Representative Consumer's Risk Aversion and Efficient Risk-Sharing Rules

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Abstract

We investigate the representative consumer's risk attitude and efficient risk-sharing rules in a single-period, single-good economy in which consumers have homogeneous probabilistic belief but heterogeneous risk attitudes. If all consumers exhibit non-increasing relative risk aversion, then so does the representative consumer. If, moreover, the individual consumers' cautiousness, defined as the derivative of the reciprocal of the Arrow-Pratt measure of absolute risk aversion, are unequal, then the representative consumer exhibits strictly decreasing relative risk aversion, ranging from the most risk averse individual consumer's counterpart to the least risk averse consumer's as the aggregate consumption level increases. When every consumer's risk tolerance is linear, much sharper results on the curvature of the risk-sharing rules can be obtained. Extensions of these results to multi-period economies, implications on asset pricing and portfolio insurance, consequences of background risks, and numerical examples are also presented.

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1 Introduction

We consider an exchange economy under uncertainty with a single good and a single consumption period, in which all consumers hold common probability assessments over the state space and yet differing expected utility functions. Two things are well known for Pareto efficient allocations in such an economy. First, every consumer's consumption level is uniquely determined by the aggregate consumption level. Hence every consumer's state-contingent consumption levels can be specified as a function, called the *risk sharing rule*, of aggregate consumption levels. Second, there exists a *representative consumer*, in the sense that the support price of the single-consumer economy consisting solely of the representative consumer is also the support price for the Pareto efficient allocation of the original, multi-consumer economy. Hence, knowing the representative consumer's risk attitude is sufficient to price all assets in financial markets.

The benchmark result on this subject matter is the *mutual fund theorem*: Define *risk tolerance* as the reciprocal of the Arrow-Pratt measure of absolute risk aversion, and call its first derivative *cautiousness*. If all consumers have the same, constant cautiousness, then the representative consumer also has the same constant cautiousness and all individuals' risk-sharing rules are linear (or, to be more precise, affine). In this paper, we drop the assumption of a constant, common cautiousness and analyze the implication of heterogeneous cautiousness on the risk-sharing rule and the representative consumer's risk attitude. As can be inferred from existing results dispersed in the wide range of literature, the mutual fund theorem would not hold without the assumption of a constant, common cautiousness. The contribution of this paper is, in short, to provide a detailed description of the way in which the representative consumer's cautiousness is *not* constant and the risk-sharing rules are *not* linear in this environment. Some of our results require specific functional forms, such as log or power functions for expected utility functions, but others do not.

We should stress that the investigation of the way in which the mutual fund theorem fails to hold is no less important than the theorem itself. Also, just like the theorem, we establish qualitative properties on the representative consumer's risk attitude and the risk-sharing rules, which are true regardless of the choice of Pareto efficient allocations, or, equivalently, of utility weights in the maximization problem characterizing the efficient allocations. If financial markets are complete, then the equilibrium allocations are Pareto efficient, and hence this is equivalent to saying that the qualitative properties we establish are true for all specifications of initial wealth distributions.

In the rest of this introduction, we list our results and explain how they clarify, generalize, and refine the preceding results.

1.1 Representative Consumer's Risk Attitude

We obtain two results on the representative consumer's risk attitude, one local and the other global. The local result (Theorem 19) is that if every individual consumer exhibits non-increasing relative risk aversion, then so does the representative consumer. If, moreover, there is at least one consumer who exhibits decreasing relative risk aversion or there are two consumers whose cautiousness differ from each other, then the representative consumer exhibits strictly decreasing relative risk aversion. In short, even the slightest heterogeneity in individual consumers' risk attitudes destroys the constancy of the representative consumer's relative risk aversion. The global result (Proposition 24) is that the representative consumer's relative risk aversion converges to the relative risk aversion of the least risk averse individual in the economy as the aggregate consumption level diverges to infinity; and it converges to that of the most risk averse individual as the aggregate consumption level converges to zero.

Combining these two results, we can say that the representative consumer exhibits strictly decreasing relative risk aversion, ranging from the most risk averse individual consumer's counterpart to the least risk averse consumer's as the aggregate consumption level increases. This result generalizes those obtained by Benninga and Mayshar (2000), who assumed that all consumers have the same discount rate and constant but differing constant coefficients of relative risk aversion in a two-period economy, and by Wang (1998), who considered a continuous-time economy with two consumers which have the same properties as assumed by Benninga and Mayshar (2000). Our result is particularly relevant to the equity premium puzzle of Mehra and Prescott (1985), who tried to reconcile the U.S. empirical data with the representative consumer model where the representative consumer exhibits constant relative risk aversion. Often the reason given for adopting this assumption is that it seems a moderately reasonable assumption for individual consumers. The above result, however, demonstrates that the assumption is not at all reasonable for the representative consumer, since the aggregation tends to induce the representative consumer to exhibit strictly decreasing relative risk aversion given the heterogeneity in the individual consumers' risk attitudes.

In terms of the organization of the paper, we first establish results on the representative consumer's cautiousness (Theorem 7 and Propositions 10 to 13), and the results referred to above (Theorem 19 and Proposition 24) are derived from them. We find that the representative consumer's cautiousness is much easier to work with than his relative risk aversion. In particular, Theorem 7 shows that if every consumer exhibits non-decreasing cautiousness (which is equivalent to convex risk tolerance), then so does the representative consumer; and heterogeneity leads to strictly increasing cautiousness (which implies strictly convex risk tolerance).

1.2 Asset Pricing

We also investigate what kind of mis-pricing of derivative assets would occur if we erroneously assumed that the representative consumer exhibits constant relative risk aversion or linear risk tolerance, when he in fact exhibits strictly decreasing relative risk aversion or strictly convex risk tolerance as a result of heterogeneity in consumer's risk attitude. We concentrate on derivative assets of the aggregate endowment of the economy, whose state-contingent payoffs are determined by the realization of the aggregate endowment. We show (Proposition 21) that the approximation by linear risk tolerance would underestimate the price, relative to the risk-free bond, of these derivative assets whose payoffs are increasing functions of the aggregate endowment, such as call options but not put options; and (Proposition 9) that the approximation by constant relative risk aversion would underestimate the price, relative to the risk-free bond, of those derivative assets which pay off only when the aggregate endowment turns out to be very high or very low, such as call options with very high exercise prices and put options with very low exercise prices. The approximation of the second type was considered by Benninga and Mayshar (2000) for the case where all consumers exhibit constant relative risk aversion and the aggregate endowment is log-normally distributed. The approximation of the first type is interesting because linear risk tolerance is a weaker requirement than constant relative risk aversion, and hence the approximation of the first type is superior to that of the second type; and yet the two have opposing directions of mis-pricing for put options.

1.3 Risk-Sharing Rules

The crucial result, which builds on results of Wilson (1968), for analyzing the shape of risk sharing rules is Proposition 6, which relates the curvature of an individual consumer's risk-sharing rule to how the individual's cautiousness compares to the cautiousness of the representative consumer. Indeed a risk-sharing rule is locally convex, concave, or linear if and only if the individual's cautiousness is locally greater than, smaller than, or equal to the representative consumer's cautiousness. The result also allows us to rank the curvature of the individual consumers' risk-sharing rules according to their cautiousness.

The behavior of the risk-sharing rules as the aggregate consumption level diverges to infinity or converges to zero (or, more generally, the minimum subsistence level) is described by Propositions 10 through 13. Roughly, the results state that as the aggregate consumption level diverges to infinity, the most cautious consumers' share of consumption as well as their marginal increment in consumption converge to one; and that as the aggregate consumption level converges to zero, the same is true for the least cautious consumers. Hence the distribution of the individual consumers' consumption levels are more biased when the realization of the aggregate endowment is very large or very small than when it is of a modest value.

Much stronger results can be obtained when all individual consumers exhibit linear risk tolerance, or, equivalently, constant cautiousness, and the constants differ across them. We show (Theorem 16) that an individual consumer's risk-sharing rule can be only of three types, depending on the individual's cautiousness. Each least cautious consumer has an everywhere strictly concave risk-sharing rule. Each most cautious consumer has an everywhere strictly convex risk-sharing rule. Any of the other consumers has a risk-sharing rule that is initially convex up to a unique inflection point and concave thereafter. Such a risk-sharing rule looks as in Figure 1 but no risk-sharing rule in this economy can be as in Figure 2, initially concave and eventually convex.

The results for the most and least cautious consumer have been established by Leland

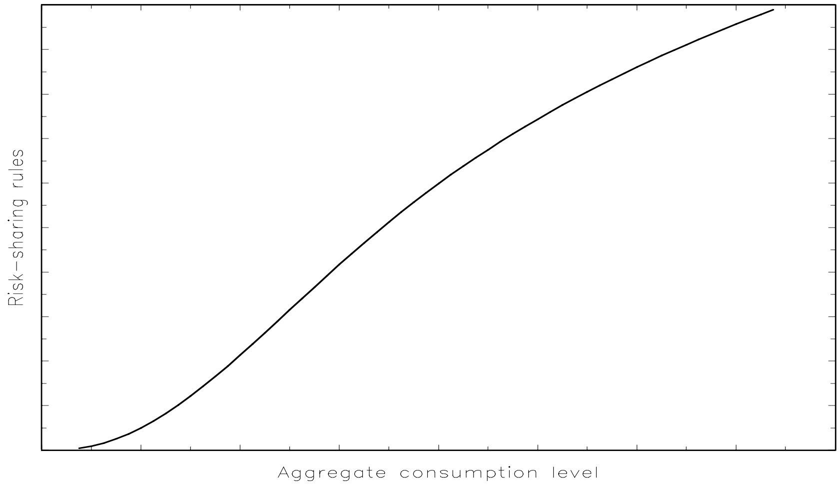

Figure 1: An example of an intermediately cautious consumer's risk-sharing rule in an economy consisting of consumers with (possibly heterogeneous) linear risk-tolerance.

(1980) and Brennan and Solanki (1981), who considered the expected-utility maximization problem of a consumer who chooses over state-contingent claims of a reference portfolio. Holding the underlying asset and a put option is equivalent to holding cash and a call option of the same exercise price, often called the call-put parity, but these are also equivalent to having a portfolio insurance. In all of these cases, the generated return is a convex function of the values of the portfolio. They were thus led to identify conditions on the consumer's utility function for his optimal choice of return to be a convex function of the value of the portfolio. Among the differences between this work and theirs, the most important one is that they took the representative consumer's risk aversion as given, while we *derive* it as a result of efficient risk-sharing among heterogeneous consumers. In particular, the situation Leland (1980) envisaged on page 589, where the individual and the representative consumers exhibit constant but differing relative risk aversion, is in fact impossible, if all the other consumers also exhibit constant relative risk aversion. Even if the representative consumer's relative risk aversion is allowed to decline, the result still has little theoretical relevance because the risk-sharing rule is everywhere convex only for the most cautious consumers.

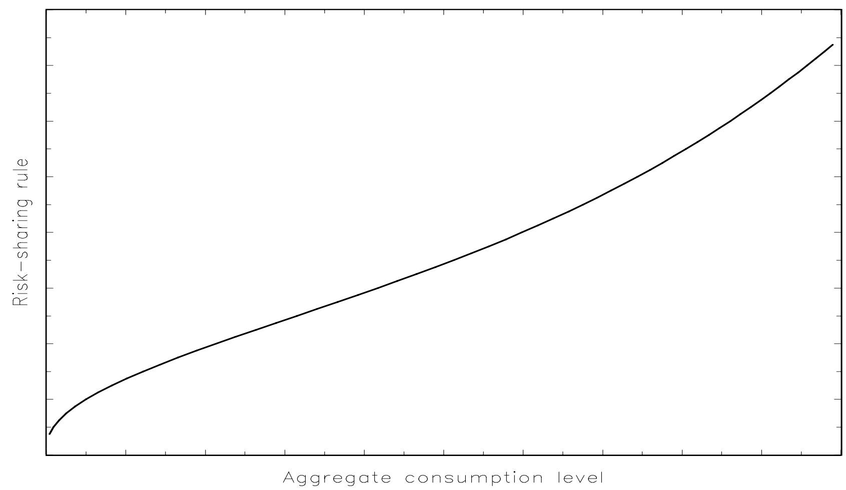

Figure 2: An example of an impossible risk-sharing rule in an economy consisting of consumers with (possibly heterogeneous) linear risk-tolerance.

1.4 Background Risks

We also investigate an economy where individuals, in addition to having heterogeneous risk attitudes, are exposed to additive non-hedgeable background risks. Franke, Stapleton, and Subrahmanyam (1998) showed that if all consumers have the common, constant cautiousness, then it is the size of the background risk which determines the curvature of the risk-sharing rule. We show that this result is not robust to even a small heterogeneity in consumers' risk attitudes. More specifically (Lemma 30 and Proposition 32), if the background risk may take only finitely many values, then it is the difference in cautiousness that dictates the curvature of the risk-sharing rule, not the difference in background risks, when the aggregate consumption level is very large or very small. While we do not have any analytical result for continuously distributed background risks, a numerical exercise in Section 9 indicates that in this case the curvature of the risk-sharing rules is determined by the difference in cautiousness for large aggregate consumption levels, and by the difference in the size of the background risk for small aggregate consumption levels.

1.5 Multiple Periods

Throughout the paper we establish our results for the static, one-period model. We give a lemma (Lemma 33) in Section 8 to show that all the results can be extended to the multi-period case provided all consumers have time-homogeneous and time-separable expected utility functions and the same time-discount rate. Hence, our results are directly comparable with dynamic models such as the ones of Mehra and Prescott (1985), Campbell and Cochrane (1999), Wang (1998), and Benninga and Mayshar (2000), where there are multiple (possibly continuous and infinite) consumption periods and the assumption of a common discount rate is made.

1.6 Organization of the Paper

This paper is organized as follows. Section 2 states the model and gives a few preliminary results on the representative consumer's risk attitude, most of them due to Wilson (1968). In Section 3 the curvature of an individual's risk-sharing rule is related to the difference between his cautiousness and the cautiousness of the representative consumer. In addition it is shown there that convex consumer risk tolerance and heterogeneous cautiousness together imply strictly convex risk tolerance for the representative consumer. To conclude the section, we analyze the implication for mis-pricing of derivative assets of erroneously assuming the representative consumer to have linear risk tolerance, when in fact his risk tolerance is strictly convex. Section 4 investigates the limiting behavior of the representative consumer's cautiousness and of the risk-sharing rules when aggregate consumption tends to infinity and the minimum subsistence level. In Section 5 sharper results on the risk-sharing rules and the representative consumer's risk attitudes are obtained for the case when all consumers exhibit linear risk tolerance, or, equivalently, constant cautiousness. General results on the representative consumer's relative risk aversion are gathered in Section 6. Also the mis-pricing of derivative assets is investigated when the representative consumer is erroneously assumed to have constant relative risk aversion when in fact he has strictly decreasing relative risk aversion. Section 7 analyzes the case when the individual consumers are not only heterogeneous with respect to their risk attitude but also face non-hedgeable background risks. A case is being made that our results extend to the multi-period setting in Section 8, while Section 9 illustrates some of our results graphically using numerical examples. Section 10 concludes, suggesting directions of future research.

2 Model and Preliminary Results

There are I consumers, $i \in \{1, \ldots, I\}$. Consumer i has a von-Neumann Morgenstern (also known as Bernoulli) utility function $u_i : \left(\underline{d}_i, \overline{d}_i\right) \to \mathbb{R}$, where $\underline{d}_i \in \mathbb{R} \cup \{-\infty\}$, $\overline{d}_i \in \mathbb{R} \cup \{\infty\}$, and u_i is smooth and satisfies $u_i'(x_i) > 0$ and $u_i''(x_i) < 0$ for every $x_i \in \left(\underline{d}_i, \overline{d}_i\right)$. The Arrow-Pratt measure of absolute risk aversion is defined as

$$a_i(x_i) = -\frac{u_i''(x_i)}{u_i'(x_i)} > 0.$$

The uncertainty of the economy is described by a probability measure space $(\Omega, \mathcal{F}, P)$. The probability measure P specifies the common (objective) belief on the likelihood of the states. Denote by E the expectation with respect to P. The aggregate endowment of the economy and each consumer's consumption are both random variables on the probability measure space.

For each consumer i, we define his *consumption set* Z_i to be $\left\{\zeta_i \in L^1(\Omega, \mathcal{F}, P) \mid \underline{d}_i < \zeta_i < \overline{d}_i \text{ almost surely}\right\}$. Define $Z_i^* = \left\{\zeta_i \in Z_i \mid u_i\left(\zeta_i\right) \in L^1(\Omega, \mathcal{F}, P)\right\}$. Then Z_i^* is the set of random variables ζ_i for which the expected utility $E\left(u_i\left(\zeta_i\right)\right)$ is finite. Note that since u_i is strictly concave, Z_i^* is a convex set. Moreover, for every $x_i \in \left(\underline{d}_i, \overline{d}_i\right)$, $u_i\left(\zeta_i\right) \leq u_i'\left(x_i\right)\left(\zeta_i - x_i\right) + u_i\left(x_i\right)$. The right hand side of this inequality is integrable, and hence the positive part $u_i\left(\zeta_i\right)^+$ of $u_i\left(\zeta_i\right)$ is integrable for every $\zeta_i \in Z_i$. Hence, $\zeta_i \in Z_i^*$ if and only if the negative part $u_i\left(\zeta_i\right)^-$ is integrable.

Define a binary relation $\succsim_i$ on Z_i by letting, for each $\zeta_i \in Z_i$ and $\eta_i \in Z_i$, $\zeta_i \succsim_i \eta_i$ if and only if either of the following two conditions is met: $\eta_i \notin Z_i^*$; or $\zeta_i \in Z_i^*$, $\eta_i \in Z_i^*$, and $E\left(u_i\left(\zeta_i\right)\right) \geq E\left(u_i\left(\eta_i\right)\right)$. Then $\succsim_i$ is reflexive, transitive, and complete. Denote its strict part by $\succ_i$ and symmetric part by $\sim_i$, then $\zeta_i \succ_i \eta_i$ for every $\zeta_i \in Z_i^*$ and every $\eta_i \notin Z_i^*$ and $\zeta_i \sim_i \eta_i$ for every $\zeta_i \notin Z_i^*$ and every $\eta_i \notin Z_i^*$. Thus the random variables ζ_i for which $u_i\left(\zeta_i\right)$ is not integrable are the least preferable ones. Since $u_i\left(\zeta_i\right)$ is not integrable if and only if the negative part $u_i\left(\zeta_i\right)^-$ is not integrable, the way we have defined $\succsim_i$ is intuitively consistent with the expected utility calculation.

We say that a consumption allocation $(\zeta_1, \cdots, \zeta_I) \in Z_1 \times \cdots \times Z_I$ is *feasible* for a aggregate endowment ζ if $\sum \zeta_i = \zeta$ almost surely. We say that a feasible consumption allocation $(\zeta_1^*, \cdots, \zeta_I^*) \in Z_1 \times \cdots \times Z_I$ is *efficient* (in the sense of Pareto) for an aggregate endowment ζ if there is no other feasible consumption allocation $(\zeta_1, \cdots, \zeta_I) \in Z_1 \times \cdots \times Z_I$ for ζ such that $\zeta_i \succsim_i \zeta_i^*$ for every i, and $\zeta_i \succ_i \zeta_i^*$ for some i. While we shall not give a formal proof, it is easy to check that, for every aggregate endowment ζ, if there exists a feasible allocation $(\zeta_1, \cdots, \zeta_I)$ for ζ such that $\zeta_i \in Z_i^*$ for some i and if $(\zeta_1^*, \cdots, \zeta_I^*)$ is an efficient allocation of ζ, then $\zeta_i^* \in Z_i^*$ for every i. In short, if the aggregate endowment is sufficiently far away from the aggregate minimum subsistence level $\underline{d}$ so that some consumer can attain a finite utility level, then every consumer attains a finite utility level at every efficient allocation. This, in particular, implies that when the aggregate endowment is sufficiently far away from the minimum level $\underline{d}$, an allocation is efficient if and only if it is efficient when the comparison is restricted to Z_i^*.

As is well known (Wilson (1968), for example), the assumption of a common probabilistic belief and expected utility allows the efficient allocations to be represented in terms of risk-sharing rules. Write $\underline{d} = \sum \underline{d}_i$ and $\overline{d} = \sum \overline{d}_i$. A *risk-sharing rule* is a smooth function $f : (\underline{d}, \overline{d}) \to (\underline{d}_1, \overline{d}_1) \times \cdots \times (\underline{d}_I, \overline{d}_I)$ that satisfies $\sum f_i(x) = x$ for every $x \in (\underline{d}, \overline{d})$, where f_i is the i-th coordinate function of f. Note that if ζ is an aggregate endowment with $\underline{d} < \zeta < \overline{d}$ almost surely and if $f_i(\zeta) \in L^1(\Omega, \mathcal{F}, P)$ for every i, then $(f_1(\zeta), \ldots f_I(\zeta))$ is a feasible consumption allocation for ζ.

For each $\lambda = (\lambda_1, \cdots, \lambda_I) \in \mathrm{I\!R}_{++}^I$ and each $x \in (\underline{d}, \overline{d})$, consider the following maximization problem:

$$\begin{array}{cc} \max & \sum \lambda_i u_i(x_i), \\ (x_1, \cdots, x_I) \in (\underline{d}_1, \overline{d}_1) \times \cdots \times (\underline{d}_I, \overline{d}_I) & \\ \text{subject to} & \sum x_i = x. \end{array} \tag{1}$$

By strict concavity for each x, there exists at most one solution to this problem, which we denote by $f_\lambda(x)$. In general, there may not be any solution for some values of x and λ, because the intervals $(\underline{d}_i, \overline{d}_i)$ are open. In particular, it is possible that for every $\lambda \in \mathrm{I\!R}_{++}^I$ there exist some x for which the maximization problem has no solution. In such a case, there may not exist any efficient allocation at all. However, if the u_i satisfy the Inada condition, that is, $u_i'(x_i) \to \infty$ as $x_i \to \underline{d}_i$ and $u_i'(x_i) \to 0$ as $x_i \to \overline{d}_i$, then, for every

λ and x, there exists a solution. This is proved in Appendix A. Then, for every λ, the mapping $f_\lambda : \left(\underline{d}, \overline{d}\right) \to \left(\underline{d}_1, \overline{d}_1\right) \times \cdots \times \left(\underline{d}_I, \overline{d}_I\right)$ is well defined. We shall assume this throughout the paper. Since f_λ is smooth by the implicit function theorem, it is a risk-sharing rule. It is straightforward to show that, for every λ, $(\zeta_1^*, \cdots, \zeta_I^*) \in Z_1^* \times \cdots \times Z_I^*$ is a solution to the maximization problem

$$\begin{array}{cc} \max & \sum \lambda_i E\left(u_i(\zeta_i)\right), \\ (\zeta_1, \cdots, \zeta_I) \in Z_1^* \times \cdots \times Z_I^* & \\ \text{subject to} & \sum \zeta_i = \zeta \text{ almost surely.} \end{array} \tag{2}$$

if and only if $\zeta_i^* = f_{\lambda i}(\zeta)$ for every i. Given this, the following lemma is a standard result, which is based on the supporting hyperplane theorem and which can be traced back to Borch (1962, p. 428) and Wilson (1968). The proof is omitted.

Lemma 1 *If $(\zeta_1^*, \cdots, \zeta_I^*) \in Z_1^* \times \cdots \times Z_I^*$ is an efficient allocation of the aggregate endowment ζ, then there exists a $\lambda \in \mathbb{R}^I_{++}$ such that $\zeta_i^* = f_{\lambda i}(\zeta)$ for every i. Conversely, for every $\lambda \in \mathbb{R}^I_{++}$, if $f_{\lambda i}(\zeta) \in Z_i^*$ for every i, then $(f_{\lambda 1}(\zeta), \ldots, f_{\lambda I}(\zeta))$ is an efficient allocation of ζ.*

As pointed out earlier, if the aggregate endowment ζ is sufficiently far away from the aggregate minimum subsistence level $\underline{d}$, then the conditions $\zeta_i^* \in Z_i^*$ and $f_{\lambda i}(\zeta) \in Z_i^*$ are redundant. By virtue of this lemma, we say that a risk-sharing rule f is *efficient* if there exists a $\lambda \in \mathbb{R}^I_{++}$ such that $f = f_\lambda$.

Let f be an efficient risk-sharing rule. Denote the maximum attained in the problem (1), with the same λ as corresponds to f, by $u(x)$. We are thereby defining a function $u : \left(\underline{d}, \overline{d}\right) \to \mathbb{R}$, which is the value function of the problem. Since

$$\sum \lambda_i E\left(u_i(f_i(\zeta))\right) = E\left(\sum \lambda_i u_i(f_i(\zeta))\right) = E\left(u(\zeta)\right)$$

if $f_{\lambda i}(\zeta) \in Z_i^*$ for every i, the function u can be interpreted as the von-Neumann Morgenstern utility function of the representative consumer corresponding to the efficient risk-sharing rule f. Note that the assumption of the common probabilistic belief can be seen to be crucial for this interpretation of u. By the implicit function theorem, u is smooth and its induced Arrow-Pratt measure of absolute risk aversion is equal to $-u''(x)/u'(x)$, which we denote by $a(x)$. The function $a : \left(\underline{d}, \overline{d}\right) \to \mathbb{R}_{++}$ is smooth. Bear in mind that a depends on the choice of an efficient risk-sharing rule f and hence on that of the weights λ, although none of our analytical results depends on the choice of λ. To contrast with the representative consumer, we sometimes refer the I consumers as *individual* consumers.

As is well known, any positive multiple of the marginal utility $u'(\zeta)$ is a state price deflator (also known as the state price density and as the pricing kernel) that may support the efficient allocation $f(\zeta)$ as an equilibrium. Since

$$\frac{u'(x)}{u'(y)} = \exp\left(-\int_y^x a(s)\, ds\right),$$

for every x and y, the random variable $\exp\left(-\int_y^\zeta a(s)\, ds\right)$ is a state price deflator. If it is integrable, define

$$\pi = \frac{\exp\left(-\int_y^\zeta a(s)\, ds\right)}{E\left(\exp\left(-\int_y^\zeta a(s)\, ds\right)\right)}. \tag{3}$$

Then the relative price of the derivative asset $\varphi(\zeta)$ with respect to the risk-free bond equals $E\left(\pi\varphi(\zeta)\right)$. Since $\pi > 0$ and $E(\pi) = 1$, π has the property of a density function.

The subsequent results of this paper are based on the following lemma.

Lemma 2 *Let f be a risk-sharing rule. Then it is efficient if and only if*

$$f_1'(x)a_1\left(f_1(x)\right) = \cdots = f_I'(x)a_I\left(f_I(x)\right) \tag{4}$$

for every $x \in \left(\underline{d}, \overline{d}\right)$. Moreover, if either (and hence both) condition is satisfied, then

$$a(x) = f_i'(x)a_i\left(f_i(x)\right) \tag{5}$$

for every i and $x \in \left(\underline{d}, \overline{d}\right)$.

This lemma implies that $f_i'(x) > 0$ for every i and x, so that every individual risk-sharing rule is strictly increasing.

Proof of Lemma 2 Wilson (1968, Theorem 5) showed that if f is efficient, then equality (4) and (5) hold. Suppose conversely that equality (4) holds. Note that

$$f_i'(x)a_i\left(f_i(x)\right) = -\frac{d}{dx}\log u_i'\left(f_i(x)\right) \tag{6}$$

for every i and x. Thus, for every i, j, and x,

$$\frac{d}{dx}\log\frac{u_i'(f_i(x))}{u_j'(f_j(x))}=\frac{d}{dx}\log u_i'(f_i(x))-\frac{d}{dx}\log u_j'(f_j(x))=0. \tag{7}$$

Thus the ratio of marginal utilities of consumer 1 and consumer i, $\frac{u_1'(f_1(x))}{u_i'(f_i(x))}$, does not depend on x. Hence, by defining $\lambda \in \mathbb{R}^I_{++}$ by making this constant number its i-th coordinate λ_i, we obtain

$$\lambda_1 u_1'(f_1(x)) = \cdots = \lambda_I u_I'(f_I(x)) \tag{8}$$

for every x. But this is the first-order sufficient condition for the solution of the maximization problem (1). Hence f is efficient. Then (5) follows as well. ■

In addition to Wilson's celebrated result, by establishing its converse, we have shown that equality (4) exhausts all the implications of an efficient risk-sharing rule for general utility functions. As can be seen from the proof, it is equivalent to the better known first-order condition (8). In the following analysis, however, (4) will turn out to be more convenient than (8), since the former is independent of the utility representations u_i, which are unique only up to positive affine transformations, and the utility weights λ_i, which are merely a particular parametrization of efficient allocations, do not have to be explicitly incorporated.

The reciprocal of the absolute risk aversion, $1/a_i(x_i)$, is called *risk tolerance* and denoted by $t_i(x_i)$. The risk tolerance t of the representative consumer having absolute risk aversion a is defined by $t(x) = 1/a(x)$. Then $t_i : (\underline{d}_i, \overline{d}_i) \to \mathbb{R}_{++}$ and $t : (\underline{d}, \overline{d}) \to \mathbb{R}_{++}$ are smooth functions. Subsequent results of this paper involve the first derivative of risk tolerances, which Wilson (1962, page 129) called *cautiousness.* According to this terminology, if two consumers exhibit constant but differing *absolute* risk aversion, then they are equally cautious; but if they exhibit constant but differing *relative* risk aversion, then the one with the smaller relative risk aversion is more cautious. This might sound a bit confusing, but we stick to Wilson's terminology. Consumer i's and the representative consumer's cautiousness are $t_i'(x_i)$ and $t'(x)$.

Lemma 3 (Wilson (1968)) *Let f be an efficient risk-sharing rule, then*

$$t(x) = \sum t_i(f_i(x))$$

and

$$t'(x) = \sum f_i'(x) t_i' \left(f_i(x) \right) \tag{9}$$

for every $x \in \left(\underline{d}, \overline{d} \right)$.

Proof of Lemma 3 The first result can be proven by taking the reciprocals of both sides of equality (5), multiplying $f_i'(x)$ on both sides, and taking the summation over i. The second can be obtained by differentiating both sides of the first with respect to x. ■

Lemma 3 gives the ranges of the risk tolerance and cautiousness of the representative consumer.

Proposition 4 *Let* f *be an efficient risk-sharing rule, then*

$$\max \left\{ \max_i t_i \left(f_i(x) \right), I \min_i t_i \left(f_i(x) \right) \right\} \leq t(x) \leq I \max_i t_i \left(f_i(x) \right), \tag{10}$$

$$\min_i t_i' \left(f_i(x) \right) \leq t'(x) \leq \max_i t_i' \left(f_i(x) \right). \tag{11}$$

The two weak inequalities in (10) hold as strict inequalities if there exist i *and* j *such that* $t_i \left(f_i(x) \right) \neq t_j \left(f_j(x) \right)$. *The two inequalities in (11) hold as strict inequalities if there exist* i *and* j *such that* $t_i' \left(f_i(x) \right) \neq t_j' \left(f_j(x) \right)$.

An immediate corollary of this proposition (in particular, inequality (11)) is a sufficient condition for t to be increasing.

Corollary 5 *If* t_i *is non-decreasing for every* i, *then so is* t. *If, moreover,* t_i *is strictly increasing for some* i, *then so is* t.

Another corollary of this proposition is the mutual fund theorem, which we will come back to in Section 5.

3 Convex Risk Tolerance

Throughout this section, we let $f : \left(\underline{d}, \overline{d} \right) \to \left(\underline{d}_1, \overline{d}_1 \right) \times \cdots \times \left(\underline{d}_I, \overline{d}_I \right)$ be an efficient risk-sharing rule and denote by $t : \left(\underline{d}, \overline{d} \right) \to \mathbb{R}_{++}$ the representative consumer's risk tolerance corresponding to f.

The following proposition is rich in interpretations.

Proposition 6 *For every i and $x \in (\underline{d}, \overline{d})$,*

$$\frac{f_i''(x)}{f_i'(x)} = \frac{1}{t(x)} \left(t_i'(f_i(x)) - t'(x) \right). \tag{12}$$

Proof of Proposition 6 By equality (5),

$$t_i(f_i(x)) = t(x) f_i'(x) \tag{13}$$

for every $x \in (\underline{d}, \overline{d})$. Differentiating both sides with respect to x, we obtain

$$t_i'(f_i(x)) f_i'(x) = t'(x) f_i'(x) + t(x) f_i''(x). \tag{14}$$

Rearranging this, we complete the proof. ■

Note that equalities (12) and (13) together imply that the first two derivatives of consumer i's risk-sharing rule f_i is determined by his own and the representative consumer's risk tolerance and cautiousness.

The first implication of Proposition 6 is that for every $x \in (\underline{d}, \overline{d})$ and every i, $f_i''(x) > 0$ if $t_i'(f_i(x)) > t'(x)$; $f_i''(x) = 0$ if $t_i'(f_i(x)) = t'(x)$; and $f_i''(x) < 0$ if $t_i'(f_i(x)) < t'(x)$. This is similar to Proposition II of Leland (1980) but differs from it in that the risk tolerance t is derived from the efficient risk-sharing rule f rather than exogenously given. Its message is otherwise the same: an individual consumer's risk-sharing rule is (locally) convex if he is more cautious than the representative consumer; (locally) concave if he is less cautious; and (infinitesimally) linear if they are equally cautious. In the context of portfolio insurance of Leland (1980) and Brennan and Solanki (1981), it implies that only those who are more cautious at every level x of aggregate consumptions than the representative consumer would purchase portfolio insurances.

The second, finer, implication of the proposition is that for every $x \in (\underline{d}, \overline{d})$ and all i and j,

$$t_i'(f_i(x)) \gtreqless t_j'(f_j(x))$$

if and only if

$$\frac{f_i''(x)}{f_i'(x)} \gtreqless \frac{f_j''(x)}{f_j'(x)}.$$

To appreciate this, recall that the ratios of the first and second derivatives, such as $f_i''(x)/f_i'(x)$ and $f_j''(x)/f_j'(x)$, often appear in expected utility theory. They measure the curvatures of the individual risk-sharing functions f_i and f_j. For example, $f_i''(x)/f_i'(x) \geq f_j''(x)/f_j'(x)$ for every x if and only if f_i is a convex function of f_j. The above implication then means that the degree of convexity of f_i is positively related to cautiousness. That is, the marginal consumption that consumer i receives as the aggregate endowment increases grows at a rate higher than its counterpart for consumer j if consumer i is more cautious than consumer j. What it means in the context of portfolio insurance is that consumer i purchases more portfolio insurance (or options) relative to the size of the reference portfolio he holds than consumer j does. Although both Leland (1980) and Brennan and Solanki (1981) were concerned with the second derivatives $f_i''(x)$ and $f_j''(x)$, rather than the ratios $f_i''(x)/f_i'(x)$ and $f_j''(x)/f_j'(x)$, we believe that the latter is a better notion of convexity. In addition, our result provides a complete ordering of all consumers' curvatures according to their cautiousness. In particular, it shows that the *levels* of risk tolerance do not matter for the curvatures of the risk-sharing rules, although they do matter for the slopes.[1] This is an important point, especially in the analysis of the background risk in Section 7.

The following theorem is the main result of this section. It establishes that if every consumer exhibits *non-decreasing* risk cautiousness, then so does the representative consumer. Moreover, even the slightest heterogeneity in their cautiousness would cause the representative consumer's counterpart to be *strictly increasing.*

Theorem 7 *Let $x \in \left(\underline{d}, \overline{d}\right)$. If $t_i''\left(f_i(x)\right) \geq 0$ for every i, then $t''(x) \geq 0$. If, moreover, either there exists a consumer i such that $t_i''\left(f_i(x)\right) > 0$ or there exist two consumers i and j such that $t_i'\left(f_i(x)\right) \neq t_j'\left(f_j(x)\right)$, then $t''(x) > 0$.*

This theorem tells us how his cautiousness varies within this range as the aggregate consumption level increases. Note that his risk-tolerance may well be decreasing while cautiousness is increasing. Note also that the representative consumer may well exhibit increasing cautiousness, while every individual consumer exhibits constant cautiousness. This latter case will be elaborated on in Section 5.

It is perhaps useful to put on record an immediate, simple corollary of this proposition, albeit ignoring the heterogeneity consequence of the theorem.

[1] We thank Christian Gollier for clarifying this point.

Corollary 8 *If t_i is a convex function for every i, then so is t.*

Proof of Theorem 7 Differentiate both sides of equality (9), then we obtain

$$t''(x) = \sum f_i''(x) t_i'(f_i(x)) + \sum \left(f_i'(x)\right)^2 t_i''(f_i(x)).$$

Use equality (14) to eliminate $t_i'(f_i(x))$, then we obtain

$$\begin{aligned} t''(x) &= \sum f_i''(x) t'(x) + \sum t(x) \frac{\left(f_i''(x)\right)^2}{f_i'(x)} + \sum \left(f_i'(x)\right)^2 t_i''(f_i(x)) \\ &= t(x) \sum \frac{\left(f_i''(x)\right)^2}{f_i'(x)} + \sum \left(f_i'(x)\right)^2 t_i''(f_i(x)) \end{aligned}$$

because $\sum f_i''(x) = 0$. Since both terms of the far right hand side are non-negative, we have $t''(x) \geq 0$.

If there exists a consumer i such that $t_i''(f_i(x)) > 0$, then the second term of the far right hand side is strictly positive and hence $t''(x) > 0$. On the other hand, if there exist two consumers i and j such that $t_i'(f_i(x)) \neq t_j'(f_j(x))$, then, by equality (12), either $f_i''(x) \neq 0$ or $f_j''(x) \neq 0$. Thus the first term is strictly positive and hence $t''(x) > 0$. ■

The proof of the theorem reveals that even if all consumers exhibit concave, rather than convex, risk tolerance, the representative consumer may exhibit *convex* risk tolerance. We can therefore say that the aggregation over heterogeneous consumers tends to induce the representative consumer to exhibit convex risk tolerance.

We now give an implication of convex risk tolerance on derivative asset pricing. By a derivative asset, we mean a derivative asset of the random aggregate endowment ζ, which promises to pay some amounts of the good contingent on the realization of ζ. It can therefore be characterized by a Lebesgue measurable function $\varphi : \left(\underline{d}, \overline{d}\right) \to \mathbb{R}$, so that the derivative asset pays $\varphi(\zeta)$. We analyze how derivative assets are mis-priced when the risk tolerance t is approximated by a linear (or, to be precise, affine) risk tolerance. This is similar to but different from the approximation considered by Benninga and Mayshar (2000). Specifically, for any choice of $y \in \left(\underline{d}, \overline{d}\right)$, if we define $\widehat{t} : \left(\underline{d}, \overline{d}\right) \to \mathbb{R}$ by $\widehat{t}(x) = t'(y)(x - y) + t(y)$, then $\widehat{t}$ is the best linear approximation of t at y. Let π be as in (3) and define

$$\widehat{\pi} = \frac{\exp\left(-\int_y^\zeta \frac{ds}{\widehat{t}(s)}\right)}{E\left(\exp\left(-\int_y^\zeta \frac{ds}{\widehat{t}(s)}\right)\right)}. \tag{15}$$

This would be the state-price deflator if the representative consumer's risk tolerance were $\widehat{t}$. The following proposition states that this linear approximation $\widehat{\pi}$ underestimates the price of every derivative asset $\varphi(\zeta)$ whenever φ is an increasing function and all individual consumers exhibit convex risk tolerance.

Proposition 9 *Suppose that t_i is convex for every i and that either there exists a consumer i such that $t_i''(f_i(x)) > 0$ or there exist two consumers i and j such that $t_i'(f_i(x)) \neq t_j'(f_j(x))$. If a derivative asset $\varphi : (\underline{d}, \overline{d}) \to \mathbb{R}$ is non-constant and non-decreasing, then*

$$E(\pi\varphi(\zeta)) > E(\widehat{\pi}\varphi(\zeta)). \tag{16}$$

This proposition provides a sufficient condition for the estimate of the price of the derivative asset using $\widehat{t}$ to be strictly less than its true price. The derivative asset $\varphi(\zeta)$ can be a call option or indeed the aggregate endowment ζ itself. Since a put option is non-increasing and non-constant, the proposition implies that its price is overestimated by $\widehat{t}$.

Proof of Proposition 9 By Theorem 7, $t''(y) > 0$ and hence $\widehat{t}(x) < t(x)$ for every $x \neq y$. Note that

$$\frac{\pi}{\widehat{\pi}} = \frac{E\left(\exp\left(-\int_y^\zeta \frac{ds}{\widehat{t}(s)}\right)\right)}{E\left(\exp\left(-\int_y^\zeta \frac{ds}{t(s)}\right)\right)} \exp\left(\int_y^\zeta \left(\frac{1}{\widehat{t}(s)} - \frac{1}{t(s)}\right)\right). \tag{17}$$

Thus $\pi/\widehat{\pi}$ is a strictly increasing function of ζ. Hence the distribution of ζ with respect to the probability measure whose Radon-Nikodym derivative is π first-order stochastically dominates the distribution of ζ with respect to the probability measure whose Radon-Nikodym derivative is $\widehat{\pi}$. Then the strict inequality (16) follows from the assumption that φ is non-constant and non-decreasing. ■

4 Limit Behavior

In this section, we investigate the limit behavior of the representative consumer's cautiousness and the risk-sharing rules. More specifically, we show that the representative consumer's cautiousness converges to the limit of the most cautious consumers' counterpart as the aggregate consumption level diverges to infinity, if the limit indeed exists; and, moreover, these consumers' share of both the consumption levels, out of the aggregate consumption level, and of marginal consumptions converges to one. We also provide an analogous result when the aggregate consumption level converges to the minimum subsistence level, but the dominant consumers are then the least cautious ones. This result is useful for the analysis of the qualitative properties of the limit behavior of the representative consumer's cautiousness and the risk-sharing rule when all consumers exhibit linear risk tolerance, which will be formally defined in the next section.

Define $\overline{I}$ to be the set of consumers i for whom $\overline{d}_i = \infty$ and there is no other consumer j such that $\overline{d}_j = \infty$ and

$$\limsup_{x_i \to \infty} t_i'(x_i) < \liminf_{x_j \to \infty} t_j'(x_j).^{2} \tag{18}$$

Since there are only finitely many consumers, if $\overline{d}_i = \infty$ for some i, then $\overline{I} \neq \emptyset$. The consumers in $\overline{I}$ are those who are not unambiguously less cautious than any other when the consumption levels are very large. The first proposition of this section states that those consumers' share in the aggregate consumption level, as well as in the marginal consumptions, converges to one as the aggregate consumption level diverges to infinity, and that the representative consumer's cautiousness eventually lies between the minimum and maximum of those consumers' cautiousness.

Proposition 10 *If $\overline{d}_i = \infty$ for some i, then $\sum_{i \in \overline{I}} f_i(x)/x \to 1$ and $\sum_{i \in \overline{I}} f_i'(x) \to 1$ as $x_i \to \infty$. If, moreover, the set $\left\{ t_j'(x_j) \mid \underline{d}_j < x_j < \overline{d}_j \right\}$ is bounded for every consumer j, then*

$$\min_{i \in \overline{I}} \liminf_{x_i \to \infty} t_i'(x_i) \leq \liminf_{x \to \infty} t'(x) \leq \limsup_{x \to \infty} t'(x) \leq \max_{i \in \overline{I}} \limsup_{x_i \to \infty} t_i'(x_i).$$

Define $\underline{I}$ to be the set of the consumers i for whom $\underline{d}_i > -\infty$, $t_i(x_i) \to 0$ as $x_i \to \underline{d}_i$, and there is no other consumer j such that $\underline{d}_j > -\infty$, $t_j(x_i) \to 0$ as $x_j \to \underline{d}_j$, and

$$\limsup_{x_j \to \underline{d}_j} t_j'(x_j) < \liminf_{x_i \to \underline{d}_i} t_i'(x_i). \tag{19}$$

[2] As a convention of this paper, we allow lim sup and lim inf to be ∞ or $-\infty$, while lim is always finite.

Since there are only finitely many consumers, if $\underline{d}_i > -\infty$ and $t_i(x_i) \to 0$ as $x_i \to \underline{d}_i$ for some i, then $\underline{I} \neq \emptyset$. Consumers in $\underline{I}$ have the following two characteristics. First, their risk-tolerance converges to zero as the consumption levels converge to the minimum subsistence levels. And second, for consumption levels very close to the minimum subsistence levels, they are not unambiguously more cautious than any other consumer whose risk tolerance also converges to zero as the consumption levels converge to the minimum subsistence levels. The following proposition states that, if the zero convergence assumption on the risk tolerance holds for every consumer, then the following is true: The same consumers' (in $\overline{I}$) share in the extra consumption level in excess of the minimum subsistence level converges to one as the aggregate consumption level converges to the minimum subsistence level. Also the representative consumer's cautiousness eventually lies between the minimum and maximum of those consumers' cautiousness.

Proposition 11 *If, for every consumer j, $\underline{d}_j > -\infty$ and $t_j(x_j) \to 0$ as $x_j \to \underline{d}_j$, then $\dfrac{\sum_{i\in\underline{I}}(f_i(x) - \underline{d}_i)}{x - \underline{d}} \to 1$ and $\sum_{i\in\underline{I}} f_i'(x) \to 1$ as $x \to \underline{d}$. Moreover, if the set $\left\{t_j'(x_j) \mid \underline{d}_j < x_j < \overline{d}_j\right\}$ is bounded for every j, then*

$$\min_{i\in\underline{I}} \liminf_{x_i\to\underline{d}_i} t_i'(x_i) \leq \liminf_{x\to\underline{d}} t'(x) \leq \limsup_{x\to\underline{d}} t'(x) \leq \max_{i\in\underline{I}} \limsup_{x_i\to\underline{d}_i} t_i'(x_i). \tag{20}$$

The proofs of these two propositions are given in Appendix B. Although we shall not prove them, we can obtain similar results on the boundary behavior for the cases where $\overline{d}_i < \infty$ for every i and where $\underline{d}_i = -\infty$ for some i. These results are useful for the analysis of the limit behavior of the representative consumer's cautiousness and the risk sharing rule when consumers have utility functions exhibiting decreasing and linear risk tolerance, such as quadratic ones.

Define $\overline{H}$ to be the set of the consumers i for whom $\overline{d}_i < \infty$, $t_i(x_i) \to 0$ as $x_i \to \overline{d}_i$, and there is no other consumer j such that $\underline{d}_j < \infty$, $t_j(x_j) \to 0$ as $x_j \to \underline{d}_j$, and

$$\limsup_{x_i\to\overline{d}_i} t_i'(x_i) < \liminf_{x_j\to\underline{d}_j} t_j'(x_j)$$

Proposition 12 *If, for every consumer j, $\overline{d}_j < \infty$ and $t_j(x_j) \to 0$ as $x_j \to \overline{d}_j$, then $\dfrac{\sum_{i\in\overline{H}}(\overline{d}_i - f_i(x))}{\overline{d} - x} \to 1$ and $\sum_{i\in\overline{H}} f_i'(x) \to 1$ as $x \to \overline{d}$. If, moreover, the set*

$\left\{t_j'(x_j) \mid \underline{d}_j < x_j < \overline{d}_j\right\}$ *is bounded for every* j, *then*

$$\min_{i\in\overline{H}} \liminf_{x_i\to\overline{d}_i} t_i'(x_i) \le \liminf_{x\to\overline{d}} t'(x) \le \limsup_{x\to\overline{d}} t'(x) \le \max_{i\in\overline{H}} \limsup_{x_i\to\overline{d}_i} t_i'(x_i).$$

Define $\underline{H}$ to be the set of the consumers i for whom $\underline{d}_i = -\infty$ and there is no other consumer j such that $\underline{d}_j = -\infty$ and

$$\limsup_{x_j\to-\infty} t_j'(x_j) < \liminf_{x_i\to-\infty} t_i'(x_i).$$

Proposition 13 *If* $\underline{d}_i = -\infty$ *for some* i, *then* $\sum_{i\in\underline{H}} f_i(x)/x \to 1$ *and* $\sum_{i\in\underline{H}} f_i'(x) \to 1$ *as* $x_i \to -\infty$. *If, moreover, the set* $\left\{t_j'(x_j) \mid \underline{d}_j < x_j < \overline{d}_j\right\}$ *is bounded for every consumer* j,

$$\min_{i\in\underline{H}} \liminf_{x_i\to-\infty} t_i'(x_i) \le \liminf_{x\to-\infty} t'(x) \le \limsup_{x\to-\infty} t'(x) \le \max_{i\in\underline{H}} \limsup_{x_i\to-\infty} t_i'(x_i).$$

5 Linear Risk Tolerance

Combining the results from the last two sections, we provide more informative results on the risk-sharing rule and the representative consumer's risk aversion when all consumers' utility functions exhibit linear risk tolerance. The additional property that we can establish with the assumption of linear risk tolerance is that an individual consumers' risk-sharing rule is either everywhere concave, everywhere convex, or has a unique inflection point below which it is convex and above which it is concave.

Mathematically, a utility function $u_i : \left(\underline{d}_i, \overline{d}_i\right) \to \mathbb{R}$ exhibits *linear risk tolerance* if, for the corresponding risk tolerance t_i, there exist two numbers τ_i and γ_i such that

$$t_i(x_i) = \tau_i + \gamma_i x_i. \tag{21}$$

for every $x_i \in \left(\underline{d}_i, \overline{d}_i\right)$. Equality (21) can also be written as

$$a_i(x_i) = \frac{1}{\tau_i + \gamma_i x_i}$$

and hence u_i is often said to exhibit *hyperbolic absolute risk aversion*. Yet another equivalent condition for equality (21) is that

$$t_i'(x_i) = \gamma_i,$$

and hence the cautiousness t_i' is constant over the entire range $\left(\underline{d}_i, \overline{d}_i\right)$ of consumption levels.

Note that the right hand side of equality (21) is of course positive for every $x_i \in \left(\underline{d}_i, \overline{d}_i\right)$ but τ_i and γ_i may be positive, zero, or negative. However, if $\gamma_i = 0$, then $\tau_i > 0$ and we take $\underline{d}_i = -\infty$ and $\overline{d}_i = \infty$. On the other hand, if $\gamma_i > 0$ then we take $\underline{d}_i = -\tau_i/\gamma_i$ and $\overline{d}_i = \infty$ and hence $t_i(x_i) = \gamma_i\left(x_i - \underline{d}_i\right)$ and $t_i(x_i) \to 0$ as $x_i \to \underline{d}_i$. If $\gamma_i < 0$, then $\underline{d}_i = -\infty$ and $\overline{d}_i = -\tau_i/\gamma_i$ and hence $t_i(x_i) = -\gamma_i\left(\overline{d}_i - x_i\right)$ and $t_i(x_i) \to 0$ as $x_i \to \overline{d}_i$. Indeed, although we do not provide the proof here, these choices of $\underline{d}_i$ and $\overline{d}_i$ are the only ones that allows u_i to satisfy the Inada condition.

The celebrated mutual fund theorem is documented in, for example, Wilson (1968), Huang and Litzenberger (1988, Sections 5.15 and 5.26), Magill and Quinzii (1996), Gollier (2001, Section 21.3.3), and LeRoy and Werner (2001, Section 15.6)) and can be stated in our notation as follows:

Theorem 14 (Mutual Fund Theorem) *Suppose that* $\gamma_1 = \cdots = \gamma_I$ *and write* $\gamma = \gamma_1 = \cdots = \gamma_I$ *and* $\tau = \sum_{i=1}^{I} \tau_i$, *then* $t(x) = \tau + \gamma x$ *for every* $x \in \left(\underline{d}, \overline{d}\right)$. *If* $\gamma = 0$, *then there exist* I *numbers* $n_1, \ldots, n_I$ *such that* $\sum n_i = 0$ *and*

$$f_i(x) = \frac{\tau_i}{\tau} x + n_i$$

for every i *and* x. *If* $\gamma > 0$, *then there exist* I *positive numbers* $m_1, \ldots, m_I$ *such that* $\sum m_i = 1$ *and*

$$f_i(x) = m_i(x - \underline{d}) + \underline{d}_i$$

for every i *and* x. *Finally, if* $\gamma < 0$, *then there exist* I *positive numbers* $m_1, \ldots, m_I$ *such that* $\sum m_i = 1$ *and*

$$f_i(x) = -m_i(\overline{d} - x) + \overline{d}_i$$

for every i *and* x.

This theorem can be proved by applying Lemma 4. Throughout this section, we assume that every consumer i has a utility function (21) defined on the range $\left(\underline{d}_i, \overline{d}_i\right)$ satisfying the Inada condition. Denote

$$\overline{\gamma} = \max\{\gamma_1, \ldots, \gamma_I\},$$
$$\underline{\gamma} = \min\{\gamma_1, \ldots, \gamma_I\}.$$

Then, according to the notation in the previous section,

$$\overline{I} = \{i \mid \gamma_i = \overline{\gamma}\},$$
$$\underline{I} = \{i \mid \gamma_i = \underline{\gamma}\}.$$

Then $\overline{I}$ is the set of the most cautious consumers and $\underline{I}$ is the set of the least cautious consumers. All consumers are equally cautious if and only if $\overline{\gamma} = \underline{\gamma}$. Of course, this case has been fully dealt with by the mutual fund theorem, and we thus assume in the rest of this section that $\overline{\gamma} > \underline{\gamma}$. Some of our results, such as part 3 of Theorem 16, are vacuous unless $\overline{I} \cup \underline{I} \neq \{1, \ldots, I\}$, that is, there is some "intermediate" consumer who is neither the most nor the least cautious. One result, part 4 of Theorem 16, is relevant only if there are at least two intermediate consumers having differing cautiousness.

As in Section 3, let $f : (\underline{d}, \overline{d}) \to (\underline{d}_1, \overline{d}_1) \times \cdots \times (\underline{d}_I, \overline{d}_I)$ be an efficient risk-sharing rule, and denote by $a : (\underline{d}, \overline{d}) \to \mathbb{R}_{++}$ the representative consumer's absolute risk aversion, by $t : (\underline{d}, \overline{d}) \to \mathbb{R}_{++}$ his risk-tolerance, and by $b : (\underline{d}, \overline{d}) \to \mathbb{R}_{++}$ his relative risk aversion, all corresponding to f.

Below is the first result for the case of linear risk tolerance.

Proposition 15 *For every $x \in (\underline{d}, \overline{d})$, $t''(x) > 0$. As $x \to \overline{d}$, $t'(x) \to \overline{\gamma}$. As $x \to \underline{d}$, $t'(x) \to \underline{\gamma}$.*

Proof of Proposition 15 The first part of this proposition follows from Theorem 7. The second part follows from Proposition 10 or 12, depending on whether $\overline{\gamma} \geq 0$ or not. The third part follows from Proposition 11 or 13, depending on whether $\underline{\gamma} \leq 0$ or not. ■

The next result is the central one of this section.

Theorem 16

1. *$f_i''(x) > 0$ for every $i \in \overline{I}$ and $x \in (\underline{d}, \overline{d})$.*

2. *$f_i''(x) < 0$ for every $i \in \underline{I}$ and $x \in \left(\underline{d}, \overline{d}\right)$.*

3. *For every $i \notin \overline{I} \cup \underline{I}$, there exists a unique $y_i \in \left(\underline{d}_i, \overline{d}_i\right)$ such that $f_i''(x) > 0$ for every $x < y_i$ and $f_i''(x) < 0$ for every $x > y_i$.*

4. *For the y_i defined as in part 3, $y_i < y_j$ if $\gamma_i < \gamma_j$; $y_i = y_j$ if $\gamma_i = \gamma_j$; and $y_i > y_j$ if $\gamma_i > \gamma_j$.*

Proof of Theorem 16 By Proposition 15, $\underline{\gamma} < t'(x) < \overline{\gamma}$ for every $x \in \left(\underline{d}, \overline{d}\right)$. Parts 1 and 2 then follow from part 1 of Proposition 6. As for part 3, note that Proposition 15 implies that $t' : \left(\underline{d}, \overline{d}\right) \to \left(\underline{\gamma}, \overline{\gamma}\right)$ is strictly increasing and onto. Hence, for every $i \notin \overline{I} \cup \underline{I}$, there exists a unique $y_i \in \left(\underline{d}_i, \overline{d}_i\right)$ such that $\gamma_i = t'(y_i)$. Since $\gamma_i = t_i'\left(f_i(x)\right)$ for every x, Proposition 6 implies that y_i has the property of part 3. Part 4 also follows from this property of y_i and the fact that t' is strictly increasing. ■

The importance of part 3 of the above proposition cannot be overemphasized. It is exactly the point that was being missed in the analysis of Leland (1980) and Brennan and Solanki (1981): When individual consumers have differing cautiousness, the representative consumer's counterpart is strictly increasing, ranging from the smallest to the largest. If an individual consumer has neither the smallest nor the largest cautiousness, then his cautiousness must be caught up with by the representative consumer's counterpart at some aggregate consumption level. Below this level, his risk-sharing rule is convex, and, above this level, it is concave. An important implication of this result in the context of portfolio insurance, is that no consumer other than those with the largest cautiousness would buy portfolio insurance, as their risk-sharing rules would eventually become concave. In particular, say, even the consumers with the second largest cautiousness would not buy portfolio insurance. This significantly undermines the applicability of the results of Leland (1980) and Brennan and Solanki (1981). They are valid in a two-consumer economy, but do not generalize much to an economy with a large number of consumers with diverse degrees of cautiousness.

The last proposition of this section is concerned with the total proportion of consumption levels consumed by those consumers with the largest or smallest cautiousness. They immediately follow from Propositions 10, 11, 12, and 13. We thus omit the proof.

Proposition 17

1. *If* $\overline{\gamma} \geq 0$, *then, as* $x \to \infty$, $\displaystyle\sum_{i \in \overline{I}} \frac{f_i(x)}{x} \to 1$.

2. *If* $\underline{\gamma} > 0$, *then, as* $x \to \underline{d}$, $\sum_{i \in \underline{I}} \dfrac{f_i(x) - \underline{d}_i}{x - \underline{d}} \to 1$.

3. *If* $\underline{\gamma} \leq 0$, *then, as* $x \to -\infty$, $\displaystyle\sum_{i \in \underline{I}} \frac{f_i(x)}{x} \to 1$.

4. *If* $\overline{\gamma} < 0$, *then as* $x \to \overline{d}$, $\sum_{i \in \overline{I}} \dfrac{\overline{d}_i - f_i(x)}{\overline{d} - x} \to 1$.

6 Relative Risk Aversion

Based on the preceding results on the representative consumer's risk tolerance, we now give similar results on his relative risk aversion. After introducing the notation and presenting preliminary results, we show that if every consumer exhibits non-increasing relative risk aversion, so does the representative consumer, and that if there is some heterogeneity in individual consumer's risk tolerance, then his relative risk aversion is strictly decreasing. This is in the same spirit as Theorem 7. We then provide a result analogous to Propositions 10 and 11, which is concerned with the limit behavior of the representative consumer's relative risk aversion as the aggregate consumption level diverges to infinity or converges to the minimum subsistence level. As we did in the preceding section, we then combine these two sets of results to obtain a result for the case of linear risk tolerance. Although there is not much new insight involved in these propositions and proofs, it is still worthwhile to present these results to illustrate how our results are related to many existing contributions on the representative consumer's risk attitude, such as those in Aït-Sahalia and Lo (2000), Benninga and Mayshar (2000), Brennan and Solanki (1981), and Leland (1980), which are stated in terms of relative risk aversion.

For each consumer i and $x_i \in \left(\underline{d}_i, \overline{d}_i\right)$, if $x_i > 0$, then the Arrow-Pratt measure of relative risk aversion is defined by

$$b_i\left(x_i\right) = -\frac{u_i''\left(x_i\right) x_i}{u_i'\left(x_i\right)} > 0.$$

Since $b_i(x_i) = a_i(x_i)\, x_i = x_i / t_i(x_i)$,

$$b_i'(x_i) = \frac{1}{t_i(x_i)} \left(1 - \frac{t_i' x_i(x_i)}{t_i(x_i)}\right).$$

Hence $b_i'(x_i) \lesseqgtr 0$ if and only if $\dfrac{t_i'(x_i)\, x_i}{t_i(x_i)} \gtreqless 1$. That is, an individual's relative risk aversion is strictly decreasing if and only if his elasticity of risk tolerance is greater than one.

Throughout the rest of this section, let f be an efficient risk-sharing rule. For every $x \in (\underline{d}, \overline{d})$, if $x > 0$ and $f_i(x) > 0$, we define

$$e_i(x) = \frac{f_i'(x) x}{f_i(x)} > 0.$$

This is the elasticity of consumer i's consumption level with respect to the aggregate consumption. Let u be the representative consumer's utility function corresponding to f. For every $x \in (\underline{d}, \overline{d})$, if $x > 0$, then the representative consumer's Arrow-Pratt measure of relative risk aversion is defined by

$$b(x) = -\frac{u''(x) x}{u'(x)}.$$

Then $b'(x) \lesseqgtr 0$ if and only if $\dfrac{t'(x)\, x}{t(x)} \gtreqless 1$. The following proposition relates the individual consumers' relative risk aversion to the representative consumer's counterpart. This is analogous to Lemma 2 and Proposition 4.

Proposition 18 *Let $x \in (\underline{d}, \overline{d})$ and $x > 0$. For every i with $f_i(x) > 0$,*

$$b(x) = e_i(x) b_i(f_i(x)). \tag{22}$$

If $f_i(x) > 0$ for every i, then

$$b(x) = \sum_{i=1}^{I} f_i'(x) b_i(f_i(x)) \tag{23}$$

and hence

$$\min_i \ b_i(f_i(x)) \leq b(x) \leq \max_i \ b_i(f_i(x)). \tag{24}$$

These two inequalities hold as strict inequalities if there exist two consumers i and j such that $b_i(f_i(x)) \neq b_j(f_j(x))$.

The inequalities (24) generalize Proposition 1 of Benninga and Mayshar (2000), which was shown for the special case where all consumers exhibit constant relative risk aversion.

Proof of Proposition 18 By multiplying x to both sides of equality (5), we obtain

$$a(x)x = \frac{f_i'(x)x}{f_i(x)} f_i(x) a_i \left(f_i(x)\right).$$

This is nothing but equality (22). Multiply $f_i(x)/x$ to both sides of this equality, then

$$\frac{f_i(x)}{x} b(x) = f_i'(x) b_i \left(f_i(x)\right).$$

Summing both sides over i and using $\sum f_i(x)/x = 1$, we obtain equality (23). The last part follows from $f_i'(x) > 0$ for every i and $\sum f_i'(x) = 1$. ■

If $\underline{d}_i \geq 0$ for every i, then the condition that $f_i(x) > 0$ for every i is automatically met. Hence equality (4) is equivalent to

$$e_1(x) b_1 \left(f_1(x)\right) = \cdots = e_I(x) b_I \left(f_I(x)\right).$$

This is therefore an equivalent condition for a risk-sharing rule to be efficient.

6.1 Decreasing Relative Risk Aversion

The following theorem is the main result on relative risk aversion for the general case. It establishes that if every consumer exhibits non-increasing relative risk aversion, then so does the representative consumer. Moreover, even the slightest heterogeneity in their cautiousness would cause the representative consumer's counterpart to be strictly decreasing. Note that neither of the negativity of the second derivative of the risk tolerance nor its elasticity being greater than unity implies the other. For example, even if a utility function u_i exhibits increasing, linear risk tolerance, then we have $\underline{d}_i \gtreqless 0$ if and only if $b_i'(x) \lesseqgtr 0$ for every $x > \max\{\underline{d}_i, 0\}$. The following theorem thus neither implies nor is implied by Theorem 15.

Theorem 19 *Let $x \in \left(\underline{d}, \overline{d}\right)$ and assume that $f_i(x) > 0$ for every i. If $b_i'\left(f_i(x)\right) \leq 0$ for every i, then $b'(x) \leq 0$. If, moreover, either there exists a consumer i such that*

$b_i'(f_i(x)) < 0$ or there exist two consumers i and j such that $t_i'(f_i(x)) \neq t_j'(f_j(x))$, then $b'(x) < 0$.

While inequalities (24) provide the range of the representative consumer's relative risk aversion, this theorem tells us how his relative risk aversion varies within this range as the aggregate consumption level increases. It generalizes Proposition 2 of Benninga and Mayshar (2000), which was shown for the special case where all consumers exhibit constant relative risk aversion.

The important implication of this theorem is that if the representative consumer is interpreted as representing the entire economy of heterogeneous consumers rather than it literally is a single consumer, then the assumption that he has a constant relative risk aversion is implausible, as it is not compatible even with a slight heterogeneity in the individual consumers. This point is particularly relevant to the equity premium puzzle of Mehra and Prescott (1985). They showed that the classical, representative consumer model cannot explain the large expected equity premium observed in the U.S. stock market during the last one hundred years unless the representative consumer unreasonably risk-averse. There has since been a large body of literature, as surveyed, for example, by Kocherlakota (1996), that attempts to explain the large premium by introducing non-expected utility functions; transaction costs; incomplete financial markets; and other types of market imperfections. To be precise, what Mehra and Prescott showed is that, in the Lucas (1978) model, if an economic modeler wishes to match the observed equity premium by choosing a value for the *constant* relative risk aversion for the representative consumer, then the value must be chosen so high as to be incompatible with casual introspection and laboratory experiments (of individual behavior). This result, however, does not automatically negate the possibility of reconciling the observed data with the Lucas model with a utility function exhibiting reasonable, *decreasing* relative risk aversion.[3]

It is perhaps helpful to put on record a simple, immediate corollary of this theorem, without involving the heterogeneity result.

[3] But this might actually *deepen* the puzzle. We have heard this sort of argument at Kazuya Kamiya's presentation at Kobe University, Japan, in April 1999. Note also that by inequalities (24), the *level* of the relative risk aversion must still be within the range of the individual counterparts, which may significantly limit the extent of solvability of the equity premium puzzle.

Corollary 20 *If $\underline{d}_i \geq 0$ and b_i is a non-increasing function for every i, then so is b.*

The proof of Theorem 19 is given in Appendix C.

The case where every consumer has constant relative risk aversion is thus a special case of this. We now give an implication of Theorem 19 to the derivative asset pricing. Our exercise here is similar to Proposition 9, but differs from it in that we approximate the representative consumer's utility function, which exhibits decreasing relative risk aversion, by a utility function exhibiting constant relative risk aversion. The following proposition is therefore closer to the approximation considered by Benninga and Mayshar (2000). Its conclusion is accordingly different from that of Proposition 9: While the call options prices, with sufficiently large exercise prices, are still underestimated, the next proposition also states that the put options prices are also underestimated, unlike Proposition 9.

In the rest of this subsection, we assume that $\underline{d}_i = 0$ for every i and $\overline{d}_i = \infty$ for some i. We define π as in equality (3). We, however, re-define $\widehat{t}$ by $\widehat{t}(x) = x/b(y) = xt(y)/y$ for a fixed $y > 0$. That is, we use the approximation $\widehat{t}$ by taking the representative consumer's relative risk aversion as constant at its true level at y. Define $\widehat{\pi}$ in the same manner as for equality (15), but using this $\widehat{t}$.

Proposition 21 *Suppose that b_i is non-increasing for every i and that either there exists a consumer i such that $b_i'(f_i(y)) < 0$ or there exist two consumers i and j such that $t_i'(f_i(y)) \neq t_j'(f_j(y))$. Then there exist two aggregate consumption levels $\underline{x}$ and $\overline{x}$, with $\underline{x} < \overline{x}$, having the following property: If a derivative asset $\varphi : \mathbb{R}_{++} \to \mathbb{R}$ satisfies $\varphi(\zeta) \geq 0$ almost surely, $\varphi(\zeta) > 0$ with strictly positive probability, and $\varphi(\zeta) = 0$ almost surely conditional on $\underline{x} < \zeta < \overline{x}$, then*

$$E(\pi\varphi(\zeta)) > E(\widehat{\pi}\varphi(\zeta)). \tag{25}$$

This proposition establishes the existence of two aggregate consumption levels $\underline{x}$ and $\overline{x}$ such that if a derivative asset pays off only if the aggregate consumption consumption level turns out to fall outside the range $(\underline{x}, \overline{x})$, then its price is underestimated by $\widehat{t}$. Hence the prices of both call and put options, with sufficiently high and low exercise prices respectively, are underestimated by an approximation taking the representative consumer's relative risk aversion as constant.

Proof of Proposition 21 We first prove that

$$\int_y^x \left(\frac{1}{\widehat{t}(x)} - \frac{1}{t(s)}\right) ds \to \infty \tag{26}$$

as $x \to \infty$ or as $x \to 0$. Indeed, by Theorem 19, $b'(y) < 0$. Hence $b'(x) < 0$ for every x in a neighborhood of y. But this is equivalent to $xt'(x)/t(x) > 1$ and hence the function $x \mapsto t(x)/x$ is strictly increasing in the neighborhood. So let $\underline{y}$ and $\overline{y}$ be in the neighborhood and satisfy $\underline{y} < y < \overline{y}$. Since, by Corollary 20, $b'(x) \leq 0$ for *every* $x > 0$, the function $x \mapsto t(x)/x$ is non-decreasing over the entire $\mathbb{R}_{++}$. Thus, for every $s > \overline{y}$, $t(s)/s > t\left(\overline{y}\right)/\overline{y}$ and hence

$$\frac{1}{\widehat{t}(s)} - \frac{1}{t(s)} = \frac{y}{t(y)s} - \frac{1}{t(s)} > \frac{y}{t(y)s} - \frac{\overline{y}}{t\left(\overline{y}\right)s} = \left(\frac{y}{t(y)s} - \frac{\overline{y}}{t\left(\overline{y}\right)}\right)\frac{1}{s}.$$

Thus

$$\int_y^x \left(\frac{1}{\widehat{t}(x)} - \frac{1}{t(s)}\right) ds > \left(\frac{y}{t(y)} - \frac{\overline{y}}{t\left(\overline{y}\right)}\right)\int_{\overline{y}}^x \frac{ds}{s},$$

and the right hand side diverges to infinity as x goes to infinity. Thus (26) is shown for the case where x goes to infinity. On the other hand, for every $s < \underline{y}$, $t(s)/s < t\left(\underline{y}\right)/\overline{y}$ and hence $\frac{1}{\widehat{t}(s)} - \frac{1}{t(s)} < \left(\frac{y}{t(y)} - \frac{\underline{y}}{t\left(\underline{y}\right)}\right)\frac{1}{s}$. Thus

$$\int_y^x \left(\frac{1}{\widehat{t}(x)} - \frac{1}{t(s)}\right) ds > \left(\frac{\underline{y}}{t\left(\underline{y}\right)} - \frac{y}{t(y)}\right)\int_x^{\underline{y}} \frac{ds}{s},$$

and the right hand side diverges to infinity as x goes to zero. Thus (26) is shown for the case where x goes to zero.

Now, by (17) and (26), there are $\underline{x}$ and $\overline{x}$, with $\underline{x} < \overline{x}$, such that if $\zeta < \underline{y}$ or if $\zeta > \overline{y}$, then $\pi/\widehat{\pi} > 1$. Then $\underline{x}$ and $\overline{x}$ have the properties in the proposition. ■

6.2 Limit Behavior

The key observation for the analysis of the limit behavior of the representative consumer's relative risk aversion is that $\lim_{x_i \to \infty} b_i(x_i) = \lim_{x_i \to \infty} x/t_i(x_i) = \lim_{x_i \to \infty} 1/t_i'(x_i)$, when the limits indeed exist, due to L'Hopital's rule; and an analogous relation holds for limits as $x_i \to 0$. This allows us to apply Proposition 10 and 11 to the relative risk aversion. To simplify the analysis, we use the following assumption.

Assumption 22 For every consumer j, $\underline{d}_j \geq 0$, $\overline{d}_j = \infty$, t_j is a convex function, $t_j(x_j) \to 0$ as $x_j \to \underline{d}_j$, and $\limsup_{x_j \to \infty} b_j(x_j)$ is finite.

This assumption can be satisfied by utility functions exhibiting strictly increasing, linear risk tolerance, with $\underline{d}_i \geq 0$. It implies that t_i' is a non-decreasing function. Thus $t_i'(x_i)$ does not diverge to infinity as $x_i \to \underline{d}_i$. Since $t_i(x_i) > 0$ for every x_i and $t_i(x_i) \to 0$ as $x_i \to \underline{d}_i$, $t_i'(x_i)$ does not converge to any negative number or diverge to negative infinity as $x_i \to \underline{d}_i$. It thus converges to a non-negative number and hence $\lim_{x_i \to \underline{d}_i} t_i'(x_i)$ is well defined and non-negative.

The following lemma is necessary to make sure that the statement of the main result of this subsection is well defined. The proof is given in Appendix D.

Lemma 23 *Under Assumption 22, $b_i(x_i)$ and $1/t_i'(x_i)$ converge to the same (finite) non-negative number as $x_i \to \infty$. If, moreover, $\underline{d}_i = 0$ and $\lim_{x_i \to \underline{d}_i} t_i'(x_i) > 0$, then $b_i(x_i) \to \left(\lim_{x_i \to \underline{d}_i} t_i'(x_i)\right)^{-1}$ as $x_i \to \underline{d}_i$. Otherwise, $b_i(x_i) \to \infty$.*

With this lemma, we can now state the main result on the limit behavior of the representative consumer's relative risk aversion. This generalizes Proposition 3 of Benninga and Mayshar (2000).

Proposition 24 *Under Assumption 22, let consumer i be such that $\lim_{x_i \to \infty} b_i(x_i) \leq \lim_{x_j \to \infty} b_j(x_j)$ for every j, then $b(x) \to \lim_{x_i \to \infty} b_i(x_i)$ as $x \to \infty$. If, moreover, for every j, $b_j(x_j)$ converges to a non-negative (finite) number as $x_i \to \underline{d}_j$, and if consumer i is such that $\lim_{x_i \to \underline{d}_i} b_i(x_i) \geq \lim_{x_j \to \underline{d}_j} b_j(x_j)$ for every j, then $b(x) \to \lim_{x_i \to \underline{d}_i} b_i(x_i)$ as $x \to \underline{d}$. If, on the other hand, $b_j(x_j)$ diverges to infinity as $x_j \to \underline{d}_j$ for some j, then so does $b(x)$ as $x \to \underline{d}$.*

The proof of Proposition 24 is also given in Appendix D.

6.3 Linear Risk Tolerance

In this subsection, as in Section 5, we assume that

$$t_i(x_i) = \tau_i + \gamma_i x_i$$

for every i and $x_i \in \left(\underline{d}_i, \overline{d}_i\right)$. In addition, we assume that $\tau_i \leq 0$ and $\gamma_i > 0$ for every i. This sign restriction is equivalent to $\underline{d}_i \geq 0$, and implies that $\overline{d}_i = \infty$. Note that

$$b_i\left(x_i\right) = \frac{x_i}{\tau_i + \gamma_i x_i} = \frac{x_i}{\gamma_i\left(x_i - \underline{d}_i\right)}.$$

Thus if $\tau_i = \underline{d}_i = 0$, then $b_i\left(x_i\right)$ equals $1/\gamma_i$ and u_i exhibits constant relative risk aversion. If not, then $b_i\left(x_i\right) > 1/\gamma_i$, $b_i'\left(x_i\right) < 0$, $b_i\left(x_i\right) \to 1/\gamma_i$ as $x_i \to \infty$, and $b_i\left(x_i\right) \to \infty$ as $x_i \to \underline{d}_i$.

The following result is a straightforward application of Propositions 19 and 24 to the case of linear risk tolerance. We omit the proof.

Proposition 25

1. *$b'(x) \leq 0$ for every $x \in (\underline{d}, \infty)$, and $b(x) \to 1/\overline{\gamma}$ as $x \to \infty$.*

2. *If $\tau_1 = \cdots = \tau_I = 0$ and $\gamma_1 = \cdots = \gamma_I$, then $\underline{d} = 0$ and $b(x) = \gamma_i$ for any i and $x \in (0, \infty)$. Otherwise, $b'(x) < 0$ for every $x \in (0, \infty)$.*

3. *If $\tau_1 = \cdots = \tau_I = 0$, then $\underline{d} = 0$ and $b(x) \to 1/\underline{\gamma}$ as $x \to 0$. Otherwise, $b(x) \to \infty$ as $x \to 0$.*

7 Background Risks

7.1 Setup and Preliminary Results

In this section, we investigate how the presence of background risks will affect the individual consumers' risk tolerance and risk-sharing rules. Specifically, we show that if an individual consumer's utility function exhibits non-decreasing risk tolerance (so that his absolute risk aversion is non-increasing and his cautiousness is non-negative), then the introduction of background risk induces him to be more cautious. According to Proposition 6, this result suggests that consumers with background risks have convex risk-sharing rules while those without them have concave risk-sharing rules. We can only give a formal result to this effect for consumers without any background risk in an economy in which all consumers' utility functions exhibit a common constant cautiousness. This is nothing but Theorem 3 of Franke, Stapleton, and Subrahmanyam (1998). The main point of our analysis is, however, that this sort of results cannot be obtained if consumers' degrees of cautiousness are even only slightly different. This is substantiated by Proposition 32.

We first give a formal framework for the discussion of background risk. For each consumer i, let Ω_i be a probability measure space describing his idiosyncratic risks. The realization of states in Ω_i should be interpreted to affect consumer i but not the others. In addition to the consumption levels we have considered so far, consumer i has another source of consumptions, characterized by a random variable $\xi_i : \Omega_i \to \mathbb{R}$, which is his *background risk*. The distribution function of ξ_i is denoted by $G_i : \mathbb{R} \to [0,1]$. For simplicity, we impose the following assumptions.

1. The support of the distribution of G_i is bounded, that is, there are two numbers $\underline{e}_i$ and $\overline{e}_i$ such that $G_i(\underline{e}_i) = 0$ and $G_i(\overline{e}_i) = 1$.
2. ξ_i has zero mean, that is, $\int_{\underline{e}_i}^{\overline{e}_i} y_i \, dG_i(y_i) = 0$.

The first assumption guarantees that all the expected values that we consider in the subsequent analysis are well defined and Leibnitz's rule is applicable, so that the order of integration and differentiation for smooth functions can be swapped. The second is a normalization and implies that $\underline{e}_i \leq 0$ and $\overline{e}_i \geq 0$.

The underlying story here is that, unlike the aggregate random consumption ζ that we have formulated in Section 2, the background risk ξ_i cannot be shared with other consumers and thus consumer i alone must absorb all of it. Hence, when he receives the random consumption ζ_i, then his final consumption is $\zeta_i + \xi_i$, from which he obtains the expected utility level $E(u_i(\zeta_i + \xi_i))$. We assume that ζ_i is always stochastically independent of ξ_i for every i. This, in particular, implies that the aggregate endowment ζ is independent of the ξ_i, and is true when the redistribution that defined the welfare maximization problem (1) is restricted to those random variables defined on a probability space independent of the ξ_i, because, for example, the realization of ξ_i can be observed by consumer i but not by the others.

By the law of iterated expectation, the above expected utility equals

$$E(E(u_i(\zeta_i + \xi_i) \mid \zeta_i)). \tag{27}$$

Hence, if we define his *induced utility function*

$$v_i(x_i) = E(u_i(x_i + \xi_i)),$$

then the expected utility level (27) equals $E(v_i(\zeta_i))$. Hence identifying properties of the efficient allocations of the aggregate endowment ζ with respect to the original utility functions u_i in the presence of the background risks ξ_i is equivalent to identifying those with respect to the induced utility functions v_i without any background risk. Given the earlier results of this paper, to characterize efficient allocations in the present context, all we need is to find implications of the background risks ξ_i on the tolerance and cautiousness of the induced utility functions v_i. In this reformulation, the realized consumption level, inclusive of the realized background risk, must of course be in the domain $(\underline{d}_i, \overline{d}_i)$ almost surely. To guarantee this, we concentrate on the consumption levels x_i in $(\underline{d}_i - \underline{e}_i, \overline{d}_i - \overline{e}_i)$. Hence this interval is the domain of v_i. Denote the corresponding risk tolerance by $s_i : (\underline{d}_i - \underline{e}_i, \overline{d}_i - \overline{e}_i) \to \mathbb{R}_{++}$. By Leibnitz's rule,

$$s_i(x_i) = -\frac{E(u_i'(x_i + \xi_i))}{E(u_i''(x_i + \xi_i))}.$$

Propositions 2 and 3 of Gollier and Pratt (1996) give conditions under which, if ξ_i has positive variance, then $s_i(x_i) > t_i(x_i)$, that is, the background risk makes the consumer less risk tolerant (or more risk averse). They called utility functions having this property as "risk vulnerable". The following result is an analogous result for cautiousness, $s_i'(x_i)$ and $t_i'(x_i)$, and useful for our subsequent analysis.

Proposition 26 *If there exists an $m \geq 0$ such that $t_i'(x_i) \geq m$ for every $x_i \in (\underline{d}_i, \overline{d}_i)$, then $s_i'(x_i) \geq m$ for every $x_i \in (\underline{d}_i - \underline{e}_i, \overline{d}_i - \overline{e}_i)$. If, moreover, $m > 0$, then $s_i'(x_i) > m$ for every $x_i \in (\underline{d}_i - \underline{e}_i, \overline{d}_i - \overline{e}_i)$.*

This proposition says that if the cautiousness of the original utility function u_i is never less than a non-negative constant m, then the cautiousness of the induced utility function v_i has the same property. Moreover, if the lower bound m can be taken to be strictly positive, then the cautiousness of v_i is strictly larger than m. Note that the condition $m \geq 0$ corresponds to non-increasing absolute risk aversion, and the case of $m > 0$ corresponds to strictly decreasing absolute risk aversion, which have been known to be a crucial condition to derive plausible comparative statics results in choice under uncertainty with background risks.

An immediate corollary of Proposition 26 is dealing with the case of linear risk tolerance. The proof is given in Appendix E.

Corollary 27 *If u_i exhibits linear risk tolerance with positive cautiousness γ_i, then $s_i'(x_i) > \gamma_i$ for every $x_i \in \left(\underline{d}_i - \underline{e}_i, \overline{d}_i - \overline{e}_i\right)$.*

To identify the curvature of efficient risk-sharing rules, we need to know the limit behavior of the cautiousness $s_i'(x_i)$ as $x_i \to \overline{d}_i$ and as $x_i \to \underline{d}_i$. The properties that we are going to obtain would hold for a wider class of utility functions exhibiting increasing risk tolerance (decreasing absolute risk aversion), but, to simplify the proofs, in the rest of this section, we concentrate on the case of strictly increasing linear risk tolerance.

Assumption 28 There are a $\tau_i \in \mathbb{R}$ and a $\gamma_i \in \mathbb{R}_{++}$ such that $t_i(x_i) = \tau_i + \gamma_i x_i$ for every $x_i > \underline{d}_i$.

This implies that $\underline{d}_i > -\infty$ and $\overline{d}_i = \infty$. Moreover, we impose the following assumption on the background risk ξ_i.

Assumption 29 The support of ξ_i consists of finitely many values and $\underline{e}_i$ equals the minimum of these values.

This assumption implies that the induced utility function $v_i : (\underline{d}_i - \underline{e}_i, \infty) \to \mathbb{R}$ also satisfies the Inada condition. Note that the Inada condition need not hold if ξ_i can take infinitely many values. For example, if u_i has constant cautiousness greater than one and if ξ_i is uniformly distributed over some bounded interval, then $\{u'(x_i) \mid x_i > \underline{d}_i - \underline{e}_i\}$ is bounded from above. The crucial aspect of this assumption is that there is a sufficiently high probability given around the lower bound $\underline{e}_i$.

Lemma 30 *Under Assumptions 28 and 29, $s_i'(x_i) \to \gamma_i$ as $x_i \to \underline{d}_i - \underline{e}_i$ and also as $x_i \to \infty$.*

This lemma states that the consumer's cautiousness, in the presence of the background risk, is closer to the cautiousness he would have were he to have no background risk, when his consumption level is sufficiently high or low. It therefore implies that the discrepancy between the cautiousness of the original utility function u_i and the induced one v_i would be largest in the intermediate region of consumption levels but negligible when the consumption level is very high or very low. The proof is given in Appendix E.

7.2 Risk-Sharing Rules with Background Risks

We are now ready to state the implications of our results on cautiousness with background risk (Corollary 27 and Lemma 30) on individuals' risk-sharing rules. In the rest of this section, $f : (\underline{d} - \underline{e}, \infty) \to (\underline{d}_1 - \underline{e}_1, \infty) \times \cdots \times (\underline{d}_I - \underline{e}_I, \infty)$ is an efficient risk-sharing rule with respect to the induced utility functions v_i, with background risks already incorporated into the v_i.

Proposition 31 (Franke, Stapleton, and Subrahmanyam (1998)) *Under Assumption 28, suppose that $\gamma_1 = \cdots = \gamma_I$ and that some of the ξ_i have strictly positive variances. Then, for each i, if the variance of ξ_i equals zero, then $f_i''(x) < 0$ for every x.*

This proposition was proved by Franke, Stapleton, and Subrahmanyam (1998). We give an alternative proof based on Corollary 27 and Proposition 6.

Proof of Proposition 31 If ξ_i has zero variance, then, by Proposition 6 and Corollary 27, $f_i''(x)/f_i'(x) \leq f_j''(x)/f_j'(x)$ for every j, and strict inequality for some j, as some consumers assumed to have background risk with strictly positive variance. We cannot have $f_i''(x) \geq 0$ for any x, because, if we had, then $f_j''(x) \geq 0$ for every j, with strict inequality for some j, which would contradict $\sum f_j''(x) = 0$. Hence $f_i''(x) < 0$ for every x. ■

Franke, Stapleton, and Subrahmanyam (1998) inferred that consumers with no background risk would sell options to replicate their concave risk-sharing functions. We claim, however, that their result is concerned only with the degenerate case of common cautiousness across all consumers, and that even the slightest deviation from this assumption would invalidate their inference. The following proposition makes this point precise by looking at a two-consumer case.

Proposition 32 *Under Assumptions 28 and 29, suppose that $I = 2$ and that the variance of ξ_1 is strictly positive but the variance of ξ_2 is zero. Then there exists a $\overline{\gamma} > \gamma_1$ such that if $\gamma_2 \in (\gamma_1, \overline{\gamma})$, then there exist an x, a $\underline{y}$, and a $\overline{y}$ such that $f_1''(x) > 0$ and $f_1''(y) < 0$ for every $y \leq \underline{y}$ and for every $y \geq \overline{y}$.*

This proposition can be read as follows. The economy consists of two consumers, both having linear, strictly increasing risk tolerance. Only the first one has a background risk. If the second consumer's cautiousness is only slightly higher than the first ($\overline{\gamma}$ being an upper bound on the second consumer's cautiousness), then the first consumer's risk-sharing rule is convex around some intermediate aggregate consumption level x, but it is concave if the aggregate consumption level is very small (smaller than $\underline{y}$) or very large (larger than $\overline{y}$). In short, even if the consumer without any background risk is only slightly more cautious than the other, then Franke, Stapleton, and Subrahmanyam's result remains valid only for intermediate aggregate consumption levels. The upper and lower tail behavior of individual risk-sharing rules are governed by the cautiousness of the (original) utility functions, not by the degree of exposure to background risks. We should also note that there are at least two inflection points of f_1. It is therefore inappropriate to say that the first consumer's risk-sharing rule is even remotely convex over the entire range of aggregate consumption levels. We can also see that for large aggregate consumption levels, the curvature of the risk-sharing rules is determined by the individual consumers' risk attitudes, not by background risks; and, in particular, consumers with larger background risks need not buy portfolio insurance.

Proof of Proposition 32 By Corollary 27, let $x_1 > \underline{d}_1 - \underline{e}_1$, then $s_1'(x_1) > \gamma_1$. So let $\overline{\gamma} \in (\gamma_1, s_1'(x_1))$. Since v_1 and $v_2 = u_2$ satisfy the Inada condition, as shown in Appendix A, there exists an $x > \underline{d} - \underline{e}$ such that $x_1 = f_1(x)$. Thus, if $\gamma_2 < \overline{\gamma}$, then $s_1'(f_1(x)) > t_2'(f_2(x))$. Hence, by Proposition 6, $f_1''(x) > f_2''(x)$ and thus $f_1''(x) > 0$. By Lemma 30, if $\gamma_2 > \gamma_1$, there exists an $\underline{y}_1 > \underline{d}_1 - \underline{e}_1$ such that $s_1'(y_1) < \gamma_2$ for every $y_1 \leq \underline{y}_1$. Again as shown in Appendix A, there exists an $\underline{y} > \underline{d} - \underline{e}_1$ such that $f_1(\underline{y}) = \underline{y}_1$. Hence $s_1'(f_1(y)) < t_2'(f_2(y))$ for every $y \leq \underline{y}$. Hence $f_1''(y) < 0$ for every $y \leq \underline{y}$. The existence of $\overline{y}$ such that $f_1''(y) < 0$ for every $y \geq \overline{y}$ can be similarly established. ■

Our result depends crucially on Assumption 29, that there are finitely many outcomes of the background risk. In Section 9, we give an example in which the background risk is uniformly distributed over an interval and the risk-sharing rules have only one inflection point, rather than two.

8 Multi-Period Models

We have been looking into the efficient risk-sharing rules and the representative consumer's risk aversion in the model of a single consumption period. We now claim that our results can be extended to models with multiple, even infinitely many, periods *if all consumers have time-separable and time-homogeneous expected utility functions and the same discount rate*, by giving a purely mathematical result to show how the multi-period models can be reduced to the single-period model under this assumption. It is very important to establish the validity of our results in multi-period models, as many results on efficient risk-sharing rules and the representative consumer's risk attitudes, such as those in Wang (1998), Aït-Sahalia and Lo (2000), and Benninga and Mayshar (2000), have been obtained in multi-period models; and we can establish more general or sharper results in those models by extending our previous results to the multi-period case.

As formulated in Section 2, the uncertainty of the economy is described by a probability measure space $(\Omega, \mathcal{F}, P)$. We now introduce the *time span* T, a subset of the non-negative half line $\mathbb{R}_+$, which represents the timings at which consumption can take place. We shall concentrate on the continuous-time case of $T = \mathbb{R}_+$ and the discrete-time case $T = \{0, 1, \ldots\}$, both of which extend to infinity. But the case of finite time length, such as $T = [0, \bar{t}]$ and $T = \{0, 1, \ldots, \bar{t}\}$ for some finite $\bar{t}$, and even cases where the continuous and discrete times are mixed, can be incorporated in the same manner. The gradual information revelation is described by the filtration $(F^t)_{t \in T}$. Note that the timing t is specified by a superscript. Denote by $\mathcal{B}(\cdot)$ the Borel σ-field. Let L be the linear space of all real-valued, progressively measurable processes with respect to the filtration $(F^t)_{t \in T}$, that is, the set of functions $\eta : T \times \Omega \to \mathbb{R}$ such that the restriction of η onto $\{s \in T \mid s \leq t\} \times \Omega$ is $\left(\mathcal{B}(\{s \in T \mid s \leq t\}) \otimes \mathcal{F}^t, \mathcal{B}(\mathbb{R})\right)$-measurable for every $t \in T$. We shall assume that all consumption processes under consideration are progressively measurable. This is not a stringent requirement because progressive measurability is implied by adaptivity if all sample paths are right continuous or if they are left continuous (Theorem 1 of Section 1.5 of Chung (1980), for example), and Brownian motion and the most commonly used stochastic integrals in finance have this property. For each $t \in T$, denote by η^t the partial function $\eta(t, \cdot) : \Omega \to \mathrm{I\!R}$, which is the $\mathcal{F}^t$-measurable random variable representing the value that the process η takes at time t.

Let r be a positive number, representing the compounded interest rate, common for all consumers. For each i, we have a utility function $u_i : \left(\underline{d}_i, \overline{d}_i\right) \to \mathrm{I\!R}$ just as in Section 2. In place of the consumption set Z_i of integrable random variables in that section, we now let the *consumption set* Y_i be a subset of L such that $\underline{d}_i < \eta_i < \overline{d}_i$ almost surely and the function $(t, \omega) \mapsto \exp(-rt)\eta_i(t, \omega)$ is integrable on the product measure space $(T \times \Omega, \mathcal{B}(T) \otimes \mathcal{F}, \lambda \otimes P)$, where λ denotes the Lebesgue measure restricted to T when $T = \mathrm{I\!R}_+$, and the counting measure when $T = \{0, 1, \ldots\}$. By Fubini's theorem, this is equivalent to saying that for almost all $\omega \in \Omega$, the sample path $t \mapsto \exp(-rt)\eta_i(t, \omega)$ is integrable with respect to λ and the random variable $\omega \mapsto \int_T \exp(-rt)\eta_i(t, \omega)\, \lambda(dt)$ is integrable with respect to P. For the continuous-time case $T = \mathrm{I\!R}_+$, the integral could be written as $E\left(\int_0^\infty \exp(-rt)\eta_i^t \, dt\right)$. In the discrete-time case $T = \{0, 1, \ldots\}$, the integral is nothing but $E\left(\sum_{t=0}^{\infty} \rho^t \eta_i^t\right)$, where $\rho = \exp(-r)$. To simplify the subsequent exposition, we shall use only the integral expression but the results are of course applicable to the discrete-time case as well. Denote by Y_i^* the subset of Y_i that consists of those η_i such that $u_i(\eta_i)$ is $(\lambda \otimes P)$-integrable.

Define a binary relation $\succsim_i$ on Y_i by letting, for each $\zeta_i \in Y_i$ and $\eta_i \in y_i$, $\zeta_i \succsim_i \eta_i$ if and only if either of the following two conditions is met: $\eta_i \notin Y_i^*$; or $\zeta_i \in Y_i^*$, $\eta_i \in Y_i^*$, and $E\left(\int_0^\infty \exp(-rt)u_i\left(\zeta_i^t\right)\, dt\right) \geq E\left(\int_0^\infty \exp(-rt)u_i\left(\eta_i^t\right)\, dt\right)$.

We say that a consumption allocation $(\eta_1, \cdots, \eta_I) \in Y_1 \times \cdots \times Y_I$ is *feasible* for an aggregate endowment η if $\sum \eta_i = \eta$ $(\lambda \otimes P)$-almost surely. An efficient allocation $(\eta_1^*, \cdots, \eta_I^*)$ is defined in terms of the binary relation $\succsim_i$ defined in the preceding paragraph in the same manner as in Section 2. The characterization of an efficient allocation in terms of the maximization problem and a sufficient condition for $\eta_i \in Y_i^*$ stated in that section remain true in the current context, with appropriate modifications.

Our assertion that a multi-period model can be reduced to a single-period model hinges on the following lemma. It draws heavily on Section 1.5 of Chung (1980).

Lemma 33 *There exists a σ-field $\mathcal{F}^*$ on $T \times \Omega$ and a probability measure P^* on $\mathcal{F}^*$ such that:*

1. *The space L of all progressively measurable functions coincides with the space of all $\mathcal{F}^*$-measurable functions on $T \times \Omega$.*

2. *For every $\eta \in L$, the function $(t, \omega) \mapsto \exp(-rt)\eta(t, \omega)$ is integrable with respect to $\lambda \otimes P$ (so that $E\left(\int_0^\infty \exp(-rt)\eta^t \, dt\right)$ is finite) if and only if η is integrable with respect to P^*. Moreover, if either (and hence both) of the conditions is met, then*

$$E^*(\eta) = rE\left(\int_0^\infty \exp(-rt)\eta^t \, dt\right), \tag{28}$$

   where E^* is the expectation with respect to P^*.

The underlying idea of this lemma is to treat the time span T as if it were another state space. The first part of the lemma says that the set of all progressively measurable processes can be identified with the set of all measurable functions once we introduce an appropriate σ-field $\mathcal{F}^*$ on the product of two state spaces T and Ω. Its second part claims that, once an appropriate probability measure P^* is introduced on the new σ-field $\mathcal{F}^*$, the integrability is preserved under these probability measures. It also shows that the integral under one probability measure is always a constant multiple of the other; and hence, by putting $\eta = u_i(\eta_i)$, which is the utility process of consumer i from the consumption process η_i, we can see that the expected utility ordering over consumption paths is identical between the two. As can be seen clearly in the proof, the discrete case poses no difficulty; indeed, it has been known and mentioned in Gollier (2001). The only difficulty arises in the continuous-time case, since it is necessary to induce an appropriate σ-field on the new product space.

Proof of Lemma 33 Following Section 1.5 of Chung (1980), we let $\mathcal{F}^*$ be the set of all subsets H of $T \times \Omega$ such that $H \cap ([0,t] \times \Omega) \in \mathcal{B}([0,t]) \otimes \mathcal{F}^t$ for every $t \in T$. It is easy to check that $\mathcal{F}^*$ is indeed a σ-field.

A function $\eta : T \times \Omega \to \mathrm{I\!R}$ is progressively measurable if and only if $\left(\overline{\eta}^t\right)^{-1}(B) \in \mathcal{B}([0,t]) \otimes \mathcal{F}^t$ for every $B \in \mathcal{B}(\mathrm{I\!R})$ and every $t \in T$, where $\overline{\eta}^t$ denotes the restriction of η on $[0,t] \times \Omega$. But $\left(\overline{\eta}^t\right)^{-1}(B) = \eta^{-1}(B) \cap ([0,t] \times \Omega)$. The progressive measurability is thus equivalent to saying that $\eta^{-1}(B) \cap ([0,t] \times \Omega) \in \mathcal{F}^t$ for every $B \in \mathcal{B}(\mathrm{I\!R})$ and every $t \in T$, which is in turn equivalent to $\eta^{-1}(B) \in \mathcal{F}^*$ for every $B \in \mathcal{B}(\mathrm{I\!R})$. That is, η is $\mathcal{F}^*$-measurable. This completes the proof of part 1.

To define another probability measure P^* on $T \times \Omega$, note first that, according to Section 1.5 of Chung (1980), for each subset H of $T \times \Omega$, $H \in \mathcal{F}^*$ if and only if $\chi_H \in L$, where $\chi_H : T \times \Omega \to \{0,1\}$ is the indicator function of H. Hence the function $(t,\omega) \mapsto \exp(-rt)\chi_H(t,\omega)$ is integrable with respect to $\lambda \otimes P$. So define $P^* : \mathcal{F}^* \to \mathrm{I\!R}_+$ by $P^*(H) = rE\left(\int_0^\infty \exp(-rt)\chi_H^t \, dt\right)$. It is easy to show that P^* is σ-additive and satisfies $P^*(T \times \Omega) = rr^{-1} = 1$. Hence P^* is a probability measure on $\mathcal{F}^*$.

It remains to prove equality (28). But, by the definition of P^*, it is true for every simple function of $\mathcal{F}^*$. Hence, by continuity of the integral, it is true for every P^*-integrable η. ■

By this lemma, the time-separable, time-homogeneous intertemporal expected utility function $E\left(\int_0^\infty \exp(-rt)u_i\left(\eta_i^t\right) \, dt\right)$ on Y_i can be identified with the expected utility function $E^*\left(u_i\left(\eta_i\right)\right)$ on a single consumption period with σ-field $\mathcal{F}^*$ and probability measure P^*. Since Lemma 1 is applicable to the probability measure space $(T \times \Omega, \mathcal{F}^*, P^*)$, the efficient allocations of the multi-period models can be expressed by means of the same risk-sharing rules as defined in Section 2, which is now regarded as time-separable and time-homogeneous. This implies that the representative consumer has a time-separable, time-homogeneous intertemporal expected utility function with the same discount rate r, and all the results in the preceding sections apply to the multi-period models with appropriate modifications. Some of its economic interpretations, however, need some care. For example, a risk-free asset in the new probability space $(T \times \Omega, \mathcal{F}^*, P^*)$ would be the perpetual bond, which pays one unit under any circumstance *at any point in time*, rather than just at a point in time. As another example, the mutual fund theorem in $(T \times \Omega, \mathcal{F}^*, P^*)$ would also imply that the *intertemporal* sharing rule is linear.

9 Numerical Examples

This section serves to illustrate some of our results graphically. Throughout this section all individual consumers' utility functions are assumed to exhibit constant relative risk aversion, and are of the form

$$u_i(x_i) = \frac{x_i^{1-\beta_i}}{1-\beta_i},$$

where $\beta_i = 1/\gamma_i > 0$. Their risk tolerance is then given by $t_i(x_i) = \gamma_i x_i = x_i/\beta_i$. The weights λ_i in the maximization problem (1) are all set equal to one. Re-scaling the individual utility functions or choosing a different set of the weights λ_i would change the quantitative results but not much of the qualitative results, except that the individual risk-sharing rules all intersect at exactly the same point in Figures 3 and 4. The figures below are numerically calculated and then plotted using the constrained optimization package in GAUSS. The values of the risk tolerance and weights are chosen to enhance graphical effects, not to fit to empirical findings.

Figure 3 shows the risk-sharing rules of an economy with three consumers having differing constant coefficients β_i of relative risk aversion. The most and the least risk averse consumers have a concave and a convex risk-sharing rule, respectively, just as suggested by Leland (1980) and Brennan and Solanki (1981). The risk-sharing rule of the, in terms of risk aversion, intermediate consumer is convex for the lower range of total consumption levels and becomes concave after the inflection point. This is precisely part 3 of Theorem 16.

The risk-sharing rules of a four-consumer economy are depicted in Figure 4. Their first and second derivatives are also given. Figure 5 shows the curvature, the ratio of second to first derivative, of risk-sharing rules for the same economy. Again, as in the three-consumer economy, the risk-sharing rules of the most and least risk averse consumers are concave and convex, respectively. Intermediately risk averse consumers have sharing rules which turn from convex for lower aggregate consumption levels to concave for higher ones. The validity of part 4 of Proposition 16 is confirmed: the inflection point of the individual risk sharing rule is higher for the less risk-averse intermediate consumer. This is better seen in the graphs of the two derivatives of the risk-sharing rules. Note also that while the rankings of the first and second derivatives change as the aggregate consumption

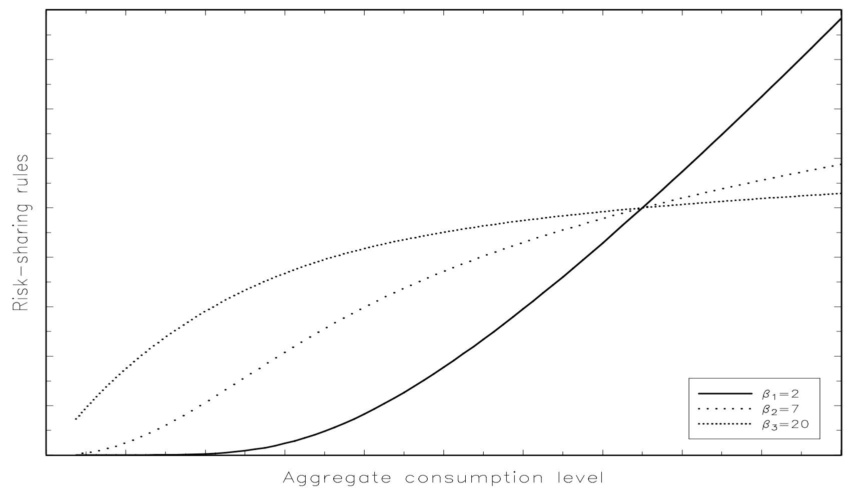

Figure 3: The risk-sharing rules in a three-consumer economy. Consumers have differing constant coefficients of relative risk aversion.

level varies, the ranking of the curvature never does. This is exactly what is implied by Proposition 6.

Figure 6 contains a collection of graphs of the second derivative of the risk-sharing rules in a two-consumer economy, where they have possibly differing constant coefficients β_i of relative risk aversion and a background risk ε_i, which can take $-\sigma_i$ and σ_i, each with probability 1/2. We assume that $\sigma_i = 1$ for one consumer and $\sigma_i = 0$ for the other, that is, only one consumer is exposed to the background risk. The consumers' induced utility function is therefore given by

$$v_i(x_i) = E\left(u_i(x_i + \varepsilon_i)\right) = \frac{1}{2(1-\beta_i)}\left((x_i - \sigma_i)^{1-\beta_i} + (x_i + \sigma_i)^{1-\beta_i}\right).$$

The top picture of Figure 6 is the case considered by Franke, Stapleton and Subrahmanyam (1998), where the two consumers have the same constant coefficient β_i of relative risk aversion. The consumer with a background risk has a risk-sharing rule that is convex everywhere. The consumer without any background risk necessarily has an everywhere concave risk-sharing rule. The next two pictures in Figure 6 show the second derivatives of the risk-sharing rules, where the consumer with the background risk is also slightly more risk averse than the other consumer, who has no background risk. The resulting risk-sharing rule is concave for low ($\leq \underline{y}$) and for high ($\geq \bar{y}$) levels of aggregate consumption levels and convex for intermediate levels. This is a particular instance of Proposition 32. The dominant force which shapes the risk-sharing rules is then the difference in the coefficients of risk aversion rather than in the background risks.

Figure 7 explores the second derivatives of the risk-sharing rules in a two-consumer economy, which differs from the preceding one, only in that the background risk ε_i is now uniformly distributed on the interval $[-\sigma_i, \sigma_i]$. The consumers' induced utility function is then given by

$$v_i(x_i) = E\left(u_i(x_i + \varepsilon_i)\right) = \frac{1}{2\sigma_i(1-\beta_i)(2-\beta_i)}\left((x_i+\sigma_i)^{2-\beta_i} - (x_i-\sigma_i)^{2-\beta_i}\right). \tag{29}$$

The top picture of Figure 7 is again the case considered by Franke, Stapleton and Subrahmanyam (1998), where the two consumers have the same degree of risk-aversion in their original utility. Just as in the discrete case, the consumer with a background risk has a convex risk-sharing rule and the consumer without any background risk has a concave one. The next two pictures in Figure 7 shows the second derivatives of the risk-sharing rules, where the consumer with the background risk is also slightly more risk averse than the other. The resulting risk-sharing rule is convex for low levels and concave for high levels of the aggregate consumption levels, with exactly one inflection point to separate the two regions. The difference in the second derivatives of the risk-sharing rules between the discrete and continuous cases is striking. It seems that even a slight difference in the tail behavior of the distribution of the background risk may lead to a large qualitative difference in the risk-sharing rules.

10 Conclusion

We have presented detailed properties of the efficient risk-sharing rules and the representative consumer's risk attitude in an economy under uncertainty where consumers have a homogeneous probabilistic belief over the state space but heterogeneous risk attitudes. In particular, we showed that heterogeneity in the consumers' cautiousness, the derivative of the reciprocal of the Arrow-Pratt measure of absolute risk aversion, is a key factor for the curvature of the risk-sharing rules and the speed at which the representative consumer's risk aversion decreases as the aggregate consumption level increases.

Although our results are mostly theoretical and qualitative, they have strong coherence with empirical findings. For example, Aït-Sahalia and Lo (2000) found from option prices that the representative consumer exhibits strictly decreasing relative risk aversion. The result by Ogaki and Zhang (2001) highlights the importance of identifying the qualitative properties of the efficient risk-sharing rules for a wide class of utility functions: Using a three-year data set on the food consumption in rural villages in India and Pakistan and an efficiency condition equivalent to our equality (22), they showed that the hypothesis that the intra-village risk-sharing is efficient is rejected if the utility functions are constrained to exhibit constant relative risk aversion, but not if they are allowed to exhibit decreasing relative risk aversion, albeit retaining linear risk tolerance. To estimate how large the consumers' risk aversions are and test whether the equilibrium allocation is efficient in a heterogeneous group of consumers, our results will serve as a useful theoretical foundation. Exploration of such empirical facts should be an interesting direction of future research.

A Existence of a Solution to the Maximization Problem (1)

In this appendix, we prove that for every λ and x, there exists a solution to the maximization problem (1), that is, $f_\lambda : (\underline{d}, \overline{d}) \to (\underline{d}_1, \overline{d}_1) \times \cdots \times (\underline{d}_I, \overline{d}_I)$ is well defined.

Indeed, for each i, the function $\lambda_i u_i' : (\underline{d}_i, \overline{d}_i) \to \mathbb{R}_{++}$ is strictly decreasing and onto. Hence it has an inverse, which we denote by $\varphi_i : \mathbb{R}_{++} \to (\underline{d}_i, \overline{d}_i)$. Then φ_i is also strictly decreasing and onto. Define $\varphi : \mathbb{R}_{++} \to (\underline{d}, \overline{d})$ by $\varphi = \sum \varphi_i$. Then φ is also strictly decreasing and onto. Then the composite mapping $\varphi \circ (\lambda_i u_i') : (\underline{d}_i, \overline{d}_i) \to (\underline{d}, \overline{d})$ is well defined. It is easy to check that the inverse of this mapping equals $f_{\lambda i}$. Note that we have also shown that $f_{\lambda i}'(x) > 0$ for every x and $f_{\lambda i}(x) \to \underline{d}$ as $x \to \underline{d}$ and $f_{\lambda i}(x) \to \overline{d}$ as $x \to \overline{d}$.

B Proof of Propositions 10 and 11

To prove Propositions 10 and 11, we need two lemmas. The first one is concerned with the ratio of two individual consumers' risk-sharing rules and their derivatives.

Lemma 34 *Let i and j be two consumers such that $\overline{d}_i = \infty$, $\overline{d}_j = \infty$, and inequality (18) holds. Then $f_i(x)/f_j(x) \to 0$ and $f_i'(x)/f_j'(x) \to 0$ as $x \to \infty$.*

Proof of Lemma 34 Let two real numbers δ_i and δ_j be such that

$$\limsup_{x_i \to \infty} t_i'(x_i) < \delta_i < \delta_j < \liminf_{x_j \to \infty} t_j'(x_j).$$

Since $\overline{d}_i = \infty$ and $t_i(x_i) > 0$ for every x_i, $\limsup_{x_i \to \infty} t_i'(x_i) \geq 0$. Hence $\delta_i > 0$ and $\delta_j > 0$. Then let $\underline{x} > \underline{d}$ be such that $t_i'(x_i) < \delta_i < \delta_j < t_j'(x_j)$ for every $x_i \geq f_i(\underline{x})$ and $x_j \geq f_j(\underline{x})$. Then, for such x_i and x_j,

$$\begin{aligned} t_i(x_i) <& \delta_i (x_i - f_i(\underline{x})) + t_i(f_i(x)), \\ t_j(x_j) >& \delta_j (x_j - f_j(\underline{x})) + t_j(f_j(x)). \end{aligned}$$

By equality (4),

$$\int_{\underline{x}}^{x} a_i(f_i(s)) f_i'(s)\, ds = \int_{\underline{x}}^{x} a_j(f_j(s)) f_j'(s)\, ds$$

for every $x \geq \underline{x}$. By integration by parts, this is equivalent to

$$\int_{f_i(\underline{x})}^{f_i(x)} a_i(s)\ ds = \int_{f_j(\underline{x})}^{f_j(x)} a_j(s)\ ds. \tag{30}$$

Thus

$$\int_{f_i(\underline{x})}^{f_i(x)} \frac{ds}{\delta_i (s - f_i(\underline{x})) + t_i(f_i(\underline{x}))} < \int_{f_j(\underline{x})}^{f_j(x)} \frac{ds}{\delta_j (s - f_j(\underline{x})) + t_j(f_j(\underline{x}))}.$$

Take the integral and then the exponential of both sides, then we obtain

$$\left(\frac{\delta_i (f_i(x) - f_i(\underline{x})) + t_i(f_i(\underline{x}))}{t_i(f_i(\underline{x}))}\right)^{1/\delta_i} < \left(\frac{\delta_j (f_j(x) - f_j(\underline{x})) + t_j(f_j(\underline{x}))}{t_j(f_j(\underline{x}))}\right)^{1/\delta_j},$$

because $0 < \delta_i < \delta_j$. Thus

$$f_i(x) - f_i(\underline{x}) + \frac{t_i(f_i(\underline{x}))}{\delta_i} < k\left(f_j(x) - f_j(\underline{x}) + \frac{t_j(f_j(\underline{x}))}{\delta_j}\right)^{\delta_i/\delta_j},$$

where

$$k = \frac{t_i(f_i(\underline{x}))}{\delta_i}\left(\frac{\delta_j}{t_j(f_j(\underline{x}))}\right)^{\delta_i/\delta_j} > 0.$$

Since $0 < \delta_i/\delta_j < 1$,

$$\frac{f_i(x) - f_i(\underline{x}) + \dfrac{t_i(f_i(\underline{x}))}{\delta_i}}{f_j(x) - f_j(\underline{x}) + \dfrac{t_j(f_j(\underline{x}))}{\delta_j}} \to 0 \tag{31}$$

as $x \to \infty$. Hence $f_i(x)/f_j(x) \to 0$ as $x \to \infty$.

By equality (2),

$$\frac{f_i'(x)}{f_j'(x)} = \frac{t_i\left(f_i(x)\right)}{t_j\left(f_j(x)\right)} < \frac{\delta_i}{\delta_j} \frac{f_i(x) - f_i\left(\underline{x}\right) + \dfrac{t_i\left(f_i\left(\underline{x}\right)\right)}{\delta_i}}{f_j(x) - f_j\left(\underline{x}\right) + \dfrac{t_j\left(f_j\left(\underline{x}\right)\right)}{\delta_j}}.$$

By (31), the far right hand side converges to 0. Hence $f_i'(x)/f_j'(x) \to 0$. ■

The next lemma is concerned with the limit behavior of the risk-sharing rules when the aggregate consumption levels converge to the minimum subsistence level.

Lemma 35 *Let i and j be two consumers such that $\underline{d}_i > -\infty$, $\underline{d}_j > -\infty$, $t_i\left(x_i\right) \to 0$ as $x_j \to \underline{d}_j$, $t_j\left(x_j\right) \to 0$ as $x_j \to \underline{d}_j$, and inequality (19) holds, then $\left(f_i(x) - \underline{d}_i\right) / \left(f_j(x) - \underline{d}_j\right) \to 0$ and $f_i'(x)/f_j'(x) \to 0$ as $x \to \underline{d}$.*

Proof of Lemma 35 Let two real numbers δ_i and δ_j be such that

$$\limsup_{x_j \to \underline{d}_j} t_j'\left(x_j\right) < \delta_j < \delta_i < \liminf_{x_i \to \underline{d}_i} t_i'\left(x_i\right).$$

Since $t_j\left(x_j\right) \geq 0$ for every x_j and $t_j\left(x_j\right) \to 0$ as $x_j \to \underline{d}_j$, we have $\limsup_{x_j \to \underline{d}_j} t_j'\left(x_j\right) \geq 0$. Hence $\delta_j > 0$ and $\delta_i > 0$. Then let $\overline{x} > \underline{d}$ be such that $t_j'\left(x_j\right) < \delta_j < \delta_i < t_i'\left(x_i\right)$ for every $x_i \leq f_i\left(\overline{x}\right)$ and $x_j \leq f_j\left(\overline{x}\right)$. Thus, for such x_i and x_j, $t_i\left(x_i\right) > \delta_i\left(x_i - \underline{d}_i\right)$ and $t_j\left(x_j\right) < \delta_j\left(x_j - \underline{d}_j\right)$. Since, for every $x \in \left(\underline{d}, \overline{x}\right)$,

$$\int_{f_j(x)}^{f_j(\overline{x})} \frac{ds}{t_j\left(s\right)} = \int_{f_i(x)}^{f_i(\overline{x})} \frac{ds}{t_i\left(s\right)},$$

we have

$$\int_{f_j(x)}^{f_j(\overline{x})} \frac{ds}{\delta_j\left(s - \underline{d}_j\right)} < \int_{f_i(x)}^{f_i(\overline{x})} \frac{ds}{\delta_i\left(s - \underline{d}_i\right)} \tag{32}$$

Thus

$$\left(\frac{f_j\left(\overline{x}\right) - \underline{d}_j}{f_j\left(x\right) - \underline{d}_j}\right)^{1/\delta_j} < \left(\frac{f_i\left(\overline{x}\right) - \underline{d}_i}{f_i\left(x\right) - \underline{d}_i}\right)^{1/\delta_i}.$$

Hence there exists a positive number k such that

$$f_i\left(x\right) - \underline{d}_i < k\left(f_j\left(x\right) - \underline{d}_j\right)^{\delta_i/\delta_j}. \tag{33}$$

Recall that both $f_i : \left(\underline{d}, \overline{d}\right) \to \left(\underline{d}_i, \overline{d}_i\right)$ and $f_j : \left(\underline{d}, \overline{d}\right) \to \left(\underline{d}_j, \overline{d}_j\right)$ are smooth, one-to-one, and onto, and have strictly positive derivatives. Hence there exists a $\varphi : \left(0, \overline{d}_j - \underline{d}_j\right) \to$

$\left(0, \overline{d}_i - \underline{d}_i\right)$ that is smooth, one-to-one, and onto, has strictly positive derivatives, and satisfies $f_i(x) - \underline{d}_i = \varphi\left(f_j(x) - \underline{d}_j\right)$. Thus, also by inequality (33), $0 < \varphi(z) < kz^{\delta_i/\delta_j}$ for every $z \in \left(0, \overline{d}_j - \underline{d}_j\right)$. Hence, by $\delta_j/\delta_i > 1$, $\varphi(z)/z \to 0$ and $\varphi'(z) \to 0$ as $z \to 0$. If z and x satisfy $z = f_i(x) - \underline{d}_i$, then $z \to 0$ if and only if $x \to \underline{d}$. Hence $(f_i(x) - \underline{d}_i) / \left(f_j(x) - \underline{d}_j\right) \to 0$ as $x \to \underline{d}$. Moreover, since $\varphi'(z) = f_j'(x)/f_i'(x)$, $f_j'(x)/f_i'(x) \to 0$ as $x \to \underline{d}$. ■

We can now turn to the proofs of Propositions 10 and 11

Proof of Proposition 10 Let J be the set of those consumers i for whom $\overline{d}_i = \infty$, then $J \supseteq \overline{I} \neq \emptyset$. For every $j \notin J$, $f_j(x)/x \to 0$ and $f_j'(x) \to 0$ as $x \to \infty$. For every $j \in J \setminus \overline{I}$, there exists a consumer $i \in J$ for whom inequality (18) holds. Since there are only finitely many consumers, this implies that there exists a consumer $i \in \overline{I}$ for whom inequality (18) holds. Since $\limsup_{x\to\infty} f_i(x)/x \leq 1$,

$$0 \leq \limsup_{x\to\infty} \frac{f_j(x)}{x} \leq \limsup_{x\to\infty} \frac{f_j(x)}{f_i(x)} \limsup_{x\to\infty} \frac{f_i(x)}{x} \leq \limsup_{x\to\infty} \frac{f_i(x)}{f_j(x)}.$$

By Lemma 34, the far right hand side equals zero. Thus $f_j(x)/x \to 0$. Since this is true for every $j \notin J$ and $\sum_{i=1}^{I} f_i(x)/x = 1$, we must have $\sum_{i\in\overline{I}} f_i(x)/x \to 1$ as $x \to \infty$.

Also, since $0 < f_i'(x) < 1$,

$$0 < f_j'(x) < \frac{f_j'(x)}{f_i'(x)}$$

and, for such i and j as in the preceding paragraph, the far right hand side converges to zero as $x \to \infty$. Hence $f_j'(x) \to 0$ as $x \to \infty$. If $\left\{t_j'(x_j) \mid \underline{d}_j < x_j < \overline{d}_j\right\}$ is bounded, then $t_j'(f_j(x)) f_j'(x) \to 0$ as $x \to \infty$ for every $j \notin \overline{I}$. Thus, by Lemma 3 and $0 < \sum_{i\in\overline{I}} f_i'(x) \leq 1$, we have

$$\begin{aligned}
&\limsup_{x\to\infty} t'(x) \\
&= \limsup_{x\to\infty} \sum_{i=1}^{I} t_i'(f_i(x)) f_i'(x) \\
&= \limsup_{x\to\infty} \sum_{i\in\overline{I}} t_i'(f_i(x)) f_i'(x) \\
&\leq \limsup_{x\to\infty} \max_{i\in\overline{I}} t_i'(f_i(x)) \\
&\leq \max_{i\in\overline{I}} \limsup_{x_i\to\infty} t_i'(x_i).
\end{aligned}$$

The other inequality,

$$\min_{i\in\overline{I}} \liminf_{x_i\to\infty} t_i'(x_i) \leq \liminf_{x\to\infty} t'(x),$$

can be shown analogously. ■

Note that the above proposition also implies that if $\underline{d}_i > -\infty$ for every i, then $\dfrac{\sum_{i\in\underline{I}}(f_i(x)-\underline{d}_i)}{x-\underline{d}} \to 1$ as $x\to\infty$.

Proof of Proposition 11 Let $j \notin \underline{I}$, then there exists a consumer $i \in \underline{I}$ for whom inequality (19) holds. Since $0 < \dfrac{f_i(x)-\underline{d}_i}{x-\underline{d}} < 1$ for every x,

$$0 \leq \limsup_{x\to\underline{d}} \frac{f_j(x)-\underline{d}_j}{x-\underline{d}} \leq \limsup_{x\to\underline{d}} \frac{f_j(x)-\underline{d}_j}{f_i(x)-\underline{d}_i} \limsup_{x\to\underline{d}} \frac{f_i(x)-\underline{d}_i}{x-\underline{d}} \leq \limsup_{x\to\underline{d}} \frac{f_j(x)-\underline{d}_j}{f_i(x)-\underline{d}_i}.$$

By Lemma 35, the far right hand side equals zero. Thus $\limsup_{x\to\underline{d}} \dfrac{f_j(x)-\underline{d}_j}{x-\underline{d}} = 0$. Hence $\dfrac{\sum_{i\in\underline{I}}(f_i(x)-\underline{d}_i)}{x-\underline{d}} \to 1$ as $x\to\underline{d}$.

Inequalities (20) can be shown in the same manner as in the proof of Proposition 10.

■

C Proof of Theorem 19

We need the following lemma to establish Theorem 19.

Lemma 36 *Let $x \in (\underline{d},\overline{d})$ and assume that $f_i(x) > 0$ for every i. If $b_i'(f_i(x)) = 0$ and $e_i(x) = 1$ for every i, then*

$$t_1'(f_1(x)) = \cdots = t_I'(f_I(x)). \tag{34}$$

Proof of Lemma 36 Since, for every i, $t_i'(f_i(x))\, f_i(x)/t_i(f_i(x)) = 1$ and $t_i(f_i(x)) = t(x)f_i'(x)$,

$$\frac{t_i'(f_i(x))\, f_i(x)}{t(x)f_i'(x)} = 1.$$

Since $f_i(x)/f_i'(x) = x$, the left hand side equals

$$\frac{t_i'(f_i(x))\, x}{t(x)}.$$

Hence $t_i'(f_i(x)) = t(x)/x$. The conclusion of the lemma thus follows. ■

Proof of Theorem 19 By equalities (9) and (13),

$$\frac{t'(x)x}{t(x)} = \sum f_i'(x)t_i'(f_i(x))\frac{x}{t(x)} \tag{35}$$

$$= \sum f_i'(x)t_i'(f_i(x))\, x\frac{f_i'(x)}{t_i(f_i(x))} \tag{36}$$

$$= \sum f_i'(x)e_i(x)\frac{t_i'(f_i(x))\, f_i(x)}{t_i(f_i(x))}. \tag{37}$$

We now show that

$$\sum f_i'(x)e_i(x) \geq 1. \tag{38}$$

Indeed, note that

$$\sum f_i'(x)\,(e_i(x))^{-1} = \sum \frac{f_i(x)}{x} = 1.$$

Since the weighted harmonic mean does not exceed the weighted arithmetic mean,

$$\left(\sum f_i'(x)\,(e_i(x))^{-1}\right)^{-1} \leq \sum f_i'(x)e_i(x).$$

Inequality (38) is thus proved. Since, as noted just before Theorem 19, $t_i'(f_i(x))\, f_i(x)/t_i(f_i(x)) \geq 1$, it implies that $t'(x)x/t(x) \geq 1$. Hence, $b'(x) \leq 0$.

As for the second part, if there exists a consumer i such that $b_i'(f_i(x)) < 0$, then $t_i'(f_i(x))\, f_i(x)/t_i(f_i(x)) > 1$. Hence, by (37) and (38), $t'(x)x/t(x) > 1$. Suppose, on the other hand, that $b_i'(f_i(x)) = 0$ for every i. If, in addition, $e_i(x) = 1$ for every i, then Lemma 36 implies equality (34), which contradicts our hypothesis that there exist two consumers i and j such that $t_i'(f_i(x)) \neq t_j'(f_j(x))$. Hence $e_i(x) \neq 1$ for some i. Since $\sum (f_i(x)/x)\, e_i(x) = 1$ and $\sum f_i(x)/x = 1$, this implies that $e_i(x) < 1 < e_j(x)$ for some i and j. Hence the weighted harmonic mean is strictly less than the weighted arithmetic mean:

$$\left(\sum f_i'(x)\,(e_i(x))^{-1}\right)^{-1} < \sum f_i'(x)e_i(x).$$

Since the left hand side equals one, $\sum f_i'(x)e_i(x) > 1$. Since $t_i'(f_i(x))\, f_i(x)/\, t_i(f_i(x)) \geq 1$ for every i, this implies that $t'(x)x/t(x) > 1$ and hence $b'(x) < 0$. ■

D Proof of Lemma 23 and Proposition 24

Proof of Lemma 23 If $t_i'(x_i) \leq 0$ for every $x_i \geq \underline{d}_i$, then $t_i(x_i)$ is non-increasing and hence $b_i(x_i) = x_i/t_i(x_i)$ diverges to infinity as $x_i \to \infty$, which contradicts the assumption that the set $\limsup_{x_i \to \infty} b_i(x_i)$ is finite. Hence $t_i'(x_i) > 0$ for some x_i. Since t_i' is a non-decreasing function, this implies that $\lim_{x_i \to \infty} 1/t_i'(x_i)$ exists (and is finite) and $t_i(x_i)$ diverges to infinity as $x_i \to \infty$. Thus, by L'Hopital's rule, $\lim_{x_i \to \infty} x_i/t_i(x_i)$ exists and equals $\lim_{x_i \to \infty} 1/t_i'(x_i)$. Since $b_i(x_i) = x_i/t_i(x_i)$, this completes the proof of the first part.

As for the second part, if $\underline{d}_i = 0$ and $\lim_{x_i \to \underline{d}_i} t_i'(x_i) > 0$, then we can apply L'Hopital's rule as in the proof of the first part to $\lim_{x_i \to \underline{d}_i} x/t_i(x_i)$ to show that $b_i(x_i) \to \left(\lim_{x_i \to \underline{d}_i} t_i'(x_i)\right)^{-1}$. The divergence to infinity when $\underline{d}_i = 0$ and $\lim_{x_i \to \underline{d}_i} t_i'(x_i) = 0$ can be obtained by extending t_i to $x_i = 0$ by letting $t_i(0) = 0$ and noticing then that $t_i'(0) = 0$. The divergence when $\underline{d}_i > 0$ follows immediately from $t_i(x_i) \to 0$ as $x_i \to \underline{d}_i$. ■

Proof of Proposition 24 By Lemma 23, if $\lim_{x_j \to \infty} b_j(x_j) > 0$ for every j, then $t_j'(x_j)$ converges to a positive (finite) number for every i. If consumer i is such that

$$\lim_{x_i \to \infty} b_i(x_i) \leq \lim_{x_j \to \infty} b_j(x_j) \tag{39}$$

for every j, then $\lim_{x_i \to \infty} t_i'(x_i) \geq \lim_{x_j \to \infty} t_j'(x_j)$ for every j. If, on the other hand, consumer i is such that $\lim_{x_i \to \infty} b_i(x_i) = 0$, then $t_i'(x_i) \to \infty$ and hence $\liminf_{x_i \to \infty} t_i'(x_i) \geq \liminf_{x_j \to \infty} t_j'(x_j)$ for every j. Thus, by Proposition 10, regardless of whether $\lim_{x_j \to \infty} b_j(x_j) > 0$ for every j, $f_i'(x) \to 0$ as $x \to \infty$ unless inequality (39) is satisfied for every j. Since $\lim_{x_j \to \infty} b_j(x_j)$ is well defined (finite), for every j,

$$\begin{aligned} \lim_{x \to \infty} b(x) &= \lim_{x \to \infty} \sum_{i=1}^{I} f_i'(x) b_i\left(f_i(x)\right) \\ &= \lim_{x \to \infty} \sum_{i \in \overline{I}} f_i'(x) b_i\left(f_i(x)\right) \\ &= \lim_{x \to \infty} b_i\left(x_i\right) \end{aligned}$$

for every i satisfying (39).

As for the limit as $x \to \underline{d}$, note first that, by Lemma 23, if, for every j, $b_j(x_j)$ converges to a (finite) non-negative number as $x_j \to \underline{d}_j$, then $t_j'(x_j) \to \left(\lim_{x_j \to \underline{d}_j} b_j(x_j)\right)^{-1}$. Hence $\liminf_{x_i \to \infty} b_i(x_i) \geq \liminf_{x_j \to \infty} b_j(x_j)$ if and only if $\liminf_{x_i \to \infty} t_i'(x_i) \leq \liminf_{x_j \to \infty} t_j'(x_j)$. On the other hand, again by Lemma 23, if consumer i is such that $b_i(x_i)$ diverges to infinity as $x_i \to \underline{d}_i$, then

$$\lim_{x_i \to \infty} t_i'(x_i) = 0 \leq \lim_{x_j \to \infty} t_j'(x_j)$$

for every j. Thus, by Proposition 11, regardless of whether $b_j(x_j)$ converges to a (finite) non-negative number for every j, $f_i'(x) \to 0$ as $x \to \infty$ unless inequality (39) is satisfied for every j. Thus, if $b_j(x_j)$ converges to a (finite) non-negative number for every j, then $b(x) \to \lim_{x_i \to \underline{d}_i} b_i(x_i)$ as $x \to \underline{d}$; otherwise $b(x) \to \infty$. ■

E Proof of Proposition 26 and Lemma 30

Proof of Proposition 26 Define $h_i : (\underline{e}_i, \overline{e}_i) \times \left(\underline{d}_i - \underline{e}_i, \overline{d}_i - \overline{e}_i\right) \to \mathbb{R}_{++}$ and $k_i : (\underline{e}_i, \overline{e}_i) \times \left(\underline{d}_i - \underline{e}_i, \overline{d}_i - \overline{e}_i\right) \to \mathbb{R}_{++}$ by

$$h_i(z_i, x_i) = \frac{u_i'(x_i + z_i)}{E(u_i'(x_i + \xi_i))},$$
$$k_i(z_i, x_i) = \frac{u_i''(x_i + z_i)}{E(u_i''(x_i + \xi_i))},$$

then

$$\int_{\underline{e}_i}^{\overline{e}_i} h_i(z_i, x_i)\ dG(z_i) = \int_{\underline{e}_i}^{\overline{e}_i} k_i(z_i, x_i)\ dG(z_i) = 1 \tag{40}$$

and hence the partial functions $h_i(\cdot, x_i)$ and $k_i(\cdot, x_i)$ have the property of a density function with respect to the distribution function G_i. Since

$$\frac{k_i(z_i, x_i)}{h_i(z_i, x_i)} = s_i(x_i)\, a_i(x_i + z_i),$$

the fraction of the left hand side is a non-increasing function of z_i and in fact strictly decreasing if $m > 0$. Hence either $h_i(\cdot, x_i) = k_i(\cdot, x_i)$ or $h_i(\cdot, x_i)$ first-order stochastically dominates $k_i(\cdot, x_i)$.[4]

Denote by $p_i(x_i)$ the prudence of u_i of Kimball (1990):

$$p_i(x_i) = -\frac{u_i'''(x_i)}{u_i''(x_i)}.$$

Also denote by q_i the prudence of the induced utility function v_i, then we have

$$q_i(x_i) = -\frac{E(u_i'''(x_i + \xi_i))}{E(u_i''(x_i + \xi_i))}.$$

[4] To be exact, we should say that the distribution whose Radon-Nikodym derivative with respect to the probability measure on $\mathbb{R}$ having the distribution function G_i is equal to h_i is equal to or first-order stochastically dominates the distribution whose Radon-Nikodym derivative with respect to the same probability measure on $\mathbb{R}$ is equal to k_i.

Moreover,

$$\frac{1}{s_i(x_i)} = \int_{\underline{e}_i}^{\overline{e}_i} a_i(x_i + z_i) h_i(z_i, x_i) \; dG_i(z_i), \tag{41}$$

$$q_i(x_i) = \int_{\underline{e}_i}^{\overline{e}_i} p_i(x_i + z_i) k_i(z_i, x_i) \; dG_i(z_i). \tag{42}$$

Since $s_i'(x_i) = s_i(x_i) q_i(x_i) - 1$, it suffices to show that $1/s_i(x_i) \leq q_i(x_i)/(m+1)$, with a strict inequality if $m > 0$. On the other hand, since $t_i'(x_i + z_i) = \dfrac{p_i(x_i + z_i)}{a_i(x_i + z_i)} - 1$, $a_i(x_i + z_i) \leq \dfrac{p_i(x_i + z_i)}{m+1}$. Hence, by the first-order stochastic dominance,

$$\begin{aligned}
\frac{1}{s_i(x_i)} &\leq \int_{\underline{e}_i}^{\overline{e}_i} a_i(x_i + z_i) k_i(z_i, x_i) \; dG_i(z_i), \\
&\leq \frac{1}{m+1} \int_{\underline{e}_i}^{\overline{e}_i} p_i(x_i + z_i) k_i(z_i, x_i) \; dG_i(z_i) \\
&= \frac{1}{m+1} q_i(x_i),
\end{aligned}$$

where the first weak inequality holds as a strict inequality if $m > 0$. This completes the proof. ■

It is clear from the proof that we can guarantee $s_i'(x_i) \geq m$ if only $t_i'(x_i + z_i) \geq m$ for every $z_i \in (\underline{e}_i, \overline{e}_i)$. That is, the latter inequality needs to hold only over the range of possible consumption levels.

Proof of Lemma 30 By equations (41), $s_i(x_i) \leq t_i(x_i + \overline{e}_i)$ and, since the prudence p_i is a strictly decreasing function, $q_i(x_i) \leq p_i(x_i + \underline{e}_i)$. Thus

$$\begin{aligned}
s_i'(x_i) &= s_i(x_i) q_i(x_i) - 1 \\
&\leq t_i(x_i + \overline{e}_i) p_i(x_i + \underline{e}_i) - 1 \\
&= \frac{t_i(x_i + \overline{e}_i)}{t_i(x_i + \underline{e}_i)} \big(t_i'(x_i + \underline{e}_i) + 1\big) - 1 \\
&= \frac{\tau_i + \gamma_i(x_i + \overline{e}_i)}{\tau_i + \gamma_i(x_i + \underline{e}_i)} (\gamma_i + 1) - 1.
\end{aligned}$$

Since the fraction of the far right hand side converges to one as $x_i \to \infty$, along with Corollary 27, we obtain $s_i'(x_i) \to \gamma_i$.

As for the limit of $s_i'(x_i)$ as $x_i \to \underline{d}_i - \underline{e}_i$, we first denote the finitely many values that ξ_i can take by $z_i^1, z_i^2, \ldots, z_i^N$ and their probabilities by $\theta_i^1, \theta_i^2, \ldots, \theta_i^N$. Without loss of generality, we can assume that z_i^1 is the minimum value that ξ_i can take, so that $z_i^1 = \underline{e}_i$ by Assumption 29. Then

$$s_i'(x_i) = s_i(x_i)\, q_i(x_i) - 1 \tag{43}$$

$$= \frac{\left(\sum_{n\geq 1} \theta_i^n u_i'(x_i + z_i^n)\right)\left(\sum_{n\geq 1} \theta_i^n u_i'''(x_i + z_i^n)\right)}{\left(\sum_{n\geq 1} \theta_i^n u_i''(x_i + z_i^n)\right)^2} - 1. \tag{44}$$

Divide the denominator and numerator of the fraction in equality (44) by both sides of

$$\frac{u_i'\left(x_i + z_i^1\right) u_i'''\left(x_i + z_i^1\right)}{t_i'\left(x_i + z_i^1\right) + 1} = \left(u_i''\left(x_i + z_i^1\right)\right)^2,$$

then we obtain

$$s_i'(x_i) = \left(t_i'(x_i + z_i) + 1\right) \frac{\left(\sum_{n\geq 1} \theta_i^n \frac{u_i'(x_i + z_i^n)}{u_i'\left(x_i + z_i^1\right)}\right)\left(\sum_{n\geq 1} \theta_i^n \frac{u_i'''(x_i + z_i^n)}{u_i'''\left(x_i + z_i^1\right)}\right)}{\left(\sum_{n\geq 1} \theta_i^n \frac{u_i''(x_i + z_i^n)}{u_i''\left(x_i + z_i^1\right)}\right)^2} - 1. \tag{45}$$

It is easy to show by the Inada condition that $u_i''\left(x_i + z_i^1\right) \to -\infty$ and $u_i'''\left(x_i + z_i^1\right) \to \infty$ as $x_i \to \underline{d}_i - \underline{e}_i$. Hence the ratios of the derivatives of the same order in equality (45) converges to 0 for every $n \neq 1$. Thus its right hand side converges to

$$(\gamma_i + 1)\frac{\left(\theta_i^1\right)^2}{\left(\theta_i^1\right)^2} - 1 = \gamma_i.$$

This completes the proof. ■

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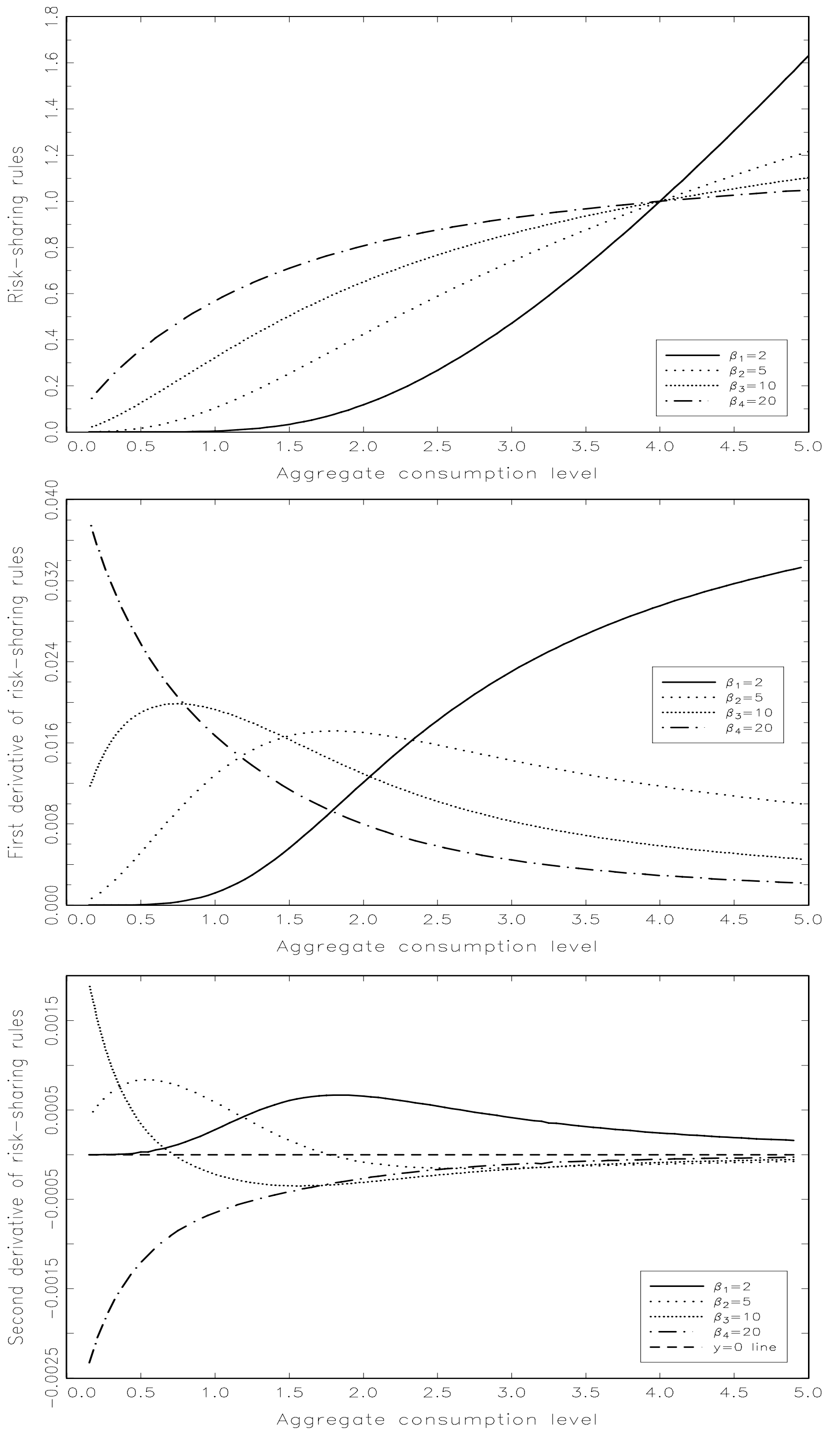

Figure 4: The risk-sharing rules and their first and second derivatives in a four-consumer economy. Consumers have differing constant coefficients of relative risk aversion.

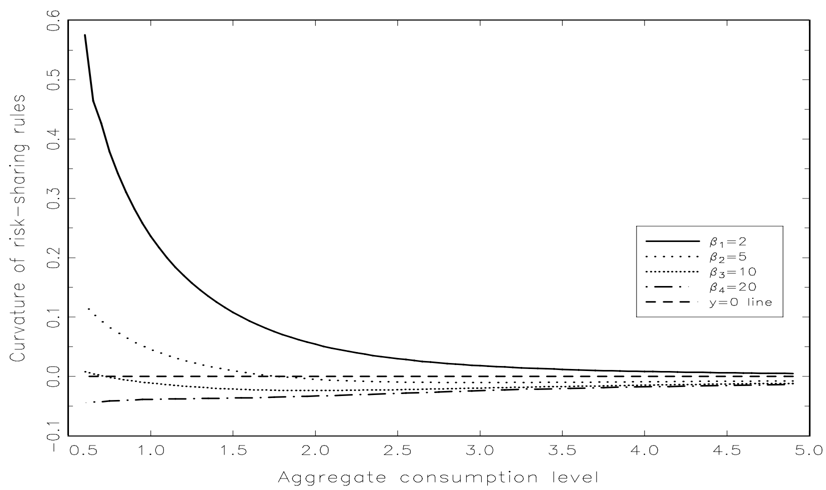

Figure 5: The curvature (ratio of second to first derivative) of risk-sharing rules in a four-consumer economy. Consumers have differing constant coefficients of relative risk aversion.

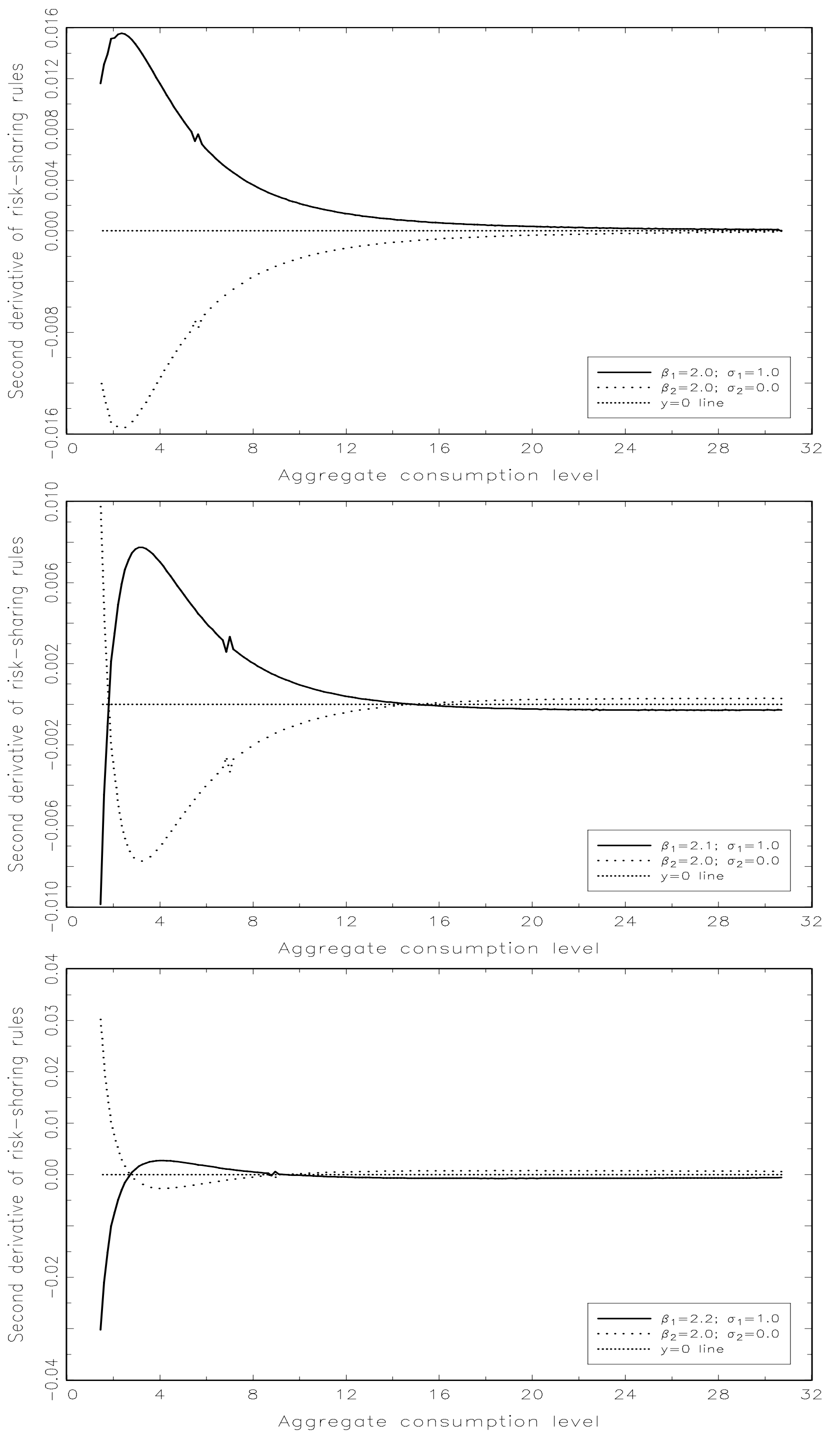

Figure 6: Second derivatives of risk-sharing rules in a two-consumer economy. They have possibly differing constant coefficients of relative risk aversions and only one of them has a *discrete* background risk.

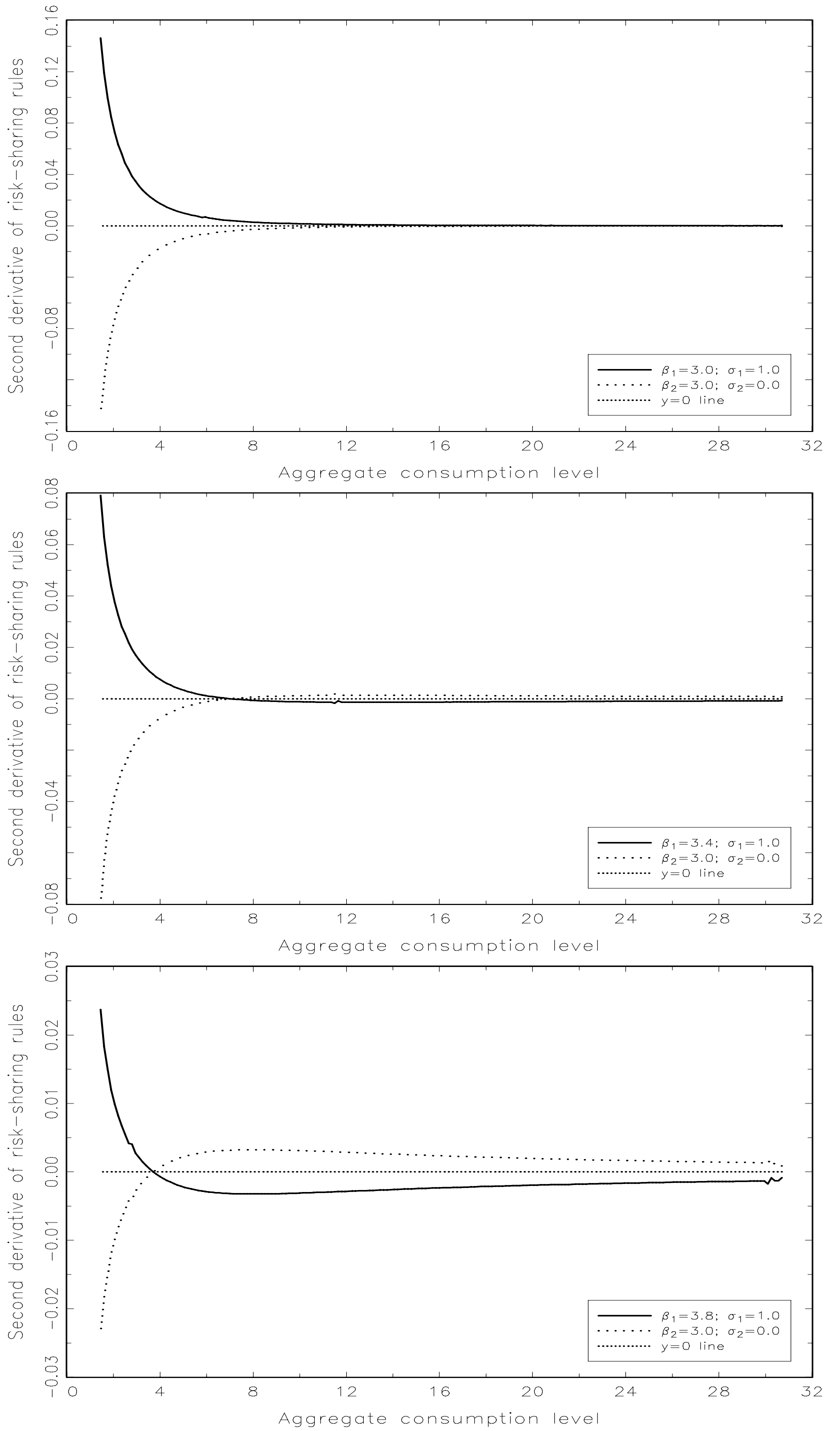

Figure 7: Second derivatives of risk-sharing rules in a two-consumer economy. They have (possibly) differing constant coefficients of relative risk aversion and only one of them has a *continuous* background risk.