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The Economics of Controlling a Biological Invasion

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ABSTRACT

The paper develops a simple economic model of biological invasion. The natural growth of the invasion is non-convex and the immediate cost of controlling the invasion depends on the level of current control as well as the current size of the invasion. Increased current control may not only escalate control cost today while avoiding potential damage - now and in the future, but may also increase the marginal cost of control in the future. The optimal path of invasion is not necessarily monotonic. When the marginal control cost declines sharply with the size of invasion, it may be optimal to allow the invasion to grow before reducing its extent. We characterize the conditions under which it is optimal to eradicate an invasive species (immediately and eventually) and the condition under which it is optimal to manage the invasion without complete eradication

1. Introduction

Throughout history the spread of plants, animals and other organisms has been governed by natural ecological processes and has accompanied the movement of humans. As human beings have become more mobile there has been an increase in the introduction of species to areas outside their native habitat. Sometimes such introductions are harmless, or even beneficial. In other instances, introduced species become invaders with few natural enemies to limit their growth.

The invasion of ecological systems by non-indigenous species is now recognized as a growing global problem that imposes significant economic and ecological damages. In the United States alone, the total costs of non-indigenous species have been estimated to be at least \$137 billion per year [Pimentel, et. al., 2000]. Approximately one-fourth of the value of the country 's agricultural output is lost to non-indigenous plant pests or the costs of controlling them [Simberloff, 1996]. Invasive species also cause significant ecological harm. They can alter ecosystem processes, act as vectors of disease, and reduce biodiversity [Vitousek, et. al., 1996]. Worldwide, out of 256 vertebrate extinctions with an identifiable cause, 109 are known to be due to biological invaders. In comparison, 70 such extinctions are known to be caused by human exploitation [Cox, 1993]. It is estimated that forty percent of the threatened or endangered species in the United States are at risk due to pressures from invading species [The Nature Conservancy, 1996; Wilcove et. al., 1998].

Although the number of harmful invasive species in the United States is in the thousands [Pimentel, et. al, 2000] a single invasive species can cause tremendous economic and/or ecological damage. For example, the costs of controlling the zebra mussel (*Dreissena polymorpha*) in the Great Lakes are expected to reach \$5 billion in 2001 [USGS, 2000] while the Russian wheat aphid (*Diuraphis noxia*) caused an estimated \$600 million (1991\$) in crop damages between 1987 and 1989 [OTA, 1993]. The invasion of Guam by the brown tree snake (*Boiga irregularis*) demonstrates the ecological harm an invader can cause. Since its introduction in the mid-twentieth century it has caused the extinction of 12 of the island 's bird species [Savidge, 1987; Rodda, Fritts and Chiszar, 1997].

Biological invasions occur almost everywhere. They comprise a significant component of global environmental change [Vitousek et. al., 1996] and are viewed by some as one of the most important issues in natural resource management today [Williams and Meffe, 2000]. As a consequence, the past decade has seen a sizeable growth in public policy directed toward invasive species. In 1990 the U.S. Congress passed the Nonindigenous Aquatic Nuisance Prevention and Control Act to deal with the increasing problems of invasive species in waterways. This Act was subsequently amended in 1996 to become the National Invasive Species Act. In 1992 the Alien Species Prevention and Enforcement Act was passed to protect Hawaii from the introduction of prohibited plants, plant pests and injurious animals that may be contained in the mail. On Feb 3, 1999, President Clinton signed Executive Order 13112 which created the National Invasive Species Council. Subsequently, the Council drafted the National Invasive Species Management Plan to develop a national strategy for combating problems of invasive species. There has also been increasing concern at the international level. In 1997, the Global Invasive Species Programme was established by the Scientific Committee for Problems of the Environment (SCOPE), in collaboration with the United Nations Environment Programme, The World Conservation Union, DIVERSITAS (an international programme on biodiversity science) and the Commonwealth Agricultural Bureau International.

In spite of the growing concern with invasive species, scientific understanding of the relation between economics and ecology in the control of biological invasions is not well-developed. The purpose of this paper is to examine the economics of controlling a biological invasion. We analyze the optimal control policy in a dynamic model of invasion by a single species. The aim is to understand the intertemporal economic and biological tradeoffs that determine the extent to which an invasion should be controlled at any point of time. The paper focuses on the conditions under which it is optimal to eradicate an invasive species and conditions under which eradication is not optimal. We characterize both of these possible outcomes using verifiable conditions on the

biological growth of the invasion, control costs, damages and the social discount rate. This involves a dynamic cost-benefit analysis.

Most introductions of non-indigenous species occur as a result of commerce, travel, agriculture or other human activity. The majority (between 80 and 95 percent) of non-native species never become established in their new environment [Williamson, 1996]. Once established, however, the spread of an invasive species is typically characterized by three distinct phases [Shigesada and Kawasaki, 1997, pp 26-27]. The first is an initial establishment phase during which little or no expansion occurs. This is followed by an expansion phase where the population and range of the invasive species increases. Finally, there is a saturation phase as the invasion approaches geographical, climatic, or ecological limits to its range.¹

The classic ecological model of the spread of an invasion is the reaction-diffusion model of Kolmogorov, Petrovsky, and Piskounov [1937], Fisher [1937] and Skellam [1951]. The model of a biological invasion developed in this paper is an aggregate model that abstracts from spatial considerations inherent in reaction-diffusion models. The variable that defines the state of the invasion at each point in time is its size. Depending on the context, this aggregate variable may be either the area contained within the frontal boundary of the invasion, or the population or biomass of the invasive species. The aggregation adopted here is analogous to the way that the standard bioeconomic models of resource harvesting aggregate over spatial and other life history

¹In the reaction-diffusion model growth and spread jointly determine the density, $n(y,z,t)$, of a species at location (y,z) at time t according to the partial differential equation $\partial n/\partial t = G(n) + D(\partial^2 n/\partial y^2 + \partial^2 n/\partial z^2)$. Local growth is governed by the growth function $G(n)$ while the coefficient D that determines how fast the species disperses in space. Fisher [1937] analyzed the case of logistic growth, $G(n) = rn(1-n/K)$, while Skellam [1951] considered Malthusian growth, $G(n) = rn$. Kolmogorov, Petrovsky, and Piskounov [1937] considered the general class of growth functions that satisfy $G(0) = G(1) = 0$, $G(n) > 0$ for $0 < n < 1$, $G'(0) > 0$, $G'(n) < G'(0)$ for $0 < n < 1$. The frontal boundary of an invasion is defined to be the radial distance at which the species density equals a critical threshold, n^* . For example, n^* may define a detection threshold below which the species density is low enough to avoid detection. For our purposes it is useful to think of n^* as an economic threshold below which the species density is not sufficient to cause damage. In what has become a classic result in the ecological theory of biological invasions Kolmogorov, Petrovsky, and Piskounov [1937] showed that the frontal boundary of an invasion governed by (3.1) expands asymptotically (as $t \rightarrow \infty$) at a constant rate $2\sqrt{G'(0)D}$. For the case of Malthusian growth this was proved by Kendall [1948]. Mollison [1977] provides a useful review.

characteristics that influence resource growth. At the same time, the model of this paper reflects the central implications of the ecological theory of invasions and historical evidence on their spread. Consequently, the growth and spread of an invasion is assumed to be governed by a nonconvex biological growth function. Initially, as the invasion grows in size it expands at a faster rate; but eventually, the expansion rate slows down as the invasion approaches its environmental carrying capacity. A nonconvex invasion growth function is consistent with invasions that follow a pattern of establishment, expansion, and saturation.

Control of an invasive species takes the form of reducing the size of the invasion by chemical, biological, manual, or other means. The evidence from historical attempts to eradicate invasive species suggests that unit control costs often escalate as the size of the invasion becomes smaller and that it may cost as much to remove the last one to ten percent of an invasion as it does to control the initial ninety to ninety-nine percent [Myers, et. al., 1998]. Hence, it is assumed that the costs of control depend on the amount of control as well as the size of the invasion at the time control is undertaken.

The nonconvex nature of invasion growth and the fact that control costs may depend on the size of the invasion have important consequences for solving the optimal control problem and the resulting policy implications. Nonconvex growth makes the feasible set for the dynamic optimization problem non-convex. From a technical standpoint this means that the “first order conditions” may not define a global optimum and corner solutions are more likely to emerge. From a policy perspective, the optimal control of invasion under nonconvex growth is not necessarily increasing in the size of invasion and there may be multiplicities or discontinuities in the optimal policy.

When control costs depend on the size of the invasion, an increase in the current control has two opposing effects on future net benefits. At the margin it reduces future damages, but it also increases future control costs due to the stock effect that reductions in the size of the invasion have on costs. The latter creates an economic incentive to postpone control and can lead to outcomes where the invasion follows a nonmonotonic time path under an optimal policy. As a result, there are

circumstances where the optimal management of an invasion exhibits cyclical or complex dynamics, as happens when the optimal policy is one that allows an invasion to grow unchecked for a number of periods and only after it becomes large enough is it reduced or eradicated.

In contrast, if the cost of control is independent of the size of the invasion (as with some forms of chemical control), then along every optimal path the size of the invasion is monotonic over time and convergent. This implies that if eradication is optimal from one invasion size, it must necessarily be optimal for any invasion of smaller size.

The analysis in this paper is related to the literature on optimal management of renewable resources (see, Clark, 1990) where the objective is to maximize the discounted sum of social welfare obtained from harvesting a useful biological resource. Particularly relevant to our problem is the analysis of conservation and extinction of renewable resources on optimal paths. The literature includes models where the growth function is non-concave (non-convex feasible set for the optimization problem) as well as models where the utility depends not only on the size of the harvest but also on the total stock size (see, for example, Olson and Roy, 1996). More generally, the literature on optimal economic growth has also analyzed related dynamic optimization problems including ones where the feasible set is non-convex (among others, Majumdar and Mitra, 1982, Dechert and Nishimura, 1983) as well as models where the utility function depends on both consumption and capital stock (see, for example, Nyarko and Olson, 1991, and the collection of papers in Majumdar, Mitra, and Nishimura, 2000). There is a key distinction between our analysis and the literatures on economic growth and renewable resources. In the latter, the physical or biological capital stock contributes to the production of a good that yields positive social welfare. Hence, conservation of the resource is positively correlated with productivity and extinction is not optimal if the productivity of the resource is higher than the discount rate. With a biological invasion, the capital stock or size of the invasion contributes in a negative way to social welfare. This difference has the effect of reversing the relationship between productivity and incentives to

preserve the invasive species. As our analysis shows, the faster the growth rate of an invasion, the greater is the incentive to eradicate it.

The paper is organized as follows. Section two develops the model. The basic properties of a controlled invasion are discussed in section three. Section four studies the economic and ecological conditions under which it makes sense to eradicate an invasive species. Circumstances under which eradication does not make sense are examined in section 5. All proofs are given in the appendix.

2. The model

Let y_t represent the size of the biological invasion at the beginning of time t and let a_t represent the amount of control at time t . The invasion that remains at the end of period t is given by $x_t = y_t - a_t$. The invasion is assumed to grow and spread according to an invasion growth function $y_{t+1} = f(x_t)$. The invasion growth function is assumed to satisfy the following properties:

- A1. $f(x)$ has a continuous derivative, $f'_x(x)$.
- A2. $f(0) = 0$.
- A3. $f'_x(x) \geq 0$.
- A4. $f'_x(0) > 1$.
- A5. (a) There exists a $k > 0$ such that $f(x)$ is convex on $[0, k]$.
(b) There exists an $K \in (k, \infty)$ such that $f(x) < x$ for all $x > K$, $f(x)$ is concave on (k, K) , and $f'_x(K) > 0$.

Assumption A2 implies that once an invasion is eradicated it cannot recur. This paper does not address situations where re-invasion is a serious concern. Assumption A3 says that the invasion growth function is increasing in the size of the invasion. To be successful an invasive species must necessarily be able to sustain an invasion. Assumption A4 implies that an invasion can be sustained from an isolated occurrence of the species. Assumption A5 is the main assumption about the

behavior of an invasion.² The first part implies that the growth function governing the spread of an invasion is convex when the size of the invasion is between zero and some positive neighborhood of the minimum size required to sustain the invasion. The second part implies that the spread of the invasion is bounded by climatic, geological or ecological factors and that the growth function eventually becomes concave as it approaches this upper bound.

The costs of control and damages caused by the invasion are denoted by $C(a,y)$ and $D(x)$, respectively. Let $\Omega \subset \mathfrak{R}_+^2$ be the set defined by $\{(a,y)|0 \leq a \leq y \leq K\}$. Costs and damages are assumed to satisfy the following:

- B1. C and D are twice continuously differentiable.³
- B2. $C(0,y) = 0$ for all y and $D(0) = 0$.
- B3. $C_a(a,y) \geq 0$, $C_y(a,y) \leq 0$, and $C_a(a,y) + C_y(a,y) \geq 0$ on Ω . $D_x(x) \geq 0$.
- B4. C and D are convex. $C(y-x,y)+D(x)$ is strictly convex in x .

Assumption B2 rules out fixed costs and it also implies that $C_y(0,y) = 0$. Assumption B3 implies that damages are increasing in the size of the invasion, the costs of control increase as control increases, and that a given amount of control is cheaper to achieve from larger invasions. The assumption that $C_a(a,y) + C_y(a,y) \geq 0$ means that if $y \leq y'$ it is less costly to reduce the size of the invasion from y to x than it is to reduce the size of the invasion from y' to x . Throughout the paper it is assumed that A1-A5 and B1-B4 hold.

²This model is isomorphic to a model of radial spread, provided the relation between the area of invasion and the radial distance to the frontal boundary is known. For example, if x is the area contained behind the frontal boundary of an invasion in a homogenous space, and r is the radial distance from the point of invasion to the frontal boundary then the spread of the invasion can be expressed as $r_{t+1} = F(r_t)$, where $F(r) = \sqrt{f(\pi r^2)/\pi}$. In this case, A5 is equivalent to assuming that the elasticity of marginal radial growth is greater than one when the radius of invasion is small, and less than one when the radius of invasion is large. In other words, $F''(r)r/F'(r) \geq 1$ for all r in some interval $[0, r_k]$, while $F''(r)r/F'(r) \leq 1$ for all $r \in [r_k, \bar{r}]$.

³Derivatives are indicated by relevant subscripts, e.g. C_a represents the partial derivative of C with respect to a .

A *policy*, $\pi = (\pi_1, \pi_2, \dots)$, is a sequence of decision rules, π_t , that specify a plan for controlling the biological invasion as a function of the previous history, $h_t = (y_0, a_0, x_0, \dots, a_{t-1}, x_{t-1}, y_t)$. That is, $a_t = \pi_t(h_t)$ and $x_t = y_t - \pi_t(h_t)$. A *stationary Markov policy* is associated with a pair of decision rules that specify the control and the size of the invasion that remains at the end of each period as a function of the size of the invasion at the beginning of the period. Associated with each initial state, y_0 , and each policy is a discounted sum of social costs $V_\pi(y_0) = \sum_{t=0}^{\infty} \delta^{t-1} [C(a_t, y_t) + D(x_t)]$, where the sequence $\{a_t, x_t\}$ is generated by the invasion growth function, f , and the policy, π , in an obvious manner. The objective of the dynamic optimization problem is to minimize the discounted sum of costs and damages over time subject to the transition equation that governs the growth and spread of the invasion. The optimal value satisfies:

$$V(y_0) = \text{Min} \sum_{t=0}^{\infty} \delta^{t-1} [C(a_t, y_t) + D(x_t)] \text{ subject to } y_t = a_t + x_t \text{ and } y_{t+1} = f(x_t). \quad (2.1)$$

Under A1-A5 and B1-B4, standard dynamic programming arguments imply that there exists a stationary optimal value that satisfies the recursion $V(y) = \text{Min} C(a_t, y_t) + D(x_t) + \delta V(f(x_t))$ subject to $y_t = a_t + x_t$ and $y_{t+1} = f(x_t)$, and that there exists a *stationary Markov optimal policy* whose decision rules are $X(y) = \text{Arg Min} C(y - x, y) + D(x) + \delta V(f(x))$ and $A(y) = y - X(y)$. A sequence $(y_t, x_t, a_t)_0^\infty$ that solves (2.1) is an optimal program from y_0 . Given an initial invasion of size $y_0 = y$ and a selection from the stationary optimal policy $X(y)$, an optimal program is defined recursively by $y_t = f(X(y_{t-1}))$, $x_t = X(y_t)$, $a_t = A(y_t)$, $t = 1, 2, \dots$

3. Controlled Invasions and their Basic Properties

This section characterizes the basic properties of an optimal policy and the optimal value. The initial results characterize the sensitivity of the optimal value $V(y)$ and optimal policy $X(y)$ to the size of the invasion.

Lemma 1. $V(y)$ is continuous and non-decreasing.

Lemma 1 formalizes the intuitive notions that incremental changes in the size of an invasion are associated with small changes in social cost and that larger invasions involve higher social costs.

Sensitivity of the optimal policy depends on how the costs of control vary with control and the invasion size. Our characterization is based on the following properties of a correspondence.

Let $x \in X(y)$ and $x' \in X(y')$ where $y \leq y'$.

Definition. A correspondence $X(y)$ is an *ascending* correspondence if $\min[x, x'] \in X(y)$ and $\max[x, x'] \in X(y')$. $X(y)$ is *descending* if $\max[x, x'] \in X(y)$ and $\min[x, x'] \in X(y')$.

Lemma 2. (a) If $C_{aa}(a, y) + C_{ay}(a, y) \geq 0$ on Ω , then $X(y)$ is an ascending correspondence and the maximal and minimal selections from X are non-decreasing functions. If the inequality is strict then every selection from X is non-decreasing. (b) Assume $C_{aa}(a, y) + C_{ay}(a, y) \leq 0$ on $\text{int } \Omega$. If there exists some $y < K$ such that $0 < X(y) < y$ then there is a neighborhood $N(y)$ of y such that $X(\cdot)$ is descending on $N(y)$ and the maximal and minimal selections from X are non-increasing functions on $N(y)$.

The economic requirement of the first part of Lemma 2 is that a change in control has a larger effect on marginal costs than a change in the size of the invasion. This provides an economic criterion for the optimal size of the invasion to evolve monotonically over time. Since the optimal invasion size is bounded, every invasion with a monotonic time path must necessarily converge to a positive steady state or zero (eradication). If two invasions differ only in their initial size, then the invasion that is larger today will be (weakly) larger at all points in the future.

When the size of the invasion has a large effect on the marginal cost of control, as in part (b), this may result in a non-monotonic optimal policy for the size of the invasion. An example of this

occurs when the marginal costs of control for a small invasion are sufficiently high that the optimal policy involves no control while the invasion is small. As the invasion grows larger, marginal costs decrease and at some point it may become optimal to reduce the invasion back to very small levels, from which no control is once again optimal.

Some invasions cause minimal damage and control is not cost effective. As a consequence, we first try to identify the circumstances under which control makes sense. There are different ways to view the control of an invasion. One may be interested in control from an invasion of a particular size or control of an invasion of any size, or one may be concerned about control of the invasion immediately or at some future date. This motivates the following definitions.

Definition. (a) An invasion is a *controlled invasion from y* if there exists some t such that $A(y_t) > 0$, where y_t is optimal from y . (b) An invasion is *currently controlled from y* if $A(y) > 0$. (c) An invasion is *controlled globally* if $A(y) > 0$ for all y . (d) An invasion is *interior* if it is controlled globally and $X(y) > 0$ for all y .

Each successive definition of control is more restrictive in the sense that $(d) \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$.

The next result characterizes the economic conditions that are sufficient for each of these types of control. Define the t^{th} iterate of $f(\cdot)$ and its derivative by $f^0(y) = y$, $f^t(y) = f^{t-1}(f(y))$, $t = 1, \dots$ and $f'_x(y) = df^t(x)/dx$.

Lemma 3. a. If there is an $n \geq 0$ such that $C_a(0, f^n(y)) < \sum_{i=n}^{\infty} \delta^i D_x(f^i(y)) f_x^i(y)$ then the invasion is a controlled invasion from y .

b. If $C_a(0, y) < D_x(y)$ then the invasion is currently controlled from y .

c. If $C_a(0, y) < D_x(y) + \delta[\inf_a \{C_a(a, f(y)) + C_y(a, f(y))\}] f_x(y)$ for all $y > 0$ then the invasion is controlled globally.

Two special cases of Lemma 3c are worth noting. First, if $C_a(0,y) = 0$ and $D_x(y) > 0$ for all $y > 0$, then the optimal policy is to control the invasion from any size. Second, when a change in control has a larger effect on marginal costs than a change in the invasion size, the requirement of Lemma 3c simplifies to $C_a(0,y) < D_x(y) + \delta C_a(0,f(y))f_x(y)$.

In many instances the marginal costs of control will be decreasing in y and the conditions for controlling an invasion in Lemma 3 will be more likely to hold, the larger the invasion.

In the following let $(y_t, x_t, a_t)_0^\infty$ be an optimal program from y_0 . The next result characterizes the intertemporal tradeoffs between marginal costs and damages along an optimal program.

Lemma 4. a. If $a_t > 0$ then $C_a(a_t, y_t) \leq D_x(x_t) + \delta[C_a(a_{t+1}, y_{t+1}) + C_y(a_{t+1}, y_{t+1})]f_x(x_t)$.

b. If $x_t > 0$ and $a_{t+1} > 0$ then $C_a(a_t, y_t) \geq D_x(x_t) + \delta[C_a(a_{t+1}, y_{t+1}) + C_y(a_{t+1}, y_{t+1})]f_x(x_t)$.

c. If $0 < x_t < y_t$ and $a_{t+1} > 0$ then

$$C_a(a_t, y_t) = D_x(x_t) + \delta[C_a(a_{t+1}, y_{t+1}) + C_y(a_{t+1}, y_{t+1})]f_x(x_t). \quad (3.1)$$

Since the value function in nonconvex models may not be differentiable, Lemma 4 cannot be obtained by applying standard envelope theorem arguments such as those of Benveniste and Scheinkman [1979]. Majumdar and Mitra [1982] use a variational approach to obtain the Euler equation. Here, we develop an alternative approach based on the principle of optimality and the fact that Dini derivatives of V exist everywhere.

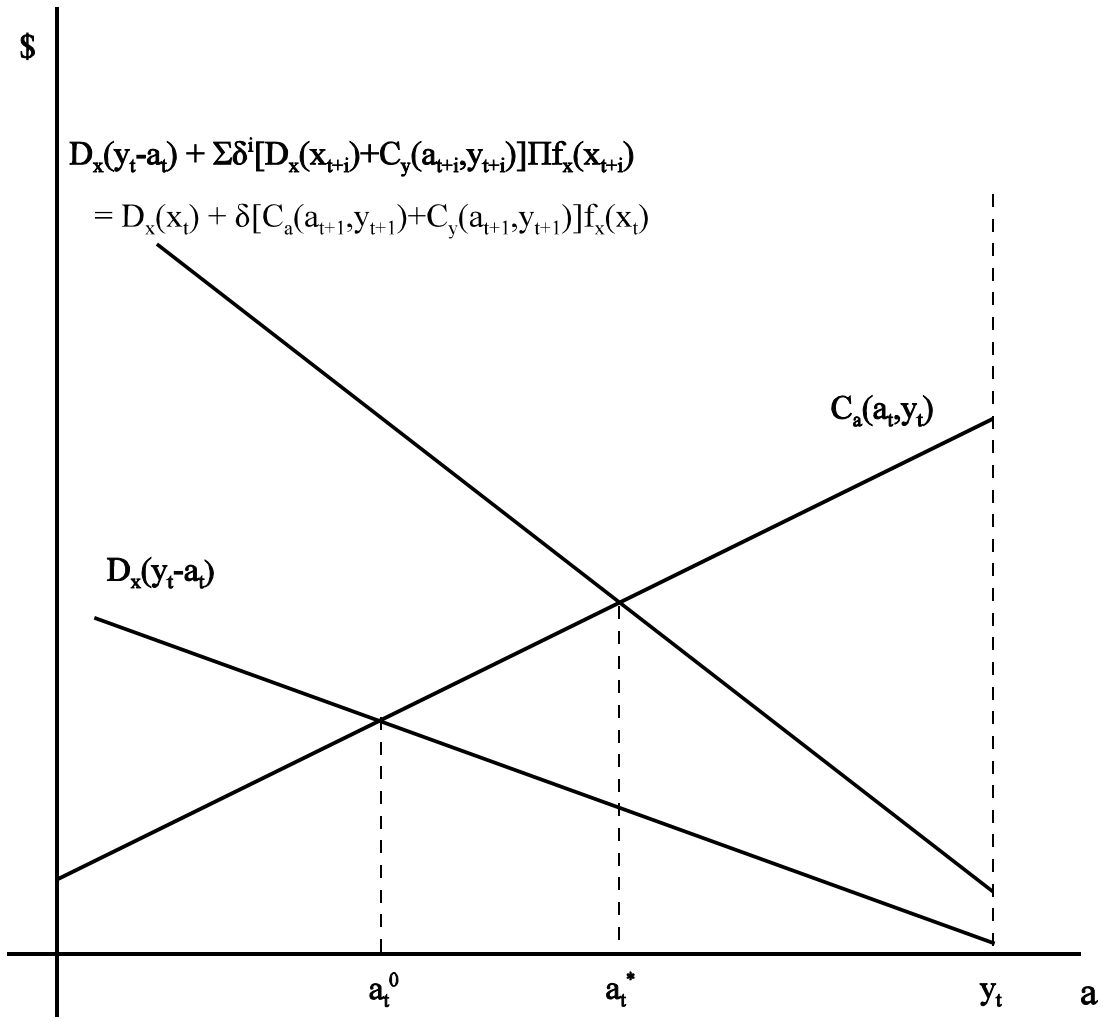
Corollary to Lemma 4c. If $0 < x_t < y_t$ and $0 < x_{t+1} < f(x_t)$ for all t then

$$C_a(a_t, y_t) = D_x(x_t) + \sum_{i=1}^{\infty} \delta^i \left[D_x(x_{t+i}) + C_y(a_{t+i}, y_{t+i}) \right] \prod_{j=0}^{i-1} f_x(x_{t+j}).$$

This has a simple interpretation when the costs of control are independent of the size of the invasion. For an interior policy the optimal control equates the marginal costs of control with the discounted sum of marginal damages over time multiplied by the compounded marginal growth of the

invasion. This is a simple cost-benefit criterion which balances the cost of removing a unit of the invasion against the discounted sum of current and future damages associated with that increment of the invasion. When the costs of control depend on the size of the invasion, the stream of future damages must be adjusted to account for the influence of the invasion size on future control costs.

The diagram below illustrates the tradeoffs involved in the dynamic cost-benefit analysis. Points along the horizontal axis represent feasible amounts of control, ranging between 0 and y_t . As control increases, so do marginal control costs. At the same time, more control lowers the current and future marginal damages caused by the invasion. The point a_t^0 represents the static optimum that equates marginal control costs with current marginal damages. The point a_t^* is the dynamic optimum that equates marginal control costs with current marginal damages plus the marginal effect of the invasion on future damages and costs. Given that it is less costly to reduce the size of the invasion to x from y than it is from $y' > y$ (see A3), the dynamic optimum always involves at least as much control as the static optimum and $a_t^* \geq a_t^0$.



3.b. The Convex Model

This subsection considers the modified dynamic optimization problem where the non-convex growth function f is replaced by its convex hull defined as follows:

$$F(x) = \begin{cases} f(x) & \text{for } x \in (0, k] \\ f(k) + f'_x(k)(x-k) & \text{for } x > k. \end{cases}$$

Note that F is continuously differentiable and satisfies A1-A4 and A5(a). This modified dynamic optimization problem is well defined and has a unique stationary optimal policy. Let $W(y)$ be the value function in the modified problem. That is, W solves the functional equation $W(y_t) = \text{Min } C(a_t, y_t) + D(x_t) + \delta W(F(x_t))$ subject to $y_t = x_t + a_t$ and $y_{t+1} = F(x_t)$. It can be shown that $W(y)$ is non-decreasing, continuous and strictly convex. Define $\eta(y)$ to be the optimal future size of the invasion from the current state x , i.e., $x_t = \eta(y_t)$, and let $\alpha(y) = y - \eta(y)$ be the optimal control in state y . The next lemma summarizes the basic properties of the optimal solution to the modified problem of controlling an invasion with a convex growth function.

Lemma 5. In the modified convex dynamic optimization problem where f is replaced by F :

- a.** the optimal value $W(y)$ is continuous, nondecreasing and convex in y , and the optimal policies $\eta(y)$ and $\alpha(y)$ are single-valued and continuous in y .
- b.** given y_0 , if the optimal program, $(y_t, x_t, a_t)_0^\infty$ in the modified problem is feasible in the original problem under the invasion growth function f , then it is the unique optimal program from y_0 in the original problem.
- c.** if $\alpha(y) > 0$, then $W(y)$ is differentiable and $W_y(y) = C_a(a, y) + C_y(a, y)$.
- d.** given y_0 , if $(y_t, x_t, a_t)_0^\infty$ is an optimal program then dynamic optimality conditions in Lemma 4 hold and (3.1) can be expressed as

$$W_y(y_t) = D_x(x_t) + C_y(a_t, y_t) + \delta W_y(F(x_t))F'_x(x_t). \quad (3.2)$$

e. if $(y_t, x_t, a_t)_{t=0}^{\infty}$ is a feasible program in the modified problem such that $\{x_t\}$ is bounded, $0 < x_t < F(x_t)$, and (3.1) holds for all $t \geq 0$, then $(y_t, x_t, a_t)_{t=0}^{\infty}$ is the optimal program from y_0 in the modified problem.

Lemma 5b provides conditions for the existence of an equivalence between an optimal program in the modified problem and an optimal program in the original problem. Since the modified problem is easier to solve, such an equivalence is useful in characterizing the optimal policy for the original problem. This is done in the next two sections.

4. The Economics of Eradication.

In this section, we consider the conditions under which it makes sense to eradicate an invasive species. The term eradication can have two meanings. In general it applies when the species is eradicated in the long run and the invasion is controlled in a manner that reduces its size to zero in the limit. It can also have a narrower meaning in cases where the species is fully exterminated in the current period. Eradication in the general sense includes both immediate eradication and the possibility that the species is fully eliminated within a finite number of periods.

Before beginning a formal analysis it is useful to think about the possibilities for eradication. First, whether eradication is optimal or not may depend on the size of the invasion. For example, the conventional wisdom from historical eradication efforts is that it is best to attempt eradication when an invasion is small. Eradication may prove too costly if an invasion becomes large. This was the rationale behind the recent, widely publicized effort by the state of Maryland to eradicate the snakehead fish from a pond near Baltimore. Less intuitive, but within the realm of possibilities allowed by our model, is an outcome where eradication is optimal for small invasions, but not for medium size invasions. If the invasion grows large enough eradication may become optimal once again. Such an outcome requires a nonmonotonic optimal policy.

We now characterize the economic and biological conditions under which eradication is optimal in the general sense where the size of invasion is reduced to zero in the long run.

Proposition 1. If there exists a $\zeta \in (0, k]$ such that the invasion is currently controlled from every $y \in (0, f(\zeta))$, and if

$$C_a(f(x) - x, f(x)) < D_x(x) + \delta f_x(x) C_a(f(x) - x, f(x)) \left[1 + \inf_{0 \leq a \leq f(x) - x} \frac{C_y(a, f(x))}{C_a(a, f(x))} \right]$$

for all $x \in (0, f^{-1}(\zeta)]$, then eradication is optimal from any invasion of size $y_0 \in (0, \zeta]$.

To interpret this result, first consider the static case where $\delta = 0$. From any small invasion, if the marginal marginal damages from the invasion exceed the cost of marginal cost of reducing the invasion size then it is optimal to reduce the size of the invasion. Since this holds for every invasion that is small, the optimal program eventually leads to eradication. In the dynamic case, the marginal costs of controlling the invasion are compared to the current and future marginal damages, adjusted for the effect of the invasion size on future control costs. The second term on the right hand side of the inequality in Proposition 1 is a lower bound on the effect that a reduction in the invasion today has on future damages and control costs. Hence, the condition implies that for any small invasion, it is less costly to reduce the size of the invasion than to incur the current and future damages should the invasion be allowed to grow. When discounted growth of all small invasions exceeds one then eradication is optimal even if marginal damages are low. The rationale is that if a fast growing invasion increases future damages and control costs more rapidly than the rate of discount then it makes sense not to allow the invasion to grow and to eradicate it when it is small.

The next lemma provides a condition under which it is economic for the current control to reduce the size of the invasion.

Lemma 6. Suppose that for any $y_0 = y > 0$, the invasion is currently controlled from $y' \in [y, f(y)]$

and for all $x \in (f^{-1}(y), y)$

$$C_a(y - x, y) < D_x(x) + \delta \left[\inf_{0 \leq a \leq f(x) - x} C_a(a, f(x)) + C_y(a, f(x)) \right] f_x(x).$$

Then every optimal program $(y_t, x_t, a_t)_{t=0}^{\infty}$ satisfies $y_1 = f(x_0) < y_0$.

The inequality in Lemma 6 is somewhat stronger than that in Proposition 1. This is because Lemma 6 applies to either small or large invasions whereas Proposition 1 applies only to small invasions whose size is in the region where growth in the invasion is convex. Our next result combines the two sets of conditions and provides a criterion for eradication to be optimal from an invasion of any size.

Proposition 2. Assume that the invasion is globally controlled. Further suppose that

$$(i) \ C_a(f(x) - x, f(x)) < D_x(x) + \delta f_x(x) C_a(f(x) - x, f(x)) \left[1 + \inf_{0 \leq a \leq f(x) - x} \frac{C_y(a, f(x))}{C_a(a, f(x))} \right]$$

for all $x \in (0, f^{-1}(k)]$, and

$$(ii) \ C_a(y - x, y) < D_x(x) + \delta \left[\inf_{0 \leq a \leq f(x) - x} C_a(a, f(x)) + C_y(a, f(x)) \right] f_x(x)$$

for every $y \in (k, K]$ and for all $x \in (f^{-1}(y), y)$. Then, eradication is globally optimal.

Proposition 2 applies to invasions of any size. The stronger requirement in part (ii) only needs to hold for invasions larger than k . This is precisely the region where it is more likely to hold due to the fact that marginal damages are increasing in x . When the optimal policy for controlling the invasion is monotone, the conditions for global eradication can be simplified.

Proposition 3. If $C_{aa}(a, y) + C_{ay}(a, y) \geq 0$ on Ω and if

$$C_a(f(x) - x, f(x)) < D_x(x) + \delta [C_a(f(x) - x, f(x)) + C_y(f(x) - x, f(x))] f_x(x)$$

for all $x \in (0, K]$, then every optimal program converges to zero and eradication is globally optimal.

Next, we characterize the circumstances under which immediate eradication is optimal, i.e., where the invasion is fully eradicated in the current period.

Proposition 4. a. Suppose that for some $y \in (0, k]$, the invasion is currently controlled from every $y_0 \in (0, f(y))$ and that

$$C_a(y, y) < D_x(0) + \delta \left[\inf_{0 \leq a \leq f(x)-x, 0 \leq x \leq y} C_a(a, f(x)) + C_y(a, f(x)) \right] f_x(0).$$

Then, immediate eradication is optimal from y .

b. Suppose that for some $y \in (k, K]$, the invasion is currently controlled from every $y_0 \in (0, f(y))$ and that

$$C_a(y, y) < D_x(0) + \delta \left[\inf_{0 \leq a \leq f(x)-x, 0 \leq x \leq y} C_a(a, f(x)) + C_y(a, f(x)) \right] \min\{f_x(0), f_x(K)\}.$$

Then, immediate eradication is optimal from y .

The criterion for immediate eradication balances the costs of removing the last unit of the invasion against the current and future damages that would be caused should the invasion be allowed to remain. The second term on the right hand side of the inequalities is a lower bound on the future damages associated with an arbitrarily small invasion.

These results on the economics of eradication lead to the following insights. First, eradication is more likely to be an optimal policy for invasions that have a higher discounted growth rate than it is for invasions that grow slowly. This is because the benefits from control today are higher when an invasion expands rapidly. In addition, the benefits from control today are magnified further into the future when the discount rate is lower. Second, for some invasions economic considerations may favor eradication when the invasion is small, but not when the invasion is large. Finally, in the special case where the marginal costs of control at $a=0$ are insignificant, the criteria for eradication in Proposition 2, part ii, and in Proposition 4 essentially involve static benefit cost considerations. This is a consequence of the fact that $C_a(0, y) = 0$ implies $\inf_{0 \leq a \leq f(x)-x} C_a(a, f(x)) +$

$C_y(a, f(x)) = 0$.⁴ Hence, the lower bound on future marginal social costs is relatively weak when marginal control costs are insignificant.

5. The Economics of Noneradication.

In this section we characterize the economic and biological environments under which it is optimal for society to allow the invasion to be sustained. Under these conditions eradication is incomplete.

Proposition 5. a. If $D_x(0) + \delta C_a(0,0)f_x(0) < C_a(y,y)$ then $X(y) > 0$.

b. If $C_{aa}(a,y) + C_{ay}(a,y) \geq 0$ on Ω and $D_x(0) < C_a(0,0)[1 - \delta f_x(0)]$ then $X(y) > 0$ for all y and, in addition, for all y sufficiently close to zero $A(y) = 0$ and $X(y) = y$.

c. If $C_{aa}(a,y) + C_{ay}(a,y) < 0$ on Ω and $D_x(y) + \delta \sup_{0 \leq x \leq y} C_a(0, f(x))f_x(x) < C_a(0,y)$ then $X(y) = y$.

Part a of the proposition rules out immediate eradication as an optimal strategy. If the damages from an arbitrarily small invasion are less than the marginal costs of removing the entire invasion, then it is always optimal to allow some of the invasion to remain. The second part of the proposition is the counterpart of Proposition 1. For an arbitrarily small invasion, if the damages compounded indefinitely at the discounted expected intrinsic growth rate are less than the marginal costs of eradicating the invasion then the optimal policy is not to control the invasion at all when it is sufficiently small. This in turn, implies that eventual eradication is not an optimal strategy from an invasion of any size.

⁴Recall that $C_a(a,y) + C_y(a,y) \geq 0$.

Appendix.

Proof of Lemma 1. The cost functions C and D are bounded continuous functions on their relevant domains. Define the operator $\Gamma V(y) = \inf C(a,y) + D(x) + \delta V(f(x))$ subject to $y = x + a$. By the contraction mapping theorem Γ maps the set of bounded continuous functions into itself. Hence, V is continuous. We now show that V maps non-decreasing functions into non-decreasing functions.

Suppose V is non-decreasing. Let x and x' be optimal from y and y' , respectively where $y < y'$.

Suppose $x' < y$. Then x' is feasible from y and $\Gamma V(y) = C(y-x,y) + D(x) + \delta V(f(x)) \leq C(y-x',y) + D(x') + \delta V(f(x')) \leq C(y'-x',y') + D(x') + \delta V(f(x')) = \Gamma V(y')$, where the first inequality is due to optimality while the second is due to B3. Now suppose that $x' \geq y$. Then $\Gamma V(y) = C(y-x,y) + D(x) + \delta V(f(x)) \leq C(0,y) + D(y) + \delta V(f(y)) = C(0,y') + D(y) + \delta V(f(y)) \leq C(y'-x',y') + D(x') + \delta V(f(x'))$. The first inequality is due to optimality, the equality follows from B2, and the second inequality is implied by B3 and the fact that V is nondecreasing. ■

Proof of Lemma 2. a. Let $x \in X(y)$ and $x' \in X(y')$ for $y \leq y'$. Then $\max[x,x'] \leq y'$ and $\min[x,x'] \leq y$.

Note that $D(\max[x,x']) + D(\min[x,x']) = D(x) + D(x')$, and likewise $V(f(\max[x,x'])) + V(f(\min[x,x'])) = V(f(x)) + V(f(x'))$. Since $C_{aa}(a,y) + C_{ay}(a,y) \geq 0$ it follows that $C(y-x,y) - C(y-\min[x,x'],y) \geq C(y'-\max[x,x'],y') - C(y'-x',y')$. Hence, $0 \geq C(y-x,y) + D(x) + \delta V(f(x)) - [C(y-\min[x,x'],y) + D(\min[x,x']) + \delta V(f(\min[x,x']))] \geq C(y'-\max[x,x'],y') + D(\max[x,x']) + \delta V(f(\max[x,x'])) - [C(y'-x',y') + D(x') + \delta V(f(x'))] \geq 0$, where the first and last inequalities follow from the principle of optimality.

This sequence of inequalities implies that $\min[x,x'] \in X(y)$ and $\max[x,x'] \in X(y')$.

b. The argument is similar to the proof of part a. Let $x \in X(y)$ and $x' \in X(y') < y'$ for $y \leq y'$. We want to show that $\max[x,x'] \in X(y)$ and $\min[x,x'] \in X(y')$. This follows immediately if $x \geq x'$ so suppose that $x < x'$. Since $x' < y'$ it must be the case that $x' \leq y$ for all y sufficiently close to y' . The assumption $C_{aa}(a,y) + C_{ay}(a,y) \leq 0$ implies $C(y-x,y) + C(y'-x',y') \geq C(y'-x,y) + C(y-x',y)$. This in turn yields $0 \geq C(y-x,y) + D(x) + \delta V(f(x)) - [C(y-x',y) + D(x') + \delta V(f(x'))] \geq C(y'-x,y) + D(x) + \delta V(f(x)) - [C(y'-x',y') + D(x') + \delta V(f(x'))] \geq 0$, where the first and last inequalities follow from optimality. This

sequence of inequalities implies that $x' \in X(y)$ and $x \in X(y')$. Hence $\max[x, x'] \in X(y)$ and $\min[x, x'] \in X(y')$. The assumption that $X(y') < y'$ is necessary to insure that x' is feasible from $y \in N(y')$. ■

Proof of Lemma 3. a. Without loss of generality we can take $n = 0$. Let $\{x_t, a_t\}$, $t = 0, \dots$ be optimal from $y_0 = y$. We first show that there exists some $t \geq 0$ for which $a_t > 0$. If not, then $a_t = 0$ for all t and $x_t = y_t = f^t(y)$. It follows that for all $y \in (0, K)$, $f^t(y) \in (0, K)$, $f^t(y) > y$, and $f_x^t(y) \geq f_x(K)^t$ for all $t \geq 1$. Consider an alternative sequence $\{\hat{x}_t, \hat{a}_t\}$ where $\hat{a}_0 = \varepsilon$ and $\hat{a}_t = 0$ for all $t \geq 1$. Then $\hat{x}_t = \hat{y}_t = f^t(y - \varepsilon)$ for all $t \geq 1$. As $\{x_t, a_t\}$ is optimal,

$$\begin{aligned} 0 &\geq \left[\sum_{t=0}^{\infty} \delta^t \{C(0, f^t(y)) + D(f^t(y))\} \right] - \left[C(\varepsilon, y) + D(y - \varepsilon) + \sum_{t=1}^{\infty} \delta^t \{C(0, f^t(y - \varepsilon)) + D(f^t(y - \varepsilon))\} \right] \\ &= C(0, y) - C(\varepsilon, y) + \sum_{t=1}^{\infty} \delta^t [C(0, f^t(y)) - C(0, f^t(y - \varepsilon))] + \sum_{t=0}^{\infty} \delta^t [D(f^t(y)) - D(f^t(y - \varepsilon))]. \end{aligned}$$

Dividing by ε and taking the limit as $\varepsilon \rightarrow 0$ implies:

$$C_a(0, y) \geq D_x(y) + \sum_{t=1}^{\infty} \delta^t [C_y(0, f^t(y)) + D_x(f^t(y))] f_x^t(y).$$

Since $C_y(0, f^t(y)) = 0$ this contradicts the condition of the proposition. Thus, there exists some t for which $a_t > 0$.

b. Since $\inf_a C_a(a, f(y)) + C_y(a, f(y)) \leq C_a(0, f(y)) + C_y(0, f(y)) = C_a(0, f(y))$ it follows from the condition in part b that $C_a(0, y) < D_x(y) + \delta C_a(0, f(y)) f_x(y)$ for all y . Since the condition in part b holds for all y , $C_a(0, f(y)) < D_x(f(y)) + \delta [\inf_a \{C_a(a, f^2(y)) + C_y(a, f^2(y))\}] f_x(f(y))$. Substituting this in the previous inequality yields $C_a(0, y) < D_x(y) + \delta [D_x(f(y)) + \delta [\inf_a \{C_a(a, f^2(y)) + C_y(a, f^2(y))\}] f_x(f(y))] f_x(y)$.

Iterating forward and repeating a similar substitution yields $C_a(0, y) < D_x(y) +$

$\sum_{t=1}^{\infty} \delta^t [C_y(0, f^t(y)) + D_x(f^t(y))] f_x^t(y)$, which is the condition in part a. Hence, the condition in part b implies part a and the invasion is a controlled invasion from y .

Let T be the first t such that $a_t > 0$ and suppose that $T > 0$. Then $a_t = 0$ for $t = 0, \dots, T-1$, while $0 < a_T \leq y_T = f(x_{T-1}) = f^T(y)$ and $x_T = y_T - a_T = f^T(y) - a_T$. Since $a_T > 0$ there exists an $\varepsilon > 0$ such that

$f(y_{T-1}) - f(y_{T-1} - \varepsilon) < a_T$. Consider the alternative feasible sequence $\hat{a}_{T-1} = \varepsilon$, $\hat{a}_T = a_T - [f(y_{T-1}) - f(y_{T-1}-\varepsilon)]$, and $\hat{a}_t = a_t$ for all other t . Then $\hat{x}_{T-1} = y_{T-1} - \varepsilon$ and $\hat{x}_t = x_t$ for $t \neq T-1$. Since the sequence $\{x_t, a_t\}$ is optimal,

$$\begin{aligned} 0 &\geq \sum_{t=0}^{\infty} \delta^t [C(a_t, y_t) + D(x_t) - C(\hat{a}_t, \hat{y}_t) - D(\hat{x}_t)] \\ &= \delta^{T-1} [C(0, y_{T-1}) + D(y_{T-1}) - C(\varepsilon, y_{T-1}) - D(y_{T-1}-\varepsilon)] \\ &\quad + \delta^T [C(a_T, f(y_{T-1})) + D(x_T) - C(a_T + f(y_{T-1}-\varepsilon) - f(y_{T-1}), f(y_{T-1}-\varepsilon)) - D(x_T)] \end{aligned}$$

Dividing by ε and letting $\varepsilon \rightarrow 0$ implies

$$0 \geq -C_a(0, f^{T-1}(y)) + D_x(f^{T-1}(y)) + \delta [C_a(a_T, f^T(y)) + C_y(a_T, f^T(y))] f_x(f^{T-1}(y))$$

Since this contradicts the condition in the proposition it must be that $T = 0$ and $A(y) > 0$ for all y . ■

Proof of Lemma 4. For purposes of exposition the proof is divided into a sequence of subsidiary lemmas. Define the lower, right and left Dini derivatives of V at y by

$$D_+ V(y) = \liminf_{\varepsilon \downarrow 0} \frac{V(y+\varepsilon) - V(y)}{\varepsilon} \quad \text{and} \quad D_- V(y) = \liminf_{\varepsilon \downarrow 0} \frac{V(y) - V(y-\varepsilon)}{\varepsilon}.$$

Lemma 4.1. $-C_a(y_0 - x_0, y_0) + D_x(x_0) + \delta D_- V(f(x_0)) \leq 0$.

Proof. Since $x_0 - \varepsilon$ is feasible from y_0 , the principle of optimality implies $C(y_0 - x_0 + \varepsilon, y_0) + D(x_0 - \varepsilon) + \delta V(f(x_0 - \varepsilon)) - C(y_0 - x_0, y_0) - D(x_0) - \delta V(f(x_0)) \geq 0$. Dividing by ε and taking $\liminf_{\varepsilon \downarrow 0}$ establishes the result. ■

Lemma 4.2. If $x_0 < y_0$ then $-C_a(y_0 - x_0, y_0) + D_x(x_0) + \delta D_+ V(f(x_0)) \geq 0$.

Proof. If $x_0 < y_0$ then $x_0 + \varepsilon$ is feasible from y_0 for sufficiently small ε . By the principle of optimality it follows that $\liminf_{\varepsilon \downarrow 0} [C(y_0 - x_0 - \varepsilon, y_0) + D(x_0 + \varepsilon) + \delta V(f(x_0 + \varepsilon)) - C(y_0 - x_0, y_0) - D(x_0) - \delta V(f(x_0))]/\varepsilon \geq 0$.

The result follows immediately. ■

Lemma 4.3. $D_+V(f(x_0)) \leq [C_a(a_1, y_1) + C_y(a_1, y_1)]f_x(x_0)$.

Proof. Since x_1 is feasible from $f(x_0 + \epsilon)$ the principle of optimality implies $V(f(x_0 + \epsilon)) - V(f(x_0)) \leq C(f(x_0 + \epsilon) - x_1, f(x_0 + \epsilon)) + D(x_1) + \delta V(f(x_1)) - C(f(x_0) - x_1, f(x_0)) - D(x_1) - \delta V(f(x_1))$. Dividing by ϵ and taking the $\liminf_{\epsilon \downarrow 0}$ on both sides and simplifying completes the proof. ■

Lemma 4.4. If $a_1 > 0$ then $D_-V(f(x_0)) \geq [C_a(a_1, y_1) + C_y(a_1, y_1)]f_x(x_0)$.

Proof. Since $a_1 > 0$, x_1 is feasible from $f(x_0 - \epsilon)$ for sufficiently small ϵ . By the principle of optimality it follows that $V(f(x_0)) - V(f(x_0 - \epsilon)) \geq C(f(x_0) - x_1, f(x_0)) + D(x_1) + \delta V(f(x_1)) - C(f(x_0 - \epsilon) - x_1, f(x_0 - \epsilon)) - D(x_1) - \delta V(f(x_1))$. Once again, dividing by ϵ and taking the $\liminf_{\epsilon \downarrow 0}$ on both sides and simplifying completes the proof. ■

The proof of part a of Lemma 4 follows from Lemmas 4.2 and 4.3 while combining Lemmas 4.1 and 4.4 yields part b. Part c is a joint implication of all four lemmas. ■

Proof of Lemma 5. a. As F is convex it is not bounded and the modified problem is a classical negative⁵ dynamic programming problem, a time stationary dynamic cost minimization problem with costs bounded from below [Strauch, 1966]. The existence of a stationary optimal policy follows from standard arguments in this literature (e.g. Schäl [1975]). As in the proof of Lemma 1 define the operator $\Gamma W(y) = \inf C(a, y) + D(x) + \delta W(F(x))$ subject to $y = x + a$. The proof that W is nondecreasing follows identical arguments to those in the proof of Lemma 1. That Γ maps the set of convex functions into itself follows from the convexity of C , D , and F , and the fact that $\{x \mid 0 \leq x \leq y\}$ is convex. The fact that an optimal policy is single-valued and continuous follows from the convexity of the cost and transition functions and the strict convexity in assumption B4.

⁵The term “negative” comes from the equivalent problem of maximizing a sum of negative rewards over time.

- b. The proof follows from the fact that the from any initial state, the set of feasible paths in the modified optimization problem includes all feasible paths in the original problem.
- c. Let (x,a) be optimal from y in the modified, convex problem. Then x is feasible from $y+\epsilon$ for $\epsilon>0$. By the principle of optimality, $W_{y^+}(y) = \lim_{\epsilon \downarrow 0} [W(y+\epsilon) - W(y)]/\epsilon \leq \lim_{\epsilon \downarrow 0} [C(y+\epsilon-x, y+\epsilon) - C(y-x, y)]/\epsilon = C_a(a, y) + C_y(a, y)$. As $a > 0$, x is feasible from $y-\epsilon$ for ϵ sufficiently close to zero. It follows that $W_{y^-}(y) = \lim_{\epsilon \downarrow 0} [W(y) - W(y-\epsilon)]/\epsilon \geq [C(y-x, y) - C(y-\epsilon-x, y-\epsilon)]/\epsilon = C_a(a, y) + C_y(a, y)$. Since W is convex $W_{y^+}(y) \geq W_{y^-}(y)$. Hence W is differentiable at y and $W_y(y) = C_a(a, y) + C_y(a, y)$.
- d. Since W is differentiable the first order necessary conditions can be expressed as $C_a(a_t, y_t) = D_x(x_t) + \delta W_y(F(x_t))F_x(x_t)$. Adding $C_y(a_t, y_t)$ to each side and then substituting $W_y(y_t)$ for the left hand side gives the result.
- e. The proof follows from the fact that the modified problem is convex and the size of the invasion is bounded.

Proof of Proposition 1. Consider invasions of initial size lying in $(0, \zeta]$ and the modified dynamic optimization problem where we replace f by its convex hull F . Note that f coincides with F on $[0, \zeta]$. Recall that from any invasion of size y , there is a unique optimal invasion size in the next period given by $F(\eta(y))$. We claim that $F(\eta(y)) < y$ for all $y \in (0, \zeta]$ so that in this modified dynamic optimization problem, the unique optimal program from $y_0 \in (0, \zeta]$ lies below η and converges to zero. To see this, suppose there exists $y \in (0, \zeta]$ such that $F(\eta(y)) \geq y$. Let $x \in (0, f^1(\zeta)]$ be defined by $y = f(x) = F(x)$. Then, $\eta(y) \geq x > 0$ and $f(x) - \eta(y) \leq f(x) - x$. Note that $F(\eta(y)) \leq f(y) \leq f(\zeta)$, so that the invasion is currently controlled from both y and $F(\eta(y))$. Lemma 5 implies $C_a(f(x) - \eta(y), f(x)) = D_x(\eta(y)) + \delta W_y(F(\eta(y)))F_x(\eta(y))$. The convexity of F , D , and W and the fact that $k \geq \eta(y) \geq x$ and $F(\eta(y)) \geq y$ implies

$$\begin{aligned} C_a(f(x) - \eta(y), f(x)) &\geq D_x(x) + \delta W_y(y)F_x(x) = D_x(x) + \delta W_y(f(x))f_x(x) \\ &= D_x(x) + \delta f_x(x)[C_a(f(x) - \eta(y), f(x)) + C_y(f(x) - \eta(y), f(x))]. \end{aligned}$$

This yields

$$1 \geq \frac{D_x(x)}{C_a(f(x) - \eta(y), f(x))} + \delta f_x(x) \left[1 + \frac{C_y(f(x) - \eta(y), f(x))}{C_a(f(x) - \eta(y), f(x))} \right]$$

$$\geq \frac{D_x(x)}{C_a(f(x) - x, f(x))} + \delta f_x(x) \left[1 + \inf_{0 \leq a \leq f(x) - x} \frac{C_y(f(x) - \eta(y), f(x))}{C_a(f(x) - \eta(y), f(x))} \right]$$

a contradiction. Thus, in the modified dynamic optimization problem, every optimal path from $y_0 \in (0, \zeta]$ lies below η and converges to zero.

Every feasible path in the original optimization problem is also feasible in the modified problem. Since the unique optimal path from any initial state $y_0 \in (0, \zeta]$ in the modified problem is also feasible in the original problem, it must be the unique optimal path from every such y_0 in the original problem. ■

Proof of Lemma 6. Suppose not. Then there exists an optimal program $(y_t, x_t, a_t)_0^\infty$ where $y_1 = f(x_0) \geq y$, i.e., $x_0 \geq f^{-1}(y)$. Since, $y_1 \in [y, f(y)]$, it follows that $x_1 < y_1$. Therefore, using Lemma 4, we have

$$C_a(y - x_0, y) \geq D_x(x_0) + \delta [C_a(f(x_0) - x_1, f(x_1)) + C_y(f(x_0) - x_1, f(x_1))] f_x(x_0)$$

which violates the inequality in the statement of the lemma. ■

Proof of Proposition 2. The proof follows from Proposition 1 and Lemma 6. ■

Proof of Proposition 3. Suppose not. Then there is an optimal program bounded away from zero that converges to a strictly positive optimal steady state $y^* = f(x^*)$. At this optimal steady state there is an optimal program where the size of the invasion is managed at a constant level every period. Note that $y_0 \in (0, K)$ implies that every optimal program is bounded above by K so that x^* and y^* lie in $[0, K]$. If $x^* \in (0, K)$ then equation (3.1) implies $C_a(f(x^*) - x^*, f(x^*)) = D_x(x^*) +$

$\delta [C_a(f(x^*) - x^*, f(x^*)) + C_y(f(x^*) - x^*, f(x^*))] f_x(x^*)$ which contradicts the inequality in the proposition.

Also, $y^* = K = f(K)$ is not an optimal steady state as the inequality in the proposition implies $C_a(0, K) < D_x(K) + \delta [C_a(0, K) + C_y(0, K)] f_x(K) = D_x(K) + \delta C_a(0, K) f_x(K)$ which implies $C_a(0, K) < [D_x(K) / (1 -$

$\delta f_x(K)]$. The latter can be used to show that a program where the control is infinitesimal but greater than zero in period 0 and equal to zero every period thereafter dominates a program where $a_t=0$ in every period. Hence, K is not an optimal steady state. Thus, it must be the case that every optimal path converges to 0. ■

Proof of Proposition 4. a. Suppose not. Then there exists an optimal program $(y_t, x_t, a_t)_0^\infty$, $y_0 = y$, where $x_0 > 0$ so that $y_1 = f(x_0) > 0$. Since the invasion is currently controlled from every $y_0 \in (0, f(y))$, it follows from Lemma 3 that $C_a(y-x_0, y) \geq D_x(x_0) + \delta[C_a(f(x_0)-x_1, f(x_0))+C_y(f(x_0)-x_1, f(x_0))]f_x(x_0)$.

Hence, the convexity of C and D imply

$$\begin{aligned} C_a(y, y) &\geq D_x(0) + \delta[C_a(f(x_0)-x_1, f(x_0))+C_y(f(x_0)-x_1, f(x_0))]f_x(x_0) \\ &\geq D_x(0) + \delta[C_a(f(x_0)-x_1, f(x_0))+C_y(f(x_0)-x_1, f(x_0))]f_x(0) \end{aligned}$$

where the last inequality follows from the fact that $x_0 \in (0, k)$. This contradicts the inequality in the antecedent of the proposition. The proof of part b is almost identical except that in the last inequality $\min\{f_x(0), f_x(K)\}$ acts as a lower bound on $f_x(x_0)$. ■

Proof of Proposition 5. a. Suppose $X(y) = \{0\}$. Consider the alternative of increasing the remaining invasion to ε and then eradicating it in the following period. By the principle of optimality

$$C(y, y) + D(0) + \delta[C(0, 0) + D(0)] + \delta^2 V(0) \leq C(y - \varepsilon, y) + D(\varepsilon) + \delta C(f(\varepsilon), f(\varepsilon)) + D(0) + \delta^2 V(0). \text{ This implies}$$

$$C_a(y, y) \leq D_x(0) + \delta[C_a(0, 0) + C_y(0, 0)]f_x(0) = D_x(0) + \delta C_a(0, 0)f_x(0), \text{ where the equality follows from}$$

B3. This is a contradiction to the condition in X_a .

b. $C_{aa} + C_{ay} \geq 0$ implies $C_a(0, 0) \leq C_a(y, y)$ for all y . Hence the condition in X_a holds for all y and $X(y) > 0$ for all y . To prove the second part we want to show that there exists a ξ sufficiently close to zero, such that $X(y) = y$ for all $y \in (0, \xi)$. Let $x \in X(y)$ and suppose that $x < y$. By Lemma 4a

$$\begin{aligned} 0 &\leq -C_a(y-x, y) + D_x(x) + \delta[C_a(A(f(x)), f(x)) + C_y(A(f(x)), f(x))]f_x(x) \\ &\leq -C_a(0, x) + D_x(x) + \delta[C_a(f(x), f(x)) + C_y(f(x), f(x))]f_x(x). \quad (\text{using } C_{aa} + C_{ay} \geq 0 \text{ twice}) \end{aligned}$$

Define $H(x) \equiv -C_a(0,x) + D_x(x) + \delta[C_a(f(x),f(x))+C_y(f(x),f(x))]f_x(x) \geq 0$. The condition in Xb implies $H(0) < 0$ and by the continuity of H one can pick an ξ sufficiently close to 0 such that $H(x) < 0$. This yields a contradiction.

c. The second condition in Xc implies $0 > -C_a(0,y) + D_x(y) + \delta \sup_{0 \leq x \leq y} [C_a(0,f(x))+C_y(0,f(x))]f_x(x) \geq -C_a(y-x,y) + D_x(x) + \delta[C_a(A(f(x)),f(x))+C_y(A(f(x),f(x))]f_x(x)$ where the last inequality is due to the convexity of C and D, and $C_{aa}(a,y) + C_{ay}(a,y) < 0$. This is the contrapositive of Proposition 2a. ■

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