Forthcoming in *Journal of Economic Theory*

A nonsmooth, nonconvex model of optimal growth*

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July 22, 2005

Abstract

This paper analyzes the nature of economic dynamics in a one-sector optimal growth model in which the technology is generally nonconvex, nondifferentiable, and discontinuous. The model also allows for irreversible investment and unbounded growth. We develop various tools to overcome the technical difficulties posed by the generality of the model. We provide sufficient conditions for optimal paths to be bounded, to converge to zero, to be bounded away from zero, and to grow unboundedly. We also show that under certain conditions, if the discount factor is close to one, any optimal path from a given initial capital stock converges to a small neighborhood of the golden rule capital stock, at which sustainable consumption is maximized. If it is maximized at infinity, then as the discount factor approaches one, any optimal path either grows unboundedly or converges to an arbitrarily large capital stock.

JEL classification: C61; D90; O41
Keywords: Nonconvex, nonsmooth, and discontinuous technology; optimal growth; unbounded growth; extinction; neighborhood turnpike.

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*We would like to thank an anonymous referee and an associate editor of this journal for helpful comments and suggestions. This research was initiated while Roy was a visiting associate professor at RIEB, Kobe University, in 2002, and part of it was conducted while he was a COE research fellow at RIEB, Kobe University, in 2004. Financial support from RIEB, Kobe University, and the 21st Century COE Program at GSE and RIEB, Kobe University, is gratefully acknowledged.
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1 Introduction

The vast majority of growth models in the economic literature assume smooth technologies. In most cases, smoothness, or differentiability, is assumed purely for analytical convenience even though aggregate technologies in reality are most likely nonsmooth and even discontinuous. Upward discontinuities can be regarded as technological breakthroughs, and are often associated with threshold effects (e.g., Azariadis and Drazen [2]).

Discontinuities are special cases of nonconvexities, the implications of which have been studied rather extensively in the literature on one-sector optimal growth models with nonconvex technologies. A fairly complete characterization of optimal paths is available for the case of an S-shaped production function (e.g., Clark [3], Skiba [28], Majumdar and Mitra [18, 19], Dechert and Nishimura [4]), while various results have been shown on models with more general production functions (e.g., Majumdar and Nermuth [20], Mitra and Ray [24], Amir et al. [1]). To our knowledge, however, there has been no formal analysis of an optimal model with a discontinuous production function.[1] Furthermore, the literature on nonconvex optimal growth models has ruled out unbounded growth by assuming the existence of a maximum sustainable capital stock.

This paper provides a comprehensive analysis of a nonconvex one-sector growth model in which unbounded growth is possible and the production function is allowed to be discontinuous. Aside from technical conditions required to rule out trivial cases or to ensure the existence of optimal paths, we only assume that the utility function is strictly increasing and strictly concave,[2] that the production function is strictly increasing, and that the lower bound on next period's capital is nondecreasing in current capital. The last assumption, which is trivially satisfied in the standard case of reversible investment, allows for a general form of irreversible investment. In terms of generality, the stationary framework used by Mitra and Ray [24] and Amir et al. [1] is the closest to ours. In their framework, however, discontinuities, irreversible investment, and unbounded growth are ruled out.

The generality of our model poses several technical challenges. The absence of differentiability makes the standard Euler equation invalid. The

[1]See Dutta and Mitra [6] for an example of a convex model in which the feasible correspondence is not continuous. The discrete-choice problems studied by Kamihigashi [13, 14] have discontinuous features.

[2]The case of linear utility is studied in Kamihigashi and Roy [17].

discontinuity of the technology implies that the value function is generally discontinuous, and that the optimal policy correspondence is generally not upper hemi-continuous. The irreversibility of investment implies that the value function is not necessarily increasing. These difficulties make various familiar techniques inapplicable, but for this very reason help gain a deeper insight into the fundamental mechanisms of economic dynamics.

We develop four essential tools for overcoming these difficulties. The first is an extension of the monotonicity arguments used by Dechert and Nishimura [4] and Mitra and Ray [24]. In fact, we impose only the very minimum set of assumptions under which their arguments go through. The second tool is an argument based on what we call the gain function, which measures discounted net returns on investment. The same function has been used in the literature to study the properties of steady states (Majumdar and Nermuth [20], Dechert and Nishimura [4], Mitra and Ray [24]). We show that optimal paths never move in a direction in which higher discounted net returns on investment, or higher "gains," will never be available. This result helps determine the directions in which an optimal path possibly moves. The third tool is our finding that a bounded optimal path converges to an optimal steady state despite the discontinuity of the technology. The fourth tool is Euler inequalities derived using generalized one-sided derivatives (called Dini derivatives) that are well-defined even for nondifferentiable or discontinuous functions.[3] We use the Euler inequalities to obtain necessary conditions for a steady state.

With these four tools, we provide sufficient conditions for optimal paths to be bounded, to converge to zero, to be bounded away from zero, and to grow unboundedly. These conditions unify and generalize various conditions known in the literature for special cases of our model. Our analysis reveals that extinction (i.e., convergence to zero) and unbounded growth are symmetrical phenomena, so are boundedness and avoidance of extinction.

We also show that under certain conditions, the model exhibits the neighborhood turnpike property, which is well-known for convex models (e.g., McKenzie [21, 22], Yano [30], Montrucchio [25, 26], Guerrero-Lechtenberg [9]). It is the property that as the discount factor approaches one, any optimal path from a given initial capital stock "converges" to a small neighborhood of the golden rule capital stock,[4] at which sustainable consumption

[3]Dini derivatives were used in Kamihigashi [15] to obtain transversality conditions for general stochastic problems.

[4]This is in fact slightly more general than the standard statement of the neighborhood

is maximized. If it is maximized at infinity, then as the discount factor approaches one, any optimal path either grows unboundedly or converges to an arbitrarily large capital stock. Our turnpike results build on some of the arguments used by Scheinkman [27] and Majumdar and Nermuth [20] as well as the tools developed in this paper.

The rest of the paper is organized as follows. Section 2 presents the model along with the assumptions that are maintained throughout the paper. Section 3 develops the essential tools discussed above and shows some results of independent interest. Section 4 offers sufficient conditions for various dynamic properties. Section 5 establishes neighborhood turnpike results. Longer proofs are relegated to the appendices.

2 The model

Consider the following maximization problem:

$$\max_{\{c_t,x_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{2.1}$$

$$\text{s.t.} \quad \forall t \in \mathbb{Z}_+, c_t + x_{t+1} = f(x_t), \tag{2.2}$$

$$c_t \geq 0, \tag{2.3}$$

$$x_{t+1} \geq r(x_t), \tag{2.4}$$

$$x_0 \text{ given}, \tag{2.5}$$

where c_t is consumption in period t, x_t is the capital stock at the beginning of period t, δ is the discount factor, u is the utility function, f is the production function, and $r(x_t)$ is the lower bound on x_{t+1}. Every infinite sum is understood as a Lebesgue integral in this paper.

Except for (2.4), the structure of the model is that of a standard one-sector growth model. In the standard case, $r(x) = 0$ for all $x \geq 0$, and $f(x)$ can be written as

$$f(x) = \tilde{f}(x) + (1 - d)x \tag{2.6}$$

for some function $\tilde{f}$ and constant d, where $\tilde{f}$ is the net production function and d is the depreciation rate (possibly equal to one). In models with irreversible investment, it is typically assumed that $r(x) = (1 - d)x$ and

turnpike theorem.

f satisfies (2.6). Our formulation allows for nonlinear depreciation, which seems natural but has not received attention in the literature.[5]

Aside from technical conditions required to rule out trivial cases or to ensure the existence of optimal paths, we only assume that u is strictly increasing and strictly concave, that f is strictly increasing with $f(0) = 0$, and that r is nondecreasing. The precise assumptions are stated and discussed in what follows. They are maintained throughout this paper.

Assumption 2.1. (i) $u : \mathbb{R}_+ \to [-\infty, \infty)$ is continuous, strictly increasing, and strictly concave. (ii) $\delta \in (0, 1)$.

The utility function u is not required to be differentiable. Since the case $u(0) = -\infty$ is permitted,[6] u can be logarithmic or, more generally, of the CRRA class.

Assumption 2.2. (i) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and upper semicontinuous. (ii) $f(0) = 0$.

The production function f is required to be neither continuous nor differentiable. Part (ii) has two roles. The first is to ensure that $k = 0$ is a steady state. This implication is used to claim that if an optimal path converges to zero, it converges to a steady state (in the proof of Proposition 3.1). The second role is to ensure that consumption is small when capital is small. This relation between capital and consumption is used to show our local extinction result (Proposition 4.2).

To state our assumption on the lower bound function r, for $h : \mathbb{R}_+ \to \mathbb{R}$, we define

$$h_-(x) = \lim_{y \uparrow x} h(y), \tag{2.7}$$

$$h_+(x) = \lim_{y \downarrow x} h(y), \tag{2.8}$$

[5] We use the following standard definitions. A path $\{c_t, x_t\}_{t=0}^{\infty}$ is *feasible* if it satisfies (2.2)–(2.4). A capital path $\{x_t\}$ is *feasible* if there is a consumption path $\{c_t\}$ such that $\{c_t, x_t\}$ is feasible. A path *from* x_0 is a path $\{c'_t, x'_t\}$ such that $x'_0 = x_0$. A capital path *from* x_0 is defined similarly. A feasible path $\{c_t, x_t\}$ is *optimal* (*from* x) if it solves the maximization problem (2.1)–(2.5) (with $x_0 = x$). An *optimal* capital path $\{x_t\}$ is defined similarly. A *stationary* (capital) path is a constant (capital) path. A pair (c, x) is a *steady state* if the stationary path $\{c_t, x_t\}$ such that $c_t = c$ and $x_t = x$ for all $t \in \mathbb{Z}_+$ is optimal. A capital stock $x \geq 0$ is a *steady state* if (c, x) is a steady state for some $c \geq 0$.

[6] In this case, continuity at $c = 0$ means $\lim_{c \downarrow 0} u(c) = -\infty$.

provided that the right-hand sides are well-defined. Any nondecreasing function h clearly satisfies $h_-(x) \leq h(x) \leq h_+(x)$.

Assumption 2.3. (i) $r : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and lower semicontinuous. (ii) $\forall x > 0, r_+(x) < x$ and $r(x) < f(x)$.

Like the production function f, the lower bound function r is required to be neither differentiable nor continuous. The inequality $r_+(x) < x$ basically means that the irreversibility constraint (2.4) is never binding at a steady state. It is easy to see that if $r_+(x) \geq x$ for some $x > 0$, and if r is strictly increasing, then any feasible capital path from $x_0 > x$ is bounded below by x. Such a possibility is ruled out here. The inequality $r(x) < f(x)$ means that strictly positive consumption is feasible at x. This is necessary for the maximization problem (2.1)–(2.5) to make sense in the case $u(0) = -\infty$. Assumption 2.3 is satisfied if $r(x) = 0$ for all $x > 0$, or if f satisfies (2.6) with $d \in (0, 1]$, $r(x) = (1 - d)x$, and $\tilde{f}(x) > 0$ for all $x > 0$.

Assumption 2.4. $\forall x > 0$, there exists a feasible path $\{c_t, x_t\}$ from x such that $\sum_{t=0}^{\infty} \delta^t u(c_t) > -\infty$.

This assumption is satisfied, for example, if u is bounded below, or if $u(c) \geq \ln c$ for small c and $f(x) \geq Ax$ and $r(x) \leq B$ for small x for some constants $A, B > 0$ with $A > B$.[7] Assumption 2.4 is required for the maximization problem (2.1)–(2.5) to make sense.

Assumption 2.5. $\forall x > 0, \sum_{t=0}^{\infty} \delta^t u(f^t(x)) < \infty$.[8]

The only role of this assumption is to ensure the existence of optimal paths and the upper semicontinuity of the value function. It is satisfied, for example, if u is bounded above, or if $u(c) \leq \ln(c)$ for large c and $f(x) \leq Ax$ for large x for some constant $A > 0$.

In general Assumptions 2.4 and 2.5 are joint restrictions on u, f, r, and δ, and there are various other cases in which they are satisfied. The assumptions made above imply the existence of an optimal path from any initial capital stock $x_0 \geq 0$ by a standard argument (e.g., Ekeland and Scheinkman [7, Proposition 4.1]).

[7] "For small x" means "for all x sufficiently small." Similar remarks apply to similar expressions.

[8] $f^2(x) \equiv f(f(x)), f^3(x) \equiv f(f(f(x)))$, etc.

3 Fundamental properties

This section establishes fundamental properties of optimal paths. In particular we show that an optimal capital path is monotone, that a bounded optimal path converges to a steady state, and that an optimal capital path never moves in a direction in which higher gains, or higher discounted net returns on investment, will never be available. We also obtain a sufficient condition for the existence of a nonzero steady state, and necessary conditions for a steady state. Many of the results here become essential tools in our subsequent analysis. Some of them are of independent interest.

3.1 Monotonicity and convergence

The Bellman equation for the maximization problem (2.1)–(2.5) is given by

$$v(x_t) = \max_{r(x_t) \le x_{t+1} \le f(x_t)} \{u(f(x_t) - x_{t+1}) + \delta v(x_{t+1})\}. \tag{3.1}$$

Let $K : \mathbb{R}_+ \to 2^{\mathbb{R}_+}$ denote the policy correspondence:

$$K(x_t) = \{x_{t+1} \in [r(x_t), f(x_t)] \mid v(x_t) = u(f(x_t) - x_{t+1}) + \delta v(x_{t+1})\}. \tag{3.2}$$

We begin by showing a monotonicity property of K.

Lemma 3.1. $\forall x_0 \ge 0, \forall y_0 > x_0, \forall x_1 \in K(x_0), \forall y_1 \in K(y_0), x_1 \le y_1$.

Proof. Let $0 \le x_0 < y_0$, $x_1 \in K(x_0)$, and $y_1 \in K(y_0)$. If $x_0 = 0$, we trivially have $x_1 = 0 \le y_1$. Suppose $x_0 > 0$ and $x_1 > y_1$. Then

$$r(x_0) \le r(y_0) \le y_1 < x_1 \le f(x_0) < f(y_0). \tag{3.3}$$

Hence y_1 is feasible from x_0, and x_1 is feasible from y_0. The rest of the proof is the same as the first paragraph of the proof of Dechert and Nishimura [4, Theorem 1].[9] □

The next result shows a monotonicity property of optimal capital paths.

Lemma 3.2. *Let $\{x_t\}$ be an optimal capital path. Then $\forall t \in \mathbb{Z}_+, x_t \le x_{t+1}$ or $\forall t \in \mathbb{Z}_+, x_t \ge x_{t+1}$.*

[9] Given (3.3), their argument goes through as long as u is strictly concave and f is strictly increasing. Lemma 3.1 can alternatively be shown by applying Topkis [29, Theorem 6.3].

Proof. See Mitra and Ray [24] or Kamihigashi and Roy [16]. □

The same result was shown by Majumdar and Nermuth [20, Theorem 3.1] and Dechert and Nishimura [4, Corollary 1] for differentiable cases, and by Mitra and Ray [24, Lemma 5.2] for a continuous case. Lemma 3.2 follows from an argument used in the working paper version of Mitra and Ray [24] ([23]).

It is immediate from Lemma 3.2 that every bounded optimal path converges. It is not obvious, however, whether it converges to a steady state (which in our terminology means an *optimal* steady state). To see why, note that an optimal path $\{c_t, x_t\}$ satisfies $v(x_t) = u(c_t) + \delta v(x_{t+1})$ for all $t \in \mathbb{Z}_+$ by the principle of optimality. If $x_t \to x > 0$ and $c_t \to c > 0$,[10] and if

$$\lim_{t \uparrow \infty} v(x_t) = v(x), \tag{3.4}$$

then $v(x) = u(c) + \delta v(x)$ by continuity of u. This implies that (c, x) is a steady state. But since f and r are not continuous, neither is v. Thus (3.4) need not hold if $\{x_t\}$ is an arbitrary convergent sequence. Nevertheless (3.4) can be shown if $\{x_t\}$ is an *optimal* capital path. A first step toward this is the following.

Lemma 3.3. *v is upper semicontinuous.*

Proof. This can be shown by a standard argument. See Kamihigashi and Roy [16]. □

Hence $\overline{\lim}_{t\uparrow\infty} v(x_t) \leq v(x)$ for any convergent capital path $\{x_t\}$ with $x = \lim_{t\uparrow\infty} x_t$, in particular, for any bounded optimal capital path. The "reverse" inequality, $\underline{\lim}_{t\uparrow\infty} v(x_t) \geq v(x)$, can be verified by arguing that when x_t is close to x, the cost of jumping from x_t to x is small, so that the "benefit," $v(x) - v(x_t)$, must be likewise small.

Proposition 3.1. *Any optimal path that is bounded converges to a steady state.*

Proof. See Appendix A. □

[10] If $x = c = 0$, then (c, x) is trivially a steady state.

Except for immediate consequences of our assumptions, upper semicontinuity is the only property of v that we use. In fact, v need not even be monotone. If $r(x) = 0$ for all $x \geq 0$, then a higher capital stock is always better because it expands the set of feasible capital stocks for the next period. But if r is not constant, then a higher capital stock, which does not always expand the feasible set, is not necessarily better.[11] The non-monotonicity of v complicates some of our proofs, but only slightly.

The next result is immediate from Lemma 3.1. We state it here for easy reference.

Lemma 3.4. *Let $y_0 > 0$. If every optimal capital path from y_0 is nonincreasing, then every optimal capital path from $x_0 \in [0, y_0]$ is bounded above by y_0. Likewise, if every optimal capital path from y_0 is nondecreasing, then every optimal capital path from $x_0 \geq y_0$ is bounded below by y_0.*

3.2 The gain function

For $x \geq 0$, define

$$\Gamma(x) = \delta f(x) - x. \tag{3.5}$$

We call this function the *gain function* for the following reason. If x units of capital are invested today, it generates $f(x)$ units of output tomorrow. Thus the discounted net return, or "gain," is $\delta f(x) - x$.

The gain function plays a central role in our analysis. The same function was used by Majumdar and Nermuth [20, p. 358], Dechert and Nishimura [4, Lemmas 2, 3], and Mitra and Ray [24, p. 160, 164] to examine the properties of steady states. We use it to determine the directions in which an optimal path possibly moves.

[11] For example, suppose $r(x) = 0$ for $x \in [0, z]$ and $r(x) = f(x) - \epsilon$ for $x > z$, where $z > 0$ and $\epsilon \in (0, f(z))$. Assume $u(0) = 0$, and $\delta = 0$ for the moment. Then $v(x) = u(f(x))$ for $x \leq z$, but $v(x) = u(\epsilon) < v(f(z))$ for $x > z$. It is easy to see that v has a similar structure for $\delta > 0$ close to zero.

It is useful to note that for any feasible path $\{c_t, x_t\}$ and $T \in \mathbb{N}$,

$$\sum_{t=0}^{T} \delta^t c_t = \sum_{t=0}^{T} \delta^t [f(x_t) - x_{t+1}] \tag{3.6}$$

$$= f(x_0) - x_1 + \delta[f(x_1) - x_2] + \cdots + \delta^T [f(x_T) - x_{T+1}] \tag{3.7}$$

$$= f(x_0) + \sum_{t=0}^{T-1} \delta^t \Gamma(x_{t+1}) - \delta^T x_{T+1}. \tag{3.8}$$

Thus $\Gamma(x_{t+1})$ is the contribution of x_{t+1} to the present discounted value of consumption. The following lemma is shown by generalizing an argument used by Majumdar and Nermuth [20, p. 358] and Dechert and Nishimura [4, Lemma 2].

Lemma 3.5. *Let $\{x_t\}$ be an optimal capital path that is nonstationary. Then $\exists t \in \mathbb{N}, \Gamma(x_0) < \Gamma(x_t)$.*

Proof. See Appendix A. □

Therefore a nonstationary optimal capital path must always achieve a higher gain at some point in the future. In other words, an optimal capital path moves in a direction in which higher gains will eventually be available. If the highest gain is available at the current capital stock, then it is optimal to stay there forever. Thus the following result is an immediate consequence of Lemma 3.5.

Proposition 3.2. *Suppose $\exists \hat{x} > 0, \Gamma(\hat{x}) = \sup_{x \geq 0} \Gamma(x)$. Then $\hat{x}$ is a steady state.*

As a consequence, we obtain a sufficient set of conditions for the existence of a nonzero steady state.

Proposition 3.3. *Suppose* (i) *$\exists \tilde{x} > 0, \Gamma(\tilde{x}) \geq 0$ and* (ii) *$\overline{\lim}_{x \uparrow \infty} \Gamma(x) < \sup_{x>0} \Gamma(x)$. Then there exists a nonzero steady state.*

Proof. Since f is upper semicontinuous, so is Γ. This together with (ii) implies that $\exists \hat{x} \geq 0, \Gamma(\hat{x}) = \sup_{x \geq 0} \Gamma(x) \equiv s$. If $s > 0$, then $\hat{x} > 0$ since $\Gamma(0) = 0$. If $s = 0$, then $\hat{x}$ can be chosen to be strictly positive by (i). Thus the conclusion follows by Proposition 3.2. □

If f satisfies (i), then it is called δ-productive in Mitra and Ray's [24, p. 164] terminology. Condition (ii) holds if there is a maximum sustainable capital stock, i.e.,

$$\exists \underline{x} > 0, \forall x > \underline{x}, \quad f(x) < x. \tag{3.9}$$

For then $\Gamma(x) = \delta f(x) - x < f(x) - x < 0$ for $x > \underline{x}$. Hence Proposition 3.3 extends Mitra and Ray [24, Theorem 4.2] to our general model.[12] Our argument is similar to theirs, but more direct since we do not consider support prices.

3.3 Euler inequalities

Even in the absence of differentiability, generalized versions of derivatives are available. For $h : (a, b) \to \mathbb{R}$ with $a < b$, define

$$h'_-(x) = \varliminf_{\epsilon \downarrow 0} \frac{h(x) - h(x - \epsilon)}{\epsilon}, \tag{3.10}$$

$$h'_+(x) = \varlimsup_{\epsilon \downarrow 0} \frac{h(x + \epsilon) - h(x)}{\epsilon}. \tag{3.11}$$

These generalized derivatives are called the lower left and the upper right Dini derivative of h evaluated at x. They allow us to obtain "Euler inequalities" instead of an Euler equation.

Lemma 3.6. *Let $\{c_t, x_t\}$ be an optimal path. Let $t \in \mathbb{Z}_+$. If $c_t > 0$ and $x_{t+2} > r_+(x_{t+1})$, then*

$$u'_-(c_t) \geq \delta u'_+(c_{t+1}) f'_+(x_{t+1}). \tag{3.12}$$

If $x_{t+1} > r(x_t)$ and $c_{t+1} > 0$, then

$$u'_+(c_t) \leq \delta u'_-(c_{t+1}) f'_-(x_{t+1}). \tag{3.13}$$

Proof. See Appendix A. □

[12] Strictly speaking, since they did not assume the strict concavity of u for their corresponding result, Proposition 3.3 does not generalize their result. But in fact Proposition 3.3 holds even if u is only concave. This is because $\hat{x}$ as given in the proof of Proposition 3.3 is a steady state by (A.11) even if u is only concave.

If u and f are differentiable, (3.12) and (3.13) imply the Euler equation $u'(c_t) = \delta u'(c_{t+1}) f'(x_{t+1})$. The cost of not assuming differentiability is that the Euler equation must be replaced by the two Euler inequalities above. This cost is rather small here since we use them only to obtain necessary conditions for a steady state. For this purpose, for $x \geq 0$, define

$$g(x) = f(x) - x, \tag{3.14}$$

which is the stationary consumption level associated with capital stock x. For $x > 0$ with $g(x) > 0$, define

$$\Phi(x) = \delta \frac{u'_+(g(x))}{u'_-(g(x))} f'_+(x), \quad \Psi(x) = \delta \frac{u'_-(g(x))}{u'_+(g(x))} f'_-(x). \tag{3.15}$$

If one divides (3.12) through by $u'_-(c_t)$ and sets $x_{t+1} = x$ and $c_t = c_{t+1} = g(x)$, then the resulting right-hand side is $\Phi(x)$. One obtains $\Psi(x)$ similarly. Hence the Euler inequalities (3.12) and (3.13) evaluated at a steady state imply (3.16) below.

Lemma 3.7. *If $x > 0$ is a steady state, then $g(x) > 0$ and*

$$\Phi(x) \leq 1 \leq \Psi(x). \tag{3.16}$$

Proof. See Appendix A. □

Since u is concave, $u'_- \geq u'_+$; thus

$$\forall x > 0, \quad \Phi(x) \leq \delta f'_+(x), \quad \delta f'_-(x) \leq \Psi(x). \tag{3.17}$$

If u is differentiable, $\Phi(x) = \delta f'_+(x)$ and $\Psi(x) = \delta f'_-(x)$. If u and f are differentiable, $\Phi(x) = \Psi(x) = \delta f'(x)$. In this case, (3.16) implies $\delta f'(x) = 1$, a well-known necessary condition for a steady state in the differentiable case. The following result gives useful relationships among the functions Φ, Ψ, f'_-, f'_+, and Γ.

Lemma 3.8. *Let $x > 0$ and $0 \leq a < b$.*

(i) *If $\Phi(x) > 1$, then $\delta f'_+(x) > 1$.*

(ii) *$\delta f'_+ \geq 1$ on $[a, b)$ iff Γ is nondecreasing on $[a, b]$.*

(iii) *If $\Psi(x) < 1$, then $\delta f'_-(x) < 1$.*

(iv) *$\delta f'_- \leq 1$ on $(a, b]$ iff Γ is nonincreasing on $[a, b]$.*

Proof. See Appendix A. □

4 Conditional properties of optimal paths

This section provides sufficient conditions for optimal paths to be bounded, to converge to zero, to be bounded away from zero, and to grow unboundedly. Let us begin by giving a condition under which optimal paths from small capital stocks are bounded.

Proposition 4.1. *Suppose*

$$\exists \underline{x} > 0, \forall x > \underline{x}, \quad \Gamma(\underline{x}) \geq \Gamma(x). \tag{4.1}$$

Then every optimal capital path from $x_0 \in [0, \underline{x}]$ is bounded above by $\underline{x}$.

Proof. By (4.1) and Lemma 3.5, any optimal capital path from $\underline{x}$ is nonincreasing. Hence the conclusion follows by Lemma 3.4. □

A sufficient condition for (4.1) is that Γ is nonincreasing on $[\underline{x}, \infty)$, or equivalently $\delta f'_{-} \leq 1$ on $(\underline{x}, \infty)$, for some $\underline{x} > 0$ (recall Lemma 3.8). In this case, every optimal capital path is bounded by Proposition 4.1. If $\Psi < 1$ on $(0, \underline{x}]$ in addition to (4.1), there is no steady state in $(0, \underline{x}]$ by Lemma 3.7, so every optimal capital path from $x_0 \in (0, \underline{x}]$ converges to zero by Propositions 4.1 and 3.1. The same conclusion can be obtained without (4.1) if u satisfies the Inada condition at zero.

Proposition 4.2. *Suppose*

$$u'_{+}(0) = \infty, \tag{4.2}$$

$$\exists z > 0, \forall x \in (0, z], \quad \Psi(x) < 1. \tag{4.3}$$

Then $\exists \overline{x} \in (0, z]$, every optimal capital path from $x_0 \in (0, \overline{x}]$ converges to zero.

Proof. See Appendix B. □

Proposition 4.2 generalizes Dechert and Nishimura [4, Lemma 3]. Unlike their proof, which relies extensively on the Euler equation, our argument uses the Euler inequalities (through Lemma 3.7) only to rule out steady states in $(0, z]$. The following result gives a condition for extinction to occur globally.

Proposition 4.3. *Suppose*

$$\forall x > 0, \quad \Psi(x) < 1. \tag{4.4}$$

Then every optimal capital path converges to zero.

Proof. By (4.4) and Lemma 3.8, Γ is nonincreasing on $\mathbb{R}_{++}$. Thus every optimal capital path is nonincreasing by Lemma 3.5. Since there is no nonzero steady state by (4.4) and Lemma 3.7, every optimal capital path converges to zero by Proposition 3.1. $\square$

If u and f are differentiable, (4.4) reduces to the condition that $\delta f'(x) < 1$ for all $x > 0$. This condition was obtained by Majumdar and Mitra [18, p. 122] and Dechert and Nishimura [4, p. 346] for the S-shaped case. Proposition 4.3 is a direct generalization of their result.

The following result gives a condition under which optimal paths from large capital stocks are bounded away from zero.

Proposition 4.4. *Suppose*

$$\exists \overline{x} > 0, \forall x \in [0, \overline{x}), \quad \Gamma(x) \leq \Gamma(\overline{x}). \tag{4.5}$$

Then every optimal capital path from $x_0 \geq \overline{x}$ is bounded below by $\overline{x}$.

Proof. Similar to the proof of Proposition 4.1 $\square$

A sufficient condition for (4.5) is that Γ is nondecreasing on $[0, \overline{x}]$, or equivalently $\delta f'_+ \geq 1$ on $[0, \overline{x})$, for some $\overline{x} > 0$. In this case, every optimal capital path is bounded away from zero by Proposition 4.4. If $\Phi > 1$ on $[\overline{x}, \infty)$ in addition to (4.5), there is no steady state in $[\overline{x}, \infty)$ by Lemma 3.7, so every optimal capital path from $x_0 \geq \overline{x}$ goes to infinity. The same conclusion can be obtained without (4.5) under an additional condition on u.

Proposition 4.5. *Suppose*

$$\varlimsup_{c \uparrow \infty} u'_+(c)c < \infty, \tag{4.6}$$

$$\exists z > 0, \forall x \geq z, \quad \Phi(x) > 1. \tag{4.7}$$

Then $\exists \underline{x} \geq z$, every optimal capital path from $x_0 \geq \underline{x}$ goes to infinity.

Proof. See Appendix B. $\square$

Condition (4.6) holds if u is bounded above, as shown in the proof. Another important case in which (4.6) holds is when $u(c) = \ln c$, which implies $u'(c)c = 1$ for all $c > 0$. Condition (4.6) means that marginal utility declines relatively fast as consumption increases. Hence, when the stationary level of consumption is already high (as implied by (4.7) and Lemma B.4 for large capital stocks), it is not attractive to choose a decreasing path, which entails even higher current consumption and lower future consumption.

The following result gives a condition for unbounded growth to occur globally.

Proposition 4.6. *Suppose*

$$\forall x > 0, \quad \Phi(x) > 1. \tag{4.8}$$

Then every optimal capital path goes to infinity.

Proof. Similar to the proof of Proposition 4.3. □

For a differentiable convex model, Jones and Manuelli [10, p. 1014] showed that unbounded growth occurs if $\delta f'$ is bounded below away from 1.[13] If u and f are differentiable, (4.8) reduces to the condition that $\delta f'(x) > 1$ for all $x > 0$. By a standard argument this condition is necessary for unbounded growth in the differentiable convex case (Jones and Manuelli [11, p. 78]). Proposition 4.6 shows that the condition is also sufficient for global unbounded growth even without convexity.[14]

5 The neighborhood turnpike property

Propositions 4.2 and 4.5 indicate the possibility that extinction occurs from small stocks, while unbounded growth occurs from large stocks. This is because (4.2), (4.3), (4.6), and (4.7) can all be satisfied simultaneously. Likewise various other path-dependent phenomena are also possible.

Despite such nonclassical features, as δ approaches one, the model essentially returns to the classical world. More specifically, this section shows that in many cases, for δ close to one, any optimal capital path from a given initial stock "converges" to a small neighborhood of what we define as the golden rule capital stock. We begin with the assumption that maximum sustainable consumption, the largest possible value of $g(x) = f(x) - x$, is strictly positive.

[13] The sufficient conditions for unbounded growth used by Dolmas [5, Assumption (P)] and Kaganovich [12, Assumption 7] reduce to Jones and Manuelli's [10] condition in the one-sector case with a single capital good.

[14] The following statement can be added to Propositions 4.5–4.6: the associated consumption path also goes to infinity. This can easily be shown by using (3.12), Lemma B.4, and the fact that $v(x) \geq u(g(x))/(1-\delta)$ whenever $g(x) \geq 0$.

Assumption 5.1. $g^* \equiv \sup_{x \geq 0} g(x) \in (0, \infty]$.

It is easy to see that if $g^* \leq 0$, every optimal capital path converges to zero.[15] The role of Assumption 5.1 is to rule out this trivial case. Note that we do not require g^* to be finite.

We define the *golden rule capital stock* x^* as follows.

$$x^* = \begin{cases} \min\{x \geq 0 \mid g(x) = g^*\} & \text{if } \exists x \geq 0, g(x) = g^*, \\ \infty & \text{otherwise.} \end{cases} \tag{5.1}$$

By Assumption 5.1, $x^* > 0$. The case $x^* = \infty$ means that sustainable consumption is maximized at infinity.

This section maintains all the assumptions stated in Section 2 for each $\delta \in (0, 1)$. The following is our last assumption.

Assumption 5.2. $\forall x \in (0, x^*] \cap (0, \infty), g_-(x) > 0$.[16]

It is easy to see that if $g_-(x) \leq 0$, i.e., $f_-(x) \leq x$, for some $x \in (0, x^*]$, then no feasible capital path from $x_0 < x$ can reach x. Hence Assumption 5.2 is a minimum requirement for the neighborhood turnpike property to hold globally.

For the rest of this section, we take an arbitrary initial capital stock $x_0 \in \mathbb{R}_{++}$ as given. For each $\delta \in (0, 1)$, let $\{x_t^\delta\}$ be an optimal capital path from x_0 with the discount factor equal to δ. The neighborhood turnpike property is now expressed as follows.

$$\lim_{\delta \uparrow 1} \lim_{t \uparrow \infty} x_t^\delta = x^*.\text{[17]} \tag{5.2}$$

This equation means that for δ close to one, $\{x_t^\delta\}$ converges to a small neighborhood of x^*.

The first step to establishing (5.2) is the following result, which is similar to Scheinkman's [27] "visit lemma."

[15] To see this, suppose $\forall x > 0, g(x) \leq 0$. Let $\{x_t\}$ be an optimal capital path from $x_0 > 0$. Then $\{x_t\}$ is nonincreasing since $\forall t \in \mathbb{Z}_+, x_{t+1} - x_t \leq f(x_t) - x_t = g(x_t) \leq 0$. By Lemma 3.7, there is no nonzero steady state. Since $\{x_t\}$ converges to a steady state by Proposition 3.1, it converges to zero.

[16] If $x^* < \infty$, then $x \leq x^*$; if $x^* = \infty$, then $x < x^*$. Recall (2.7) for the definition of g_-.

[17] Of course, we do not assume that the expression on the left-hand side is well-defined a priori. The same remark applies to Lemma 5.1.

Lemma 5.1. $\lim_{\delta\uparrow 1}\sup_{t\in\mathbb{Z}_+} g(x_{t+1}^{\delta}) = g^*$.

Proof. See Appendix C. □

This lemma implies that if x^* is a unique maximizer of $g(x)$ (or, more precisely, under (5.6) and (5.7) below), then for δ close to one, $\{x_t^{\delta}\}$ "visits" a small neighborhood of x^* at least once. This implication was shown by Scheinkman [27] for a convex multi-sector model. In our case, $\{x_t^{\delta}\}$ approaches some maximizer of $g(x)$ at least once for δ close to one. The following result shows that if $r(x_0) \le x^*$, then $\{x_t^{\delta}\}$ in fact converges to a small neighborhood of x^* for δ close to one.

Proposition 5.1. *Suppose*

$$r(x_0) \le x^*. \tag{5.3}$$

Then (5.2) holds.

Proof. See Appendix C. □

An obvious sufficient condition for (5.3) is

$$\forall x \ge 0, \quad r(x) = 0, \tag{5.4}$$

i.e., the irreversibility constraint (2.4) is effectively absent. In this case, (5.3) holds regardless of x_0, so does (5.2). Another immediate implication of Proposition 5.1 is an "unbounded growth" version of the neighborhood turnpike theorem.

Corollary 5.1. *Suppose* $x^* = \infty$. *Then*

$$\lim_{\delta\uparrow\infty}\lim_{t\uparrow\infty} x_t^{\delta} = \infty. \tag{5.5}$$

Proof. Immediate from Proposition 5.1. □

Equation (5.5) means that for δ close to one, $\{x_t^{\delta}\}$ either goes to infinity or converges to an arbitrarily large steady state.

The only situation that is not covered by Proposition 5.1 is when $x^* < \infty$ and $r(x_0) > x^*$. In this case, (5.2) need not hold since $\{x_t^{\delta}\}$ could converge to a neighborhood of some $x \in (x^*, \infty]$ with $g(x) = g^*$. However this ambiguity disappears when x^* is the "unique maximizer" of g.

Proposition 5.2. *Suppose $x^* < \infty$. Suppose*

$$\forall x \in (0, \infty) \setminus \{x^*\}, \quad g(x) < g^*, \tag{5.6}$$

$$\varlimsup_{x \uparrow \infty} g(x) < g^*. \tag{5.7}$$

Then (5.2) holds.

Proof. See Appendix C. □

Proposition 5.2 can be thought of as a generalization of Majumdar and Nermuth [20, Theorem 3.4]. To be specific, assume (5.6). Suppose (i) there is a maximum sustainable capital stock, (ii) there is a neighborhood of zero on which $\Phi > 1$ for δ close to one, and (iii) there is a neighborhood of x^* that contains only one steady state $x(\delta)$ for δ close to one. In this case, (5.7) holds by (i) (recall (3.9)), and Proposition 5.2 along with Propositions 3.1 and 4.4 implies that for δ close to one, all optimal capital paths from all initial stocks converge to $x(\delta)$. This result was shown by Majumdar and Nermuth [20] for a differentiable case using the argument of Scheinkman's [27] visit lemma.

Appendix A Proofs of Section 3 results

A.1 Proof of Proposition 3.1

Lemma A.1. *Let $\{x_t\}$ be a convergent feasible capital path such that $x \equiv \lim_{t\uparrow\infty} x_t \in (0, \infty)$. Then $\exists \nu \in \mathbb{R}$, for large t,*

$$\frac{u(0)}{1-\delta} < \nu < v(x_t). \tag{A.1}$$

Proof. Let $z \in (r_+(x), x)$. This interval is nonempty by Assumption 2.3. Let $\underline{c} \in (0, x - z)$. Since $\lim_{t\uparrow\infty} x_t = x$, for large t,

$$r(x_t) < z < z + \underline{c} < x_{t+1} \le f(x_t). \tag{A.2}$$

Let $t \in \mathbb{Z}_+$ be large enough to satisfy (A.2). By Assumption 2.4, there is a feasible path $\{\hat{c}_s, \hat{x}_s\}$ from z with $\sum_{s=0}^{\infty} \delta^s u(\hat{c}_s) > -\infty$. Define $\{\tilde{c}_s, \tilde{x}_s\}$ as follows.

$$\tilde{x}_0 = x_t, \tilde{x}_1 = z, \tilde{c}_0 = f(x_t) - z, \quad \forall s \in \mathbb{N}, \tilde{x}_{s+1} = \hat{x}_s, \tilde{c}_s = \hat{c}_{s-1}. \tag{A.3}$$

Then by (A.2), $\tilde{x}_1 \in (r(x_t), f(x_t))$ and $\tilde{c}_0 > \underline{c}$. Hence by feasibility of $\{\hat{c}_s, \hat{x}_s\}$ from $\tilde{x}_1$ $(= z)$, $\{\tilde{c}_s, \tilde{x}_s\}$ is feasible from x_t. Thus

$$v(x_t) \geq \sum_{s=0}^{\infty} \delta^s u(\tilde{c}_s) > u(\underline{c}) + \delta \sum_{s=0}^{\infty} \delta^s u(\hat{c}_s) \equiv \nu. \tag{A.4}$$

Since the last sum is finite and $\underline{c} > 0$, (A.1) follows. □

Lemma A.2. *Let $\{x_t\}$ be an optimal capital path such that $x \equiv \lim_{t\uparrow\infty} x_t \in (0, \infty)$. Then $\lim_{t\uparrow\infty} v(x_t) = v(x)$.*

Proof. Let $\{c_t\}$ be the corresponding consumption path. We claim

$$c \equiv \lim_{t\uparrow\infty} c_t > 0. \tag{A.5}$$

Note that $\lim_{t\uparrow\infty} c_t$ exists since $\{f(x_t)\}$ is monotone in t and thus

$$\lim_{t\uparrow\infty} c_t = \lim_{t\uparrow\infty} [f(x_t) - x_{t+1}] = \lim_{t\uparrow\infty} f(x_t) - x. \tag{A.6}$$

Suppose $\lim_{t\uparrow\infty} c_t = 0$. Then

$$\overline{\lim_{t\uparrow\infty}}\, v(x_t) = \overline{\lim_{t\uparrow\infty}} \sum_{i=0}^{\infty} \delta^i u(c_{t+i}) \leq \frac{u(0)}{1-\delta}, \tag{A.7}$$

where the inequality holds by Fatou's lemma since $\{c_t\}$ is bounded. But (A.7) contradicts Lemma A.1. We have verified (A.5).

To prove the lemma, it suffices to show

$$v(x) \leq \underline{\lim_{t\uparrow\infty}}\, v(x_t), \tag{A.8}$$

since $\overline{\lim}_{t\uparrow\infty} v(x_t) \leq v(x)$ by Lemma 3.3. By Assumption 2.3, (A.5), and (A.6), $r_+(x) < x < \lim_{t\uparrow\infty} f(x_t)$. Hence for large t, $r(x_t) < x < f(x_t)$. Thus

$$u(f(x_t) - x) + \delta v(x) \leq u(c_t) + \delta v(x_{t+1}). \tag{A.9}$$

Since $\lim_{t\uparrow\infty} u(c_t) = \lim_{t\uparrow\infty} u(f(x_t) - x) = u(c)$ by (A.6) and continuity, applying $\underline{\lim}_{t\uparrow\infty}$ to (A.9) yields (A.8). □

Let us now prove Proposition 3.1. Let $\{c_t, x_t\}$ be an optimal path that is bounded. Let $x = \lim_{t\uparrow\infty} x_t$ and $c = \lim_{t\uparrow\infty} c_t$. By upper semicontinuity of f,

$$c + x \leq f(x). \tag{A.10}$$

If $x = 0$, then $c = 0$, so (c, x) is trivially a steady state. Suppose $x > 0$. For $t \in \mathbb{Z}_+$, we have $v(x_t) = u(c_t) + \delta v(x_{t+1})$. By Lemma A.2 and continuity of u, $v(x) = u(c) + \delta v(x)$. Since $x > r(x)$ by Assumption 2.3, it remains to verify $c+x = f(x)$. Recall (A.10). If $c+x < f(x)$, then $v(x) \geq u(f(x)-x)+\delta v(x) > u(c) + \delta v(x) = v(x)$, a contradiction.

A.2 Proof of Lemma 3.5

Lemma A.3. *For any feasible path $\{c_t, x_t\}$ with $x_0 > 0$,*

$$\sum_{t=0}^{\infty} \delta^t u(c_t) \leq \frac{u((1-\delta)f(x_0) + \overline{\Gamma})}{1-\delta}, \tag{A.11}$$

where $\overline{\Gamma} = \sup_{t\in\mathbb{Z}_+} \Gamma(x_{t+1})$. The inequality is strict if $\{c_t\}$ is not constant.

Proof. If $\sum_{t=0}^{\infty} \delta^t u(c_t) = -\infty$ or $\overline{\Gamma} = \infty$, then (A.11) trivially holds with strict inequality. Suppose $\sum_{t=0}^{\infty} \delta^t u(c_t) > -\infty$ and $\overline{\Gamma} < \infty$. It follows from (3.6)–(3.8) that $\forall T \in \mathbb{N}$,

$$\sum_{t=0}^{T} \delta^t c_t \leq f(x_0) + \sum_{t=0}^{T-1} \delta^t \overline{\Gamma}. \tag{A.12}$$

Hence

$$\sum_{t=0}^{\infty} \delta^t c_t \leq f(x_0) + \frac{\overline{\Gamma}}{1-\delta}. \tag{A.13}$$

Multiplying through by $1 - \delta$ and recalling that u is increasing, we get

$$u\left((1-\delta)\sum_{t=0}^{\infty} \delta^t c_t\right) \leq u\left((1-\delta)f(x_0) + \overline{\Gamma}\right). \tag{A.14}$$

Applying Jensen's inequality to the left-hand side yields (A.11). Since u is strictly concave, (A.11) holds with strict inequality if $\{c_t\}$ is not constant. □

Lemma A.4. *Let* $\{c_t, x_t\}$ *be a nonstationary optimal path with* $x_0 > 0$ *such that*

$$\forall t \in \mathbb{Z}_+, \quad \Gamma(x_{t+1}) \leq \Gamma(x_0). \tag{A.15}$$

Then $\{c_t\}$ *is not constant.*[18]

Proof. Suppose $\{c_t\}$ is constant. By Lemma A.3 and (A.15),

$$u(c_0) \leq u((1-\delta)f(x_0) + \Gamma(x_0)) = u(g(x_0)), \tag{A.16}$$

where g is defined by (3.14). It follows that $c_0 \leq g(x_0)$. If $c_0 < g(x_0)$, then this contradicts optimality since the stationary path from x_0 is feasible (recall Assumption 2.3(ii)). Suppose $c_0 = g(x_0)$. Then $x_1 = f(x_0) - g(x_0) = x_0$. Since $\{c_t\}$ is constant, it follows that $\forall t \in \mathbb{N}, x_t = x_0$, contradicting the nonstationarity of $\{c_t, x_t\}$. □

Let us now prove Lemma 3.5. Let $\{c_t\}$ be the associated consumption path. Assume (A.15). Since $\{x_t\}$ is nonstationary, $\{c_t\}$ is not constant by Lemma A.4. Thus by Lemma A.3 and (A.15),

$$\sum_{t=0}^{\infty} \delta^t u(c_t) < \frac{u((1-\delta)f(x_0) + \Gamma(x_0))}{1-\delta} = \frac{u(g(x_0))}{1-\delta}. \tag{A.17}$$

This requires $g(x_0) > 0$, which together with Assumption 2.3(ii) implies that the stationary path from x_0 is feasible. But this contradicts the optimality of $\{c_t, x_t\}$ by (A.17) again.

A.3 Proof of Lemma 3.6

Proof. We only prove (3.13). The proof of (3.12) is similar. Suppose

$$x_{t+1} > r(x_t), \quad c_{t+1} > 0. \tag{A.18}$$

If f is not left continuous at x_{t+1}, then $f'_-(x_{t+1}) = \infty$, so (3.13) trivially follows. Suppose f is left continuous at x_{t+1}. Consider increasing c_t by ϵ, decreasing x_{t+1} by ϵ, and decreasing c_{t+1} by $\mu(\epsilon) \equiv f(x_{t+1}) - f(x_{t+1} - \epsilon)$,

[18] The nonstationarity of $\{c_t, x_t\}$ only implies that $\{x_t\}$ is not constant. It is possible that $\{c_t\}$ is constant while $\{x_t\}$ is not constant. On the other hand, if $\{x_t\}$ is constant, $\{c_t\}$ is obviously constant.

while keeping the rest of the path unchanged. By (A.18), this perturbation is feasible for small $\epsilon > 0$. For small $\epsilon > 0$, by optimality,

$$u(c_t + \epsilon) + \delta u(c_{t+1} - \mu(\epsilon)) \leq u(c_t) + \delta u(c_{t+1}). \tag{A.19}$$

Rearranging and dividing through by ϵ, we get

$$\frac{u(c_t + \epsilon) - u(c_t)}{\epsilon} \leq \delta \frac{u(c_{t+1}) - u(c_{t+1} - \mu(\epsilon))}{\mu(\epsilon)} \frac{\mu(\epsilon)}{\epsilon}. \tag{A.20}$$

By concavity the left-hand side is monotone in ϵ, so is $[u(c_{t+1}) - u(c_{t+1} - \mu(\epsilon))]/\mu(\epsilon)$. Since $\lim_{\epsilon \downarrow 0} \mu(\epsilon) = 0$ by left continuity of f at x_{t+1}, applying $\underline{\lim}_{\epsilon \downarrow 0}$ to both sides of (A.20) yields (3.13). □

A.4 Proof of Lemma 3.7

If $g(x) < 0$, the stationary path from x is not feasible. Thus $g(x) \geq 0$. To verify $g(x) > 0$, it suffices to show that there is a feasible path $\{c'_t, x'_t\}$ from x such that $c'_0 > 0$ and $\sum_{t=0}^{\infty} \delta^t u(c'_t) > -\infty$; for this implies that a feasible path along which consumption is zero every period cannot be optimal.

Let $c'_0 = (f(x) - r(x))/2 > 0$ and $x'_1 = f(x) - c'_0 > r(x)$, where the first inequality holds by Assumption 2.3. By Assumption 2.4, there is a feasible path $\{\hat{c}_i, \hat{x}_i\}$ from x'_1 with $\sum_{i=0}^{\infty} \delta^i u(\hat{c}_i) > -\infty$. For $t \in \mathbb{N}$, let $c'_t = \hat{c}_{t-1}$ and $x'_{t+1} = \hat{x}_t$. Then $\{c'_t, x'_t\}$ is feasible and has the desired property. It follows that $g(x) > 0$.

From this and Assumption 2.3, $r_+(x) < x < f(x)$. Hence (3.12) and (3.13) hold with $c_t = c_{t+1} = g(x)$ and $x_{t+1} = x$. Both inequalities in (3.16) now follow.

A.5 Proof of Lemma 3.8

Lemma A.5. *Let $h : [a, b] \to \mathbb{R}$ be upper semicontinuous, where $-\infty < a < b < \infty$.*

(i) *If h is nondecreasing (nonincreasing), then $h'_- \geq (\leq)\ 0$ on $(a, b]$ and $h'_+ \geq (\leq)\ 0$ on $[a, b)$.*[19]

(ii) *If $h'_+ \geq 0$ on $[a, b)$, then h is nondecreasing on $[a, b]$.*

(iii) *If $h'_- \leq 0$ on $(a, b]$, then h is nonincreasing on $[a, b]$.*

[19]This part does not require upper semicontinuity.

Proof. See Giorgi and Komlósi [8, pp. 13–14]. □

Let us prove Lemma 3.8. Parts (i) and (iii) are immediate from (3.17). Parts (ii) and (iv) hold by Lemma A.5 because Γ is upper semicontinuous, $\Gamma'_+ = \delta f'_+ - 1$, and $\Gamma'_- = \delta f'_- - 1$.

Appendix B Proofs of Section 4 results

B.1 Proof of Proposition 4.2

Lemma B.1. *Let $\{c_t, x_t\}$ be an optimal path satisfying*

$$\text{(i) } g(x_0) > 0, \quad \text{(ii) } \exists T \in \mathbb{N}, \forall t \leq T-1, \quad \Gamma(x_{t+1}) \leq \Gamma(x_0). \tag{B.1}$$

Then

$$\delta[v(x_{T+1}) - v(x_0)] \geq u'_+(g(x_0))(x_{T+1} - x_0). \tag{B.2}$$

Proof. By (3.6)–(3.8) and (B.1)(ii),

$$\sum_{t=0}^{T} \delta^t c_t \leq f(x_0) + \sum_{t=0}^{T-1} \delta^t \Gamma(x_0) - \delta^T x_{T+1} \tag{B.3}$$

$$= \sum_{t=0}^{T} \delta^t [f(x_0) - x_0] + \delta^T x_0 - \delta^T x_{T+1}. \tag{B.4}$$

Since $g(x_0) = f(x_0) - x_0$ (recall (3.14)), it follows that

$$\sum_{t=0}^{T} \delta^t (c_t - g(x_0)) \leq -\delta^T (x_{T+1} - x_0). \tag{B.5}$$

Define $\{\tilde{x}_t\}$ as follows: $\tilde{x}_t = x_0$ for $t \leq T+1$ and $\tilde{x}_t = x_{t-T-1}$ for $t \geq T+2$. By (B.1)(i), $\{\tilde{x}_t\}$ is feasible and $\{\tilde{x}_{T+1+i}\}_{i=0}^{\infty}$ $(= \{x_t\})$ is optimal from x_0. Thus

$$\sum_{t=0}^{T} \delta^t u(g(x_0)) + \delta^{T+1} v(x_0) \leq \sum_{t=0}^{T} \delta^t u(c_t) + \delta^{T+1} v(x_{T+1}). \tag{B.6}$$

This rearranges to

$$
\begin{aligned}
-\delta^{T+1}[v(x_{T+1}) - v(x_0)] &\leq \sum_{t=0}^{T} \delta^t [u(c_t) - u(g(x_0))] && \text{(B.7)} \\
&\leq \sum_{t=0}^{T} \delta^t u'_+(g(x_0))(c_t - g(x_0)) && \text{(B.8)} \\
&\leq -u'_+(g(x_0))\delta^T (x_{T+1} - x_0), && \text{(B.9)}
\end{aligned}
$$

where (B.8) holds by concavity, and (B.9) by (B.5). Now (B.2) follows. □

Lemma B.2. *Assume (4.2). Suppose* $\exists z > 0$, *there exists a sequence* $\{z_i\}_{i=1}^{\infty}$ *in* $(0, z)$ *such that*

$$
\text{(i) } \lim_{i\uparrow\infty} z_i = 0, \quad \text{(ii) } \forall i \in \mathbb{N}, \forall x \in (z_i, z],\ \Gamma(z_i) \geq \Gamma(x). \tag{B.10}
$$

Then $\exists i \in \mathbb{N}$, *any optimal capital path from* z_i *is bounded above by* z.

Proof. Suppose the conclusion is false. Then $\forall i \in \mathbb{N}$, there is an optimal path $\{c_t^i, x_t^i\}$ with $x_0^i = z_i$ such that $\lim_{t\uparrow\infty} x_t^i > z$. Let $\underline{z} > 0$ be such that

$$
f(\underline{z}) < z, \quad \frac{\underline{z}}{1-\delta} < z. \tag{B.11}
$$

Without loss of generality, assume $\forall i \in \mathbb{N}, x_0^i < \underline{z}$ (recall $x_0^i = z_i$). Note that

$$
\lim_{i\uparrow\infty} g(x_0^i) = 0, \tag{B.12}
$$

$$
\forall i \in \mathbb{N}, \quad g(x_0^i) = f(x_0^i) - x_0^i > f(x_0^i) - x_{t_i}^i \geq 0, \tag{B.13}
$$

where t_i is the first t with $x_t^i > x_0^i$.

Let $i \in \mathbb{N}$. Let T_i be the first $t \in \mathbb{Z}_+$ with $x_{t+1}^i > z$. Note that

$$
x_0^i < \underline{z} < x_{T_i}^i \leq z < x_{T_i+1}^i \leq f(x_{T_i}^i) \leq f(z). \tag{B.14}
$$

By Lemma B.1, (B.10)(ii), and (B.13),

$$
\delta v(x_{T_i+1}^i) \geq u'_+(g(x_0^i))(x_{T_i+1}^i - x_0^i) + \delta v(x_0^i). \tag{B.15}
$$

Recalling (B.14) and (B.13), we get

$$
m \equiv \delta \max_{y\in[z,f(z)]} v(y) \geq u'_+(g^i)(z - \underline{z}) + \frac{\delta u(g^i)}{1-\delta}, \tag{B.16}
$$

where $g^i = g(x_0^i)$. By concavity, $u(g^i) \geq u(\underline{z}) - u'_+(g^i)(\underline{z} - g^i) \geq u(\underline{z}) - u'_+(g^i)\underline{z}$. Thus

$$\infty > m \geq u'_+(g^i)(z - \underline{z}) + \frac{\delta}{1-\delta}[u(\underline{z}) - u'_+(g^i)\underline{z}] \tag{B.17}$$

$$= u'_+(g^i)\left[z - \frac{\underline{z}}{1-\delta}\right] + \frac{\delta u(\underline{z})}{1-\delta}. \tag{B.18}$$

Since the expression in square brackets is strictly positive by (B.11), and since $\lim_{i\uparrow\infty} u'_+(g^i) = \infty$ by (B.12) and (4.2), the right-hand side of (B.18) goes to ∞ as $i \uparrow \infty$, a contradiction. □

To complete the proof of Proposition 4.2, let $\{z_i\}_{i=1}^{\infty}$ be a sequence in $(0, z)$ satisfying (B.10)(i). By (4.3) and Lemma 3.8, Γ is nonincreasing on $(0, z]$. Thus $\{z_i\}$ also satisfies (B.10)(ii). By Lemma B.2, $\exists i \in \mathbb{N}$, any optimal capital path from z_i is bounded above by z. Let $\overline{x} = z_i$. By (4.3) and Lemma 3.7, there is no steady state in $(0, z]$. Thus any optimal capital path from $\overline{x}$ converges to zero by Proposition 3.1. Hence every optimal capital path from $x_0 \in (0, \overline{x}]$ converges to zero by Lemma 3.4 and Proposition 3.1.

B.2 Proof of Proposition 4.5

Lemma B.3. *If u is bounded above, then* $\lim_{c\uparrow\infty} u'_+(c)c = 0$.

Proof. By concavity, $\forall c, \hat{c} > 0, u(c) \leq u(\hat{c}) + u'_+(\hat{c})(c - \hat{c})$. Thus

$$\forall c, \hat{c} > 0, \quad u'_+(\hat{c})\hat{c} \leq u(\hat{c}) - u(c) + u'_+(\hat{c})c. \tag{B.19}$$

Since u is bounded above, $\lim_{\hat{c}\uparrow\infty} u'_+(\hat{c}) = 0$. It follows that

$$\forall c > 0, \quad \overline{\lim_{\hat{c}\uparrow\infty}}\, u'_+(\hat{c})\hat{c} \leq \lim_{\hat{c}\uparrow\infty} u(\hat{c}) - u(c). \tag{B.20}$$

Applying $\lim_{c\uparrow\infty}$ yields $\lim_{\hat{c}\uparrow\infty} u'_+(\hat{c})\hat{c} = 0$. □

Lemma B.4. *Suppose $\exists z > 0$, there exists a sequence $\{z_i\}_{i=1}^{\infty}$ in (z, ∞) such that*

$$\text{(i) } \lim_{i\uparrow\infty} z_i = \infty, \quad \text{(ii) } \forall i \in \mathbb{N}, \forall x \in [z, z_i),\ \Gamma(x) \leq \Gamma(z_i). \tag{B.21}$$

Then $\exists \theta > 0, \exists \underline{i} \in \mathbb{N}, \forall i \geq \underline{i}, z_i \leq \theta g(z_i)$.

Proof. By (B.21), $\forall i \in \mathbb{N}, \delta f(z_i) - z_i \geq \delta f(z) - z$; equivalently,

$$\forall i \in \mathbb{N}, \quad f(z_i) - z_i \geq f(z) - \frac{z}{\delta} + \mu z_i, \tag{B.22}$$

where $\mu = 1/\delta - 1$. From (B.22), $\forall i \in \mathbb{N}, g(z_i)/z_i \geq [f(z) - z/\delta]/z_i + \mu$. Let $\theta > 1/\mu$. Then since $\lim_{i\uparrow\infty} z_i = \infty$, $g(z_i)/z_i \geq 1/\theta$ for large i. □

Lemma B.5. *Assume (4.6). Suppose* $\exists z > 0$, *there exists a sequence* $\{z_i\}_{i=1}^{\infty}$ *in* (z, ∞) *satisfying (B.21). Then* $\exists i \in \mathbb{N}$, *any optimal capital path from* z_i *is bounded below by* z.

Proof. Suppose the conclusion is false. Then $\forall i \in \mathbb{N}$, there is an optimal path $\{c_t^i, x_t^i\}$ with $x_0^i = z_i$ such that $\lim_{t\uparrow\infty} x_t^i < z$. Let $\theta > 0$ and $\underline{i} \in \mathbb{N}$ be as given by Lemma B.4. Without loss of generality, assume $\underline{i} = 1$. Then

$$\forall i \in \mathbb{N}, \quad x_0^i \leq \theta g(x_0^i). \tag{B.23}$$

(Recall $x_0^i = z_i$.) By (B.21)(i) and (B.23),

$$\lim_{i\uparrow\infty} g(x_0^i) = \infty. \tag{B.24}$$

Let $i \in \mathbb{N}$. Let T_i be the first $t \in \mathbb{Z}_+$ with $x_{t+1}^i < z$. By Lemma B.1 and (B.23),

$$\delta[v(x_0^i) - v(x_{T_i+1}^i)] \leq u'_+(g^i)(x_0^i - x_{T_i+1}^i) \leq u'_+(g^i)\theta g^i, \tag{B.25}$$

where $g^i = g(x_0^i)$. Since the stationary path from x_0^i is feasible by (B.23),

$$\frac{u(g^i)}{1-\delta} \leq v(x_0^i). \tag{B.26}$$

Since $x_{T_i+1}^i < z$,

$$v(x_{T_i+1}^i) \leq \max_{y\in[0,z]} v(y). \tag{B.27}$$

It follows from (B.25)–(B.27) that

$$\delta\left[\frac{u(g^i)}{1-\delta} - \max_{y\in[0,z]} v(y)\right] \leq \theta u'_+(g^i)g^i. \tag{B.28}$$

Suppose u is bounded above. Then by (B.24), the left-hand side of (B.28) tends to

$$\delta\left[\frac{\sup_{c\geq 0} u(c)}{1-\delta} - \max_{y\in[0,z]} v(y)\right] > 0 \tag{B.29}$$

as $i \uparrow \infty$. On the other hand, by (B.23) and Lemma B.3, the right-hand side of (B.28) goes to zero as $i \uparrow \infty$, a contradiction.

Now suppose u is unbounded above. Then by (B.23), the left-hand side of (B.28) goes to ∞ as $i \uparrow \infty$, while by (4.6) the right-hand side is uniformly bounded above for large i, again a contradiction. $\square$

The proof of Proposition 4.5 can now be completed by an argument similar to the last paragraph of the proof of Proposition 4.2.

Appendix C Proofs of Section 5 results

C.1 Proof of Lemma 5.1

Lemma C.1. (i) $\forall \underline{x} > 0, \forall \overline{x} \geq \underline{x}, \inf_{x \in [\underline{x}, \overline{x}]}(x - r(x)) > 0$. (ii) $\forall \underline{x} \in (0, x^*), \forall \overline{x} \in [\underline{x}, x^*] \cap [\underline{x}, \infty)$,[20] $\inf_{x \in [\underline{x}, \overline{x}]} g(x) > 0$.

Proof. We prove (ii) first. Suppose $\inf_{x \in [\underline{x}, \overline{x}]} g(x) = 0$. Then there exists a sequence $\{x^i\} \subset [\underline{x}, \overline{x}]$ such that $\lim_{i \uparrow \infty} g(x^i) = 0$. Taking a subsequence, we may assume that $\{x^i\}$ converges to some $y \in [\underline{x}, \overline{x}]$. Now we obtain the following contradiction: $0 = \lim_{i \uparrow \infty} g(x^i) \geq g_-(y) > 0$, where the last inequality holds by Assumption 5.2. To prove (i), replace $g(x)$ by $(x - r(x))$ and note from Assumption 2.3(ii) that $y - r_+(y) > 0$. $\square$

Lemma C.2. *Let $y \in (0, x^*] \cap (0, \infty)$. Then $\forall x > 0$, there exists a feasible path $\{c_t, x_t\}$ from x such that*

$$\exists T \in \mathbb{Z}_+, \quad \forall t \leq T, c_t > 0, \quad \forall t \geq T + 1, c_t = g(y), x_t = y. \tag{C.1}$$

Proof. Suppose $x \geq y$ first. We construct $\{c_t, x_t\}$ recursively as follows:

$$x_0 = x, \quad \forall t \in \mathbb{Z}_+, x_{t+1} = \max\{r(x_t), y\}, c_t = f(x_t) - x_{t+1}. \tag{C.2}$$

It is easy to see that $\{x_t\}$ is nonincreasing. Let $a = \inf_{z \in [y,x]}(z - r(z))$. By Lemma C.1, $a > 0$. If $\exists t \in \mathbb{Z}_+, r(x_t) > y$, then $x_{t+1} - x_t = r(x_t) - x_t \leq -a$. Thus the inequality $r(x_t) > y$ holds only finitely many times or never holds. Let T be the first $t \in \mathbb{Z}_+$ such that $r(x_t) \leq y$. If $T \geq 1$, then for $t \leq T - 1, c_t = f(x_t) - r(x_t) > 0$ by Assumption 2.3. We have $x_{T+1} = y$ and $c_T = f(x_T) - y > f(y) - y = g(y) \geq g_-(y) > 0$, where the last inequality

[20] Recall Footnote 16.

holds by Assumption 5.2. For $t \geq T+1, c_t = g(y)$. It follows that $\{c_t, x_t\}$ is feasible and satisfies (C.1).

Now suppose $x < y$. Let $\bar{c} \equiv \inf_{z \in [x,y]} g(z)$. By Lemma C.1, $\bar{c} > 0$. Let $\underline{c} \in (0, \bar{c})$. We construct $\{c_t, x_t\}$ recursively as follows:

$$x_0 = x, \quad \forall t \in \mathbb{Z}_+, x_{t+1} = \min\{f(x_t) - \underline{c}, y\}, c_t = f(x_t) - x_{t+1}. \tag{C.3}$$

It is easy to see that $\{x_t\}$ is nondecreasing. Note that

$$\forall t \in \mathbb{Z}_+, \quad c_t = \max\{\underline{c}, f(x_t) - y\} \geq \underline{c}. \tag{C.4}$$

If $\exists t \in \mathbb{Z}_+, f(x_t) - \underline{c} < y$, then $x_{t+1} - x_t = f(x_t) - \underline{c} - x_t = g(x_t) - \underline{c} \geq \bar{c} - \underline{c} > 0$. Thus the inequality $f(x_t) - \underline{c} < y$ holds only finitely many times or never holds. Since $\{x_t\}$ is nondecreasing, the irreversibility constraint (2.4) always holds. It follows that $\{c_t, x_t\}$ is feasible and satisfies (C.1). $\square$

Lemma C.3. *Let $y \in (0, x^*] \cap (0, \infty)$. Then*

$$\varliminf_{\delta \uparrow 1} \sup_{t \in \mathbb{Z}_+} g(x_{t+1}^{\delta}) \geq g(y). \tag{C.5}$$

Proof. Let $\eta > 0$. We show that for δ close to one, any feasible path $\{\tilde{c}_t, \tilde{x}_t\}$ from x_0 is nonoptimal if

$$\sup_{t \in \mathbb{Z}_+} g(\tilde{x}_{t+1}) \leq g(y) - \eta. \tag{C.6}$$

Since $\forall x \geq 0, \Gamma(x) = \delta f(x) - x \leq f(x) - x = g(x)$, it follows by Lemma A.3 that

$$\forall \delta \in (0,1), \quad (1-\delta) \sup \sum_{t=0}^{\infty} \delta^t u(\tilde{c}_t) \leq u((1-\delta) f(x_0) + g(y) - \eta), \tag{C.7}$$

where the supremum is taken over all feasible paths $\{\tilde{c}_t, \tilde{x}_t\}$ satisfying (C.6). By Lemma C.2, there is a feasible path $\{c_t, x_t\}$ from x_0 satisfying (C.1). Hence

$$\forall \delta \in (0,1), \quad (1-\delta) \sum_{t=0}^{T} \delta^t u(c_t) + \delta^{T+1} u(g(y)) \leq (1-\delta) v(x_0). \tag{C.8}$$

Note that T does not depend on δ and that v implicitly depends on δ. As $\delta \uparrow 1$, the left-hand side of (C.8) goes to $u(g(y))$, while the right-hand side of (C.7) goes to $u(g(y)-\eta) < u(g(y))$. It follows that for δ close to one, any feasible path $\{\tilde{c}_t, \tilde{x}_t\}$ satisfying (C.6) is nonoptimal.

Since $\{x_t^\delta\}$ is optimal for each $\delta \in (0,1)$, we must have $\sup_{t\in\mathbb{Z}_+} g(x_{t+1}^\delta) > g(y)-\eta$ for δ close to one. Since this is true for any $\eta > 0$, (C.5) follows. $\square$

Now we prove Lemma 5.1. By definition of g^*, $\overline{\lim}_{\delta\uparrow 1} \sup_{t\in\mathbb{Z}_+} g(x_{t+1}^\delta) \leq g^*$. Thus it suffices to show

$$\underline{\lim}_{\delta\uparrow 1} \sup_{t\in\mathbb{Z}_+} g(x_{t+1}^\delta) \geq g^*. \tag{C.9}$$

If $x^* < \infty$, (C.9) follows from Lemma C.3 with $y = x^*$. If $x^* = \infty$, applying $\sup_{y\in\mathbb{R}_{++}}$ to (C.5) yields (C.9).

C.2 Proof of Proposition 5.1

Lemma C.4. *Suppose $\exists \hat{x} > 0, g(\hat{x}) = g^*$. Then for any $\delta \in (0,1)$, any optimal capital path from $\hat{x}$ is nonincreasing.*

Proof. Note that $\forall x \geq \hat{x}, \Gamma(x) = \delta f(x) - x = \delta(f(x)-x) - (1-\delta)x \leq \delta g(\hat{x}) - (1-\delta)\hat{x} = \Gamma(\hat{x})$. Thus the conclusion holds by Lemma 3.5. $\square$

Lemma C.5. *If $x_0 \leq x^*$, then $\overline{\lim}_{\delta\uparrow 1} \lim_{t\uparrow\infty} x_t^\delta \leq x^*$.*

Proof. The inequality is trivial if $x^* = \infty$. If $x^* < \infty$, it holds by Lemmas 3.4 and C.4 with $\hat{x} = x^*$. $\square$

Lemma C.6. *If $x_0 < x^*$, then (5.2) holds.*

Proof. By Lemma C.5, it suffices to show $\underline{\lim}_{\delta\uparrow 1} \lim_{t\uparrow\infty} x_t^\delta \geq x^*$. Suppose this inequality does not hold; i.e., $\exists \overline{x} \in (x_0, x^*)$, there is a sequence $\{\delta_i\}_{i=1}^\infty$ in $(0,1)$ with $\lim_{i\uparrow\infty} \delta_i = 1$ such that $\forall i \in \mathbb{N}, \lim_{t\uparrow\infty} x_t^{\delta_i} \leq \overline{x}$. By monotonicity, $\forall i \in \mathbb{N}, \forall t \in \mathbb{Z}_+, x_{t+1}^{\delta_i} \leq \overline{x}$. Thus $\forall i \in \mathbb{N}, \sup_{t\in\mathbb{Z}_+} g(x_{t+1}^{\delta_i}) \leq \max_{y\in[0,\overline{x}]} g(y) < g^*$, contradicting Lemma 5.1. $\square$

Lemma C.7. *Let $y \in (0, x^*)$. Then for δ close to one, every optimal capital path from y is nondecreasing.*

Proof. Note that Lemma C.6 holds for any $x_0 \in (0, x^*)$ and any set of optimal capital paths $\{\{x_t^\delta\}\}_{\delta\in(0,1)}$ from x_0 such that each $\{x_t^\delta\}$ is optimal when the discount factor equals δ. Thus the conclusion follows from monotonicity and Lemma C.6 with $x_0 = y$. $\square$

Lemma C.8. *If $x^* \leq x_0$, then $x^* \leq \underline{\lim}_{\delta \uparrow 1} \lim_{t \uparrow \infty} x_t^\delta$.*

Proof. By Lemmas C.7 and 3.4, $\forall y \in (0, x^*), y \leq \underline{\lim}_{\delta \uparrow 1} \lim_{t \uparrow \infty} x_t^\delta$. Letting $y \uparrow x^*$ gives the desired inequality. □

Lemma C.9. *If $x_0 = x^*$, then (5.2) holds.*

Proof. By Lemmas 3.4 and C.4 with $\hat{x} = x^*$, $\overline{\lim}_{\delta \uparrow 1} \lim_{t \uparrow \infty} x_t^\delta \leq x^*$. This together with Lemma C.8 shows (5.2). □

Lemma C.10. *Suppose $x^* < \infty$ and $x^* < x_0$. Assume (5.3). Let $z \in (x^*, x_0]$. Then for any $\delta \in (0, 1)$, there exists no optimal capital path from z that is nondecreasing.*

Proof. Suppose there is an optimal path $\{c'_t, x'_t\}$ from z such that $\{x'_t\}$ is nondecreasing. Then

$$\forall t \in \mathbb{Z}_+, \quad c'_t = f(x'_t) - x'_{t+1} \leq f(x'_t) - x'_t = g(x'_t). \tag{C.10}$$

Define $\{\tilde{x}_t, \tilde{c}_t\}$ as follows.

$$\tilde{x}_0 = z, \tilde{c}_0 = f(z) - x^*, \quad \forall t \geq 1, \tilde{x}_t = x^*, \tilde{c}_t = g(x^*). \tag{C.11}$$

Since $x'_1 > x^*$,

$$\tilde{c}_0 > f(z) - x'_1 = c'_0 \geq 0. \tag{C.12}$$

This together with (5.3) implies that $\{\tilde{c}_t, \tilde{x}_t\}$ is feasible. By (C.10) and (C.11), $\forall t \in \mathbb{N}, \tilde{c}_t \geq c'_t$. This together with (C.12) contradicts the optimality of $\{c'_t, x'_t\}$. □

Let us now complete the proof of Proposition 5.1. Lemmas C.6 and C.9 cover the case $x_0 \leq x^*$. Suppose $x^* < \infty$ and $x^* < x_0$. By Lemma C.10, $\forall \delta \in (0, 1), \lim_{t \uparrow \infty} x_t^\delta \leq x^*$. Thus $\overline{\lim}_{\delta \uparrow 1} \lim_{t \uparrow \infty} x_t^\delta \leq x^*$. This together with Lemma C.8 shows (5.2).

C.3 Proof of Proposition 5.2

If $x_0 \leq x^*$, (5.2) holds by Proposition 5.1. Suppose $x_0 > x^*$. By Lemma C.8, it suffices to verify $\overline{\lim}_{\delta \uparrow 1} \lim_{t \uparrow \infty} x_t^\delta \leq x^*$. Suppose $\overline{\lim}_{\delta \uparrow 1} \lim_{t \uparrow \infty} x_t^\delta > x^*$. Then $\exists \underline{x} \in (x^*, x_0)$, there is a sequence $\{\delta_i\}_{i=1}^\infty$ in $(0, 1)$ with $\lim_{i \uparrow \infty} \delta_i = 1$ such that $\forall i \in \mathbb{N}, \lim_{t \uparrow \infty} x_t^{\delta_i} \geq \underline{x}$. By monotonicity, $\forall i \in \mathbb{N}, \forall t \in \mathbb{Z}_+, x_t^{\delta_i} \geq \underline{x}$. Hence $\forall i \in \mathbb{N}, \sup_{t \in \mathbb{Z}_+} g(x_{t+1}^{\delta_i}) \leq \sup_{x \geq \underline{x}} g(x) < g^*$, where the last inequality holds by (5.6) and (5.7). But this contradicts Lemma 5.1.

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