Coordination and Evolutionary Network Formation with Asymmetric Link Costs: Part One

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Direct Link Case

Abstract

We analyze a dynamic implication of an evolutionary process in a population where both actions and network structures change over time.

At every period, a coordination game is played by players who are linked with each other. An asymmetric cost of a link is incorporated. Under this setting each player myopically adapts with its circumstance consisting of the network structure and the action profile.

In a stochastically stable state there are link cost patterns such that all players play a pareto dominant equilibrium strategy of a coordination game. This is the most different result from a standard stochastic evolutionary models that selects a risk dominant equilibrium.

Key words: Network formation, coordination game, stochastic evolution, asymmetric link cost, pareto dominant equilibrium, risk dominant equilibrium

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1 Introduction

Network structeres are observed in many socio-economic organizations. Market is one of organizations that has a netwok structure. Firms have various network structures which affect their production technology. A coalition is interpreted as a network because networks are established by individual incentives for linking. Financial network systems, network externalities, friendships, cartel of firms are examples of entities that have network structures.

In these organizations it is essential that there are two distinct choices. Of course each individual must decides whom he should link with. But for playing games with linked companions the individual must also choices an action of a game.

A transaction in a market occurs only if two players-a buyer and a seller- meet and play a game. Standarization of technologies or languages in a population depends on the network or coalitional structure. But network structures or coalitional structures also depend on the action profile in the population. We should deal with these two phenomenon simultaneously.

The reasonableness of different network structures is the basic motivation for our model. By assuming that the network structure itself is subject to evolutionary pressure, we depart from the conventional models of evolutionary equilibrium selection models. Endogenous interaction structures are itself an object of study. Individually each player adds or servers links in comformity with its own interest. As related works for engogenous interactin structures, there are Bala and Goyal (2000), Jackson and Wolinsky (1996). Both study deal with an individual incentive to form a link using a directed graph. They study a dynamic stability of the network formation. They show emergence of inefficient network structure.

Our model in this paper investigate a dynamic stability of an interaction scheme. But our model is more general than Bala and Goyal, Jackson and Wolinsky because we deal with an equilibrium selection problem in a coordination game. In a nongeneric two by two coordination game, two strict Nash equilibria exists. When bulk of a population plays an action of the strict Nash equilibria, it is best for an individual player to accomodate own behavior to the action. Therefore the action of strict Nash equilibrium is called a *conventionin* in the population who plays the coordination game. To select an equilibrium, Harsanyi and Selten (1988) presents concepts such that *pareto domination* and *risk domination.* In some cases, we select a distinct equilibrium(convention) due to each concept. Models of a *dynamic equilibrium selection* is called an evolutionary game theory or a learning theory. One of strict Nash equilibria is selected as a stable state of some adaptation process. By introducing *perpetual* experimental actions or mutations into the adaptation process, Kandori, Mailath and Rob (1993) and Young (1993) shows that such an adaptation process selects a risk dominant equilibrium of a two by two coordination game. They express the adaptation process through irreducible Markov chains on a finite state space.[1] An asymptotic frequency distribution of the visiting states is a selection criterion:over the long run, the process selects those states on which the asymptotic frequency ditribution puts positive probability. Those states are called *stochastically stable states* (Young (1993)).[2] Both studies assume that players are *radomly matched* at every period.[3] Random matching moddels do not entertain incentives for choices of adversaries. In our model allowing

[1]Due to perpetual mutations, the process becomes irreducible.

[2]Kandori et,al. call those states *long run equilibria.*

[3]Precisely Young (1993) investigate a *recurrent game.* A population is partitioned into n classes. At every period one player is drawn at random from each of the n populations to play a game. Under adequate assumptions, this process coincides with a random matching model.

individual choices for link formations, incentives for choices of adversaries is incorporated into an evolutionary equilibrium selection model well. It is assumed that each player can play a coordination game *only if* they are linked with a network. That is, a play of the coordination game occurs only if there is a link by which players connect with each other. We assume that it costs for players to establish an active link. But if one side player of a link incurs the link cost, then both side players can play the game. This assumption peculiarises an individual incentive for a connection. We are interested in the influence of the endogenous link formation and the costs for establishing links on individual behavior in games of coordination. Blume (1993) and Ellison (1993) study evolutionary equilibrium selection models with an *exogenous* network structure. Addresses of players are fixed on a circle or a lattice. Because adversaries are less than random matching models, their models are called local interaction models. Ellison argues the time for the evolutionary system to converge to an stable state and shows that in a local interaction structure, more rapidly the system converges. They shows also that the stable state is the risk dominant equilibrium. But our model shows that all individuals play the pareto dominant action in a stochastically stable state by introducing an evolutionary process of network formations.

Overview of the Model

A two by two symmetric coordination game is repeatedly played by anonymous individuals. Because of the anonymity, a strategic interaction like a repeated game is not allowed for. That is, we suppose that individuals forms a large population or society. The game is played by two players only if at least one of them must incurs the link cost. Therefore players must decide about *both* actions and links, given an environment. The environment is a pair of a network formation and an action profile of other players. This decision is called an *adaptation*. Best response dynamics (Matsui(1992)) is exerted for the adaptation process[4]. The state space of the dynamics is the set of all pairs of network formations and action profiles of the coordination game. Given a pair of a network formation and an action profile of the last period(the state of the last period), a player *myopically* adjusts its behavior subject to best response to the pair(state). The myopic adjustment is justified by small degree of friction for revisions of behaviors [5] or *bounded rationality* of players.

As for the adaptation, a period is divided into four steps. At the first step a pair of players is randomly choosen and one side player decides to *establishes*(maintains) or severs an active link with the other player *given a state of the last* $t-1$ *period*. After this decision, an experimentation of a link formation occurs with a small probability γ. If a link is established at the first step, then *the link* is severed with the probability γ. At the end of this second step, the network formation of the t period is determined. At the third step, a player is randomly choosen and adjusts the action of the two by two coordination game, *given the network formed at the second step*. After this decision, an experimentation of an action choice occurs with a small probability ε. If an action is chosen at the third step, the other action is tried to choose with the probability ε. At the end of this fourth step, an action profile of the period t is determined. After these four steps, the coordination games are played by players who are linked with the other players in the network established at the period. This sequential adaptation rule is similar to Jackson,M and A.Watts (2001).

[4]Precisely our dinamics is called better reply dynamics.

[5]A perfect foresight dynamics coincides with best response dynamics as the friction rate becomes small. See Matusi and Matsuyama (1995).

A homogenous Markov process on the state space is defined due to the above adaptation process. Since this process is irreducible and aperiodic, there is a unique stationary distribution and has an *ergodic property.* As the probability of both experimentations vanishes, states with the strict positive probability of the stationary ditributation, that is, the *stochastically stable states*, are investigated.

The most different assumption from other models of this reserach field is that we itroduce an *asymmetricity* of link costs. The population is partitioned into two types of link costs-high type or low type. Aggresive firms for standarization of a technology are applicable to the low type. A person who follows the philosophy of isolationism may be the high type. This asymmetricity generates the pareto dominant equilibrium action in the stochastically stable state.

Results of Part One (Direct Link Case)

All cases of costs parameters are investigated. For each link cost level, all recurrent(or absorbing) states of our adaptation process without experiments are shown. We call the adaptation process without experiments the *unperturbed process.* From these states, we select the stochastically stable network formations and conventions according to an algorithm shown at Kandori et,al (1993) and Young (1993,1998).

At almost every cases, two strict Nash equilibria(conventions) emerge in recurrent states of the unperturbed process. If the low link cost parameter is sufficiently small, the only low type players support the whole network. At almost every cases, fully connected networks are stoichastically stable.

For some cost parameter cases, the pareto dominant convention emerges. If each cost level is enough different, then high cost players free ride on low types for network formation. Then a density of links is coarser than full connected networks. Risk dominant convention maintains at this coarser network because a gain from risk dominant action is smaller than pareto dominant one. This state is less robust to perpetual mutations if the number of low cost type players is small. Since only a few players support the network, the convention of the society depends more strongly on the few players' behavior than random matching models.

Precedences of this field of research

In this paper links are costly for establishing and maintaining. As for evolutionary network formations and coordination games, there are several precedences. Droste,E. R. Gilles, and C.Jhonson (2000), Goyal,S and F.Vega-Redondo (2000), Skyrms,B. and R.Pemantle (2000), Jackson,M. and A.Watts (2001). All these works introduce a link cost which leads us to study the incentives of players to establish a network. Droste *et al.* fix players on a circle assigning its address. Cost considerations of social interaction are incorporated by considering endogenous neighborhoods on the circle. That is, players can create their own costly communication network in some neighborhood. As a stochastically stable point, only the risk dominance strategy is sustainable on the circle. Goyal and Vega-Redondo shows if a link cost is sufficiently high then pareto dominant equilibrium becomes stochastically stable. They study the case also that two distinct players linked with each other through a path of links can play a coordination game. In this case a center-sponsered network emerges in the long run. Their work generates some near conclusions. Jacson and Watts assumes the adaopation process to be sequential-in the first step a link is adapted and in the second step actions-and introduce a convex link cost function on the number of linked players. The latter assumption stops the growth of the network size at some levels. They show also that there are cases that generates the pareto dominant convention.

This paper is construct as follows. In section 2 networks and a social game(coordination game)used in our model are defined. An adaptation rule and corresponding Markov chain are explaied in section 3. In section 4 recurrent states of our unperturbed process and stochastically stable states are investigated. Section 5 concludes this Part One (Direct Link Case).

2 Networks and social game

2.1 Networks

We consider a population which consists of finite players. The set of the players is $N = \{1, 2, \dots, n\}$. It is assumed that $n \geq 3$ since we are interested in emergence of convention and evolutionary stable network formation patterns in a *large* population. Two players are connected with each other by so called *links*. Potential links which a player i can add or servers are denoted by $g_i = (g_{i1}, g_{i2}, \dots, g_{ii-1}, g_{ii+1}, \dots, g_n)$ where $g_{ij} \in \{0, 1\}$. If the player i incurs the cost of maintaining/establishing the link between i and j, g_{ij} becomes 1. If the player i does not incurs the cost of maintaining/establishing the link between i and j, then g_{ij} becomes 0. Note that even if g_{ij} is 0, g_{ji} may be 1. This case ($g_{ij} = 0$ and $g_{ji} = 1$) means that there is only one link which connects player i and j, and the cost of this connection is incured only by player j. A link g_{ij} is called an *active link* for i and a *passive link* for j.

Definition 1. *Player i **establishes** an active link with player j if $g_{ij} = 1$.*
*Player i **severs** an active link with player j if $g_{ij} = 0$.*

We call a $(n-1) \times n$ dimensional vector g consisting of 0 and 1 such that $g = (g_1, g_2, \dots, g_n)$ a *network*. The set of all networks is denoted by G. That is, $G = \{\{0, 1\}^{n-1}\}^n$.

A player can play a coordination game descrived at next Section 3 and get his(her) payoff only if he(she) must be connected/linked with other players. There are two ways of connection which are called as *direct* links and *indirect* links.

Definition 2. *Distinct players $i, j \in N$ are **directly linked** if* $\max\{g_{ij}, g_{ji}\} = 1$.

Let $\bar{g}_{ij} \equiv max\{g_{ij}, g_{ji}\}$. If player i and j are directly linked, then $\bar{g}_{ij} = 1$. A player plays a game and gets a positive payoff only if there must be an other player j such that $\bar{g}_{ij} = 1$.

To define indirect connection, we use a concept of a *path* between player i and j.

Definition 3. *There is a **path** in g between player i and player j if $\bar{g}_{ij} = 1$ or there exist some players $j_1, j_2, \dots, j_m \in N$ such that $\bar{g}_{ij_1} = \bar{g}_{j_1 j_2} = \dots = \bar{g}_{j_{m-1} j_m} = \bar{g}_{j_m j} = 1$.*

When there is a *path* between i and j we represent it as $i \overset{\bar{g}}{\longleftrightarrow} j$. Without confusion, player i and j is *indirectly* linked if there exist some paths between them. In section 5, we investigate the case where player i can play a coordination game with player j if they are linked indirectly.

Here we prepare some notations to define payoffs concluding a cost for establishing a link (at Subsection 2.3 below). $N^d(i; g)$ denotes the set of all players in network g with whom player i has established links. Precisely $N^d(i; g) \equiv \{j \in N \mid g_{ij} = 1\}$. The cardinality of the set $N^d(i; g)$ is $\nu^d(i; g) \equiv |N^d(i; g)|$. The set of all players directly linked with player i is $N^d(i; \bar{g})$. Precisely $N^d(i; \bar{g}) \equiv \{j \in N \mid \bar{g}_{ij} = 1\}$. The cardinality of the set $N^d(i; \bar{g})$ is $\nu^d(i; \bar{g}) \equiv |N^d(i; \bar{g})|$. The set of all players indirectly linked with a player i is denoted by $N(i;\bar{g}) \equiv \{j \in N \mid i \overset{\bar{g}}{\longleftrightarrow} j\}$. We also define $\nu(i;\bar{g}) \equiv |N(i;\bar{g})|$.

To confirm above definitions and notations, we use an example as Figure 1 below. Network formations are characterized by graph theory. A network g is a directed graph. Its node is a player and its directed edge is a link between two distinct players. A player at a root of an edge incurs the cost for the link.

In this example, $N = \{1,2,3,4\}$. $g = ((g_{12}, g_{13}, g_{14}), (g_{21}, g_{23}, g_{24}), (g_{31}, g_{32}, g_{34}), (g_{41}, g_{42}, g_{43})) = ((1,0,0),(0,0,1),(0,1,0),(1,0,1))$, $N^d(1;g) = \{2\}$, $N^d(1;\bar{g}) = \{2,4\}$, $N(1;\bar{g}) = \{1,2,3,4\}$.

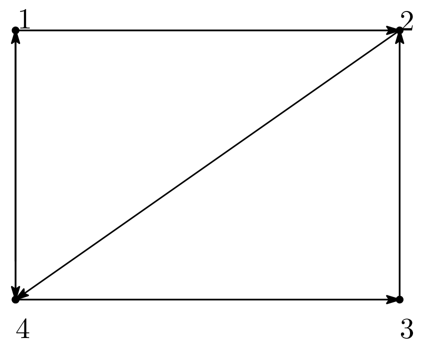


Figure 1.

Thus we can regard a network g as a *graph*. We identify the set of networks G as the set of graphs.

Definition 4. *Subgraph*[6] *$g' \subset g$ is a **component** of g if for any nodes(players) i, j of a graph g' there exists some paths between the node(player) i and the node(player) j such that if i is a node of g', j is a node of g and $g_{ij} = 1$ then $g'_{ij} = 1$.*

Definition 5. *A network with only one component is called **connected**. A connected network g is said to be **minimally connected** if the network obtained by deleting any single link is not connected.*

Our model predicts emergence of following fully connected networks in many cases.

Definition 6. *A network g is **fully connected** if $\bar{g}_{ij} = 1$ for any i,$j \in N$.*

One sided link is more efficient than two sided one. It will be shown that our evolutionary model often selects following efficient networks.

Definition 7. *A network g is **efficiently formed** if for any i, $j \in N$ $(g_{ij} = 1 \rightarrow g_{ji} = 0)$.*

Let G_e denote the set of *efficiently formed and fully connected* networks.

Using examples drawn at Figure 2, we confirm these characterization. Figure 2(a) is an example of minimally and efficiently connected graphs. It is called a *star network* graphs. Figure 2(b) is not a efficiently but fully connected network. Figure 2(c) is an efficiently connected network.

[6] $g' \subset g$ if all nodes of g' belongs to g.

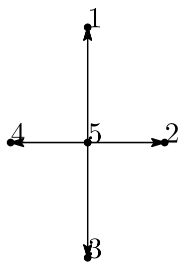


Figure 2(a).

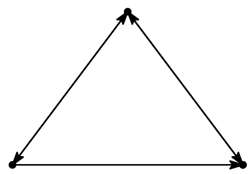


Figure 2(b).

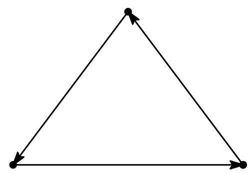


Figure 2(c).

2.2 Link costs

Suppose that players must incure cost for maintaining and adding their active links. To play a game or to transact with other players, each player may incur the cost of links. By introducing the cost of links, it becomes possible to analyze individual incentives for playing or participanting to a game.

Off course, if for a link between player i and j only player i incurs its cost (that is, g_{ij} is 1 and g_{ji} is 0) then without incuring the cost player j *can* play a game with player i. Thus if any pair of players play a game then at least one player of the pair must incur the cost for connection ($\bar{g}_{ij} = 1$). Hereafter we call the unit cost for establishing/maintaing any active link a *link cost*. The link cost does not depend on the opponent player linked with. Let n_i be the number of active links that player i establishes/maintains. Player i pays an unit cost $k(n_i)$ for maintaining/establishing his active link, leading to a total cost of $n_i k(n_i)$ for player i.

Generally the unit cost function, $k(n_i)$, is depending on the number of active links for the concerning player. But at the begining a constant cost is assumed for convenience of analysis. That is, for any number of active links the cost of a link is k.[7]

The most important difference from preceeding works, Goyal and Vega-Redondo(1999) and Jackson and Watts(2001) is that we introduce an *asymmetricity* of the link cost. The link cost may depends with whether a player has invested much resource to a link technology yet. The amount of the investment is regarded as the main factor which decides whether a player is high type or not. For example, maintenance technology of a traffic,.... To represent this two types of link costs players are assumed to be partitioned into two groups. *High link cost type* and *Low link cost type*. Type h player has a high cost techbology for his active link and type l has low one. Any player in the subset N_x of N has unit constant link cost k_x for maintaing/establishing an active link, where x is h or l. Let $k_h > k_l$. [8]

Assumption 1. *All players are partitioned into two types of link costs. That is,* $N = N_h \cup N_l$, $N_h \cap N_l = \phi$, $N_h \neq \phi$ *and* $N_l \neq \phi$.

[7]In section, we investigate the case that a cost of a link has convexity property depending on the number of active links.

[8]The number of active links becomes more, the unit cost for maintaining its active links may grow higher. Friendship relation is a good example for this situation. This is modeled as convexity of a cost function of the number of activelinks. This case is compared with our model at Section 5 of Part Two(coupled paper) .

2.3 Social games

Two players linked with each other play a symmetric coordination game called as a Social Game. Each player has two *alternatives* or *actions*, named as α, β. Let A be the set of actions. This symmetric coordination game is drawn at Table 1 below.

1 \ 2	α	β
α	d	e
β	f	b

Table 1

Note that the payoffs in the Table 1 is not necessary real payoffs of the players because of the existence of the link cost. If a player has an active link then the player gets a net payoff which is less than the payoff as descrived at the Table.

One of the most interesting things is the equilibrium selection problem with an endogenous network formation.

Assumption 2. *In the symmetric coordination game, payoffs satisfy*

$$d > f, \quad b > e, \quad d > b \quad \textit{and} \quad d + e < b + f.$$

Lemma 2.1. *If the symmetric coordination game satisfies Assumption 2, two action pairs, (α, α) and (β, β) are strict Nash equilibria*[9] *in the coordination game.*

Proof. Since $d > f$, α is a unique best response to itself. β is best response to itsself. Thus they are strict Nash equilibria. ∎

If there is no confusion we call these strict Nash equilibria the *conventions* in the Social Game according to Young(1993).

Since $d + e < b + f$, the former pair (α, α) is a *pareto dominant* equilibrium and the latter pair (β, β) is a *risk dominant* equilibrium. Intuitively the former pair is efficient but more risky in coordination. As for formal definitions, see Harsanyi and Selten(1988).

Lemma 2.2. *If the symmetric coordination game satisfies Assumption 2 then $e < f < b < d$ or $e < b < f < d$.*

Proof. Since $d > f$ and $d > b > e$, d is the maximal payoff. By $d + e < b + f$, $e - f < b - d$. Since d is the maximal payoff, $e - f < b - d < 0$. Therefore $e < f < d$ and $e < b$. ∎

[9] A Nash equilibrium strategy profile x is *strict* if the best reply to the x is only x itself.

We call $e < f < b < d$ Case 1 and $e < b < f < d$ Case 2 in Section 4.

Definition 8. *1.* ***Direct Link Case*** *: Distinct players, i and j play the coordination game when $\bar{g}_{ij} = 1$.*

2. ***Indirect Link Case*** *: Distinct players, i and j play the coordination game when $i \overset{\bar{g}}{\longleftrightarrow} j$.*

This assumption is called as *one sided links* because it is no need for the realization of the play that both players establish the link between them. Due to this assumption, the high link cost players may *freely ride* on low cost players for playing the Social game.

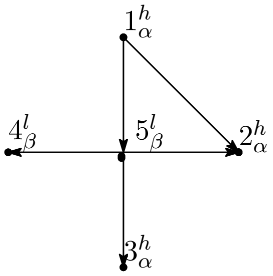
Net payoffs

Let players be located on a fixed network $g \in G$. Given the network g, each player plays an action in A. A *net* payoff is a value of a function on the set $G \times A$.

Based on Assumption 1,3 and Table 1, for each $(g, a) \in G \times A$,[10] the net payoff for a type x player i is defined as follows.[11]

1. Direct Links case. $\Pi_i(g, a) = \sum_{j \in N^d(i;\bar{g})} v_i(a_i, a_j) - k_x \cdot \nu^d(i; g)$
2. Indirect Links case. $\hat{\Pi}_i(g, a) = \sum_{j \in N(i;\bar{g})} v_i(a_i, a_j) - k_x \cdot \nu^d(i; g)$

In these equations, k_x is the cost function of player i's unit active link. $v_i(a_i, a_j)$ represents player i's *gross* payoff depending on only actions chosen. By figure $v_i(\alpha, \alpha) = d$, $v_i(\beta, \beta) = b$, $v_i(\alpha, \beta) = e$, and $v_i(\beta, \alpha) = f$. Note that in the indirect link case the gross payoffs occur at the game with players indirectly linked. At Figure 3 $N^d(1, \bar{g}) = \{2, 5\}$ and $N(1, \bar{g}) = \{2, 3, 4, 5\}$. $N_l = \{4, 5\}$ and $N_h = \{1, 2, 3\}$. $a = (\alpha, \alpha, \alpha, \beta, \beta)$. 1^α_h represents that player 1 is a high link cost player and plays α.



$$\Pi_1(g, a) = v(\alpha, \alpha) + v(\alpha, \beta) - k_h \cdot 1$$
$$= d + e - k_h \cdot 1$$

$$\hat{P}i_1(g, a) = 2v(\alpha, \alpha) + 2v(\alpha, \beta) - k_h \cdot 1$$
$$= 2d + 2e - k_h \cdot 1$$

Figure 3.

[10] (g, a) represents a pair of a network form and an action profile of all players. That is, $g = (g_1, g_2, \ldots, g_n) \in \{0, 1\}^{(n-1)^n}$ and $a = (a_1, a_2, \ldots, a_n) \in \{\alpha, \beta\}^n$.

[11] When the link cost is a function of the number of active link, the net payoff is defined as follows where x is in $\{h, l\}$.

1. Direct Links case. $\Pi_i(g, a) = \sum_{j \in N^d(i;\bar{g})} v_i(a_i, a_j) - \nu^d(i; g) \cdot k_x(\nu^d(i; g))$
2. Indirect Links case. $\hat{\Pi}_i(g, a) = \sum_{j \in N(i;\bar{g})} v_i(a_i, a_j) - \nu^d(i; g) \cdot k_x(\nu^d(i; g))$.

Note that the link cost k_x is a function over the number of directly linked players.

3 Endogenous networks and stochastic stability

3.1 Unperturbed Adaptation Rule

Players play the above coordination game repeatedly but the network formation may change over the time. Follwing adaptation behaviors of players cause a dynamic process. Each player adjusts the action and the links *myopically*, which is interpred as one of the best response dynamics (Matsui(1992)). Let g^{t-1} is the network and a^{t-1} is the action profile at the end of a period $t-1$. Similary g^t_{ij} represents a link ij at period t. In the next period t, an adaptation occurs through two steps.

In conditions below, $g + ij$ represents a graph of which g_{ij} is replaced with 1. Of course if $g^{t-1}_{ij} = 1$, then no replacement occurs. $g - ij$ represents a graph of which g_{ij} replaced with 0. If $g^{t-1}_{ij} = 0$, then no replacement occurs. For example, if $g = ((g_{12}, g_{13}), (g_{21}, g_{23}), (g_{31}, g_{32})) = ((1,1),(1,0),(0,1))$ then $g + 12 = 1$, $g - 12 = 0$, $g + 31 = 1$ and $g - 31 = 0$.

1. One pair i,j of players is randomly drawn with probability $\{p_{ij}\}$ where $\sum_{(i,j)\in N\times N, i\neq j} p_{ij} = 1$ and for any i, $j \in N$, $p_{ij} > 0$. This probabilities are determined exogenously.

   If $\Pi_i(g^{t-1} + ij, a^{t-1}) > \Pi_i(g^{t-1}, a^{t-1})$ then $g^t_{ij} = 1$.

   If $\Pi_i(g^{t-1} - ij, a^{t-1}) > \Pi_i(g^{t-1}, a^{t-1})$ then $g^t_{ij} = 0$.

   If each equation holds with equality then g^{t-1} remains.

   At this point g^t is determined.

2. One player k is randomly drawn with strict positive probability $\{q_k\}$ where $\sum_{k\in N} q_k = 1$ and for any $k \in N$, $q_k > 0$. If the picked up player k has been linked with at least one another player in the network g^t formed at step 1, an adaptation for actions occurs. In the condition below, a^{t-1}_k represents an action profile except the player k at period $t-1$. Therefor $(\cdot, a^{t-1}_k) \in A$.:

   If $\Pi_k(g^t, (\alpha, a^{t-1}_{-k})) > \Pi_k(g^t, (\beta, a^{t-1}_{-k}))$ then k chooses an action α.

   If $\Pi_k(g^t, (\alpha, a^{t-1}_{-k})) < \Pi_k(g^t, (\beta, a^{t-1}_{-k}))$ then k chooses an action β.

   If each equation holds with equality then a^{t-1} remains.

   If the picked up player k is not linked with no player of the netwok g^t, then he choose an action that maximizes his payoff under a condition that all the other players would establish links with the player k.

   At this point a^t is determined.

At the end of the period t, *All* players play the coordination game with directly(indirectly) linked players *if such players exist*. Then they get net payoffs of the period t defined at the last section.

At the step 1, given a network and an action profile of the last period $t-1$, a randomly selected player i decides whether or not he(she) maintains/establishes a randomly selected his(her) connection with a player j. At the step 2, given a network formed at the step 1 and an action profile of the last period $t-1$, a randomly selected player chooses the best action of the coordination game.At the step 3, payoffs realizes. These steps in a period is summerized at Figure 4 below.

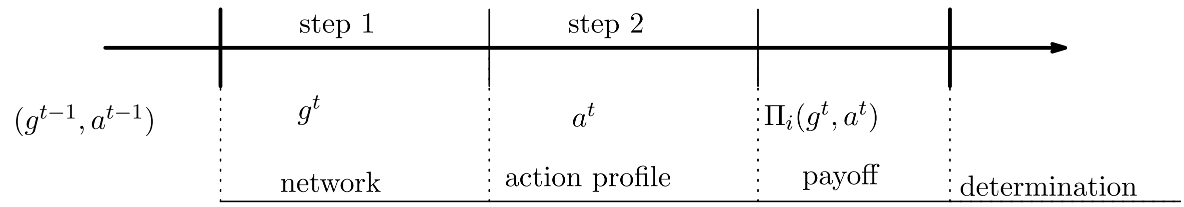


Figure 3. unperturbed adaptation rule

3.2 Markov chain

By this adaptation rule for each period t, we can define a *discrete time Markov process* P on a *finite* state space $S = G \times A$.[12] This process specifies the probability of transiting to each state in S in the next period, given that the process is currently in some given state s. For every pair of states s, $s' \in S$, and every time $t \geq 0$, let $P_{ss'}$ be the transition probability of moving to state s' at time $t+1$ conditional on being in state s at period t. Since $P_{ss'}(t)$ is independent of t due to our definition of the adaptation process, the transition probabilities are called as *time homogenous*. We identify the process P with the corresponding matrix $(P_{ss'})_{s \in S, s' \in S}$.

Lemma 3.1. *Our adaptation rule defines a time homogenous Markov process.*

Proof. By definition of our adaptation rule, for any states s,$s' \in S$ we can define a unique transition probability. Thus a Markov process is defied.

Assume that there are periods t,t' such that $P_{ss'}(t) \neq P_{ss'}(t')$. Since our adaptation rule defines a unique transition probability from any state, this is a contradiction. ■

See an example ($n = 3$) at Appendix 2.

A state s' is *accessible* from s, if there is a positive probability of moving from s to s' in a finite number of periods. States s and s' are *communicate*, written $s \sim s'$, if each is accessible from the other.

Lemma 3.2. *The relation $s \sim s'$ is an equivalence relation.*

Proof. By the definition directly if $s \sim s'$ then $s' \sim s$. If $s \sim s'$ and $s' \sim t$ then t is accessible from s by finite steps.

■

From this lemma, the state space S is partitioned into equivalent classes, called as *communication classes.*

[12] If a population size is n then the cardinality of S is $(n-1)^n \times 2^n$.

Definition 9. *A* ***recurrent class*** *of a Markov process is a communication class such that no state outside the class is accessible from any state inside it. A state is* ***recurrent*** *if it is contained in one of the recurrent classes. A state is* ***absorbing*** *if it is in a recurrent class and the class is singleton.*

Definition 10. *If the process has exactly one recurrent class, which consists of the whole state space, the process is said to be* ***irreducible****.*

Let $s_1, s_2, \ldots, s_M$ be an enumeration of the states and $\mu = (\mu(s_1), \mu(s_2), \ldots, \mu(s_M))$ where $M = \mid S \mid$.

Definition 11. *A* ***stationary distribution*** μ *of our process is the solution of following system of liner equations.*
$\mu P = \mu$, *where* $\mu \geq 0$ *and* $\sum_{s \in S} \mu(s) = 1$.

The first equation of this definition is called a *stationary equation* which represent that the stationary distribution μ is a fixed point of the Markov process. About the existence and its uniqueness of the stationary distribution the next lemma is well known.

Lemma 3.3. *The stationary equation in above definition has a unique solution if and only if the process is irreducible.*

Lemma 3.4. *The Markov process defined by our unperturbed adaptation rule needs not has a unique stationary distribution.*

Proof. Consider two states such that (g_1, a_1), $g_1 \in G_e$ and $a_1 = \alpha^n$, (g_2, a_2), $g_2 \in g_e$ and $a_2 = \beta^n$. If a high type link cost is lower than b, then there emerges no players $\bar{g_i}j = 0$ in both state. Since both action profiles are strict Nash, no players have incentive to change actions. Therefore each of (g_1, a_1) and (g_2, a_2) is a sigleton recurrent class. Because the process needs not be irreducible, it may not have a unique stationary distribution.

■

3.3 Perturbed adaptation rule

Consider our Markov process P defined on a finite state space S. A *perturbation* of P is a Markov Process whose transition probabilities are slightly perturbed or distorted versions of the transition probabilities $P_{s,s'}$. This perturbation for our Markov Process is derived from modification of the above adaptation rules.

Let g^{t-1} is the network and a^{t-1} the action profile at the end of a period $t-1$. In the next period t, through following four steps the adaptation occurs. In conditions below, $g + ij$ represents a graph of which g_{ij} replaced with 1. Of course if $g_{ij}^{t-1} = 1$, then no replacement occurs. $g - ij$ represents a graph of which g_{ij} replaced with 0. If $g_{ij}^{t-1} = 0$, then no replacement occurs.

1. One pair i,j of players is randomly drawn with probability $\{p_{ij}\}$ where $\sum_{(i,j) \in N \times N, i \neq j} p_{ij} = 1$ and for any i,$j \in N$ $p_{ij} > 0$. This probability is determined exogenously.

   If $\Pi_i(g^{t-1} + ij, a^{t-1}) > \Pi_i(g^{t-1}, a^{t-1})$ then the player i *establishes* the link ij.

   If $\Pi_i(g^{t-1} - ij, a^{t-1}) > \Pi_i(g^{t-1}, a^{t-1})$ then player i *severes* the link ij.

   If each equation holds with equality then g^{t-1} remains.

2. (*Experiments, mutations for links*)After this link formation by players, Nature adds the link which was served at step 1 of this period t by *player* i with small probability $\gamma > 0$ and let them keep unestablished with probability $1 - \gamma$. Nature also serves the link which was established at step 1 of this period t by *players* i with small probability $\gamma > 0$ and remains with probability $1 - \gamma$.

   At this point, a network of the period t, g^t, is determined.

3. One player k is randomly drawn with strict positive probability $\{q_k\}$ where $\sum_{k \in N} q_k = 1$ and $q_k > 0$ for any $k \in N$. If the picked up player k is linked with at least one another player in the network g^t, an adaptation for an action choice occurs.

   If $\Pi_k(g^t, (\alpha, a_{-k}^{t-1})) > \Pi_k(g^t, (\beta, a_{-k}^{t-1}))$ then player k chooses an action α.
   If $\Pi_k(g^t, (\alpha, a_{-k}^{t-1})) < \Pi_k(g^t, (\beta, a_{-k}^{t-1}))$ then player k chooses an action β.
   If each equation holds with equality then a^{t-1} remains.

   If the picked up player k is not linked with no player of the netwok g^t, then he choose an action that maximizes his payoff under a condition that all the other players would establish links with the player k.

4. (*Experiments, mutations for actions*)After this choice of the action by player k, with small probability $\varepsilon > 0$ Nature makes the player k choose an action which was not choosen by the player k at the step 3 of this period. With probability $1 - \varepsilon > 0$ the choice made by the player k is unchanged by Nature.

   At this point, an action profile of the period t, a^t, is determined.

At the end of period all players play the coordination game with directly connected players if such players exist. Then they get net payoffs of the period defined at the last section.

Note that at the step 2 and 4 small probabilities of mutations are introduced. The former γ may be interpred as the probability of an experimentation for link formation. The latter ε is interpreted as the probability for action choice.

3.4 Perturbed Markov process

Lemma 3.5. *Our perturbed adaptation rule at each period define a homogeneous Markov process* $P(\varepsilon, \mu)$ *depending on* ε, μ.

See an example at Appendix 2.

Lemma 3.6. *For our perturbed Markov Process there exists an unique stationary distribution* $\mu(\varepsilon, \gamma)$ *such that* $\mu(\varepsilon, \gamma)P(\varepsilon, \gamma) = \mu(\varepsilon, \gamma)$.

Proof. By the definition of our perturbed adaptation rule, the corresponding Markov process is irreducible. From the lemma 3, this process has a unique stationary distribution. ■

To simplify our analysis the perturbed adaptation and corresponding Markov process is restricted a regular process defined below. For our Markov process to be the regular perturbed Markov process about *only* ϵ, we suppose that a convergence of ε and γ becomes a same rate.

Assumption 3. *There is a constant $r > 0$ such that $\varepsilon = r\gamma$.*

$\gamma(\varepsilon)$ represents that γ depends on ϵ due to above Assuption.

Definition 12 (Young(1993,1998)). *A Markov Process $P(\varepsilon, \gamma(\varepsilon))$ is a* ***regular perturbed Markov process*** *if $P(\epsilon, \gamma(\varepsilon))$ is irreducible for every $\epsilon \in [0, \bar{\varepsilon}]$ and for every s,$s' \in S$, following two conditions hols.*

$\lim_{\varepsilon \to 0} P_{s,s'}(\varepsilon, \gamma) = P_{s,s'}$,

and

if $P_{s,s'}(\varepsilon, \gamma(\varepsilon)) > 0$ for some $\varepsilon > 0$, then $0 < \lim_{\varepsilon \to 0} \dfrac{P_{s,s'}(\varepsilon, \gamma(\varepsilon))}{\varepsilon^{r(s,s')}} < \infty$ for some $r(s, s') \geq 0$.

The real number $r(s, s')$ is called as the *resistence* of the transition from s to s'. Note also that $P_{s,s'} > 0$ if and only if $r(s, s') = 0$. For convenience, we shall adopt the convention that $r(s, s') = \infty$ if $P_{s,s'}(\varepsilon, \mu) = P_{s,s'} = 0$ for all $\varepsilon \in [0, \bar{\varepsilon}]$.

Lemma 3.7. *Our perturbed Markov process $P(\varepsilon, \gamma)$ is a regular perturbed Markov process.*

Proof. From Asuumption 3, it holds true clearly.

∎

Definition 13 (Young(1993)). *A state $s \in S$ is a* ***stochastically stable state*** *if $\lim_{\varepsilon \to 0} \mu(\varepsilon, \gamma(\varepsilon))(s) > 0$.*

Due to Assumption 3, it is possible to define stochastically stable states by a convergence of only ε.

3.5 Computation of stochastically stable states

Based on the appendix in Young(1993) it is summerized how to compute stochastically stable states for any regular perturbed Markov process $P(\epsilon, \gamma(\epsilon))$ on the finite state space S. Let P have recurrent classes $E_1, E_2, \ldots, E_K$. For each pair of distinct recurrent classes E_i and E_j, an *ij-path* is a sequence of states $\zeta = (s_1, s_2, \ldots, s_q)$ that begins in E_i and ends in E_j. The *resistence* of this path is the sum of the resistence of its edges, that is, $r(\zeta) = r(s_1, s_2) + r(s_2, s_3) + \ldots + r(s_{q-1}, s_q)$. Let $r_{ij} = \min r(\zeta)$ be the least resistence over all ij-paths ζ. Now construct a complete directed graph with K vertices, one for each recurrent class. The vertex corresponding to a recurrent class E_j will be called j. The weight on the directed edge $i \to j$ is r_{ij}. A tree rooted at vertex j (called a *j-tree*) is a set of $K - 1$ directed edges such that, from every vertex different from j, there is a unique directed path in the tree to j.

The *resistence* of a rooted tree T is the sum of the resistences r_{ij} on the $K - 1$ edges that compose it. The stochastic potential p_j of the recurrent class E_j is defined as the minimum resistence over all trees rooted at j.

Theorem 3.1. *(Young[1993])*

Let $P(\varepsilon, \gamma(\varepsilon))$ be a regular perturbed Markov Process of P, and let $\mu(\varepsilon, \gamma(\varepsilon))$ be the unique stationary distribution of $P(\varepsilon, \gamma(\varepsilon))$ for each $\varepsilon > 0$, $\gamma = r\varepsilon$. Then $\lim_{\varepsilon \to 0} \mu(\varepsilon, \gamma(\varepsilon)) = \mu$ exists and μ is a stationary distrubution of P. The stochastically stable states are precisely those states are precisely those states that are contained in the recurrent class(es) of P having minimum stochastic potential.

Proof. See Young(1993,1998).

■

This theorem is directely used to investigate the stchastically stable convention and network formation of our model. We will find recurrent classes which are minimum over all recurrent classes of our Markov process P.

4 Stochastically stable convention and network formation in the direct link case

We call a state (a^*, g^*) which is stochastically stable as *stochastically stable convention* and *network*. Stochastcally stable conventions and network formation depends cost levels of k_h and k_l. Some cases of different cost parameters are investigated. Let $N_\alpha(a)$ be the set of players choosing α at the profile a. $|N|$,$|N_l|$,$|N_h|$ represent cardinalities of each set. For any action profile a^x, a_i^x is a player i's action under a^x. As an example of our analysis, we investigate Case 0 as following. After Case 0, we will investigate Case 1 and Case 2 which were assorted on Lemma 2.2.

4.1 Case 0: $d > k_h >$max$\{f, b\}$ and $k_l <$ min $\{f, b\}$

Only states that all players take an identical action are absorbing for the unperturbed dynamics P in this case. A uniform convention emerges. It is possible for each action α and β to become the convention based on the process P.

Lemma 4.1. *If the link cost parameters satisfy* $d > k_h > \max\{f, b\}$ *and* $k_l < \min\{f, b\}$ *then the set of absorbing states for the unperturbed process* P *is classified into two groups. The first class is the set of states such that the network is efficiently formed and fully connected, and all players play the action* α*. The second class is the set of states such that the network is established by only the low cost players and all players play the action* β*. Formally this statement is as follows.*

The first class E_1^α *of the absorbing states is a set of pairs* $\{(g, a)\}$ *such that* $g \in G_e$ *and for any* $i \in N$ $a_i = \alpha$*.*

The second class E_0^β *of the absorbing states is a set of pairs* $\{(g, a)\}$ *such that for any pair* $(i, j) \in N_l \times N_h$ $g_{ij} = 1$ *and* $g_{ji} = 0$*, for any* $i,j \in N_h$ $g_{ij} = 0$*, and for any* $i \in N$ $a_i = \beta$*.*

Proof. At any period $t-1$, if $p_{ij} > 0$ and $i \in N_l$ then $g_{ij}^t = 1$ since $k_l < \min\{f, b\}$. Since $\prod_{(i,j)\in N_l\times N} p_{ij} > 0$ and $|N|$ is finite, a state s where for any $i \in N_l$ and for any $j \in N$, $g_{ij} = 1$ is accessible from any state by finite steps. If at the state s the number k_α of α players in N_l satisfies

$$k_\alpha < \frac{(b-e)N_l}{(d-e)+(f-b)}$$

then any player will choose β at all following states. It is possible that with a probability that

$$(\prod_{k\in N} q_k)(\prod_{(k,i)\in N\times N} p_{ij}) > 0$$

a state s' such that all players choose β, for any $i \in N_l$ and $j \in N_h$ ($g_{ij} = 1$ and $g_{ji} = 0$), and for any $i, j \in N_l$ ($\bar{g_{ij}} = 1$ and $g_{ij} = 1 \rightarrow g_{ji} = 0$) is accessible from the state s. Since $k_h > \max f, b$ no high cost player establish new active links from the state s'. Therefore no state outside s' is accessible from s'.

If at state s,

$$k_\alpha \geq \frac{(b-e)N_l}{(d-e)+(f-b)}$$

then any player will choose α at all following states. It is possible that with a probability that

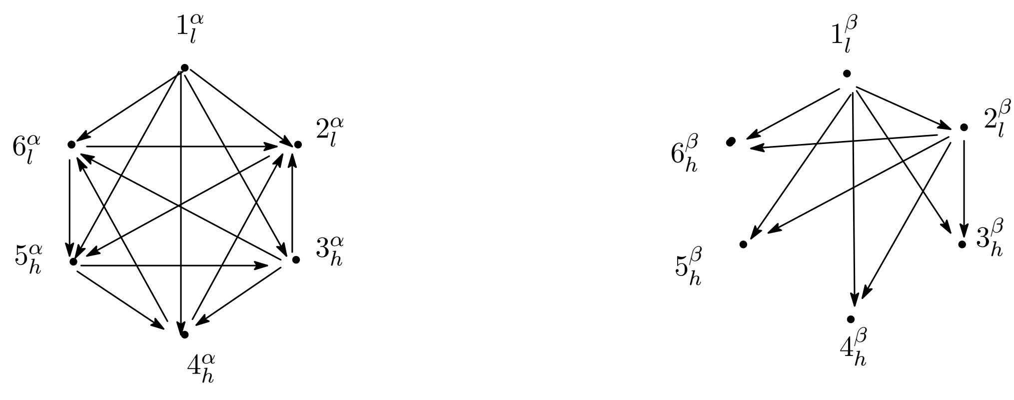
$$(\prod_{k \in N} q_k)(\prod_{(k,i) \in N \times N} p_{ij})(\prod_{i \in N_l} p_{ij}) > 0$$

a state s'' such that all plays α and $g \in G_e$ is accessible from s. Since no player will establish new active link and choose β from s'', no outside state is accessble from s''.

Because in an efficiently formed network only one side of any link must establishes it and another side must severs it, it is possible $|E_0^\beta| \geq 2$.

■

To confirm this lemma see the figure below. The the set of players N is assumed to be $\{1,2,3,4,5,6\}$, $N_l = \{1,2\}$ and $N_h = \{3,4,5,6\}$. 1_h^α represents that the player 1 is high cost player and plays α action.



E_1^α E_0^β

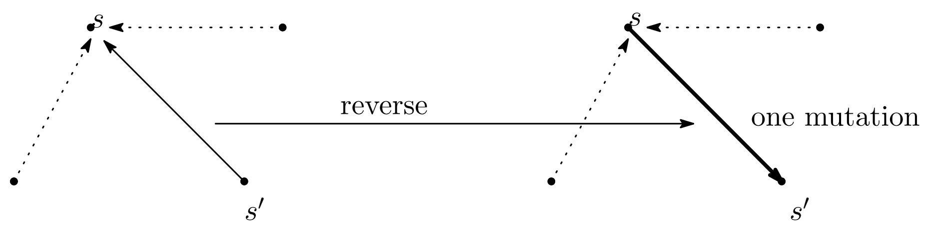
Note that each state in E_i^x can be reached each other by a *chain* of states in E_i^x. This chain is constructed as following. Assume that in a state s *one* mutation occurs and the perturbed process $P(\varepsilon, \gamma)$ transits into a state s'. From s', the process goes into a state s'' due to the unperturbed process P. Then we write $s \rightarrow s'$. The *chain* between states z and z' is a set C which consists of states $z_1, z_2, \ldots, z_{m-1}, z_m$ in recurrent states such that $z_1 = z$, $z_m = z'$ and $z_1 \rightarrow z_2 \rightarrow \ldots \rightarrow z_{m-1} \rightarrow z_m$. Denote a set of states which have some chains with a state s be $C(s)$.

Lemma 4.2. *If a state s is stochastically stable and $s' \in C(s)$ then s' is also stochastically stable.*

This idea is shown in Noldeke and Samuelson(1993).

Proof. Consider s-tree. By assumption the s has minimam stochastic potential. In the tree, reverse directions all edges between s and s'. This set of reversed edges is a chain from s to s' and constitute a path of s' of a s' tree. By the definition of a chain, this s' tree must have the same weight as s-tree. Therefore s' become also stochastically stable. (See Figure 5.)

■



s' has also the minimum stochastic potential.

Figure 5.

Lemma 4.3. *For any state s and $s' \in E_1^\alpha$ (E_0^β), there is a chain between s and s'.*

Proof. Let $g_{ij} = 1$ in s. By one mutation such that i serves an active link ij, our perturbed process goes into a state t where $g_{ij} = 0$ and other. From t our unperturbed process P can reach a state $z \in E_i^x$ such that $g_{ji} = 1$. Without confusion we can write $g_{ij} = 1 \to g_{ji} = 1$. Due to this proceedings, we can construct a chain between any states s and s' in E_i^x.

■

Since any states s and s' which are connected by a chain has the same stochastic potential, we regard a weight of a state s as a weight of a set $C(s)$. So we expand the concept of a *least weighted tree of a state* to a set E_i^x and denote it by $E_i^x - tree$.

Theorem 4.1. *In Case 0,*

$$|N_l| < \lceil \frac{|(N-1)|(d-f)}{(b-e)} \rceil$$

if and only if E_1^α is the set of stochastically stable states.

Proof. By Lemma 4.1, Lemma 4.2 and Lemma 4.3, it is sufficient to compare the stochastic potentioal of E_1^α to that of E_0^β. Let each resistence be $r_{1,\alpha}$,$r_{0,\beta}$. These correspond to the number of mutations for the process $P(\varepsilon, \gamma)$ to transit into the concerning set of absorbing sets from the other set of absorbing sets. To escape from E_1^α it is needed and sufficient for the number of mutations (β players) to exeed a threshold over the *whole* players. Because at E_1^α, players are fully connected, the threshold or the basin of attraction for E_1^α is the same value as the random matching case. Therefore

$$r_{0,\beta} = \lceil \frac{|(N-1)|(d-f)}{(b-f)+(d-e)} \rceil$$

To escape from E_0^β, the needed number of mutations is

$$\min\{\lceil \frac{|N_l|(b-e)}{(b-f)+(d-e)} \rceil, \lceil \frac{|N|(b-e)}{(b-f)+(d-e)} \rceil\}$$

The former condition in above $\min\{\cdot,\cdot\}$ represents a case that mutations occurs over only the low cost players who establish the whole network. The latter represents a case that mutations occurs over all the players.

Clearly

$$\lceil \frac{|N_l|(b-e)}{(b-f)+(d-e)} \rceil \leq \lceil \frac{|N|(b-e)}{(b-f)+(d-e)} \rceil$$

. Therefore

$$r_{1\alpha} = \lceil \frac{|N_l|(b-e)}{(b-f)+(d-e)} \rceil$$

.

By comparing $r_{0\beta}$ to $r_{1\alpha}$, we get this theorem.

■

At Figure 6 an example of Theorem 4.1 is drawn for Theorem 4.1. Bold arrows represents transitions with mutations. Normal and dashed arrows represents transitions due to best responses. At the begining low cost players mute and switch to α. This example shows that at most two mutationss bring its process to the pareto dominant convention.

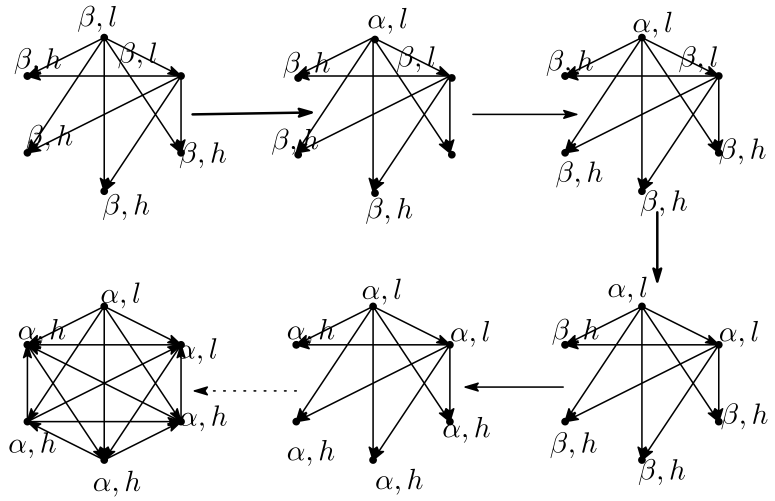


Figure 6.

This case is the one of the most interesting cases. The low link cost players who choose β are willing to add links with any miscoordinated players . Then high cost link players can *free ride* on the link established by the low cost player, and all players are connected only one network. That is, there is no isolated players.

By free riding on link formations of high cost players, pareto domonant convention emerges.

4.2 Case 1: $e < f < b < d$

We assume $e < f < b < d$. Except a pattern such that $d > k_h >\max\{f, b\}$ and $k_l < \min\ \{f, b\}$, there are following nine cases.

Lemma 4.4. *(Case 1.0)*
If k_l, $k_h < e$, then there are two classes of absorbing states E_1^α and E_2^β.
E_1^α is a set $\{(g^1, a^1)\}$ such that $g^1 \in G_e$ and for any $i \in N$, $a_i^1 = \alpha$.
E_2^β is a set $\{(g^2, a^2)\}$ such that $g^2 \in G_e$ and for any $i \in N$, $a_i^2 = \beta$.

Lemma 4.5. *(Case 1.1)*
If $k_l < e$ and $e < k_h < f$ then there are two classes of absorbing states E_1^α and E_2^β.
E_1^α is a set $\{(g^1, a^1)\}$ such that $g^1 \in G_e$ and for any $i \in N$, $a_i^1 = \alpha$.
E_2^β is a set $\{(g^2, a^2)\}$ such that $g^2 \in G_e$ and for any $i \in N$, $a_i^2 = \beta$.

Lemma 4.6. *(Case 1.2)*
If $k_l < e$ and $f < k_h < b$ then there are two classes of absorbing states E_1^α and E_2^β.
E_1^α is a set $\{(g^1, a^1)\}$ such that $g^1 \in G_e$ and for any $i \in N$, $a_i^1 = \alpha$.
E_2^β is a set $\{(g^2, a^2)\}$ such that $g^2 \in G_e$ and for any $i \in N$, $a_i^2 = \beta$.

Proof. (Case 1.0, Case 1.1, Case 1.2) In these three cases, at any period $t-1$ if $i \in N_l$ and $p_{ij} > 0$ then $g_{ij}^t = 1$ for any $jinN$. That is, if a low cost player is picked up at step 1, the player must establish an active link with any player. Therefore from any initial state, a fully connected network wii be approached by finite steps. (From any state, with a positive probability that

$$(\prod_{(i,j)\in N_l\times N} p_{ij})(\prod_{k\in N_s,|N_s|=|N_l\times N|} q_k) > 0$$

a fully connected network is approached.)

Once our process P transits into a state such that the network is fully connected, each player must coordinate on an action. With a positive probability of finite steps

$$(\prod_{(i,j)\in N\times N} p_{ij})(\prod_{k\in N_s,|N_s|=|N\times N|} q_k) > 0$$

all players necessarily coordinate on an action. ■

Lemma 4.7. *(Case 1.3)*

If $e < k_l, k_h < f$ then there are two classes of absorbing states E_1^α and E_2^β.

E_1^α is a set $\{(g^1, a^1)\}$ such that $g^1 \in G_e$ and for any $i \in N$, $a_i^1 = \alpha$.

E_2^β is a set $\{(g^2, a^2)\}$ such that $g^2 \in G_e$ and for any $i \in N$, $a_i^2 = \beta$.

Proof. (Case 1.3)

Assume $a_i^{(t-1)} = \alpha$, $a_j^{(t-1)} = \beta$, and $i \in N_l$ $j \in N_h$. Since $p_{ji} > 0$, $g_{ji}^t = 1$ occurs with the positive probability $p_{ji} > 0$ on the condition that $a_i^{(t-1)} = \alpha$, $a_j^{(t-1)} = \beta$. Assume $a_i^{(t-1)} = \alpha$, $a_j^{(t-1)} = \beta$, and $i \in N_h$ $j \in N_l$. Since $p_{ji} > 0$, $g_{ji}^t = 1$ occurs with the positive probability $p_{ji} > 0$ on the condition that $a_i^{(t-1)} = \alpha$, $a_j^{(t-1)} = \beta$. Anyway even if there is a miscoordination, one side player choosing β will linked with any player. Therefore from any state a fullly connected network will be approached with a positive probability by finite steps. Once the fully connected network is formed, if $g_{ij} = 1$ and $g_{ji} = 1$, $p_{ij} > 0$ then the player servers the link ij. Therefore efficiently network must be reached.

Once efficiently connected network is reached, all player necessarily coordinate on an action by finite steps. ■

Theorem 4.2. *In Case 1.0, Case 1.1, Case 1.2, Case 1.3, E_1^α is the set of stochastically stable states.*

Proof. Note that $r_{ss'}$ is the least resistence over all $ss'-paths$.

Let

$$r_{12}^m = \min_{s\in E_1^\alpha, s'\in E_2^\beta} r_{ij}$$

and

$$r_{21}^m = \min_{s\in E_2^\beta, s'\in E_1^\alpha} r_{ij}.$$

We call r_{ij}^m the resistence from E_i to E_j.

By Table 1,

$$r_{12}^m = \lceil \frac{(|N|-1)(d-f)}{(b-f)+(d-e)} \rceil$$

and

$$r_{21}^m = \lceil \frac{(|N|-1)(b-e)}{(b-f)+(d-e)} \rceil$$

where $\lceil \cdot \rceil$ represents the least integer larger than $\cdot$.

By Assumption 2, $r_{21}^m > r_{12}^m$. Therefore by Lemma any state in E_2^β minimize a stochastic potential.

■

Lemma 4.8. *(Case 1.4)*

If $e < k_l < f$ and $f < k_h < b$, then there are three classes of absorbing states E_1^α, E_2^β and $E_3^{\alpha,\beta}$.

E_1^α is a set $\{(g^1, a^1)\}$ such that $g^1 \in G_e$ and for any $i \in N$, $a_i^1 = \alpha$.

E_2^β is a set $\{(g^2, a^2)\}$ such that $g^2 \in G_e$ and for any $i \in N$, $a_i^2 = \beta$.

$E_3^{\alpha,\beta}$ is a set $\{(g^3, a^3)\}$ such that for any pair $(i,j) \in N_h \times N_h$ $\bar{g}_{ij} = 1$, $g_{ij} = 1 \rightarrow g_{ji} = 0$, and $a_i = a_j = \beta$, and for any pair $(k,l) \in N_l \times N_l$ $\bar{g}_{kl} = 1$, $g_{kl} = 1 \rightarrow g_{lk} = 0$, and $a_k = a_l = \alpha$.

Proof. Since $k_l, k_h < b < d$, if the action is coordinated between a pair $(i,j) \in N \times N$, then $\bar{g}_{ij} = 1$. But if $i \in N_l$, $j \in N_h$ $a_i = \alpha$ and $a_j = \beta$ then $\bar{g}_{ij} = 0$.

■

Theorem 4.3. *In Case 1.4, $|N_l| < |N_h|$ if and only if E_1^α is the set of stochastically stable states.*

Proof. By simple calculation each $E_i - tree$ has an weight as follows.(See Figure 6.)

E_1^α: r_{12}^m or $(|N|-1) - |N_h|$

E_2^β: r_{21}^m or $(|N|-1) - |N_l|$

E_3: $r_{12}^m + |N_l|$ or $r_{21}^m + |N_h|$ or $|N|$.

By comparing these weights we complete the proof.

■

Lemma 4.9. *(Case 1.5)*

If $f < k_l, k_h < b$ then there are five classes of absorbing states E_1^α, E_2^β, $E_3^{\alpha,\beta}$, $E_4^{\beta,\alpha}$ and E_5^{iso}.

E_1^α is a set $\{(g^1, a^1)\}$ such that $g^1 \in G_e$ and for any $i \in N$, $a_i^1 = \alpha$.

E_2^β is a set $\{(g^2, a^2)\}$ such that $g^2 \in G_e$ and for any $i \in N$, $a_i^2 = \beta$.

$E_3^{\alpha,\beta}$ is a set $\{(g^3, a^3)\}$ such that for any pair $(i,j) \in N_h \times N_h$ $\bar{g}_{ij} = 1$, $g_{ij} = 1 \rightarrow g_{ji} = 0$, and $a_i = a_j = \beta$, and for any pair $(k,l) \in N_l \times N_l$ $\bar{g}_{kl} = 1$, $g_{kl} = 1 \rightarrow g_{lk} = 0$, and $a_k = a_l = \alpha$.

$E_4^{\beta,\alpha}$ is a set $\{(g^4, a^4)\}$ such that for any pair $(i,j) \in N_h \times N_h$ $\bar{g}_{ij} = 1$, $g_{ij} = 1 \rightarrow g_{ji} = 0$, and $a_i = a_j = \alpha$, and for any pair $(k,l) \in N_l \times N_l$ $\bar{g}_{kl} = 1$, $g_{kl} = 1 \rightarrow g_{lk} = 0$, and $a_k = a_l = \beta$.

E_5^{iso} is a set $\{(g^5, a^5)\}$ such that there are some isolated players choosing x and other players are connected and coordinated on another action y.

Proof. The cases to note are only $E_4^{\beta,\alpha}, E_5^{\beta}$. All players can not link with others when they coordinate on the action β. Thus initially players choosing β of high link costs will change their action.

And link cost is high enough for low type players to link with miscoordinated players. Therefore there may be separeted two populations which are coordinated ondifferent actions each other.

■

To show its stochastic stability, it is sufficient to compare the numbers E_1^{α} and E_2^{β}.

Theorem 4.4. *In Case 1.5, E_2^{β} is only stochastcally stable.*

Proof. Since the link cost k_h is less than b, we can use a standard argument of Kandori et,al(1993) and Young(1993).

■

Lemma 4.10. *(Case 1.6)*

If $f < k_l < b$, $b < k_h < d$ then there are three classes of absorbing states E_1^{α}, E_6^{β} and E_5^{iso}.

E_1^{α} is a set $\{(g^1, a^1)\}$ such that $g^1 \in G_e$ and for any $i \in N$, $a_i^1 = \alpha$.

E_6^{β} is a set $\{(g^6, a^6)\}$ such that for any $i, j \in N_l$ $\bar{g}_{ij} = 1$, $g_{ij} = 1 \rightarrow g_{ji} = 0$, for any $k, l \in N_h$ $\bar{g}_{ij} = 0$, for any $i \in N_l$ $k \in N_h$ $g_{ik} = 1$ and $g_{ki} = 0$, and for any $i \ \in N i$ $a_i^6 = \beta$.

E_5^{iso} is a set $\{(g^5, a^5)\}$ such that there are some isolated players choosing x and other players are connected and coordinated on another action y.

Proof. E_6^{β} is only the case different above lemmas. In this case, since high cost players cannot link with others when coordinated on β. Thus only low cost players can establish links with all players by finite steps.

■

Theorem 4.5. *In Case 1.6, if*

$$|N_l| < \lceil \frac{(|N|-1)(d-f)}{(b-e)} \rceil$$

then

E_1^{α} and E_5^{iso} are the sets of stchastically stable states.

Otherwise E_6^{β} is the state of stchastically stable states.

Proof. Similar to Theorem 4.1. Thus it is omitted.

■

Lemma 4.11. *(Case 1.7) If $b < k_l < d$ and $d < k_h$ then the classes of absorbing states are E_1^{α} and $E_5^{iso,\beta}$.*

E_1^{α} is a set $\{(g^1, a^1)\}$ such that $g^1 \in G_e$ and for any $i \in N$, $a_i^1 = \alpha$.

$E_5^{iso,\beta}$ is a set $\{(g^5, a^{5,\beta})\}$ such that there are some isolated players choosing β and other players are connected and coordinated on another action α.

Proof. Only players coordinating on α have an incentive to establish an active link. Since a miscoordinated player chooseing β does not have establish an active link, he will be isolated. ∎

Theorem 4.6. *In Case 1.8* E_1^α *is a unique set of the stochastically stable states.*

Proof. This case is investigated by Goyal and Vega-Redondo(1999). Thus omitted. ∎

Lemma 4.12. *(Case 1.9)*
If $d < k_h, k_l$*, then the sets of absorbing states are* E_0^α *and* E_0^β.

Proof. This case is the same one of Kandori,et,al(1993) and Young(1993). Thus omitted. ∎

5 Concluding Remarks of the direct link case and An Application

5.1 Concluding Remarks

We have analyzed mainly the direct link case. Roughly speaking the performance of the indirect link case model is very similar to our main result Theorem 4. According to Goyal and Vega-Redondo, indirect link case derive the Star network as unique strict Nash equilibrium of a network formation game.[13] This fact is easily checked by definitions in our Section 2,3. Since in indirect link case each player can play the coordination game with all other players through at most one link, they have no incentives to add more links. At this Star network since the unique support(center) player has ultimately strong power to upset the convention, the stochastically stable action may be α. The role of the center player is very similar to the role of the low cost players in our model. These precise relations are one of the interesting future researchs.

We suppose link cost as a constant value, but as say at section 2, the link cost may be convex function of active links in the real communication. As Jackson and Watts shows, this link cost restricts the number of full connected players in a network. If asymmetricity of cost is introduced too, distinct convention regions may emerge. This will be more exiting futur research.

[13] Note their adaptation process is different from our model about steps in a period.

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7 Appendix

7.1 Case 2: $e < b < f < d$

We use the same notations as case 1 for the set of absorbing states. Proofsand Studying stochastically stabilities are similar to Case 1 for each case. Thus this cases are written at an Appendix and proofs are omitted.

Lemma 7.1. *(Case 2.0)*
If $k_h, k_l < e$, then the sets of absorbing states are E_1^α and E_2^β.

Lemma 7.2. *(Case 2.1)*
If $k_l < e$ and $e < k_h < b$, then the sets of absorbing states are E_1^α and E_2^β.

Proof. Low link cost players establish active links with any player. Proof is same as Case 1.0, 1.1, 1.2.

■

Lemma 7.3. *(Case 2.2)*
If $k_l < e$ and $b < k_h < f$, then the sets of absorbing states are E_1^α and
E_6^β is a set $\{(g^6, a^6)\}$ such that for any$i, j \in N_l$ $\bar{g}_{ij} = 1$, $g_{ij} = 1 \rightarrow g_{ji} = 0$, for any $k, l \in N_h$ $\bar{g}_{ij} = 0$, for any $i \in N_l$ $k \in N_h$ $g_{ik} = 1$ and $g_{ki} = 0$, and for any $\in N i$ $a_i^6 = \beta$.

Proof. Since $b < k_h < f$, in a coordination on β any high cost player serves any active link. Precise proof is shown at Case 0.

■

Lemma 7.4. *(Case 2.3)*
If $e < k_h, k_l < b$, then the sets of absorbing states are E_1^α and E_2^β.

Proof. Proof is same as Case 1.3.

■

Lemma 7.5. *(Case 2.4)*
If $e < k_l < b$ $b < k_h < f$ then the sets of absorbing states are E_1^α and E_6^β.

Proof. Since $b < k_h < f$, in a coordination on β any high cost player serves any active link. Precise proof is shown at Case 0.

■

Lemma 7.6. *(Case 2.5)*
If $b < k_l, k_h < f$ then the sets of absorbing states are E_1^α and $E_7^{iso,\beta}$.
$E_7^{iso,\beta}$ is a set $\{(g^7, a^7)\}$ such that for any $i \in N$ $a_i^7 = \beta$ and for any i,j $\in N$ $\bar{g}_{ij} = 0$.

Proof. Since $b < k_h, k_l < f$, in a coordination on β no player serves any active link. But at any miscoordination state any β player establish active link with α player.

■

Lemma 7.7. *(Case 2.6)*
If $b < k_l < f$, $f < k_h < d$ then the sets of absorbing states are E_1^α and $E_5^{iso,\beta}$.
$E_5^{iso,\beta}$ is a set $\{(g^5, a^{5,\beta})\}$ such that there are some isolated players choosing β and other players are connected and coordinated on another action α.

Lemma 7.8. *(Case 2.7)*
If $f < k_l < k_h < d$ then the sets of absorbing states are E_1^α and $E_5^{iso,\beta}$.
$E_5^{iso,\beta}$ is a set $\{(g^5, a^{5,\beta})\}$ such that there are some isolated players choosing β and other players are connected and coordinated on another action α.

Lemma 7.9. *(Case 2.8)*

If $f < k_l < d$ $d < k_h$ *then the sets of absorbing states are* E_8^{α} *and* $E_8^{iso,complex}$.

$E_8^{iso,complex}$ *is a set such that there are some isolated players choosing* β *and other players are connected and coordinated on another action* α.*Where players who are connected care low cost players.*

Lemma 7.10. *(Case 2.9)*

If $d < k_l, k_h$, *then the sets of absorbing states are* E_0^{α} *and* E_0^{β}.