A utility function can generate Giffen behavior

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Abstract

We present a specific utility function which generates Giffen behavior. The derived demand function of each good is not only continuous in its price and income but also partly increasing in its price and decreasing in income. Moreover, we show that Giffen behavior is compatible with any level of utility and an arbitrarily low share of income spent on the inferior good; thus the standard "margarine-butter paradigm" may not adequately explain Giffen's paradox.

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1 Introduction

Giffen's paradox is usually construed as a possible economic behavior acted by households with low real wealth levels. For a recent example, a standard textbook of microeconomic theory gives an intuitive explanation of Giffen behavior as follows:

> Low-quality goods may well be Giffen goods for consumers with low wealth levels. For example, imagine that a poor consumer initially is fulfilling much of his dietary requirements with potatoes because they are a low-cost way to avoid hunger. If the price of potatoes falls, he can then afford to buy other, more desirable foods

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that also keep him from being hungry. His consumption of potatoes may well fall as a result. Note that the mechanism that leads to potatoes being a Giffen good in this story involves a wealth consideration: When the price of potatoes falls, the consumer is effectively wealthier (he can afford to purchase more generally), and so he buys fewer potatoes....(Mas-Colell, Whinston, and Green, 1995, p.26).

The purpose of this paper is to give a theoretical example which clearly shows the existence of Giffen behavior to be distinguished from the standard interpretation. Specifically, we show that there exists a continuous, non-decreasing, quasi-concave utility function $u = u(x_1, x_2)$ which maps the two-dimensional positive orthant R^2_{++} onto the set of real numbers R such that it has the following properties.

Property I For any given positive income I and prices of good 1 and good 2, $p_1\ (\equiv p)$ and $p_2\ (\equiv 1)$, the textbook utility maximization problem has a unique, positive and interior optimal consumption bundle; continuous demand functions, say $x_1(p, I)$ and $x_2(p, I)$, exist for any positive pair (p, I). That is, those functions map the set of price-income pairs which is equal to R^2_{++} into the set of positive real numbers, R_{++}.

Property II The set of price-income pairs, R^2_{++}, is decomposed into three mutually exclusive subsets, Δ_g, Δ_i, and Δ_n, i.e., $R^2_{++} = \Delta_g \cup \Delta_i \cup \Delta_n$ and $\Delta_s \cap \Delta_t = \varnothing$, $s, t = g, i, n$, $s \neq t$, as follows. (i) Good 1 is a Giffen good [$x_1(p, I)$ is increasing in p and decreasing in I over the interior of Δ_g]; (ii) Good 1 is an inferior good with negatively sloped demand curve [$x_1(p, I)$ is decreasing in both p and I over the interior of Δ_i]; (iii) Good 1 is a normal good [$x_1(p, I)$ is decreasing in p and increasing in I over the interior of Δ_n].

Property III Define a set of positive pair (p, I)

$$\Omega(u) \equiv \{(p, I) : u = u(x_1(p, I), x_2(p, I))\}$$

For any utility level u there are three nonempty subsets $\Omega_g(u)$, $\Omega_i(u)$, and $\Omega_n(u)$ such that (i) $x_1(p, I)$ is increasing in p and decreasing in I over the interior of $\Omega_g(u)$; good 1 is a Giffen good; (ii) $x_1(p, I)$ is decreasing in both p and I over the interior of $\Omega_i(u)$; good 1 is an inferior good with negatively sloped demand curve; (iii) $x_1(p, I)$ is decreasing in p and increasing in I over the interior of $\Omega_n(u)$; good 1 is a normal good.

Property IV Denote a share of income spent on good 1 by θ, that is, $\theta \equiv px_1(p, I)/I$. There exist two upper bounds θ_g and θ_i such that (i) good 1 is a Giffen good if and only if $\theta < \theta_g$; (ii) good 1 is an inferior good if and only if $\theta < \theta_i$; (iii) θ_g and θ_i are independent of (p, I) and satisfy $0 < \theta_g < \theta_i$.

We emphasize that Property III means that Giffen behavior is possible, *irrespective of the standard of living,* and Property IV means that Giffen behavior is compatible with *an arbitrarily low share of income spent on the inferior good,* *not compatible with high share.* So Giffen behavior under this utility function cannot be comprehended through the standard interpretation, and reveals the theoretical possibility such that it is inappropriate to presume Giffen's paradox merely an exceptional phenomenon under extreme circumstances.

Let us briefly review the theoretical literature on Giffen's paradox. According to Weber (1997), Wold and Jureen (1953) are the first to present a specific utility function that generates Giffen behavior. Vandermeulen (1972) and Spiegel (1994) presented other specific utility functions that generate Giffen behavior.[1] On the other hand, Moffatt (2002) shows that for an arbitrarily given $(\tilde{x}_1, \tilde{x}_2)$ there exists a strictly quasi-concave utility function which generates a backward sloping price offer curve around $(\tilde{x}_1, \tilde{x}_2)$.

However, all of those specific functions lack either some standard properties as a utility function or analytical tractability due to smooth demand functions. And none of preceding researchers fully characterizes the domain of the demand functions to the extent of Property II and Property III above. As we shall show in the subsequent sections, subsets Δ_χ, $\Omega_\chi(u)$, $\chi = g, i, n$ and the upper bounds θ_g and θ_i are explicitly derived. Based on Property II, for an arbitrarily given point on the two-dimensional positive commodity space, we can draw the income expansion path and the price offer curve (see Figure 3B in this paper) both of which intersect with each other at the point. Moreover, find that on each income expansion path, Giffen behavior is accompanied by an arbitrarily high level of utility and low expenditure share on the inferior good. Thus, we can examine an asymptotic behavior of the demand for a Giffen good, which is impossible without specifying utility functions as we propose.[2]

This paper is organized as follows. Section 2 presents the specific utility function. Section 3 derives the demand functions of good 1 and good 2 from the utility function and proves Property I. Section 4 characterizes the demand functions and proves Property II, Property III, and Property IV. Section 5 concludes.

2 The utility function

2.1 The definition and the main assumption

Let

$$F(x_1, x_2) \equiv \alpha \ln x_1 + \beta \ln x_2 - \gamma x_1 x_2$$

The utility function we consider is defined as

$$u(x_1, x_2) \equiv \begin{cases} F(x_1, x_2) & \text{for } (x_1, x_2) \in R_+^2 - \Lambda_0, \\ \ln(\alpha/\gamma)^\alpha x_2^{\beta-\alpha} - \alpha & \text{for } (x_1, x_2) \in \Lambda_0, \end{cases} \tag{1}$$

[1] Vandermeulen (1972) also shows that expenditure shares are irrelevant to Giffen behavior, because it depends solely on the properties of the indifference map, that is, the slope of indifference curves along vertical lines.

[2] Indeed, none of preceding researchers does, because their specific functions lack global quasi-concavity or generate Giffen behavior only at a finite part of the commodity space.

where $\Lambda_0 \equiv \{(x_1, x_2) > (0,0) : x_1 x_2 \geq \alpha/\gamma\}$. Parameters, α, β, and γ, satisfy the following assumption.

ASSUMPTION: $\beta > \alpha > \frac{\beta}{2} > 0$ and $\gamma > 0$.

2.2 The properties

(1) has all the standard properties as a utility function.

First, both $u(x_1, x_2)$ and its partial derivatives with respect to x_1 and x_2 are continuous in (x_1, x_2) over the domain, as stated in the following lemma.

Lemma 1: *For any* $(\tilde{x}_1, \tilde{x}_2) \in R^2_{++}$,

$$\lim_{(x_1,x_2)\to(\tilde{x}_1,\tilde{x}_2)} u(x_1, x_2) = u(\tilde{x}_1, \tilde{x}_2), \tag{2}$$

$$\lim_{(x_1,x_2)\to(\tilde{x}_1,\tilde{x}_2)} u_i(x_1, x_2) = u_i(\tilde{x}_1, \tilde{x}_2), \tag{3}$$

where $u_i(x_1, x_2)$ *is the partial derivative of* $u(x_1, x_2)$ *with respect to* x_i, $i = 1, 2$.

Proof. See Appendix. (QED)

Second, we prove the following lemma, and then draw some indifference curves of the utility function (see Figure 1). Note that the slope of any indifference curve is zero in Λ_0.

Lemma 2: $u(x_1, x_2)$ *is non-decreasing and quasi-concave in* x_1 *and* x_2 *over the domain* R^2_{++}.

Proof. In fact, the function (1) is concave in x_1 and x_2: For any $(x_1, x_2) \in R^2_+ - \Lambda_0$,

$$\begin{aligned}
u_1(x_1, x_2) &= \frac{\alpha}{x_1} - \gamma x_2 > 0, \\
u_2(x_1, x_2) &= \frac{\beta}{x_2} - \gamma x_1 > 0, \\
u_{11}(x_1, x_2) &\equiv \frac{\partial^2}{\partial x_1^2} u(x_1, x_2) = -\frac{\alpha}{x_1^2} < 0, \\
u_{22}(x_1, x_2) &\equiv \frac{\partial^2}{\partial x_2^2} u(x_1, x_2) = -\frac{\beta}{x_2^2} < 0, \\
u_{12}(x_1, x_2) &\equiv \frac{\partial^2}{\partial x_2 \partial x_1} u(x_1, x_2) = -\gamma < 0,
\end{aligned}$$

$$\begin{vmatrix} u_{11} & u_{12} \\ u_{12} & u_{22} \end{vmatrix} = \frac{\alpha\beta}{x_1^2 x_2^2} - \gamma^2$$

$$= \left[\frac{\sqrt{\alpha\beta}}{x_1 x_2} + \gamma\right]\left[\frac{\sqrt{\alpha\beta}}{x_1 x_2} - \gamma\right]$$

$$> 0, \quad (\because \alpha < \sqrt{\alpha\beta}, \text{ due to } \textbf{ASSUMPTION})$$

and for any $(x_1, x_2) \in \Lambda_0$,

$$u_1 = u_{12} = u_{11} = 0,$$
$$u_2 = \frac{\beta - \alpha}{x_2} > 0, \quad u_{22} = -\frac{\beta - \alpha}{x_2^2} < 0$$

It follows that the utility function is quasi-concave over the commodity space R_+^2 and strictly concave on the subset $R_+^2 - \Lambda_0$. (QED)

Third, the utility level on its border, $\alpha = \gamma x_1 x_2$, is

$$u\left(\frac{\alpha}{\gamma x_2}, x_2\right) = \ln\left(\frac{\alpha}{\gamma}\right)^{\alpha} x_2^{\beta-\alpha} - \alpha$$

Due to **ASSUMPTION** ($\beta > \alpha$), it is monotonically increasing in x_2 along the hyperbola. Since $\lim_{x_2 \to 0} u\left(\frac{\alpha}{\gamma x_2}, x_2\right) = -\infty$ and $\lim_{x_2 \to \infty} u\left(\frac{\alpha}{\gamma x_2}, x_2\right) = \infty$, for any real number u there is (x_1, x_2) in R_{++}^2 such that $u = u(x_1, x_2)$. In a word, $u(x_1, x_2)$ is a mapping from R_{++}^2 *onto* the set of real numbers R.

Finally, considering that $u(0, x_2) = u(x_1, 0) = -\infty$, we see that any indifference curve cuts neither horizontal nor vertical axis of coordinates; both goods are indispensable.

3 The derivation of the demand function

3.1 The utility maximization problem

Let us show that for any positive price-income pair (p, I) the utility maximization problem,

$$\max_{0 \le x_1 \le I/p} u(x_1, I - px_1) \tag{4}$$

has a unique, interior and positive optimal consumption pair $(x_1(p, I), x_2(p, I))$ in $R_{++}^2 - \Lambda_0$.

First of all, note that, depending on the values of p and I, we have two types of budget constraints. Let $G(x_1) \equiv \alpha - \gamma x_1(I - px_1)$.

Type 1 budget constraints such that $G(x_1) > 0$ for any $x_1 \in (0, I/p)$. An example is BEC in Figure 1.

Type 2 budget constraints which are not of type 1; that is, each type-2 budget contstraint satisfies the condition as follows

$$^{\exists}x_1 \in (0, I/p) \text{ such that } G(x_1) \leq 0$$

An example is $B'E'C'$ in Figure 1.

Let us derive the optimal consumption pair for each type.

3.2 Type-1 budget constraint

Taking into account the definition of the utility function, we have

$$u(x_1, I - px_1) = \alpha \ln x_1 + \beta \ln(I - px_1) - \gamma x_1 (I - px_1)$$

Since

$$\lim_{x_1 \to 0} u(x_1, I - px_1) = \lim_{x_1 \to I/p} u(x_1, I - px_1) = -\infty,$$

there is an interior optimal solution to (4) which satisfies the first-order condition[3]

$$\begin{aligned} \frac{d}{dx_1} u(x_1, I - px_1) &= \frac{\alpha}{x_1} - \frac{\beta p}{I - px_1} - \gamma(I - 2px_1) \\ &= \left[\frac{\alpha}{x_1} - \gamma(I - px_1)\right] - p\left[\frac{\beta}{I - px_1} - \gamma x_1\right] \\ &= 0 \end{aligned} \tag{5}$$

To show the uniqueness of the optimal solution, let us prove the following lemma.

Lemma 3: *For any $\tilde{x}_1 \in (0, I/p)$ satisfying $G(\tilde{x}_1) > 0$, the second derivative of $u(x_1, I - px_1)$ with respect to x_1 is negative, i.e.,*

$$\left. \frac{d^2}{dx_1^2} u(x_1, I - px_1) \right|_{x_1 = \tilde{x}_1} < 0 \tag{6}$$

Proof. See Appendix. *(QED)*

[3]Note that $\frac{d}{dx_1}u(.)$ denotes the total derivative of $u(.)$ with respect to x_1. Letting $x_2 = I - px_1$, we have the frollowing identity.

$$\begin{aligned} \frac{d}{dx_1} u(x_1, x_2) &= \frac{\partial}{\partial x_1} u(x_1, x_2) + \frac{\partial}{\partial x_2} u(x_1, x_2) \cdot \frac{dx_2}{dx_1} \\ &= \frac{\partial}{\partial x_1} u(x_1, I - px_1) - p \frac{\partial}{\partial x_2} u(x_1, I - px_1) \end{aligned}$$

Lemma 3 implies that the second-order condition holds, and that the optimal solution to the maximization problem (4) is uniquely determined by the first-order condition for any given positive pair (p, I) which generates a type-1 budget constraint.

Note that the first-order condition (5) can be rewritten to the standard (*price*) = (*the marginal rate of substitution*) *condition,*

$$p = \frac{(I - px_1)[\alpha - \gamma x_1(I - px_1)]}{x_1[\beta - \gamma x_1(I - px_1)]} \tag{7}$$

Point E in Figure 1 is the unique optimal solution corresponding to a type-1 budget constraint BEC. The point satisfies (7).

3.3 Type-2 budget constraint

We find from the definition of $G(x_1)$ that $G(0) = G(I/p) = \alpha$, $G'(0) = -\gamma I < 0$, and $G'(I/p) = \gamma I > 0$. Thus, there exist $\bar{x}_1$ and $\hat{x}_1$ such that

$$0 < \bar{x}_1 < \hat{x}_1 < I/p$$

and

$$\begin{aligned} G(x_1) &> 0 \text{ for } x_1 \in (0, \bar{x}_1) \cup (\hat{x}_1, I/p), \\ G(x_1) &< 0 \text{ for } x_1 \in (\bar{x}_1, \hat{x}_1), \\ G(x_1) &= 0 \text{ for } x_1 = \bar{x}_1, \hat{x}_1 \end{aligned}$$

Based on this sign pattern of $G(x_1)$ and Lemma 1 which implies that not only $u(x_1, I - px_1)$ but also the first derivative $\frac{d}{dx_1}u(x_1, I - px_1)$ are continuous in $x_1 \in (0, I/p)$, we are now at the position to prove the unique existence of the optimal solution to the maximization problem (4) for any given type-2 budget constraint.

First, it follows from (5) that there exists $\varepsilon' > 0$ such that the first derivative, $\frac{d}{dx_1}u(x_1, I - px_1)$, is positive for any $x_1 \in (0, \varepsilon')$. Second, it is also clear from (5) that the first derivative is negative at $\bar{x}_1$ and $\hat{x}_1$. In fact, we can find that the first derivative is always negative in the interval $(\bar{x}_1, \hat{x}_1)$. Third, Lemma 3 ensures us that $\frac{d}{dx_1}u(x_1, I - px_1)$ is negative for any $x_1 > \hat{x}_1$ as well. Since Lemma 3 also holds for the interval $(0, \bar{x}_1)$, we can conclude that the graph of $u(x_1, I - px_1)$ is bell-shaped with the single peak between 0 and $\bar{x}_1$, where the first-order condition

$$\frac{d}{dx_1}u(x_1, I - px_1) = 0$$

is established. Point E' in Figure 1 is an example of the optimal solution.

3.4 The first main proposition

Based on the foregoing argument, we can assert that, whether the budget constraint is of type 1 or type 2, the optimal solution is uniquely determined and continuous in p and I. We now derive the first result of this paper.

PROPOSITION 1 (Property I): *For any given positive income I and prices of good 1 and good 2, p_1 ($\equiv p$) and p_2 ($\equiv 1$), the textbook utility maximization problem has a unique, positive, and interior optimal consumption bundle. That is, the demand functions $x_1(p, I) > 0$ and $x_2(p, I) \equiv I - px_1(p, I) > 0$ are defined for any positive pair (p, I). Moreover the demand functions are continuous in p and I.*

4 Properties of the demand functions

To characterize the demand functions, $x_1(p, I)$ and $x_2(p, I)$, we focus on the budget constraint and the equality between the price and the marginal rate of substitution (7).

$$I = px_1 + x_2, \tag{8}$$

$$0 = (\beta - z)px_1 - (\alpha - z)x_2, \tag{9}$$

where $z \equiv \gamma x_1 x_2$.[4] Solving this system of equations for x_1 and x_2, we obtain

$$x_1 = \frac{(\alpha - z)I}{p(\alpha + \beta - 2z)}, \tag{10}$$

$$x_2 = \frac{(\beta - z)I}{\alpha + \beta - 2z} \tag{11}$$

Combining (10) and (11), we have

$$\frac{p}{\gamma I^2} = \frac{(\alpha - z)(\beta - z)}{z(\alpha + \beta - 2z)^2} \equiv \Psi(z) \tag{12}$$

Figure 2 depicts the graph of $\Psi(z)$. As is clear from

$$\lim_{z \to 0+} \Psi(z) = \infty, \quad \Psi(\alpha) = 0, \tag{13}$$

and

$$\frac{\Psi'(z)z}{\Psi(z)} = -\left[1 + \frac{z(\alpha - \beta)^2}{(\alpha - z)(\beta - z)(\alpha + \beta - 2z)}\right] < -1 \text{ for any } z \in (0, \alpha), \tag{14}$$

[4]As shown in the preceding section, for any (p, I) the optimal consumption pair (x_1, x_2) exists in $R^2_{++} - \Lambda_0$, therefore, $z \in (0, \alpha)$.

z is uniquely determined for any positive $m \equiv p/\gamma I^2$.

Let us denote the inverse function of Ψ by $z(m)$. It follows from (13) and (14) that

$$\lim_{m\to\infty} z(m) = 0, \quad z(0) = \alpha,$$

and

$$\begin{aligned} \frac{z'(m)m}{z(m)} &= -\left[1 + \frac{z(m)(\alpha-\beta)^2}{(\alpha - z(m))(\beta - z(m))(\alpha+\beta-2z(m))}\right]^{-1} \\ &\in (-1,0) \text{ for any } m > 0 \end{aligned} \tag{15}$$

Making use of (10), (11) and the inverse function, we can express the demand functions as

$$x_1(p,I) = \frac{I\big(\alpha - z\big(\frac{p}{\gamma I^2}\big)\big)}{p\big(\alpha+\beta-2z\big(\frac{p}{\gamma I^2}\big)\big)}, \tag{16}$$

$$x_2(p,I) = \frac{I\big(\beta - z\big(\frac{p}{\gamma I^2}\big)\big)}{\alpha+\beta-2z\big(\frac{p}{\gamma I^2}\big)} \tag{17}$$

In what follows, we shall use these expressions.[5]

4.1 Comparative statics

Now that we obtain the demand functions, we shall check how the demand for $x_1(p,I)$ and $x_2(p,I)$ may depend on p and I.

4.1.1 Price effects

First, the demand for good 1 has a distinctive feature such that the sign of $\partial x_1(p,I)/\partial p$ depends solely on the value of $z(m)$, as proved in the following lemma.

Lemma 4.1: *For any given price-income pair,* (i) $\partial x_1(p,I)/\partial p$ *is negative if* $z(p/\gamma I^2) \in \big(0, \beta - \sqrt{\beta(\beta-\alpha)/2}\big)$; (ii) $\partial x_1(p,I)/\partial p$ *is positive if* $z(p/\gamma I^2) \in \big(\beta - \sqrt{\beta(\beta-\alpha)/2}, \alpha\big)$.

Proof. The logarithmic differentiation of $x_1(p,I)$ with respect to p yields

$$\frac{p}{x_1(p,I)} \cdot \frac{\partial x_1(p,I)}{\partial p} = -1 + \frac{z(m)(\alpha-\beta)}{(\alpha - z(m))(\alpha+\beta-2z(m))} \cdot \frac{z'(m)m}{z(m)}$$

[5] Needless to say, from the definition of $z(.)$ the equality $z(p/\gamma I^2) = \gamma x_1(p,I)x_2(p,I)$ holds.

Making use of (15), we derive

$$\frac{p}{x_1(p,I)} \cdot \frac{\partial x_1(p,I)}{\partial p}$$
$$= \frac{2(z(m)-\alpha)\left[z(m)-\left\{\beta-\sqrt{\frac{1}{2}\beta(\beta-\alpha)}\right\}\right]\left[z(m)-\left\{\beta+\sqrt{\frac{1}{2}\beta(\beta-\alpha)}\right\}\right]}{z(m)(\alpha-\beta)^2+(\alpha-z(m))(\beta-z(m))(\alpha+\beta-2z(m))}, \tag{18}$$

where the denominator is positive, due to $\beta > \alpha > z(m)$.

Let us denote the numerator as

$$\Xi(z(m)) \equiv 2(z(m)-\alpha)\left[z(m)-\left\{\beta-\sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}\right\}\right]\left[z(m)-\left\{\beta+\sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}\right\}\right]$$

It is clear that $\Xi(0) = -\alpha\beta(\alpha+\beta) < 0$. Further, we see that the inequality

$$\alpha > \beta - \sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}$$

holds, due to **ASSUMPTION**.[6] Considering that $\Xi(0) < 0$, we derive

$$\begin{aligned} \Xi(z(m)) &< 0 \quad \text{for } z(m) \in \left(0, \beta - \sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}\right), \\ \Xi(z(m)) &> 0 \quad \text{for } z(m) \in \left(\beta - \sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}, \alpha\right), \end{aligned}$$

and (18) implies

$$sign\left[\frac{\partial x_1(p,I)}{\partial p}\right] = sign\left[\Xi\left(z\left(\frac{p}{\gamma I^2}\right)\right)\right]$$

Therefore, Lemma 4.1 is established. (QED)

Second, let us partially differentiate $x_2(p,I)$ with respect to p. We obtain, from the partial differentiation of (17),

$$\frac{\partial x_2(p,I)}{\partial p} = \frac{(\beta-\alpha)z'(m)}{\gamma I(\alpha+\beta-2z(m))^2} < 0$$

[6] $(\beta-\alpha) - \sqrt{\beta(\beta-\alpha)/2} < 0$ holds for

$$\begin{aligned} &\left[(\beta-\alpha) - \sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}\right] \times \left[(\beta-\alpha) + \sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}\right] \\ = \; &(\beta-\alpha)^2 - \frac{\beta(\beta-\alpha)}{2} \\ = \; &(\beta-\alpha)\left(\frac{\beta}{2}-\alpha\right), \end{aligned}$$

which is negative due to **ASSUMPTION**.

Finally, since $z(.)$ is the inverse function of $\Psi(.)$, from Lemma 4.1 we obtain the following result.

PROPOSITION 2.1 (Property II): (i) *If a pair* (p, I) *satisfies* $\Psi^* \equiv \Psi\big(\beta - \sqrt{\beta(\beta-\alpha)/2}\big) < p/\gamma I^2$*, then* $\partial x_1(p, I)/\partial p$ *is negative at the pair;* (ii) *If they satisfy* $\Psi^* > p/\gamma I^2 > 0$*, then* $\partial x_1(p, I)/\partial p$ *is positive at the pair; good 1 is a Giffen good at the price and income levels;* (iii) $\partial x_2(p, I)/\partial p$ *is always negative.*

4.1.2 Income effects

Next, let us check income effects on the demand for good 1 and good 2. Similarly to the case of price effects, we first derive the following lemma.

Lemma 4.2: *For any given price-income pair,* (i) $\partial x_1(p, I)/\partial I$ *is positive if* $z(p/\gamma I^2) \in \big(0, \beta - \sqrt{\beta(\beta-\alpha)}\big)$*;* (ii) $\partial x_1(p, I)/\partial I$ *is negative if* $z(p/\gamma I^2) \in \big(\beta - \sqrt{\beta(\beta-\alpha)}, \alpha\big)$*.*

Proof. See Appendix. (*QED*)

Second, let us partially differentiate $x_2(p, I)$ with respect to I. We obtain, from the partial differentiation of (17),

$$\frac{\partial x_2(p, I)}{\partial I} = \frac{\beta - z(m)}{\alpha + \beta - 2z(m)} - \frac{2p(\beta-\alpha)z'(m)}{\gamma I^2(\alpha+\beta-2z(m))^2} > 0$$

From Lemma 4.2 we have the following result.

PROPOSITION 2.2 (Property II): (i) *If a pair* (p, I) *satisfies* $\Psi^{**} \equiv \Psi\big(\beta - \sqrt{\beta(\beta-\alpha)}\big) < p/\gamma I^2$*, then* $\partial x_1(p, I)/\partial I$ *is positive at the pair; good 1 is a normal good at the pair;* (ii) *If they satisfy* $\Psi^{**} > p/\gamma I^2 > 0$*, then* $\partial x_1(p, I)/\partial I$ *is negative at the pair; good 1 is an inferior good at the pair;* (iii) $\partial x_2(p, I)/\partial I$ *is always positive.*

4.1.3 The diagrammatic explanation of the comparative static analysis

Let us diagrammatically explain PROPOSITIONS 2.1 and 2.2, and then depict the income expansion path and the price offer curve. See Figure 3A, where the horizontal and vertical axes of coordinates measure p and I, respectively; curve OGC and OAH are two quadratic curves $\Psi^* = p/\gamma I^2$ and $\Psi^{**} = p/\gamma I^2$. The propositions above mean that (i) good 1 is a Giffen good at (p, I) in Region I, (ii) good 1 is an inferior good with negatively sloped demand curve at (p, I) in Region II, and good 1 is a normal good at (p, I) in Region III.

The signs of $\partial x_1(p, I)/\partial p$ and $\partial x_1(p, I)/\partial I$ depends on whether the value $z = \gamma x_1(p, I)x_2(p, I)$, which is uniquely determined by $p/\gamma I^2$, is greater or smaller than $\beta - \sqrt{\beta(\beta-\alpha)/2}$ and $\beta - \sqrt{\beta(\beta-\alpha)}$ (Lemma 4.1 and 4.2). Thus, as is shown in Figure 3B, the commodity space $(x_1, x_2) > (0, 0)$ is divided by three hyperbolas;

$$\begin{aligned}\gamma x_1 x_2 &= \alpha, \\ \gamma x_1 x_2 &= \beta - \sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}, \\ \gamma x_1 x_2 &= \beta - \sqrt{\beta(\beta-\alpha)}\end{aligned}$$

Regions I, II, and III in Figure 3B correspond to Region I, II, and III in Figure 3A, respectively. Take any positive point in the commodity space, say (x_1^e, x_2^e). There is an indifference curve and its tangent line at point e; the latter line is expressed as

$$\frac{u_1(x_1^e, x_2^e)}{u_2(x_1^e, x_2^e)} x_1 + x_2 = \frac{u_1(x_1^e, x_2^e)}{u_2(x_1^e, x_2^e)} x_1^e + x_2^e$$

(x_1^e, x_2^e) is the solution to the maximization problem (4) if the price is $p^E \equiv \frac{u_1(x_1^e, x_2^e)}{u_2(x_1^e, x_2^e)}$ and the income is $I^E \equiv \frac{u_1(x_1^e, x_2^e)}{u_2(x_1^e, x_2^e)} x_1^e + x_2^e$; we have $x_1^e = x_1(p^E, I^E)$ and $x_2^e = x_2(p^E, I^E)$. For example, suppose that point e is in Region II in Figure 3B. Then $p^E/\gamma(I^E)^2$ is in between Ψ^* and Ψ^{**}; point (p^E, I^E) has to be in Region II in Figure 3A, in which case good 1 is an inferior good with negatively sloped demand curve at (p^E, I^E), i.e., both $\partial x_1(p^E, I^E)/\partial p$ and $\partial x_1(p^E, I^E)/\partial I$ are negative. By this correspondence, points in Figures 3A and 3B have an one-to-one relationship.

4.1.4 Income expansion path and price offer curve[7]

Based on the foregoing argument, we can draw the income expansion path and the price offer curve both of which commonly cross a point, say E, in Figure 3A. Point E in Figure 3A corresponds to point e in Figure 3B, and so does the vertical line $BAECD$ in Figure 3A to the income expansion path $baecd$ in Figure 3B and the horizontal line $JHEGF$ in Figure 3A to the price offer curve $jhegf$ in Figure 3B. The income expansion path and the price offer curve bend when they cross the hyperbolas $\beta - \sqrt{\beta(\beta-\alpha)} = \gamma x_1 x_2$ and $\beta - \sqrt{\beta(\beta-\alpha)/2} = \gamma x_1 x_2$, respectively. One can also verify that the Engel curve of good 1 is bell-shaped.

Moving up along the vertical line $BAECD$ in Figure 3A, the ratio $p/\gamma I^2$ monotonically declines, which means that $z(p/\gamma I^2)$ $(= \gamma x_1(p, I) x_2(p, I))$ monotonically rises. The path goes up between the hyperbola $\alpha = \gamma x_1 x_2$ and the vertical axis of coordinates.

On the other hand, moving to the right direction along the horizontal line $FGEHJ$ in Figure 3A, the ratio $p/\gamma I^2$ monotonically rise, which means that $z(p/\gamma I^2)$ $(= \gamma x_1(p, I) x_2(p, I))$ monotonically decreases. The price offer curve converges a point on the vertical axis of coordinates, say j in Figure 3B. Note that, because of (17), $\lim_{z(p/\gamma I^2)\to 0} x_2 = I\beta/(\alpha+\beta)$.

[7] The definitions of income expansion path and price offer curve are given in standard textbooks of micro eocnomic theory. For example, see Varian (1991, pp.116-118).

4.1.5 The independency of Giffen behavior from living standard

For an arbitrarily chosen hyperbola $\zeta = \gamma x_1 x_2$ $(0 < \zeta \leq \alpha)$, the utility level on it is given by

$$u\left(\frac{\zeta}{\gamma x_2}, x_2\right) = \ln\left(\frac{\zeta}{\gamma}\right)^{\alpha} x_2^{\beta-\alpha} - \zeta, \tag{19}$$

which is monotonously increase from $-\infty$ to $+\infty$, as x_2 increase from 0 to $+\infty$. Therefore, any indifference curve crosses the hyperbola $\zeta = \gamma x_1 x_2$, which obviously means it has to cross the two border curves $\beta - \sqrt{\beta(\beta-\alpha)} = \gamma x_1 x_2$ and $\beta - \sqrt{\beta(\beta-\alpha)/2} = \gamma x_1 x_2$. Based on this fact, we derive the following result.

PROPOSITION 3 (Property III): *Define a set of positive pair* (p, I)

$$\Omega(u) \equiv \{(p, I) : u = u(x_1(p, I), x_2(p, I))\}$$

For any utility level u there are three nonempty subsets $\Omega_g(u)$, $\Omega_i(u)$, *and* $\Omega_n(u)$ *such that* (i) *for any* $(p, I) \in \Omega_g(u)$

$$\frac{\partial x_1(p, I)}{\partial p} > 0 \quad \textit{and} \quad \frac{\partial x_1(p, I)}{\partial I} < 0; \tag{20}$$

good 1 is a Giffen good at $(p, I) \in \Omega_g(u)$*;* (ii) *for any* $(p, I) \in \Omega_i(u)$

$$\frac{\partial x_1(p, I)}{\partial p} < 0 \quad \textit{and} \quad \frac{\partial x_1(p, I)}{\partial I} < 0; \tag{21}$$

good 1 is an inferior good with negatively sloped demand curve at $(p, I) \in \Omega_i(u)$*;* (iii) *for any* $(p, I) \in \Omega_n(u)$

$$\frac{\partial x_1(p, I)}{\partial p} < 0 \quad \textit{and} \quad \frac{\partial x_1(p, I)}{\partial I} > 0; \tag{22}$$

good 1 is a normal good at $(p, I) \in \Omega_n(u)$*.*

Proof: See Figure 4A, where $a'awns$ is an arbitrarily chosen indifference curve. As we proved just before the statement of PROPOSITION 3, the curve has to cross the two border hyperbolas at w and n. Corresponding to the indifference curve $awns$ in Figure 4A, we can derive the curve $AWNS$ in Figure 4B; points w and n in Figure 4A correspond to points W and N in Figure 4B, respectively. As is clear from what we have showed so far, good 1 is a Giffen good when $(x_1(p, I), x_2(p, I))$ is on aw, an inferior good with negatively sloped demand curve when it is on wn, and a normal good when it is on ns. Therefore, we can define $\Omega_g(u)$, $\Omega_i(u)$, and $\Omega_n(u)$ as follows.

$$\begin{aligned}
\Omega_g(u) &= \{(p, I) : (p, I) \text{ is on } AW \text{ in Figure 4B}\} \\
\Omega_i(u) &= \{(p, I) : (p, I) \text{ is on } WN \text{ in Figure 4B}\} \\
\Omega_n(u) &= \{(p, I) : (p, I) \text{ is on } NS \text{ in Figure 4B}\} \qquad (QED)
\end{aligned}$$

REMARK: Formally, the curve $AWNS$ in Figure 4B is expressed as follows. Let $x_2 = \varsigma(x_1; u)$ be an indifference curve with the utility level u. Then, the corresponding p and I is expressed as

$$\begin{aligned} p(x_1) &\equiv -\left.\frac{\partial \varsigma(v;u)}{\partial v}\right|_{v=x_1} \\ I(x_1) &\equiv \left(-\left.\frac{\partial \varsigma(v;u)}{\partial v}\right|_{v=x_1}\right) x_1 + \varsigma(x_1;u) \end{aligned}$$

Any point on the curve $AWNS$ in Figure 4B is expressed as $(p(x_1), I(x_1))$. For example, point A is $(p(x_1^a), I(x_1^a))$ where $p(x_1^a) = 0$. Similarly, point W and point N are $(p(x_1^w), I(x_1^w))$ and $(p(x_1^n), I(x_1^n))$, respectively.

4.1.6 The expenditure share on a Giffen good

Suppose that the consumption pair (x_1, x_2) is optimally chosen for a given (p, I), then from (8) and (9), the share of income spent on good 1 is

$$\begin{aligned} \frac{px_1}{I} &= \frac{\frac{(\alpha-z)x_2}{(\beta-z)x_1} x_1}{\frac{(\alpha-z)x_2}{(\beta-z)x_1} x_1 + x_2} \\ &= \frac{\alpha - z}{\alpha + \beta - 2z} \end{aligned}$$

Let $\vartheta(z) \equiv \frac{\alpha - z}{\alpha+\beta-2z}$, then

$$\vartheta(z) \in \left(0, \frac{\alpha}{\alpha+\beta}\right) \quad \text{and} \quad \vartheta'(z) = \frac{\alpha - \beta}{(\alpha+\beta-2z)^2} < 0 \quad \text{for } z \in (0, \alpha) \tag{23}$$

Based on the above, we obtain the following result.

PROPOSITION 4 (Property IV): *Denote a share of income spent on good 1 by θ, that is, $\theta \equiv px_1(p, I)/I$. There exist the following relations between the property of demand for good 1 and θ.* (i) *Good 1 is a Giffen good if $\theta \in (0, \theta_g)$;* (ii) *Good 1 is an inferior good with negatively sloped demand curve if $\theta \in (\theta_g, \theta_i)$;* (iii) *Good 1 is a normal good if $\theta \in \left(\theta_i, \frac{\alpha}{\alpha+\beta}\right)$, where θ_g and θ_i are defined as follows.*

$$\begin{aligned} \theta_g &\equiv \frac{\alpha - \beta + \sqrt{\frac{1}{2}\beta(\beta-\alpha)}}{\alpha - \beta + \sqrt{2\beta(\beta-\alpha)}}, \\ \theta_i &\equiv \frac{\alpha - \beta + \sqrt{\beta(\beta-\alpha)}}{\alpha - \beta + 2\sqrt{\beta(\beta-\alpha)}} \end{aligned}$$

Proof: Considering the fact that for any (p, I) the expenditure share on good 1 is given by $\vartheta(z(p/\gamma I^2))$, from (23), Lemma 4.1, and 4.2, we can conclude

$$\frac{\partial x_1(p,I)}{\partial p} > 0 \quad \text{and} \quad \frac{\partial x_1(p,I)}{\partial I} < 0 \quad \text{if } \theta \in \left(\vartheta(\alpha), \vartheta\left(\beta - \sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}\right)\right),$$

$$\frac{\partial x_1(p,I)}{\partial p} < 0 \quad \text{and} \quad \frac{\partial x_1(p,I)}{\partial I} < 0 \quad \text{if } \theta \in \left(\vartheta\left(\beta - \sqrt{\tfrac{1}{2}\beta(\beta-\alpha)}\right), \vartheta\big(\beta - \sqrt{\beta(\beta-\alpha)}\big)\right),$$

$$\frac{\partial x_1(p,I)}{\partial p} < 0 \quad \text{and} \quad \frac{\partial x_1(p,I)}{\partial I} > 0 \quad \text{if } \theta \in \left(\vartheta\big(\beta - \sqrt{\beta(\beta-\alpha)}\big), \vartheta(0)\right),$$

where $\vartheta(\alpha) = 0$, $\vartheta\big(\beta - \sqrt{\beta(\beta-\alpha)/2}\big) = \theta_g$, $\vartheta\big(\beta - \sqrt{\beta(\beta-\alpha)}\big) = \theta_i$, and $\vartheta(0) = \alpha/(\alpha+\beta)$. *(QED)*

Let us consider the change along an income expansion path. As households' income I goes up, which means $z(p/\gamma I^2)$ monotonously rises, the expenditure share on good 1 monotonously declines, but at the same time good 1 becomes a Giffen good. However low the expenditure share on good 1 becomes, it remains to be a Giffen good.

5 Concluding remarks

In this paper we have proposed a specific but standard utility function under which one good can be an inferior good or even a Giffen good. We show that Giffen behavior generated from the utility function is compatible with an arbitrarily high level of utility and low share of income spent on the inferior good. Thus, this behavior is clearly to be discriminated from the textbook "margarine-butter paradigm", so the theoretical possibility emerges such that it is inappropriate to presume Giffen's paradox merely an exceptional phenomenon under extreme circumstances.

Finally let us make a couple of technical remarks on the utility function we proposed in this paper. First, we can slightly modify the utility function in such a way that it becomes increasing and strictly quasi-concave in good 1 and good 2. See Figure 5. The curve $ABCD$ is an arbitrarily chosen indifference curve of the utility function. Choose a small positive number ε that satisfies $\alpha - \left[\beta - \sqrt{\beta(\beta-\alpha)/2}\,\right] > \varepsilon$. We can draw a curve BFG such that (i) it is tangent to the original indifference curve at B, (ii) it is a convex curve, and (iii) it asymptotically converge to the horizontal part of the original indifference curve $ABCD$. Define $ABFG$ as a modified indifference curve. Making such modification to each indifference curve, we can derive a modified utility function which is increasing and strictly quasi-concave in good 1 and good 2.

Second, one may wonder whether there is a family of utility functions that has the same properties as (1). The answer is affirmative. The following utility function does.

$$u(x_1, x_2) \equiv \begin{cases} \frac{\alpha}{1-\sigma}(x_1^{1-\sigma} - 1) + \frac{\beta\sigma}{\sigma-1}(x_2^{\frac{\sigma-1}{\sigma}} - 1) - \gamma x_1 x_2 & \text{for } x_1^\sigma x_2 \le \alpha/\gamma, \\ \left[\beta - \gamma\left(\frac{\alpha}{\gamma}\right)^{\frac{1}{\sigma}}\right] \frac{\sigma}{\sigma-1} x_2^{\frac{\sigma-1}{\sigma}} + \frac{\alpha-\beta\sigma}{\sigma-1} & \text{for } x_1^\sigma x_2 > \alpha/\gamma, \end{cases}$$

where we assume that all parameters, α, β, γ, and σ, are positive and satisfy $\frac{\beta}{2\gamma} < \left(\frac{\alpha}{\gamma}\right)^{1/\sigma} < \frac{\beta}{\gamma}$. We can check that, as $\sigma \to 1$, this function converges to (1).

Appendix

Proof of Lemma 1.

Trivial from the definition if $\alpha \neq \gamma\tilde{x}_1\tilde{x}_2$. Let us focus on any point of the hyperbola $\alpha = \gamma\tilde{x}_1\tilde{x}_2$. It suffices to prove

1. $\lim_{(x_1,x_2)\to(\tilde{x}_1,\tilde{x}_2)} F(x_1, x_2) = \ln(\alpha/\gamma)^\alpha \tilde{x}_2^{\beta-\alpha} - \alpha \left\{ = u(\tilde{x}_1, \tilde{x}_2) \right\}$

2. $\lim_{(x_1,x_2)\to(\tilde{x}_1,\tilde{x}_2)} F_1(x_1, x_2) = 0 \left\{ = \frac{\partial}{\partial x_1}\left[\ln(\alpha/\gamma)^\alpha x_2^{\beta-\alpha} - \alpha\right]\Big|_{(x_1,x_2)=(\tilde{x}_1,\tilde{x}_2)} \right\}$

3. $\lim_{(x_1,x_2)\to(\tilde{x}_1,\tilde{x}_2)} F_2(x_1, x_2) = \dfrac{\beta-\alpha}{\tilde{x}_2} \left\{ = \frac{\partial}{\partial x_2}\left[\ln(\alpha/\gamma)^\alpha x_2^{\beta-\alpha} - \alpha\right]\Big|_{(x_1,x_2)=(\tilde{x}_1,\tilde{x}_2)} \right\}$

where $F_i(x_1, x_2)$ is the partial derivative of $F(x_1, x_2)$ with respect to x_i, $i = 1, 2$.

Let us prove 1 here. The others can be proved in a similar way.
Since

$$\begin{aligned} & F(x_1, x_2) - \left[\ln(\alpha/\gamma)^\alpha \tilde{x}_2^{\beta-\alpha} - \alpha\right] \\ &= \alpha \ln x_1 + \beta \ln x_2 - \gamma x_1 x_2 - \left[\alpha \ln \frac{\alpha}{\gamma\tilde{x}_2} + \beta \ln \tilde{x}_2 - \alpha\right] \\ &= \alpha \ln \frac{\gamma x_1 \tilde{x}_2}{\alpha} + \beta \ln \frac{x_2}{\tilde{x}_2} - (\gamma x_1 x_2 - \alpha), \end{aligned}$$

it is apparent from the continuity of $\gamma x_1 x_2$ that for any $\delta > 0$ there is some $\epsilon > 0$ such that

$$\max\left\{|x_1 - \tilde{x}_1|, |x_2 - \tilde{x}_2|\right\} < \epsilon \quad \Rightarrow \quad \left|F(x_1, x_2) - \left[\ln(\alpha/\gamma)^\alpha \tilde{x}_2^{\beta-\alpha} - \alpha\right]\right| < \delta$$

Therefore, 1 is established. *(QED)*

Proof of Lemma 3.
Considering (5), the differentiation of the first derivative at $\tilde{x}_1$ yields

$$\begin{aligned}
&\frac{d^2}{dx_1^2}u(x_1, I-px_1)\bigg|_{x_1=\tilde{x}_1} \\
= &-\left[\frac{\beta p^2}{(I-p\tilde{x}_1)^2} - 2\gamma p + \frac{\alpha}{\tilde{x}_1^2}\right] \\
= &-\beta\left[\frac{p}{(I-p\tilde{x}_1)} - \frac{\gamma}{\beta}(I-p\tilde{x}_1)\right]^2 - \frac{(I-p\tilde{x}_1)^2}{\beta}\left[\frac{\beta\alpha}{\tilde{x}_1^2(I-p\tilde{x}_1)^2} - \gamma^2\right]
\end{aligned}$$

Here the term $\left[\frac{\beta\alpha}{\tilde{x}_1^2(I-p\tilde{x}_1)^2} - \gamma^2\right]$ is positive if $G(\tilde{x}_1) = \alpha - \gamma\tilde{x}_1(I-p\tilde{x}_1) > 0$ and $\tilde{x}_1 \in (0, I/p)$. For,

$$\begin{aligned}
\frac{\beta\alpha}{\tilde{x}_1^2(I-p\tilde{x}_1)^2} - \gamma^2 &= \frac{\left[\frac{\sqrt{\beta\alpha}}{\tilde{x}_1} + \gamma(I-p\tilde{x}_1)\right]\left[\frac{\sqrt{\beta\alpha}}{\tilde{x}_1} - \gamma(I-p\tilde{x}_1)\right]}{(I-p\tilde{x}_1)^2} \\
&> \frac{\left[\frac{\sqrt{\beta\alpha}}{\tilde{x}_1} + \gamma(I-p\tilde{x}_1)\right]\left[\frac{\alpha}{\tilde{x}_1} - \gamma(I-p\tilde{x}_1)\right]}{(I-p\tilde{x}_1)^2} \\
&> 0,
\end{aligned}$$

where the first inequality comes from $\alpha < \sqrt{\beta\alpha}$ ($\because$**ASSUMPTION**) and the second inequality is due to $G(\tilde{x}_1) > 0$. Therefore, (6) is established for any $\tilde{x}_1 \in (0, I/p)$ if $G(\tilde{x}_1) > 0$. *(QED)*

Proof of Lemma 4.2
The logarithmic differentiation of $x_1(p, I)$ with respect to I yields

$$\begin{aligned}
&\frac{I}{x_1(p,I)} \cdot \frac{\partial x_1(p,I)}{\partial I} \\
&= 1 + \frac{2z(m)(\beta-\alpha)}{(\alpha - z(m))(\alpha+\beta-2z(m))} \cdot \frac{z'(m)m}{z(m)} \\
&= 1 - \frac{2z(m)(\beta-\alpha)(\beta-z(m))}{z(m)(\alpha-\beta)^2 + (\alpha-z(m))(\beta-z(m))(\alpha+\beta-2z(m))} \\
&= \frac{\left[\alpha\beta - 2\beta z(m) + z(m)^2\right](\alpha+\beta-2z(m))}{\left[z(m)(\alpha-\beta)^2 + (\alpha-z(m))(\beta-z(m))(\alpha+\beta-2z(m))\right]}
\end{aligned}$$

Let

$$\Phi(z(m)) \equiv \alpha\beta - 2\beta z(m) + z(m)^2$$

The sign of $\partial x_1(p, I)/\partial I$ is equal to the one of $\Phi(z(m))$. Since $\Phi(0) = \alpha\beta > 0$ and $\Phi(\alpha) = -\alpha(\beta - \alpha) < 0$, we derive

$$sign\left[\frac{\partial x_1(p,I)}{\partial I}\right] = sign\left[\Phi\Big(z\Big(\frac{p}{\gamma I^2}\Big)\Big)\right] > 0$$

$$\text{if } z\Big(\frac{p}{\gamma I^2}\Big) \text{ is in the interval } \big(0, \beta - \sqrt{\beta(\beta-\alpha)}\,\big)$$

and

$$sign\left[\frac{\partial x_1(p,I)}{\partial I}\right] = sign\left[\Phi\Big(z\Big(\frac{p}{\gamma I^2}\Big)\Big)\right] < 0$$

$$\text{if } z\Big(\frac{p}{\gamma I^2}\Big) \text{ is in the interval } \big(\beta - \sqrt{\beta(\beta-\alpha)}, \alpha\big)$$

Therefore, Lemma 4.2 is established. (QED)

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