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Almost sure convergence to zero in stochastic growth models*

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Summary: This paper considers the resource constraint commonly used in stochastic one-sector growth models. Shocks are not required to be i.i.d. It is shown that any feasible path converges to zero exponentially fast almost surely under a certain condition. In the case of multiplicative shocks, the condition means that the shocks are sufficiently volatile. Convergence is faster the larger their volatility is, and the smaller the maximum average product of capital is.

Keywords and Phrases: Stochastic growth; Inada condition; Convergence to zero; Extinction

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1 Introduction

In a seminal paper, Brock and Mirman (1972) showed that the optimal paths of a stochastic one-sector growth model converge to a unique nontrivial stationary distribution. While various cases are known in which their theorem can be extended,[1] it does not seem to be well understood when the theorem fails. Most of the extensions of the Brock-Mirman theorem assume that the production function satisfies the Inada condition at zero, i.e., that the marginal product of capital goes to infinity as capital goes to zero.[2]

Although the Inada condition at zero is widely used in economics, the only justification for its use seems to be mathematical convenience.[3] In fact it is known to have the rather unrealistic implication that each unit of capital must be capable of producing an arbitrarily large amount of output with a sufficient amount labor (e.g., Färe and Primont, 2002).

In this paper we consider the resource constraint commonly used in stochastic one-sector growth models, focusing on the case in which the Inada condition at zero is not satisfied. Our framework encompasses stochastic endogenous growth models as well as stochastic overlapping generations models. To accommodate nonconcave production functions, we assume that the maximum (stochastic) average product of capital is always finite, which is equivalent to the violation of the Inada condition at zero in the concave case. Under this assumption we show that any *feasible* path converges to zero exponentially fast almost surely if there is a negative upper bound on the long-run sample average of the logarithm of the maximum average product of capital. In the case of multiplicative shocks, this general condition means that the shocks are sufficiently volatile. Convergence is faster the larger their volatility is, and the smaller the maximum (deterministic) average product of capital is.

To our knowledge, this relationship between almost sure convergence to zero and the volatility of shocks has not been documented in the stochastic

[1]For example, see Stachurski (2002) and the references therein.

[2]Notable exceptions are Hopenhayn and Prescott (1992, Section 6.B(i)) and Nishimura and Stachurski (2005, Theorem 3.1). Our results offer partial converses to their and Boylan's (1979, Theorem 2) results.

[3]According to Barro and Sala-i-Martin (1995, p. 16), the Inada conditions $f'(0) = \infty$ and $f'(\infty) = 0$ are named after Inada (1963). But actually he used these conditions following Uzawa (1963). Neither Inada nor Uzawa provided an economic justification for the conditions.

growth literature, though technically similar results have recently been obtained independently by Mitra and Roy (2003) (MR henceforth) and Nishimura, Rudnicki, and Stachurski (2004) (NRS henceforth). Both MR and NRS study optimal stochastic growth models. The advantage of our results is twofold. First, we do not assume i.i.d. shocks, while both MR and NRS require shocks to be i.i.d. and to satisfy additional assumptions. Second, unlike MR and NRS, we establish (or notice) almost sure *exponential* convergence to zero and provide an approximate rate of convergence. We must admit however that we obtain these advantages mainly because we focus on one particular phenomenon, while MR and NRS consider various other phenomena as well.

Another closely related result was shown by Athreya (2004, Corollary 1) for a certain class of Markov processes with i.i.d. shocks. Our general result can be viewed as an extension of his result to a nonstationary setting with non-i.i.d. shocks.

Two other results in the literature are particularly relevant to this paper. First, for a parametric model with logarithmic utility, Danthine and Donaldson (1981) showed that increased uncertainty negatively affects the expected value of the long-run capital stock. Second, Rothschild and Stiglitz (1971, Section 2.A) showed that increased uncertainty may increase the savings rate, depending on the curvature of utility.[4] Our result in the case of multiplicative shocks makes it clear that regardless of the objective function, and regardless of the savings rate (even if it is 100%), convergence to zero occurs almost surely if there is sufficient uncertainty, as long as the maximum average product of capital is finite.

The rest of the paper is organized as follows. Section 2 proves a general result for nonstationary one-sector growth models with non-i.i.d. shocks. Section 3 focuses on the stationary case with multiplicative shocks, showing and discussing a consequence of the general result that is easier to interpret.

[4]See Jones et al. (2005) for a recent treatment of related problems in the context of endogenous growth.

2 The General Result

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider an infinite horizon economy in which the resource constraint in period $t \in \mathbb{Z}_+$ is given by

$$\forall \omega \in \Omega, \quad c_t(\omega) + k_{t+1}(\omega) = g_t(k_t(\omega), \omega),\,^5 \tag{2.1}$$

where $c_t(\omega)$ is consumption in period t, $k_t(\omega)$ is the capital stock at the beginning of period t, and $g_t : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is the production function in period t, which is random and may also vary over time in a deterministic way. We say that a nonnegative stochastic process $\{k_t\}_{t=0}^{\infty}$ is a *feasible path* if it satisfies (2.1) for all $t \in \mathbb{Z}_+$ for some nonnegative stochastic process $\{c_t\}$.[6]

Special cases of (2.1) are used as resource constraints in various stochastic growth models, including optimal growth models, endogenous growth models with externalities, and overlapping generations models.[7] No further equation is required for our results, which concern only feasible paths. (Additional assumptions on g_t are introduced in Section 3.) Note that optimal or equilibrium paths are required to be feasible no matter how they are defined. Thus our results apply to any model in which (2.1) is required as a resource constraint.

The following result provides a sufficient set of conditions for almost sure convergence to zero.

Theorem 2.1. *Suppose*

$$\forall t \in \mathbb{Z}_+, \forall \omega \in \Omega, \quad g_t(0, \omega) = 0, \tag{2.2}$$

$$\forall t \in \mathbb{Z}_+, \forall \omega \in \Omega, \quad a_t(\omega) \equiv \sup_{k>0} \frac{g_t(k, \omega)}{k} < \infty, \tag{2.3}$$

$$\mu \equiv \operatorname*{ess\,sup}_{\omega \in \Omega} \limsup_{T \uparrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln a_t(\omega) < 0.^8 \tag{2.4}$$

Then any feasible path $\{k_t\}$ converges to zero exponentially fast a.s. More specifically, for almost all $\omega \in \Omega, \forall \lambda \in (0, -\mu)$,

$$\exists T \in \mathbb{Z}_+, \forall t \geq T, \quad k_t(\omega) < e^{-\lambda t}. \tag{2.5}$$

[5] The quantifier "$\forall \omega \in \Omega$" may be replaced by "for almost all $\omega \in \Omega$" throughout this paper. The distinction is negligible since we are only concerned with almost sure convergence to zero.

[6] By a stochastic process $\{x_t\}_{t=0}^{\infty}$, we mean a sequence of functions $x_t : \Omega \to \mathbb{R}$.

[7] In overlapping generations models, c_t in (2.1) represents aggregate consumption.

[8] For a random variable $x : \Omega \to \overline{\mathbb{R}}$, $\operatorname{ess\,sup}_{\omega \in \Omega} x(\omega) \equiv \inf\{s \in \overline{\mathbb{R}} \,|\, x \leq s \text{ a.s.}\}$.

Proof. Let $\{k_t\}$ be any feasible path. By (2.1), (2.2), and (2.3),

$$\forall t \in \mathbb{Z}_+, \forall \omega \in \Omega, \quad k_{t+1}(\omega) \leq g_t(k_t(\omega), \omega) \leq a_t(\omega) k_t(\omega). \tag{2.6}$$

Let

$$\Omega_1 = \{\omega \in \Omega \,|\, \forall t \in \mathbb{Z}_+, a_t(\omega) > 0, k_t(\omega) > 0\}, \tag{2.7}$$

$$\Omega_2 = \left\{ \omega \in \Omega \;\middle|\; \limsup_{T \uparrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} \ln a_t(\omega) \leq \mu \right\}. \tag{2.8}$$

By (2.2) and (2.6), $\forall \omega \in \Omega \setminus \Omega_1, k_t(\omega) = 0$ for sufficiently large t. By (2.4), $P(\Omega_2) = 1$. Thus we may restrict attention to $\omega \in \Omega_1 \cap \Omega_2$. Fix $\omega \in \Omega_1 \cap \Omega_2$ for the rest of the proof. We write k_t instead of $k_t(\omega)$, etc., for notational simplicity.

Since $\omega \in \Omega_1$, it follows from (2.6) that

$$\forall t \in \mathbb{Z}_+, \quad \ln k_{t+1} \leq \ln a_t + \ln k_t. \tag{2.9}$$

Hence

$$\forall T \in \mathbb{N}, \quad \ln k_T \leq \sum_{t=0}^{T-1} \ln a_t + \ln k_0. \tag{2.10}$$

Dividing through by T, we get

$$\forall T \in \mathbb{N}, \quad \frac{\ln k_T}{T} \leq \frac{\sum_{t=0}^{T-1} \ln a_t}{T} + \frac{\ln k_0}{T}. \tag{2.11}$$

Let $\lambda \in (0, -\mu)$, i.e., $\mu < -\lambda < 0$. Since $\omega \in \Omega_2$, the right-hand side of (2.11) is strictly less than $-\lambda$ for sufficiently large T. Thus for sufficiently large T, $\ln(k_T)/T < -\lambda$, i.e., $k_T < e^{-\lambda T}$. $\square$

Condition (2.2) is a standard restriction. Condition (2.3) says that the maximum (stochastic) average product of capital is always finite, which implies the violation of the Inada condition at zero.[9] Condition (2.4) means that there is a negative upper bound on almost every long-run sample average of the logarithm of the maximum average product of capital. Since $\exp(\sum_{t=0}^{T-1} \ln a_t) = \prod_{t=0}^{T-1} a_t$, and since a_t is the maximum possible gross

[9]Given (2.4), (2.3) is essentially redundant as long as μ is well-defined. But (2.3) is needed for μ to be well-defined unless $a_t > 0$ a.s.

growth rate of capital in period t, (2.4) implies that for almost every sample path, the gross growth rate over a long horizon is less than one, i.e., the net growth rate over a long horizon is negative.

If $\{\ln a_t\}$ satisfies the law of large numbers, then (2.4) reduces to $E \ln a_t < 0$. This condition becomes easy to interpret particularly in the case of multiplicative shocks, which we consider in the next section.

3 Multiplicative Shocks

In this section we focus on the case of multiplicative shocks. In particular we assume the following in (2.1).

Assumption 3.1. There exist $f : \mathbb{R}_+ \to \mathbb{R}_+$ and a stochastic process $\{s_t\}$ such that

$$\forall t \in \mathbb{Z}_+, \forall \omega \in \Omega, \forall k \geq 0, \quad g_t(k, \omega) = s_t(\omega) f(k). \tag{3.1}$$

Hence (2.1) can now be written as $c_t + k_{t+1} = s_t f(k_t)$. Let us state and discuss our other assumptions.

Assumption 3.2. $\forall t \in \mathbb{Z}_+$, (i) $\forall \omega \in \Omega, 0 \leq s_t(\omega) < \infty$, and (ii) $E s_t = 1$.

Assumption 3.3. $f(0) = 0$, and

$$m \equiv \sup_{k>0} \frac{f(k)}{k} < \infty. \tag{3.2}$$

Assumption 3.2(ii) is merely a normalization. If f is concave and differentiable, then (3.2) is equivalent to $f'(0) < \infty$, the violation of the Inada condition at zero. More generally, (3.2) means that the maximum (deterministic) average product of capital is finite. Though m is required to be finite, it is allowed to be arbitrarily large.

Assumption 3.4. $\exists \nu \in (-\infty, \infty], \forall t \in \mathbb{Z}_+, E \ln s_t = -\nu$.

This assumption means only that $E \ln s_t$ does not depend on t. By Jensen's inequality and Assumption 3.2,

$$-\nu = E \ln s_t \leq \ln E s_t = 0. \tag{3.3}$$

Thus ν is in fact nonnegative. Since it is the difference between $\ln E s_t$ $(= 0)$ and $E \ln s_t$ due to the strict concavity of the log function, ν can be interpreted as a measure of volatility.

Assumption 3.5. We have

$$\lim_{T\uparrow\infty} \frac{1}{T}\sum_{t=0}^{T-1} \ln s_t = -\nu \quad \text{a.s.,} \tag{3.4}$$

where ν is given by Assumption 3.4.

Assumptions 3.4 and 3.5 mean that $\{\ln s_t\}$ has a constant mean and satisfies the law of large numbers. These assumptions hold if $\{\ln s_t\}$ is stationary and ergodic with $E|\ln s_t| < \infty$ (e.g., White, 2000, Theorem 3.34). For example, $\{\ln s_t\}$ may be an i.i.d. process, as typically assumed in the stochastic growth literature. More generally, it may be a stationary ARMA process.

The following result is a consequence of Theorem 2.1.

Theorem 3.1. *Let Assumptions 3.1–3.5 hold. Suppose*

$$\ln m < \nu. \tag{3.5}$$

Then any feasible path $\{k_t\}$ converges to zero exponentially fast a.s. More specifically, for almost all $\omega \in \Omega, \forall \lambda \in (0, \nu - \ln m)$, (2.5) holds.

Proof. Assumptions 3.1–3.3 imply (2.2) and (2.3). Condition (3.5) together with Assumptions 3.4 and 3.5 implies (2.4) with $\mu = \ln m - \nu$. Hence the conclusion holds by Theorem 2.1. □

If $\nu = 0$, i.e., if there is no uncertainty, then (3.5) reduces to

$$m < 1. \tag{3.6}$$

This means that the graph of f lies entirely below the 45° line, implying that all feasible paths converge to zero. Since $\nu \geq 0$ by (3.3), (3.6) implies (3.5) even in the stochastic case. However, (3.5) holds even if the graph of f lies entirely above the 45° line, provided that ν is sufficiently large. Thus almost sure convergence to zero occurs if shocks are sufficiently volatile. A simple example illustrates this point.

Suppose s_t is unconditionally log-normal.[10] Then by Assumption 3.2 and log-normality,

$$1 = Es_t = E\exp(\ln s_t) = \exp\left(E\ln s_t + \frac{Var(\ln s_t)}{2}\right). \tag{3.7}$$

[10]This is true, for example, if $\{\ln s_t\}$ is i.i.d. normal or a stationary AR process with normal innovations.

Since $\nu = -E \ln s_t$ by definition (recall Assumption 3.4), from (3.7),

$$\nu = \frac{Var(\ln s_t)}{2}. \tag{3.8}$$

Hence (3.5) holds if and only if $Var(\ln s_t) > 2 \ln m$. Thus if $Var(\ln s_t)$ is large enough, any feasible path converges to zero exponentially fast a.s. A similar example can easily be constructed in which the support of shocks is bounded and bounded away from zero, as in the original Brock-Mirman (1972) model.

Theorem 3.1 shows not only that (3.5) implies almost sure convergence to zero, but also that an approximate rate of convergence is given by $\nu - \ln m$. Hence convergence is faster the large the volatility of shocks is, and the smaller the maximum average product of capital is.

As mentioned above, Assumption 3.3 holds if f is concave and violates the Inada condition at zero. A special case of this is when f is linear. Hence in a stochastic "AK" model, under (3.5), any feasible path converges to zero exponentially fast a.s. regardless of the objective function.[11]

Though Assumption 3.1 rules out the case in which the resource constraint is given by $c_t + k_{t+1} = s_t f(k_t) + (1-\delta)k_t$ for some $\delta \in (0, 1)$, this case can be dealt with using Theorem 2.1. Other cases that can be dealt with using the general result include nonconvex stochastic growth models of the type studied by Majumdar et al. (1989) and Nishimura et al. (2004) as well as stochastic overlapping generations models.

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[11] To our knowledge, Phelps (1962) was the first to study a stochastic "AK" model. The analysis was extended by Levhari and Srinivasan (1969). These articles studied mainly the properties of the consumption policy function rather than the asymptotic properties of optimal paths.

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