N| \geq 3$. A rule f on a rich domain $\mathcal{D}$ is *reallocation-proof* if and only if there exist a function $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^N$ and $|K|$ functions $\hat{W}_1, \hat{W}_2, \dots, \hat{W}_{|K|}\colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = A_i(\bar{c}, E) + \sum_{k \in K} \hat{W}_k(c_{ik}, \bar{c}, E),$$

and for each $k \in K$, $\hat{W}_k(\,\cdot\,, \bar{c}, E)$ is additive.

Proof. Since the "if" part is straightforward, we prove the "only if" part. Let f be a *reallocation-proof* rule defined on a rich domain $\mathcal{D}$. We fix $E > 0$ and $d \in \mathbb{R}^K_{++}$, and consider problems (c, E) such that $\bar{c} = d$. Let $\mathcal{C} \equiv \{c \in \mathbb{R}^{K \times N}_+ : \bar{c} = d\}$.

First, note that *reallocation-proofness* with respect to N implies that, for each $c, c' \in \mathcal{C}$, $\sum_{i \in N} f_i(c, E) = \sum_{i \in N} f_i(c', E)$. It is then easy to see that *reallocation-proofness* also implies that, for each $S \subseteq N$ and each $c, c' \in \mathcal{C}$, if $\sum_{i \in S} c_i = \sum_{i \in S} c'_i$, then $\sum_{i \in S} f_i(c, E) = \sum_{i \in S} f_i(c', E)$, since each term is equal to $\sum_{i \in S} f_i(c'_S, c_{N \setminus S})$. ¿From this observation applied to singletons S, we can define $A_i(d, E) \equiv f_i(c, E)$ where $c \in \mathcal{C}$ and $c_i = 0$. That is, $A_i(d, E)$ is the amount that entity i receives whenever i's characteristic vector is zero, given d and E. From the same observation with respect to coalitions $S \subsetneq N$, we can also define a function $w\colon \{S \in 2^N : S \neq \emptyset, N\} \times [0, d] \to \mathbb{R}$ by

$$w(S, x) \equiv \sum_{i \in S} f_i(c, E) - \sum_{i \in S} A_i(d, E) \tag{2}$$

where $c \in \mathcal{C}$ is such that $\sum_{i \in S} c_i = x$. Although $w(S, x)$ depends on (d, E), we suppress the information to simplify notation. By the definition of $A_i(\bar{c}, E)$, we have $w(S, 0) = 0$ for each $S \subsetneq N$.

Step 1: For each $\boldsymbol{S, S' \subsetneq N}$ and each $\boldsymbol{x \in [0, d]}$, $\boldsymbol{w(S, x) = w(S', x)}$. To prove this, first consider the case when $S' \subseteq S$. Let $c \in \mathcal{C}$ be such that $\sum_{i \in S} c_i = x$ and $c_i = 0$ for each $i \in S \setminus S'$. Since $f_i(c, E) - A_i(d, E) = 0$ for each $i \in S \setminus S'$, (2) implies

$$w(S, x) = \sum_{i \in S'} f_i(c, E) - \sum_{i \in S'} A_i(d, E) = w(S', x),$$

as desired. Now, consider the case in which no inclusion holds between S and S'. Let $i \in S$ and $j \in S'$. The result just obtained implies $w(S, x) = w(\{i\}, x) = w(\{i, j\}, x) = w(\{j\}, x) = w(S', x)$.

This step enables us to write $w(S, x)$ as $w^*(x)$.

Step 2: For each $\boldsymbol{x, y \in [0, d]}$ with $\boldsymbol{x + y \in [0, d]}$, $\boldsymbol{w^*(x) + w^*(y) = w^*(x + y)}$. Let $i, j, h \in N$ be three distinct entities. Let $c \in \mathcal{C}$ be such that $c_i = x$, $c_j = y$, and $c_h = d - x - y$. Then

$$\begin{aligned} w^*(x) + w^*(y) &= w(\{i\}, x) + w(\{j\}, y) \\ &= f_i(c, E) - A_i(d, E) + f_j(c, E) - A_j(d, E) \\ &= w(\{i, j\}, x + y) = w^*(x + y). \end{aligned}$$

Step 3: Concluding. For each $i \in N$, each $k \in K$, and each $c_{ik} \in [0, d]$, we define

$$\hat{W}_k(c_{ik}, d, E) \equiv w^*(0, \ldots, 0, c_{ik}, 0, \ldots, 0),$$

where the dependence on (d, E) is written explicitly. Then, Step 2 and the definitions of A and W imply that, for each $c \in \mathcal{C}$ and each $i \in N$,

$$\begin{aligned} f_i(c, E) &= A_i(\bar{c}, E) + w(\{i\}, c_i) \\ &= A_i(\bar{c}, E) + \sum_{k \in K} \hat{W}_k(c_{ik}, \bar{c}, E). \end{aligned}$$

The additivity of $\hat{W}_k(\,\cdot\,, \bar{c}, E)$ follows from Step 2.[5] $\square$

The following result shows that *reallocation-proofness* together with *one-sided boundedness* fully characterizes the family of generalized proportional rules:

Theorem 2. Assume $|N| \geq 3$. A rule on a rich domain satisfies *reallocation-proofness* and *one-sided boundedness* if and only if it is a generalized proportional rule.

Proof. The "if" part has been discussed. The "only if" part holds since, over $[0, \bar{c}_k]$, $W_k(\,\cdot\,, \bar{c}, E)$ is additive and bounded either above or below, which implies that $W_k(\,\cdot\,, \bar{c}, E)$ is linear (e.g., Aczèl and Dhombres, 1989, Corollary 2.5). Therefore, $\hat{W}_k(c_{ik}, \bar{c}, E) = (c_{ik}/\bar{c}_k)\hat{W}_k(\bar{c}_k, \bar{c}, E)$. Letting $W_k(\bar{c}, E) \equiv \hat{W}_k(\bar{c}_k, \bar{c}, E)$, we complete the proof. $\square$

The two axioms in Theorem 2 are independent. Indeed, a number of rules studied in the literature satisfy *one-sided boundedness* but not *reallocation-proofness*. The following example shows that there exists a rule that satisfies *reallocation-proofness*, *efficiency*, *no award for null*, and *anonymity*, but not *one-sided boundedness*.

Example 3. Let $g\colon \mathbb{R} \to \mathbb{R}$ be an additive and nonlinear function; an example can be found in Aczél and Dhombres (1989, Theorem 2.2.10). Let $w \in \mathbb{R}^K_{++}$ be such that $\sum_{k \in K} w_k = 1$. Define a rule f on $\mathcal{D}$ by

$$f_i(c, E) \equiv \sum_{k \in K} \frac{g(c_{ik}) w_k}{g(\bar{c}_k)} E.$$

Since g is additive and $\bar{c}_k > 0$, we have $g(\bar{c}_k) \neq 0$ and hence the right-hand side is well defined. Clearly, f satisfies *anonymity*. By the additivity of g, $g(0) = 0$ and therefore f satisfies *no award for null*. Since g is additive, f is *reallocation-proof*. Since g is additive and $\sum_{k \in K} w_k = 1$, f is *efficient*. Since g is additive and nonlinear, it is unbounded from below and above everywhere (e.g., Aczél and Dhombres, 1989, Corollary 2.5), so f violates *one-sided boundedness*.

We can obtain necessary and sufficient conditions on (A, W) under which the *reallocation-proof* rules characterized in Theorem 1 satisfy additional basic axioms. The proofs are easy and omitted.

Proposition 1. Assume $|N| \geq 3$. Let f be a *reallocation-proof* rule on a rich domain $\mathcal{D}$, and $(A, (\hat{W}_k)_{k \in K})$ be the list of associated functions. Then

1. Rule f satisfies *no award for null* if and only if, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$A_i(\bar{c}, E) = 0. \tag{3}$$

[5] We defined $\hat{W}_k(c_{ik}, \bar{c}, E)$ only for $c_{ik} \leq \bar{c}_k$ and Step 2 shows only that $\hat{W}_k(\,\cdot\,, \bar{c}, E)$ is additive over $[0, \bar{c}_k]$. But we can easily extend the definition and the additivity to $\mathbb{R}_+$.

2. Rule f satisfies *equal treatment of equals* if and only if, for each $(c, E) \in \mathcal{D}$,

$$A_1(\bar{c}, E) = A_2(\bar{c}, E) = \cdots = A_N(\bar{c}, E), \tag{4}$$

which holds if and only if f satisfies *anonymity*. Thus f satisfies *anonymity* if and only if f satisfies *equal treatment of equals*. By (3) and (4), if f satisfies *no award for null*, then f satisfies *anonymity*.

3. Rule f satisfies *no transfer paradox* if and only if, for each $(c, E) \in \mathcal{D}$, each $k \in K$, and each $i \in N$,

$$\hat{W}_k(c_{ik}, \bar{c}, E) \geq 0, \tag{5}$$

which is the case if and only if, for each $k \in K$, $\hat{W}_k$ is non-decreasing in c_{ik}.

4. Rule f satisfies *one-sided boundedness* (hence f is a generalized proportional rule) if and only if, for each $k \in K$ and each $(c, E) \in \mathcal{D}$, $\hat{W}_k(\,\cdot\,, \bar{c}, E)$ is monotonic, i.e., either non-decreasing or non-increasing. Thus, if f satisfies *no transfer paradox*, then it also satisfies *one-sided boundedness*.

5. Rule f satisfies *order preservation in gains* if and only if f satisfies *equal treatment of equals* and *no transfer paradox* (i.e., f satisfies (4) and (5)).

6. Rule f satisfies *non-negativity* if and only if f satisfies *one-sided boundedness* and, for each $(c, E) \in \mathcal{D}$,

$$A_i(\bar{c}, E) \geq 0 \quad \text{for each } i \in N, \tag{6}$$

$$\min_{j \in N} A_j(\bar{c}, E) + E \sum_{k \in K} \min\{0, \hat{W}_k(\bar{c}_k, \bar{c}, E)\} \geq 0. \tag{7}$$

7. Rule f satisfies *efficiency* if and only if, for each $(c, E) \in \mathcal{D}$,

$$\sum_{k \in K} \hat{W}_k(\bar{c}_k, \bar{c}, E) = E - \sum_{i \in N} A_i(\bar{c}, E). \tag{8}$$

Therefore if $|K| = 1$, then f satisfies *efficiency* and *one-sided boundedness* if and only if it takes the following form:

$$f_i(c, E) = A_i(\bar{c}, E) + \frac{c_i}{\bar{c}}\big[E - \sum_{i \in N} A_i(\bar{c}, E)\big]. \tag{9}$$

This rule first allocates $A_i(\bar{c}, E)$ to each i and then divides the remainder among the entities proportional to their characteristics. This rule satisfies *non-negativity* if and only if, for each $(c, E) \in \mathcal{D}$ and each $i \in N$, (6) is satisfied and

$$\sum_{j \in N \setminus \{i\}} A_j(\bar{c}, E) \leq E. \tag{10}$$

Proof. Omitted.

We now prove a characterization of the proportional rule.

Corollary 1. Assume $|N| \geq 3$. A rule on a rich domain satisfies *reallocation-proofness*, *efficiency*, *no award for null*, and *non-negativity* (or *no transfer paradox*) if and only if it is a proportional rule.

Proof. Let f be a rule that satisfies the axioms in the statement and let $(A, \hat{W})$ be the pair associated with f. Proposition 1 (items 3, 4, and 6) implies that, since f satisfies either *non-negativity* or *no transfer paradox*, f satisfies *one-sided boundedness*. Thus f is a generalized proportional rule and $\hat{W}_k(\bar{c}_k, \bar{c}, E) = W_k(\bar{c}, E)E$. Since *no award for null* implies $A_i(\bar{c}, E) = 0$, (7) implies $W_k(\bar{c}, E) \geq 0$, which is also implied by *no transfer paradox*. Finally, (8) and $A_i(\bar{c}, E) = 0$ imply $\sum_{k \in K} W_k(\bar{c}, E) = 1$. Hence W is a weight function. $\square$

The following result is a characterization of a subfamily of generalized proportional rules and follows easily from Proposition 1 (thus we omit the proof).

Corollary 2. Assume $|N| \geq 3$. A rule f on a rich domain $\mathcal{D}$ satisfies *reallocation-proofness*, *one-sided boundedness*, *equal treatment of equals*, and *efficiency* if and only if f is a generalized proportional rule such that there exists a function $W \colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ and, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = \frac{E}{|N|}\big[1 - \sum_{k \in K} W_k(\bar{c}, E)\big] + \sum_{k \in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E)E. \tag{11}$$

This rule satisfies *non-negativity* and *no transfer paradox* if and only if, for each $(c, E) \in \mathcal{D}$, $W(\bar{c}, E) \geqq 0$ and $\sum_{k \in K} W_k(\bar{c}, E) \leq 1$.

Chun (1988, Theorem 1) shows that, for one-dimensional claim problems, a rule satisfies *reallocation-proofness*, "continuity," *anonymity*, and *efficiency* if and only if it takes the functional form (11). Corollary 2 replaces *anonymity* and "continuity" with *equal treatment of equals* and *one-sided boundedness*, respectively, and extends domains for which the characterization holds.

We now show that, for the characterization of the proportional rule in Corollary 1, *reallocation-proofness* can be weakened to its pairwise version.

Theorem 3. Assume $|N| \geq 3$. A rule on a rich domain satisfies *pairwise reallocation-proofness*, *efficiency*, *no award for null*, and *non-negativity* (or *no transfer paradox*) if and only if it is a proportional rule.

Proof. Let f be a rule on a rich domain $\mathcal{D}$ with $|N| \geq 3$ satisfying all the axioms. For each $S \subseteq N$, let $\mathcal{D}_S \equiv \{(c, E) \in \mathcal{D} : c_i = 0 \text{ for all } i \notin S\}$. Then by *no award for null*, we can treat problems in $\mathcal{D}_S$ as those in which only entities in S are present.

It is easy to see that, on $\mathcal{D}_S$ such that $|S| = 3$, *pairwise reallocation-proofness* and *efficiency* imply *reallocation-proofness*. Thus Corollary 1 implies that, on $\mathcal{D}_S$, f coincides with a proportional rule. Let W^S denote the associated weight function. For each $S, T \subseteq N$ such that $|S| = |T| = 3$ and $|S \cap T| \geq 2$, since $\mathcal{D}_S \cap \mathcal{D}_T \neq \emptyset$, it follows that $W^S = W^T$. Thus, weight functions for all triples are identical and we can write them simply by W. Hence, on $\cup_{|S| \leq 3} \mathcal{D}_S$, f coincides with the proportional rule associated with W.

To prove that f is the proportional rule on the entire domain, we use an induction argument. Given $k \geq 3$, suppose that f coincides with the proportional rule associated with W on $\cup_{|S| \leq k} \mathcal{D}_S$, and let $S \subseteq N$ contain $k+1$ entities. To prove that f also coincides with the proportional rule on $\mathcal{D}_S$, let $(c, E) \in \mathcal{D}_S$. Consider a pair $\{i, j\} \subseteq S$. Let $c' \in \mathbb{R}_+^{S \times K}$ be such that $c'_i = c_i + c_j$, $c'_j = 0$, and $c'_h = c_h$ for each $h \neq i, j$. Then by *pairwise reallocation-proofness* and *no award for null*, $f_i(c, E) + f_j(c, E) = f_i(c', E) + f_j(c', E) = f_i(c', E)$. Since $(c', E) \in \mathcal{D}_{S \setminus \{j\}}$, the induction hypothesis implies that $f_i(c, E) + f_j(c, E) = P_i^W(c', E) = P_i^W(c, E) + P_j^W(c, E)$. Since this holds for each $i, j \in S$, it follows that $f(c, E) = P^W(c, E)$. □

We can obtain a similar result by replacing *non-negativity* (and *no transfer paradox*) in Theorem 3 with *one-sided boundedness*. We can easily show that, for any rule f that satisfies the modified list of axioms, there exists a function $W \colon \mathbb{R}_{++}^K \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$, $\sum_{k \in K} W_k(\bar{c}, E) = 1$ and

$$f_i(c, E) = \sum_{k \in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) E.$$

This family of rules is larger than the family of proportional rules, since W may take negative values. However, if $|K| = 1$, (9) implies that, if a rule satisfies *reallocation-proofness*, *one-sided boundedness*, *efficiency*, and *no award for null*, then it also satisfies *non-negativity* and *no transfer paradox*. This implies that, if $|K| = 1$, *non-negativity* (and *no transfer paradox*) in Theorem 3 can be weakened to *one-sided boundedness*. Thus we obtain

Corollary 3. Assume $|N| \geq 3$ and $|K| = 1$. A rule on a rich domain satisfies *pairwise reallocation-proofness*, *one-sided boundedness*, *efficiency*, and *no award for null* if and only if it is the proportional rule.

In the literature, several papers considered the case when $|K| = 1$ and proved results similar to Corollary 1. Ching and Kakkar (2001, Corollary 1) consider bankruptcy problems and characterize the proportional rule using *non-negativity*. Moulin and Shenker (1992, p. 1012) and Moulin (2002, Theorem 2.1) also state the same characterization in the context of cost sharing problems. O'Neill (1982, Theorem C.1) and Chun (1988, Theorem 2) use *anonymity* and "continuity" in addition to the axioms considered by Ching and Kakker. Our Corollary 3 shows that these results hold for any rich domain and that the pairwise version of *reallocation-proofness* suffices to characterize the proportional rule.

4 Application I: Fixed Set of Entities

4.1 Claim Problems and Variants

This subsection presents applications of our results in the contexts of bankruptcy, surplus sharing, claim problems, and income redistribution. Throughout this subsection, $\mathcal{D}$ denotes any of these classes of problems.

We introduce four standard axioms: *resource monotonicity* says that no agent loses when the amount to divide increases; *resource additivity* says that, if the amount to divide is split into two parts and the award vector is computed separately for each part, then the sum of the award vectors should coincide with the award vector obtained from a single calculation applied to the total amount to divide; *claim monotonicity* says that no agent loses when his claim increases; *homogeneity* says that the division rule is independent of the unit with which the data of the problems are measured, i.e., the rule is linear in (c, E). Formally,

Resource Monotonicity. For each $(c, E) \in \mathcal{D}$ and each $E' > E$, if $(c, E') \in \mathcal{D}$, then for each $i \in N$, $f_i(c, E') \geq f_i(c, E)$.

Resource Additivity. For each $(c, E) \in \mathcal{D}$ and each $(c, E') \in \mathcal{D}$ such that $(c, E + E') \in \mathcal{D}$, $f(c, E) + f(c, E') = f(c, E + E')$.

Claim Monotonicity. For each $(c, E) \in \mathcal{D}$, each $i \in N$, and each $c'_i \geq c_i$, if $(c'_i, c_{-i}, E) \in \mathcal{D}$, then $f_i(c'_i, c_{-i}, E) \geq f_i(c, E)$.

Homogeneity. For each $(c, E) \in \mathcal{D}$ and each $\lambda > 0$, $f(\lambda c, \lambda E) = \lambda f(c, E)$.

We begin by characterizing a subfamily of generalized proportional rules that satisfy *resource additivity*.

Theorem 4. Assume that $\mathcal{D}$ is the class of either bankruptcy or surplus sharing or claim problems with $|N| \geq 3$. A rule f on $\mathcal{D}$ satisfies *reallocation-proofness*, *efficiency*, *non-negativity*, *no transfer paradox*, and *resource additivity* if and only if there exists a function

$A\colon \mathbb{R}_{++} \to \mathbb{R}^N_+$ such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c,E) = E\Big[A_i(\bar{c}) + \frac{c_i}{\bar{c}}\big[1 - \sum_{j\in N} A_j(\bar{c})\big]\Big] \text{ and } 0 \leq \sum_{j\in N} A_j(\bar{c}) \leq 1.$$

Proof. Let $\mathcal{D}$ be the class of bankruptcy problems with $|N| \geq 3$ (proofs for the other classes are similar). Let f be a rule on $\mathcal{D}$ that satisfies the axioms. By Proposition 1 (equations (6) and (9)), there exists a function $A\colon \mathbb{R}^2_{++} \to \mathbb{R}^N_+$ such that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c,E) = \frac{c_i}{\bar{c}}\big[E - \sum_{j\in N} A_j(\bar{c}, E)\big] + A_i(\bar{c}, E).$$

Let $c \in \mathbb{R}^N_+$ and $i \in N$. We shall show that $A_i(\bar{c}, \cdot\,)$ is linear on $[0, \bar{c}]$. To prove this, we can assume that $c_i = 0$. Then, *resource additivity* implies that, for each $E, E' \in [0, \bar{c}]$, we have $A_i(\bar{c}, E) + A_i(\bar{c}, E') = A_i(\bar{c}, E + E')$ as long as $0 \leq E + E' \leq \bar{c}$; i.e., $A_i(\bar{c}, \cdot\,)$ is additive on $[0, \bar{c}]$. Since f satisfies *non-negativity*, a standard argument of Cauchy's equation yields (as in the proof of Theorem 2) that $A_i(\bar{c}, \cdot\,)$ is linear on $[0, \bar{c}]$. Thus, for each $E \in [0, \bar{c}]$, we can write $A_i(\bar{c}, E)$ as $A_i(\bar{c})E$. By *no transfer paradox*, $1 - \sum_{j\in N} A_j(\bar{c}) \geq 0$. □

If *non-negativity* and *no transfer paradox* are removed from the list of axioms in Theorem 4, then the restriction on the range of A is removed. An analogue of Theorem 4 in the class of income redistribution problems will be obtained later.

We can easily show that, if *homogeneity* is added to the list of axioms, then $A_i(\,\cdot\,)$ in Theorem 4 has to be constant for each $i \in N$. Thus, we obtain

Corollary 4. Let $\mathcal{D}$ be the class of either bankruptcy or surplus sharing or claim problems with $|N| \geq 3$, and let f be a rule on $\mathcal{D}$ that satisfies all the axioms in Theorem 4. Then, f also satisfies *homogeneity* and *equal treatment of equals* if and only if there exists $\alpha \in [0, 1]$ such that, for each $(c, E) \in \mathcal{D}$,

$$f_i(c,E) = \alpha \frac{1}{|N|} E + (1 - \alpha)\frac{c_i}{\bar{c}} E, \tag{12}$$

which means that f is a convex combination of the proportional rule and equal division.

Moulin (1987) characterizes the same family of rules for claim problems.[6] However, his characterization also uses *resource monotonicity*. Corollary 4 shows that his characterization holds without *resource monotonicity* and for any of the domains mentioned in the corollary.

The family of rules characterized in Corollary 4 is indexed by a single number $\alpha \in [0, 1]$. The axiom to be introduced next, called *composition down*, further contracts this family.

[6]Moulin uses *claim monotonicity* instead of *no-transfer paradox*. Our characterization in Corollary 4 also holds if we replace *no-transfer paradox* with *claim monotonicity*.

To motivate the axiom, consider a problem (c, E) and suppose that, after an award vector x is agreed upon, it is revealed that the amount to divide is actually less than expected, i.e., $E' < E$. There are at least two ways to adjust the award vector. One is simply to re-calculate the award vector for the problem with the right amount to divide. An alternative is to consider the previously agreed award vector x as the relevant claim vector and calculate the award vector for the problem (x, E'). The axiom then states that two ways of re-calculation yield the same award vector. The axiom is well defined in the classes of bankruptcy and claim problems (but not surplus sharing).[7]

Composition Down. For each $(c, E) \in \mathcal{D}$ and each $E' < E$ with $(c, E') \in \mathcal{D}$, $f(c, E') = f(f(c, E), E')$.

Corollary 5. In the classes of bankruptcy and claim problems with at least three agents, a rule satisfies all the axioms in Corollary 4 and *composition down* if and only if it is either the proportional rule or equal division.

Proof. The "if" part follows since the proportional rule and equal division satisfy *composition down*. To prove the converse, let f be a rule satisfying the axioms. By Corollary 4, f is a convex combination of the proportional rule and equal division with a weight $\alpha \in [0, 1]$ on equal division. Let $(c, E) \in \mathcal{D}$ and $E' \in (0, E)$. Notice that $(c, E') \in \mathcal{D}$ and $(f(c, E), E') \in \mathcal{D}$. By *composition down*, $f(c, E') = f(f(c, E), E')$, which implies

$$\frac{\alpha}{|N|} + (1 - \alpha)\frac{c_i}{\bar{c}} = \frac{\alpha}{|N|} + (1 - \alpha)\frac{E[\frac{\alpha}{|N|} + (1 - \alpha)\frac{c_i}{\bar{c}}]}{E}.$$

Hence $(1 - \alpha)\alpha[\frac{1}{|N|} - \frac{c_i}{\bar{c}}] = 0$. Since c and i were chosen arbitrarily, it follows that either $\alpha = 0$ or $\alpha = 1$. □

In the class of surplus sharing problems, Moulin (1987, Theorem 2) characterizes the pair of the proportional rule and equal division using "path independence" instead of *composition down*. "Path independence" is also a condition of dynamic consistency in calculating awards, but it is not well-defined in the class of bankruptcy problems.

For the class of income redistribution problems, by using an argument similar to that of Theorem 4, we can characterize the family of "proportional income taxation with an asymmetric redistribution scheme":

Corollary 6. Assume that $\mathcal{D}$ is the class of income redistribution problems with $|N| \geq 3$. A rule f on $\mathcal{D}$ satisfies *reallocation-proofness*, *efficiency*, *non-negativity*, and *no transfer paradox* if and only if there exist two functions $T\colon \mathbb{R}_{++} \to [0, 1]$ and $R\colon \mathbb{R}_{++} \to \mathbb{R}^N_+$ such

[7] This axiom, introduced by Moulin (2002), is well-defined under *efficiency* and *non-negativity*. We assume these axioms when we discuss *composition down*.

that, for each $(c, E) \in \mathcal{D}$ and each $i \in N$,

$$f_i(c, E) = (1 - T(\bar{c}))c_i + R_i(\bar{c}) \quad \text{and} \quad \sum_{j \in N} R_j(\bar{c}) = T(\bar{c})\bar{c}.$$

In these rules, T determines the proportional tax rate $T(\bar{c})$ as a function of the size of the economy, $\bar{c}$, while R determines the reallocation scheme $(R_1(\bar{c}), R_2(\bar{c}), \dots, R_{|N|}(\bar{c}))$ as a function of individuals' identities subject to the budget balance: $\sum_{j \in N} R_j(\bar{c}) = T(\bar{c})\bar{c}$. It is easy to see that these rules also satisfy *homogeneity* if and only if T is constant and each R_i is linear.

4.2 Social Choice with Transferable Utility

This subsection considers social choice problems with transferable utility. In these problems, since c_i is an agent's valuations for alternatives, it is immaterial how the vector is normalized. This motivates us to consider the following axiom of invariance. Let $\mathbf{1} \in \mathbb{R}^K$ denote the vector consisting of 1 only.

Translation Invariance. For each $(c, E) \in \mathcal{D}$, each $\lambda \in \mathbb{R}_+$, each $i \in N$, and each $j \neq i$,

$$f_i((c_i + \lambda\mathbf{1}, c_{-i}), E + \lambda) = f_i(c, E) + \lambda,$$
$$f_j((c_i + \lambda\mathbf{1}, c_{-i}), E + \lambda) = f_j(c, E).$$

For each $c \in \mathbb{R}_+^{N \times K}$, let $\bar{c}_{\max} \equiv \max_{k \in K} \bar{c}_k$. Since $E = \bar{c}_{\max}$ for each problem, we suppress E throughout this subsection. We consider the following family of rules:

Definition 4 (Equal Sharing Above a Convex Decision, ESCD). There exists a function $\rho\colon \mathbb{R}_{++}^K \to \Delta^{|K|-1}$ such that, for each $\bar{c} \in \mathbb{R}_+^K$ and each $\lambda \geq 0$,

$$\rho(\bar{c} + \lambda\mathbf{1}) = \rho(\bar{c}), \tag{13}$$

and, for each $c \in \mathbb{R}_+^{N \times K}$ and each $i \in N$,

$$f_i(c) = \frac{1}{|N|}\left[\bar{c}_{\max} - \sum_{k \in K} \bar{c}_k \rho_k(\bar{c})\right] + \sum_{k \in K} c_{ik}\rho_k(\bar{c}). \tag{14}$$

Let ES^ρ denote the ESCD rule associated with ρ.

It is easy to see that ES^ρ is *efficient* and *translation invariant.*

We now show that ES^ρ is a generalized proportional rule. Let $W_k^\rho(\bar{c}) \equiv \bar{c}_k \rho_k(\bar{c})/\bar{c}_{\max}$. Then

$$\sum_{k \in K} c_{ik}\rho_k(\bar{c}) = \sum_{k \in K} \frac{c_{ik}}{\bar{c}_k} W_k^\rho(\bar{c})\bar{c}_{\max}.$$

Thus, for each $c \in \mathbb{R}_+^{N\times K}$ and each $i \in N$,

$$ES_i^\rho(c) = A_i^\rho(\bar{c}) + \sum_{k\in K} \frac{c_{ik}}{\bar{c}_k} W_k^\rho(\bar{c})\bar{c}_{\max},$$

where

$$A_i^\rho(\bar{c}) \equiv \frac{\bar{c}_{\max}}{|N|}\big[1 - \sum_{k\in K} W_k^\rho(\bar{c})\big].$$

Then ES^ρ is a generalized proportional rule associated with (A^ρ, W^ρ).

Moulin (1985a, Theorem 1) characterizes the family of ESCD rules by *reallocation-proofness*, *efficiency*, *no transfer paradox*, *translation invariance*, and *anonymity*. The next result, which relies on Corollary 2, shows that his characterization remains valid if *anonymity* is weakened to *equal treatment of equals*.

Corollary 7. In the class of social choice problems with transferable utilities with at least three agents, a rule satisfies *reallocation-proofness*, *efficiency*, *no transfer paradox*, *translation invariance*, and *equal treatment of equals* if and only if it is an ESCD rule.

Proof. See the Appendix.

Moulin (1985a) also considers the following subfamily of ESCD rules.

Definition 5. A *utilitarian rule* is an ESCD rule whose weight function $\rho\colon \mathbb{R}_{++}^K \to \Delta^{|K|-1}$ is such that, for each $c \in \mathbb{R}_+^{N\times K}$, (13) is satisfied and

$$\rho_k(\bar{c}) = 0 \quad \text{for each } k \in K \text{ with } \bar{c}_k < \bar{c}_{\max}. \tag{15}$$

We denote by U^ρ the utilitarian rule associated with ρ. By (15), the first term of (14) is zero. Thus

$$U_i^\rho(c) = \sum_{k\in K} c_{ik}\rho_k(\bar{c}) = \sum_{k\in K} \frac{c_{ik}}{\bar{c}_k}\rho_k(\bar{c})\bar{c}_{\max}.$$

This clearly shows that utilitarian rules can be written as proportional rules. Since these rules assign zero weights on inefficient alternatives, the amount assigned to each agent is given by a weighted average of his valuations for efficient alternatives. When there exists a unique efficient alternative, each agent is assigned his valuation for the alternative. Thus, when agents have expected utility preferences, utilitarian rules can be considered as rules that simply select an efficient alternative randomly and make no side-payment.

Among all ESCD rules, only utilitarian rules satisfy *no award for null*. This suggests a characterization of utilitarian rules in the manner of Theorem 3. Indeed, Moulin (1985a, Theorem 3) characterizes this family of rules using *no award for null* together with *reallocation-proofness*, *efficiency*, *non-negativity*, and *anonymity*. However, the characterization holds without *anonymity* since *anonymity* is implied by *reallocation-proofness* and *no award for null* by Proposition 1 (Item 2). Furthermore, *reallocation-proofness* can be weakened to the pairwise version, and *non-negativity* can be replaced with *no transfer paradox*.

Corollary 8. In the class of social choice problems with transferable utilities with at least three agents (i.e., $|N| \geq 3$), a rule satisfies *pairwise reallocation-proofness*, *efficiency*, *no award for null*, *non-negativity* (or *no transfer paradox*), and *translation invariance* if and only if it is a utilitarian rule.

Proof. See the Appendix.

Remark 1. Though Corollaries 7–8 are shown on $\mathbb{R}_+^{N \times K}$, by *translation invariance*, these results can be easily extended to $\mathbb{R}^{N \times K}$, which is in fact the domain considered in Moulin (1985a).

4.3 Probability Updating and Aggregation

For probability updating problems, Theorem 3 and Corollary 1 give a characterization of Bayes' rule.

Corollary 9. In the class of probability updating problems with at least three states (i.e., $|N| \geq 3$), a rule satisfies *pairwise reallocation-proofness*, *efficiency*, *non-negativity*, and *no award for null* if and only if it is Bayes' rule.

For probability aggregation, McConway (1981) considers the following axiom. A rule f satisfies *strong setwise function property* if there is a function $h\colon [0,1]^K \to [0,1]$ such that, for each $(c, E) \in \mathcal{D}$ and each $S \subseteq N$,

$$\sum_{i \in S} f_i(c, E) = h(\sum_{i \in S} c_i).$$

Note that h does not depend on $\sum_{i \in N} c_i$ nor E, since $\sum_{i \in N} c_i = (1, \ldots, 1)$ and $E = 1$ for any problem of probability aggregation. McConway's axiom is stronger than *reallocation-proofness* since he requires h to be independent of S. Hence, we obtain the following result of McConway as a corollary.

Corollary 10 (McConway 1981, Theorem 3.3). In the class of probability aggregation problems with at least three states (i.e., $|N| \geq 3$), a rule satisfies *strong setwise function property*, *efficiency*, *non-negativity*, and *no award for null* if and only if it is a linear opinion pool.

5 Application II: Variable Set of Entities

We extend the model in the previous sections to accommodate a variable set of entities. Let $I \subseteq \{1, 2, \ldots\}$ be the set of "potential" entities, which may be finite or infinite. Let

$\mathcal{N}$ be the set of all non-empty finite subsets of I. For each $N \in \mathcal{N}$, let $\mathcal{A}^N$ be the class of all allocation problems associated with N. We retain our simplifying assumption that for each $k \in K$, $\bar{c}_k > 0$. For each $N \in \mathcal{N}$, let $\mathcal{D}^N \subseteq \mathcal{A}^N$ and $\mathcal{D} \equiv \cup_{N \in \mathcal{N}} \mathcal{D}^N$. A *rule* is now a function f that associates with each $N \in \mathcal{N}$ and each problem $(c, E) \in \mathcal{D}^N$ an award vector $f(c, E) \in \mathbb{R}^N$. We say that $\mathcal{D}$ is *rich** if, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, each $N' \subseteq N$, and each $c' \in \mathbb{R}_+^{N'}$, if $\sum_{i \in N'} c'_i = \sum_{i \in N} c_i$, then $(c', E) \in \mathcal{D}^{N'}$. Note that if $\mathcal{D}$ is rich*, then for each $N \in \mathcal{N}$, $\mathcal{D}^N$ is rich. The axioms and notions defined in the previous sections can be easily redefined in this extended setup by simply adding "*for each* $N \in \mathcal{N}$" in the definitions.

5.1 Merging-Splitting-Proofness

This subsection considers an axiom, *merging-splitting-proofness*, which is closely related to *reallocation-proofness* and also formulates immunity to strategic transfers of characteristic vectors among entities. In the context of claim problems, a rule is *merging-splitting-proof* if no group of agents can increase their total awards by merging their claims and, conversely, no single agent can increase his award by creating dummy agents and splitting his claim among these dummy agents and himself. This axiom is introduced by O'Neill (1982) in the context of bankruptcy.

Merging-Splitting-Proofness. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, each non-empty $S \subseteq N$, each $i \in S$, and each $c'_i \in \mathbb{R}_+^K$, if $c'_i = \sum_{j \in S} c_j$, then

$$f_i(c'_i, c_{N \setminus S}, E) = \sum_{j \in S} f_j(c, E). \tag{16}$$

The left-hand side of (16) is well-defined since $\mathcal{D}$ is rich*. In the context of claim problems, if the left-hand side exceeds the right-hand side, then, in problem (c, E), group S can increase its total awards by merging the members' claims and having agent i represent the total claim of the group. Conversely, if the right-hand side is larger, then, in problem $(c'_i, c_{N \setminus S}, E)$, agent i can gain by creating dummy agents $S \setminus \{i\}$ and splitting his claim among himself and these dummy agents.

We also consider a pairwise version of *merging-splitting-proofness*.

Pairwise Merging-Splitting-Proofness. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, each pair $\{i, j\} \subseteq N$ with $i \neq j$, and each $c'_i \in \mathbb{R}_+^K$, if $c'_i = c_i + c_j$, then

$$f_i(c'_i, c_{N \setminus \{i,j\}}, E) = f_i(c, E) + f_j(c, E). \tag{17}$$

The following axiom, introduced by Chun (1988), says that, if $c_i = 0$ for an entity i, then the awards to the other entities are independent of whether entity i is present:

Null Consistency. For each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, and each $i \in N$, if $c_i = 0$, then, for each $j \in N \setminus \{i\}$, $f_j(c_{N\setminus\{i\}}, E) = f_j(c, E)$.

This axiom is similar to but different from *no award for null*. When $c_i = 0$, *no award for null* says that $f_i(c, E) = 0$ but allows the other entities $j \in N \setminus \{i\}$ to receive different amounts at (c, E) and $(c_{N\setminus\{i\}}, E)$.

We first extend the characterization of generalized proportional rules in Theorem 2 to the case of variable sets of entities using *null consistency*.

Theorem 5. Assume $|I| \geq 3$ and let f be a rule on a rich* domain $\mathcal{D}$. A rule f satisfies *reallocation-proofness*, *one-sided boundedness*, and *null consistency* if and only if it is a generalized proportional rule, i.e., there exist two functions $A\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^I$ and $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K$ such that, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, and each $i \in N$,

$$f_i(c, E) = A_i(\bar{c}, E) + \sum_{k\in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E)E.$$

Proof. Let f be a rule on $\mathcal{D}$ satisfying the axioms. Theorem 2 implies that, for each $N \in \mathcal{N}$, f coincides with a generalized proportional rule on $\mathcal{D}^N$. Let (A^N, W^N) denote the associated pair. By *null consistency*, it follows that, if $S \subseteq N$, then $(A^S, W^S) = (A^N, W^N)$. Thus, for each $N \in \mathcal{N}$, (A^N, W^N) is identical. □

The following theorem characterizes the family of all *merging-splitting-proof* rules.

Theorem 6. Assume $|I| \geq 3$ and let f be a rule on a rich* domain $\mathcal{D}$. Then the following three statements are equivalent:

(i) f satisfies *merging-splitting-proofness*;

(ii) f satisfies *reallocation-proofness*, *no award for null*, and *null consistency*;

(iii) there exist $|K|$ functions $\hat{W}_1, \hat{W}_2, \ldots, \hat{W}_{|K|}\colon \mathbb{R}_+ \times \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}$ such that, for each $N \in \mathcal{N}$, each $(c, E) \in \mathcal{D}^N$, and each $i \in N$,

$$f_i(c, E) = \sum_{k\in K} \hat{W}_k(c_{ik}, \bar{c}, E),$$

and, for each $k \in K$, $\hat{W}_k(\,\cdot\,, \bar{c}, E)$ is additive.

Proof. Let f be a rule on a rich* domain $\mathcal{D}$ with $|I| \geq 3$.

(i) ⇒ (ii): Suppose that f is *merging-splitting-proof*. We show that f satisfies each axiom listed in (ii).

We first show that f is *reallocation-proof*. Let $N \in \mathcal{N}$, $S \subsetneq N$, $i \in S$, $(c, E) \in \mathcal{D}^N$, and $c'_i \in \mathbb{R}_+$ be such that $c'_i = \sum_{j\in S} c_j$. Then by *merging-splitting-proofness*,

$$f_i(c'_i, c_{N\setminus S}, E) = \sum_{j\in S} f_j(c, E).$$

This obviously implies that $\sum_{j\in S} f_j(c,E)$ is invariant under any reallocation of characteristic vectors within S.

We now show that f satisfies *no award for null* and *null consistency*. Let $N \in \mathcal{N}$ and $(c,E) \in \mathcal{D}^N$ be such that $c_h = 0$ for some $h \in N$.

We first consider the case when $|N| \geq 3$. Let $x \equiv f(c,E)$ and $y \equiv f(c_{N\setminus\{h\}}, E)$. Let $j \in N \setminus \{h\}$ and let $\alpha = f_j(\hat{c}_j, E)$ be the award for entity j in the single-entity problem where $\hat{c}_j = \sum_{\ell \in N} c_\ell$. By applying *merging-splitting-proofness* to each of (c,E) and $(c_{N\setminus\{h\}}, E)$, we obtain $\sum_{i\in N} x_i = \alpha$ and $\sum_{i\in N\setminus\{h\}} y_i = \alpha$. Hence $\sum_{i\in N} x_i = \sum_{i\in N\setminus\{h\}} y_i$. On the other hand, for each $i \in N\setminus\{h\}$, *merging-splitting-proofness* for pair $\{i,h\}$ implies $x_i + x_h = y_i$. Hence $\sum_{i\in N} x_i + (|N|-2)x_h = \sum_{i\in N\setminus\{h\}} y_i$. Since $\sum_{i\in N} x_i = \sum_{i\in N\setminus\{h\}} y_i$ and $|N| \geq 3$, it follows that $x_h = 0$. This in turn implies $x_i = y_i$ for each $i \in N \setminus \{h\}$. This proves *no award for null* and *null consistency* when $|N| \geq 3$.

Now, consider N such that $|N| = 2$, say $N = \{1,2\}$. Let $(c_1, c_2, E) \in \mathcal{D}^N$ be such that $c_2 = 0$ and let $x \equiv f(c_1, c_2, E)$. If $x_2 > 0$, then, in the three-entity problem $((c_1, c_2, 0), E)$, entities 2 and 3 can increase their total awards by merging their characteristic vectors (which are both zero) and transforming the problem into (c_1, c_2, E). Conversely, if $x_2 < 0$, then, in problem (c_1, c_2, E), entity 2 can avoid the payment by bringing entity 3 as a dummy entity. Thus $x_2 = 0$, which proves *no award for null*. Furthermore, *merging-splitting-proofness* implies $f_1(c_1, E) = x_1 + x_2 = x_1$, which proves *null consistency*.

(ii) ⇒ (iii): This follows from Theorem 1 as in the proof of Theorem 5.

(iii) ⇒ (i): Obvious. □

We extend the definition of proportional rules to the current setting. We use weight functions as in the previous sections but the weights (over K) are required to be independent of the set of entities N. The following result characterizes proportional rules as in Theorem 3.

Theorem 7. Assume $|I| \geq 3$ and let f be a rule on a rich* domain $\mathcal{D}$. Then, the following three statements are equivalent:

(i) f satisfies *pairwise merging-splitting-proofness*, *efficiency*, and *non-negativity* (or *no transfer paradox*);

(ii) f satisfies *pairwise reallocation-proofness*, *efficiency*, *non-negativity* (or *no transfer paradox*), and *null consistency*;

(iii) f is a proportional rule, i.e., there exists a weight function $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \Delta^{|K|-1}$ such that, for each $N \in \mathcal{N}$, each $(c,E) \in \mathcal{D}^N$, and each $i \in N$,

$$f_i(c,E) = \sum_{k\in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c}, E) E.$$

Proof. Clearly, (iii) implies (i) and (ii). Thus, it remains to show that (i) implies (ii) and (ii) implies (iii).

(ii) ⇒ (iii): Let f satisfy the axioms in (ii). Note that *efficiency* and *null consistency* imply *no award for null*. Theorem 3 then implies that, on $\mathcal{D}^N$ for a given $N \in \mathcal{N}$, f coincides with a proportional rule for some weight function W^N. By *null consistency*, W^N is identical for all N.

(i) ⇒ (ii): Let f satisfy the axioms in (i). To prove that f is *pairwise reallocation-proof*, we can use the argument in the proof of Theorem 6 ((i) ⇒ (ii)) for S such that $|S| = 2$.

To show that f satisfies *null consistency*, let $N \in \mathcal{N}$ and $(c, E) \in \mathcal{D}^N$ be such that $c_h = 0$ for some $h \in N$. Let $x \equiv f(c, E)$ and $y \equiv f(c_{N\setminus\{h\}}, E)$. In the proof of Theorem 6 ((i) ⇒ (ii)), we used *reallocation-proofness* with respect to coalitions with more than two entities only to obtain $\sum_{i \in N} x_i = \sum_{i \in N\setminus\{h\}} y_i$. This equality now holds by *efficiency*. We can use the remaining argument in the proof of Theorem 6 ((i) ⇒ (ii)) to show that f satisfies *null consistency* (and *no award for null*). □

Corollary 3 implies that if $|K| = 1$, then *non-negativity* (or *no transfer paradox*) in Theorem 7 can be weakened to *one-sided boundedness*.

Corollary 11. Assume $|I| \geq 3$ and $|K| = 1$, and let f be a rule on a rich* domain. Then the following three statements are equivalent:

(i) f satisfies *pairwise reallocation-proofness*, *efficiency*, *one-sided boundedness*, and *null consistency*;

(ii) f satisfies *pairwise merging-splitting-proofness*, *efficiency*, and *one-sided boundedness*;

(iii) f is the proportional rule.

Corollary 11 strengthens a number of existing results in this literature by relaxing or removing axioms, generalizing the class of problems, and allowing for the set of potential entities to be finite. Moulin (1985b, Theorem 5) and Chun (1988, Theorem 2) consider claim problems and characterize the proportional rule using *reallocation-proofness*, *efficiency*, *null consistency*, *anonymity*, and "continuity." Chun (1988, Theorem 3) proves another characterization of the proportion rule using *merging-splitting-proofness*, *efficiency*, *anonymity*, and "continuity." de Frutos (1999, Theorem 1) considers bankruptcy problems and characterizes the proportional rule using *merging-splitting-proofness*, *efficiency*, and *non-negativity*. Ju (2003) studies bankruptcy problems and shows that for rules satisfying *efficiency*, *non-negativity*, and "claim boundedness," the combination of *pairwise reallocation-proofness* and *null consistency* is equivalent to *pairwise merging-splitting-proofness*. The equivalence between (i) and (ii) strengthens Ju's result by removing "claim boundedness" and weakening *non-negativity* to *one-sided boundedness*. All the results mentioned above are proved under the assumption that there exists an infinite number of potential agents.

5.2 Consistency

This subsection examines another variable-population axiom known as *consistency* and refines a characterization of the proportional rule proved in Chambers and Thomson (2002). In this subsection, we restrict ourselves to the classes of bankruptcy, surplus sharing, and claim problems. We say that a rule is *regular* if the following conditions are satisfied.

(i) In the class of bankruptcy problems, the rule satisfies *non-negativity*, *efficiency*, and the "upper claim boundedness": for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{D}^N$, $f(c, E) \leqq c$;

(ii) In the class of surplus sharing problems, the rule satisfies *non-negativity*, *efficiency*, *no award for null*, and the "lower claim boundedness": for each $N \in \mathcal{N}$ and each $(c, E) \in \mathcal{D}^N$, $f(c, E) \geqq c$;

(iii) In the class of claim problems, the rule satisfies *non-negativity*, *efficiency*, and *no award for null* (no boundedness condition is imposed).

For bankruptcy problems, *no award for null* is not required since it is implied by *non-negativity* and the boundedness condition.

The main axiom considered in this section is *consistency*, which considers a situation in which, after awards are determined, a subset of agents "leave the scene" with their awards. A rule is *consistent* if reapplying the rule to the problem with the remaining agents and the remaining amount to divide does not change the award vector for these agents.[8]

Consistency. For each $N, N' \in \mathcal{N}$ with $N' \subseteq N$ and each $(c, E) \in \mathcal{D}^N$, $f_{N'}(c, E) = f(c_{N'}, E - \sum_{i \in N \setminus N'} f_i(c, E))$.[9]

The next axiom says that, for any two groups whose aggregate claims are equal, the aggregate awards are equal (Chambers and Thomson, 2002).

Equal Treatment of Equal Groups. For each $N \in \mathcal{N}$, each $N', N'' \subseteq N$, and each $(c, E) \in \mathcal{D}^N$, if $\sum_{i \in N'} c_i = \sum_{i \in N''} c_i$, then $\sum_{i \in N'} f_i(c, E) = \sum_{i \in N''} f_i(c, E)$.

A weaker version of this condition is obtained by applying the condition only to a pair of groups of the same *size*.

Equal Treatment of Strongly Equal Groups. For each $N \in \mathcal{N}$, each $N', N'' \subseteq N$, and each $(c, E) \in \mathcal{D}^N$, if $|N'| = |N''|$ and $\sum_{i \in N'} c_i = \sum_{i \in N''} c_i$, then $\sum_{i \in N'} f_i(c, E) = \sum_{i \in N''} f_i(c, E)$.

A logical relation between these two versions is given by the following proposition.

[8] Thomson (2003c) offers a survey of the vast literature devoted to the analysis of the consistency principle.

[9] If $E - \sum_{i \in N \setminus N'} f_i(c, E) = 0$, then the last term is not well-defined because of our simplifying assumption that $E > 0$ for all problems. Thus we complete the definition by saying that, if $E - \sum_{i \in N \setminus N'} f_i(c, E) = 0$, then $f_{N'}(c, E) = 0$.

Proposition 2. On a rich* domain with $|I| = \infty$, *equal treatment of strongly equal groups*, *no award for null*, and *null consistency* together imply *equal treatment of equal groups*.

Proof. Assume $|I| = \infty$ and let f be a rule on a rich* domain $\mathcal{D}$ satisfying the first three axioms in the proposition. Let $N, N_1, N_2 \in \mathcal{N}$ and $(c, E) \in \mathcal{D}^N$ be such that $N_1, N_2 \subseteq N$ and $\sum_{i \in N_1} c_i = \sum_{i \in N_2} c_i$. Without loss of generality, assume $|N_2| < |N_1|$ and $N_1 \cap N_2 = \emptyset$ (the case with an overlap can be proved by repeating the following argument twice). Let $S \in \mathcal{N}$ be such that $S \cap N = \emptyset$ and $|S| + |N_2| = |N_1|$ (such an S exists since $|I| = \infty$). Define $c' \in \mathcal{D}^{N \cup S}$ by $c'_i = c_i$ for all $i \in N$ and $c'_i = 0$ for all $i \in S$ (c' belongs to the domain since the domain is rich*). Since N_1 and $N_2 \cup S$ are of the same size and have the same aggregate characteristics, *equal treatment of strongly equal groups* and *no award for null* imply $\sum_{i \in N_1} f_i(c', E) = \sum_{i \in N_2 \cup S} f_i(c', E) = \sum_{i \in N_2} f_i(c', E)$. Since c differs from c' only in its zero entries, *null consistency* implies $\sum_{i \in N_1} f_i(c, E) = \sum_{i \in N_1} f_i(c', E) = \sum_{i \in N_2} f_i(c', E) = \sum_{i \in N_2} f_i(c, E)$. □

Chambers and Thomson (2002, Theorem 5) show that, in the class of bankruptcy problems with at least three potential agents, the proportional rule is the only *regular* rule satisfying *equal treatment of equal groups*, *consistency*, and "continuity in claim." As they mention (p. 250), when the number of potential agents is infinite, "continuity in claim" is redundant in their characterization. It has been an open question whether, when $|I|$ is finite, their result holds without "continuity in claim." In the next theorem, we show that "continuity in claim" is redundant when there are at least six potential agents.[10] Furthermore, *equal treatment of equal groups* can be weakened to *equal treatment of strongly equal groups*. This result holds for the classes of bankruptcy, surplus sharing, and claim problems.

Theorem 8. In the classes of bankruptcy, surplus sharing, and claim problems with at least six potential agents, the proportional rule is the only *regular* rule satisfying *equal treatment of strongly equal groups* and *consistency*.

Proof. Let f be a *regular* rule satisfying the axioms on any of the three domains with at least six potential agents.

Step 1: The restriction of f on the subdomain of three-agent problems is *reallocation-proof*. Since $|I| \geq 6$, we can assume, without loss of generality, that $\{1, 2, 3, 4, 5, 6\} \subseteq I$. To show that f is *reallocation-proof* on the subdomain of three-agent problems, let $N = \{1, 2, 3\}$ and $(c, E) \in \mathcal{D}^N$. Consider any vector $\hat{c} = (\hat{c}_4, \hat{c}_5, \hat{c}_6) \in \mathbb{R}_+^{\{4,5,6\}}$ such that

$$\hat{c}_4 + \hat{c}_5 + \hat{c}_6 = c_1 + c_2 + c_3, \tag{18}$$

$$\hat{c}_4 + \hat{c}_5 = c_1 + c_2. \tag{19}$$

[10] Whether the result holds when $3 \leq |I| \leq 5$ remains open.

Then, *equal treatment of strongly equal groups* and *efficiency* imply

$$\sum_{i=1}^{3} f_i(c, \hat{c}, 2E) = \sum_{i=4}^{6} f_i(c, \hat{c}, 2E) = E, \tag{20}$$

$$\sum_{i=1}^{2} f_i(c, \hat{c}, 2E) = \sum_{i=4}^{5} f_i(c, \hat{c}, 2E). \tag{21}$$

Then, by (20) and *consistency*, we obtain

$$f_{\{1,2,3\}}(c, \hat{c}, 2E) = f(c, E),$$
$$f_{\{4,5,6\}}(c, \hat{c}, 2E) = f(\hat{c}, E).$$

These equalities and (21) imply

$$\sum_{i=1}^{2} f_i(c, E) = \sum_{i=4}^{5} f_i(\hat{c}, E). \tag{22}$$

Since this equality is obtained for any $\hat{c}$ that satisfies (18) and (19), we can now apply the same argument replacing c by any (c'_1, c'_2, c_3) such that $c'_1 + c'_2 = c_1 + c_2$ (since then (c'_1, c'_2, c_3) and $\hat{c}$ satisfy equalities corresponding to (18) and (19)). We then obtain

$$\sum_{i=1}^{2} f_i(c'_1, c'_2, c_3, E) = \sum_{i=4}^{5} f_i(\hat{c}, E).$$

This equality and (22) yield

$$\sum_{i=1}^{2} f_i(c'_1, c'_2, c_3, E) = \sum_{i=1}^{2} f_i(c, E),$$

which proves that f is *reallocation-proof* on the subdomain of three-agent problems.

Step 2: Concluding. Since f satisfies *efficiency*, *no award for null*, and *non-negativity*, Step 1 and Corollary 1 imply that f coincides with the proportional rule on the subdomain of three-agent problems. By *consistency*, f also coincides with the proportional rule on the subdomain of two-agent problems. To show that f coincides with the proportional rule on the entire domain, let $(c, E) \in \mathcal{D}^N$ for some N such that $|N| \geq 4$, and let $x \equiv f(c, E)$. Pick $i, j \in N$ arbitrarily. By applying *consistency* to the pair $\{i, j\}$ and noting that f is the proportional rule for two-agent problems, we obtain $x_i c_j = c_i x_j$. Since f is *efficient*, it follows that $x_i \sum_{j \in N} c_j = c_i \sum_{j \in N} x_j = c_i E$, as desired. $\square$

Appendix

Proof of Corollary 7

Let f be a rule satisfying the axioms. Then by Corollary 2, f is given by (11) for some non-negative valued function $W\colon \mathbb{R}^K_{++} \times \mathbb{R}_{++} \to \mathbb{R}^K_+$. Define ρ by $\rho_k(\bar{c}) \equiv W_k(\bar{c})\bar{c}_{\max}/\bar{c}_k$. With this definition and $E = \bar{c}_{\max}$, (11) reduces to (14). It remains to show that ρ satisfies (13) and $\sum_{k\in K} \rho_k(\bar{c}) = 1$.

We first prove that ρ satisfies (13). So, let $d \in \mathbb{R}^K_{++}$ and $\lambda > 0$. Pick $h \in K$ and $j, \ell \in N$ arbitrarily, and let $c \in \mathbb{R}^{K\times N}_+$ be such that

$$\begin{aligned} \bar{c} &= d, \\ c_{jh} &> 0, \\ c_{jk} &= 0 \quad \text{for each } k \in K \setminus \{h\}, \\ c_{\ell k} &= 0 \quad \text{for each } k \in K. \end{aligned} \tag{23}$$

Since $|N| \geq 3$, there exists another agent $m \in N \setminus \{j, \ell\}$. Let $c' \in \mathbb{R}^{K\times N}_+$ be the profile defined by $c' \equiv (c_m + \lambda \mathbf{1}, c_{-m})$. By *translation invariance*, $f_j(c') = f_j(c)$ and $f_\ell(c') = f_\ell(c)$. By (23) and (14),

$$\begin{aligned} f_j(c) &= f_\ell(c) + c_{jh}\rho_h(\bar{c}), \\ f_j(c') &= f_\ell(c') + c_{jh}\rho_h(\bar{c} + \lambda\mathbf{1}). \end{aligned}$$

Since $c_{jh} > 0$, we obtain $\rho_h(\bar{c} + \lambda\mathbf{1}) = \rho_h(\bar{c})$.

We now prove that $\sum_{k\in K} \rho_k(d) = 1$ for all d. By the result we just obtained, it suffices to prove the result for d such that $d_k > 1$ for all $k \in K$. Pick two agents $j, \ell \in N$ arbitrarily, and let $c \in \mathbb{R}^{K\times N}_+$ be such that $\bar{c} = c_\ell = d$. Let $c' \in \mathbb{R}^{K\times N}_+$ be defined by $c' \equiv (c_j + \mathbf{1}, c_\ell - \mathbf{1}, c_{N\setminus\{j,\ell\}})$. Then $\bar{c}' = \bar{c}$, and *translation invariance* implies $f_j(c') = f_j(c) + 1$. Since $f_j(c)$ and $f_j(c')$ differ only in the last term of (14), we have

$$1 = f_j(c') - f_j(c) = \sum_{k\in K} (c_{jk} + 1 - c_{jk})\rho_k(\bar{c}) = \sum_{k\in K} \rho_k(\bar{c}),$$

completing the proof. □

Proof of Corollary 8

Let f be a rule satisfying the axioms in the corollary. By Theorem 3, f is a proportional rule with some weight function W. Thus f is given by

$$f_i(c) = \sum_{k\in K} \frac{c_{ik}}{\bar{c}_k} W_k(\bar{c})\bar{c}_{\max}.$$

Define ρ by $\rho_k(\bar{c}) = W_k(\bar{c})\bar{c}_{\max}/\bar{c}_k$. Then

$$f_i(c) = \sum_{k \in K} \rho_k(\bar{c}) c_{ik}.$$

We first show that ρ satisfies (13). So, let $d \in \mathbb{R}^K_{++}$ and $\lambda > 0$. Pick $j \in N$ and $h \in K$ arbitrarily, and let $c \in \mathbb{R}^{N \times K}_{+}$ be such that $\bar{c} = d$, $c_{jh} > 0$, and $c_{jk} = 0$ for all $k \in K \setminus \{h\}$. Let $\ell \in N \setminus \{j\}$ be another agent and define c' by $c' \equiv (c_\ell + \lambda \mathbf{1}, c_{-\ell})$. By *translation invariance*, $f_j(c') = f_j(c)$. Since

$$\begin{aligned} f_j(c) &= \rho_h(\bar{c}) c_{jh}, \\ f_j(c') &= \rho_h(\bar{c} + \lambda \mathbf{1}) c_{jh}, \end{aligned}$$

it follows that $\rho_h(\bar{c} + \lambda \mathbf{1}) = \rho_h(\bar{c})$.

We now prove that $\sum_{k \in K} \rho_k(d) = 1$ for all d. By the result we just proved, it suffices to consider d such that $d_k > 1$ for all $k \in K$. Then pick two agents $j, \ell \in N$ arbitrarily and let $c, c' \in \mathbb{R}^{K \times N}_{+}$ be defined by $\bar{c} = c_\ell = d$ and $c' = (c_j + \mathbf{1}, c_\ell - \mathbf{1}, c_{N \setminus \{j, \ell\}})$. By *translation invariance*, $f_j(c') = f_j(c) + 1$. Since $\bar{c} = \bar{c}'$,

$$1 = f_j(c') - f_j(c) = \sum_{k \in K} \rho_k(\bar{c})(c_{jk} + 1 - c_{jk}) = \sum_{k \in K} \rho_k(\bar{c}).$$

Finally, we prove that $\rho(\bar{c})$ satisfies (15); i.e., $\rho_k(\bar{c}) > 0$ holds only for $k \in K$ such that $\bar{c}_k = \bar{c}_{\max}$. By *efficiency*,

$$\bar{c}_{\max} = \sum_{i \in N} f_i(c) = \sum_{k \in K} \rho_k(\bar{c}) \bar{c}_k.$$

Since $\rho_k(\bar{c}) \geq 0$ and $\sum_{k \in K} \rho_k(\bar{c}) = 1$, the equality holds if and only if $\rho(\bar{c})$ satisfies (15). $\square$

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