Necessity of the Transversality Condition for Stochastic Models with CRRA Utility

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May 20, 2003

Abstract

This paper shows that the standard transversality condition (STVC) is necessary for optimality for stochastic models with constant-relative-risk-aversion (CRRA) utility under general conditions. We consider an infinite-horizon stochastic maximization problem that takes a general form of multi-sector growth model with a single consumption good and CRRA utility. We establish two results. The first result is that the STVC is necessary in the case of logarithmic utility. The second result is that the STVC is necessary in the case of non-logarithmic CRRA utility as long as lifetime utility is finite at the optimum. These results apply to various stochastic growth models, including real business cycle (RBC) models with endogenous labor supply. Our results make it clear that there is practically no issue about necessity of the STVC for stochastic models with CRRA utility.

Keywords: Transversality condition, stochastic optimization, stochastic growth, CRRA, real business cycle.

JEL Classification Numbers: C61, D90, G12

1 Introduction

Since Shell (1969) and Halkin (1974), necessity of transversality conditions (TVCs) has been an uneasy matter to economic theorists who use infinite-horizon optimization problems. The most standard TVC, which we call the standard TVC (STVC), is the condition that the value of optimal stocks at infinity must be zero. Necessity of the STVC becomes an issue particularly for models with unbounded utility, such as constant-relative-risk-aversion (CRRA) utility.

Models with CRRA utility are prominent in macroeconomics, for example, in the endogenous growth literature (e.g., Barro and Sala-i-Martin, 1995), the indeterminacy literature (e.g., Benhabib and Farmer, 1999), and the real business cycle (RBC) literature (e.g., King and Rebelo, 1999). Although some results on necessity of the STVC for models with CRRA utility are available in the literature, they only deal with deterministic cases (Alvarez and Stokey, 1998; Kamihigashi 2001). Though some of the results can perhaps be extended to stochastic cases in a more or less straightforward way, those results are basically applicable only to models with constant-returns-to-scale technologies.[1]

In this paper, we show that the STVC is necessary for stochastic models with CRRA utility under general conditions. In particular, we consider an infinite-horizon stochastic maximization problem that takes a general form of multi-sector growth model with a single consumption good and CRRA utility. Our model encompasses various stochastic growth models (e.g., Brock and Mirman, 1972), including RBC models with endogenous labor supply (e.g., King and Rebelo, 1999). Our assumptions on the technology side of the model are general and standard; they are rarely violated in applications.

We establish two results in this paper. The first result is that the STVC is necessary in the case of logarithmic utility. This result does not require any additional condition. The second result is that the STVC is necessary in the case of non-logarithmic CRRA utility as long as lifetime utility is finite *at the optimum*. It is important to emphasize that we do *not* assume the finiteness of lifetime utility for all feasible paths, for such an assumption is often violated in models with unbounded utility, particularly when utility is unbounded below.

Though these results are established based a general result shown in

[1] See Kamihigashi (2001, 2003) for the literature on TVCs.

Kamihigashi (2003), they are surprisingly clean and powerful. Because of their general applicability, they practically eliminate the issue of necessity of the STVC for models with CRRA utility.

The rest of the paper is organized as follows. Section 2 presents the general model and discusses our assumptions. Our main results are stated in Section 3. The proofs appear in Appendix B. Section 4 explains how one can apply our results to models with endogenous labor supply such as RBC models. Section 5 concludes the paper. Appendix A contains preliminary lemmas.

2 The Model

This section presents our general model and discusses our assumptions. Some of the material here is borrowed from Kamihigashi (2003), which offers further discussions on the general structure of the model.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let E denote the associated expectation operator; i.e., $Ez = \int z(\omega)dP(\omega)$ for any random variable $z : \Omega \to \overline{\mathbb{R}}$. When it is important to make explicit the dependence of z on ω, we write $Ez(\omega)$ instead of Ez. Consider the following problem:

$$(2.1) \qquad \begin{cases} \text{``}\max\limits_{\{x_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t Eu(g_t(x_t(\omega), x_{t+1}(\omega), \omega))\text{''} \\ \quad \text{s.t.} & x_0 = \overline{x}_0, \\ & \forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in X_t.^2 \end{cases}$$

In what follows, we list and discuss all our assumptions, which are maintained throughout the paper.

Let $n \in \mathbb{N}$ and F be the set of all functions from Ω to $\mathbb{R}^n$.[3] The following assumption means that each x_t is a random variable in $\mathbb{R}^n$.

Assumption 2.1. $\overline{x}_0 \in F$ and $\forall t \in \mathbb{Z}_+, X_t \subset F \times F$.

Since F consists of all functions from Ω to $\mathbb{R}^n$, and since X_t is only a subset of $F \times F$, X_t can be chosen in such a way that x_t and x_{t+1} must be measurable with respect to the information available in period t and period

[2] $\mathbb{Z}_+ \equiv \{0, 1, 2, \ldots\}$.
[3] $\mathbb{N} \equiv \{1, 2, 3, \ldots\}$.

$t+1$, respectively. Restricting X_t this way is not necessary, however, since our results require no such information structure.

The next assumption collects our general restrictions on the preference side of the model.

Assumption 2.2. (i) $\beta \in (0,1)$; (ii) $u : \mathbb{R}_+ \to [-\infty, \infty)$ is C^1 on $\mathbb{R}_{++}$, concave, and strictly increasing; and (iii) $\lim_{c \downarrow 0} u'(c) = \infty$.

Parts (ii) and (iii) above are implied by CRRA utility, which is assumed by our main results. These general restrictions are stated here since they are implicit in some of the discussions to follow.

The next assumption simply means that the expression

$$Eu(g_t(x_t(\omega), x_{t+1}(\omega), \omega)) \tag{2.2}$$

makes sense for all $(x_t, x_{t+1}) \in X_t$.

Assumption 2.3. $\forall t \in \mathbb{Z}_+, \forall (y,z) \in X_t$, (i) $\forall \omega \in \Omega, g_t(y(\omega), z(\omega), \omega) \geq 0$, (ii) the mapping $g_t(y(\cdot), z(\cdot), \cdot) : \Omega \to \mathbb{R}_+$ is measurable, and (iii) the expectation $Eu(g_t(y(\omega), z(\omega), \omega))$ exists in $[-\infty, \infty)$.

To simplify expressions like (2.2), let $g_t(x_t, x_{t+1})$ denote the random variable $g_t(x_t(\cdot), x_{t+1}(\cdot), \cdot) : \Omega \to \mathbb{R}$ for $(x_t, x_{t+1}) \in X_t$. We say that a sequence $\{x_t\}_{t=0}^{\infty}$ is a *feasible path* if $x_0 = \overline{x}_0$ and $\forall t \in \mathbb{Z}_+, (x_t, x_{t+1}) \in X_t$. Since in applications the objective function is often not guaranteed to be finite or well-defined for all feasible paths, we use weak maximality (Brock, 1970) as our optimality criterion. We say that a feasible path $\{x_t^*\}$ is *optimal* if for any feasible path $\{x_t\}$,

$$\varliminf_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t [Eu(g_t(x_t, x_{t+1})) - Eu(g_t(x_t^*, x_{t+1}^*))] \leq 0.^4 \tag{2.3}$$

Our optimality criterion (i) reduces to the standard maximization criterion whenever the latter makes sense, (ii) applies even when the standard criterion fails, and (iii) is weaker than the similar criterion with $\varlimsup$ replacing $\varliminf$ in (2.3). Using this optimality criterion does not limit the applicability

[4] To be precise, this inequality requires that the left-hand side be well-defined. This means that the left-hand side may not involve expressions like "$\infty - \infty$" and "$-\infty + \infty$." Thus optimality implies that $Eu(g_t(x_t^*, x_{t+1}^*))$ is finite for all $t \in \mathbb{Z}_+$; for otherwise the left-hand side of (2.3) is undefined for $\{x_t\} = \{x_t^*\}$.

of our results in any way since we are only interested in necessary conditions for optimality.

For the same reason, assuming the existence of an optimal path imposes no restriction on the model.

Assumption 2.4. There exists an optimal path $\{x_t^*\}$.

For $t \in \mathbb{Z}_+$ and $d : \Omega \to \mathbb{R}^n$ such that $(x_t^*, x_{t+1}^* + \epsilon d) \in X_t$ for sufficiently small $\epsilon > 0$, we define the random variable $g_{t,2}(x_t^*, x_{t+1}^*; d)$ as follows:

$$g_{t,2}(x_t^*, x_{t+1}^*; d) = \varliminf_{\epsilon \downarrow 0} \frac{g_t(x_t^*, x_{t+1}^* + \epsilon d) - g_t(x_t^*, x_{t+1}^*)}{\epsilon}, \tag{2.4}$$

where $\varliminf_{\epsilon \downarrow 0}$ is applied pointwise (i.e., for each $\omega \in \Omega$ separately). The right-hand side is called a lower Dini directional derivative. Note that if g_t is differentiable in the second argument, then

$$g_{t,2}(x_t^*, x_{t+1}^*; d) = D_2 g_t(x_t^*, x_{t+1}^*)d. \tag{2.5}$$

We use the general derivative defined in (2.4) only because assuming differentiability does not offer any simplification but, at the same time, requires additional assumptions that are somewhat cumbersome to state.

The most standard TVC, which we call the standard TVC (STVC), is the condition that the value of optimal stocks at infinity must be zero. It takes the following form in our framework:

$$\lim_{t \uparrow \infty} \beta^t E u'(g_t(x_t^*, x_{t+1}^*)) g_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) = 0. \tag{2.6}$$

Recalling (2.5), one can see that (2.6) becomes the familiar TVC in the differentiable case. The following assumption is a minimum requirement for (2.6) to make sense.

Assumption 2.5. The mapping $g_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) : \Omega \to \overline{\mathbb{R}}$ is measurable.

It is difficult to think of a situation in which this assumption is violated. It must be assumed nonetheless since the other assumptions do not imply it. It is satisfied, for example, if g_t depends on ω through a technology shock and is continuously differentiable in x_t, x_{t+1}, and the technology shock (and if these variables are measurable).

The assumptions stated thus far are preliminary requirements that are in many cases assumed implicitly. In what follows, we state three assumptions that are actually needed to prove the necessity of the STVC. Each of them is general and standard though it has a specific role. The following is our assumption on X_t.

Assumption 2.6. $\forall t \in \mathbb{Z}_+, \exists \underline{\lambda}_t \in [0,1), \forall \lambda \in [\underline{\lambda}_t, 1), (x_t^*, \lambda x_{t+1}^*) \in X_t$ and $\forall \tau \geq t+1, (\lambda x_\tau^*, \lambda x_{\tau+1}^*) \in X_\tau$.

This assumption means that the optimal path $\{x_t^*\}$ can be shifted proportionally downward starting from any period. The assumption is common to most results on the STVC, which basically means that no gain should be achieved by shifting the optimal path proportionally downward.

Assumption 2.6 is satisfied if X_t is convex, $(0,0) \in X_t$, and $(x_t^*, 0) \in X_t$ for all $t \in \mathbb{Z}_+$. We use Assumption 2.6 in order to accommodate cases in which it is not always feasible to reduce capital stocks to zero in one step, such as in models with irreversible investment (e.g., Olson, 1989).

The next two assumptions impose some structure on g_t.

Assumption 2.7. $\forall t \in \mathbb{Z}_+$, (i) $g_t(x_t^*, x_{t+1}^*) > 0$ and (ii) $\forall t \in \mathbb{N}, g_t(\lambda x_t^*, \lambda x_{t+1}^*)$ is concave in $\lambda \in [\underline{\lambda}_0, 1]$, where $\underline{\lambda}_0$ is given by Assumption 2.6.

Part (i) above is needed for TVC (2.6) to make sense. It is implied by the Inada condition at zero (Assumption 2.2(iii)) as long as such a path is feasible. Part (ii) is used to find a lower bound on $g_t(\lambda x_t^*, \lambda x_{t+1}^*)$ that depends only on λ and $g_t(x_t^*, x_{t+1}^*)$ (see Lemma A.2). This part can be weakened considerably though it is already general enough for most purposes.

The following is our last assumption.

Assumption 2.8. $\forall t \in \mathbb{Z}_+, g_t(x_t^*, \lambda x_{t+1}^*)$ is nonincreasing and continuous in $\lambda \in (\underline{\lambda}_t, 1]$, where $\underline{\lambda}_t$ is given by Assumption 2.6.

That $g_t(x_t^*, \lambda x_{t+1}^*)$ is nonincreasing in λ basically means that there is a tradeoff between consumption and investment. Such a tradeoff is typically required for the existence of an optimal path. In this paper, this restriction is used to express the STVC as an equality condition as in (2.6). The continuity of $g_t(x_t^*, \lambda x_{t+1}^*)$ in λ plays a technical role that allows us to write $g_t(x_t^*, x_{t+1}^*)$ as the argument of $u'(\cdot)$ in (2.6). Without this continuity requirement, $g_t(x_t^*, x_{t+1}^*)$ must be replaced by $\underline{\lim}_{\lambda \uparrow 1} g_t(x_t^*, \lambda x_{t+1}^*)$ in (2.6); see the proof of Lemma A.3 for details.

3 Main Results

We are now ready to state our main results, which assume all the assumptions stated in the preceding section.

Proposition 3.1. *Suppose*

$$u(\cdot) = \ln(\cdot). \tag{3.1}$$

Then TVC (2.6) holds.

Hence the STVC is always necessary in the case of logarithmic utility. A similar result is shown in Kamihigashi (2001) for a deterministic continuous-time model. Proposition 3.1 extends the result to a general stochastic environment. It makes it clear that once logarithmic utility is assumed, there is no issue about necessity of the STVC even for stochastic models.

The next result deals with the case of non-logarithmic CRRA utility.

Proposition 3.2. *Suppose*

$$\exists \alpha \in (-\infty, 1] \setminus \{0\}, \quad u(\cdot) = \frac{(\cdot)^\alpha}{\alpha}, \tag{3.2}$$

$$-\infty < \sum_{t=0}^{\infty} \beta^t E u(g_t(x_t^*, x_{t+1}^*)) < \infty. \tag{3.3}$$

Then TVC (2.6) holds.

Thus under (3.2), the STVC is guaranteed to be necessary unless lifetime utility is allowed to be infinite at the optimum. Such cases are usually ruled out in practice, and (3.3) is usually assumed or taken for granted in applied studies.

It is worth emphasizing that (3.3) requires lifetime utility to be finite only for the optimal path. It does not require lifetime utility to be finite for any other feasible path. This is important since lifetime utility is often $-\infty$ for many feasible paths when $\alpha < 0$. Proposition 3.2 makes it clear that once non-logarithmic CRRA utility is assumed, there is no issue about necessity of the STVC for all practical purposes.

4 Endogenous Labor Supply

One can easily apply the results shown in the preceding section to models with endogenous labor supply such as RBC models (e.g., King and Rebelo, 1999). To do so, one can take an optimal labor (or, equivalently, leisure) path as given and consider the maximization problem over consumption and capital paths.

To be more specific, consider the following problem:

$$(4.1) \qquad \begin{cases} \text{“}\max_{\{x_t\}_{t=0}^{\infty}} & \sum_{t=0}^{\infty} \beta^t E r(c_t, l_t)\text{”} \\ \text{s.t.} & x_0 = \overline{x}_0, \\ & \forall t \in \mathbb{Z}_+, c_t + x_{t+1} = f_t(x_t, 1 - l_t), \\ & c_t, x_{t+1} \geq 0, \end{cases}$$

where c_t is consumption, l_t is leisure, r is the utility function, and f_t is the production function (which depends on ω in the same way as g_t does).[5] In this section we assume $n = 1$, i.e., $x_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$.

First consider the case in which utility is additively separable in consumption and leisure, i.e.,

$$(4.2) \qquad r(c_t, l_t) = u(c_t) + n(l_t)$$

for some functions $u(\cdot)$ and $n(\cdot)$. In this case one can take an optimal leisure path $\{l_t^*\}$ as given and simply set $g_t(x_t, x_{t+1}) = f_t(x_t, 1 - l_t^*) - x_{t+1}$. The model then reduces to (2.1), ignoring utility from leisure (which is taken as given).

When utility is not additively separable, it is typically the case that utility can be written as

$$(4.3) \qquad r(c_t, l_t) = u(c_t e(l_t))$$

for some functions $u(\cdot)$ and $e(\cdot)$ (e.g., King and Rebelo, 1999, p. 945). Typically, $u(\cdot)$ is assumed to satisfy (3.2) (or (3.1), which leads to (4.2)), and $e(\cdot)$ is required to satisfy certain regularity conditions (King and Rebelo, 1999,

[5] More precisely, $f_t(x_t, 1 - l_t)$ denotes the random variable $f_t(x_t(\omega), 1 - l_t(\omega), \omega)$ in the same way as $g_t(x_t, x_{t+1})$ denotes the random variable $g_t(x_t(\omega), x_{t+1}(\omega), \omega)$; recall the paragraph below Assumption 2.3.

footnote 69). In the case of (4.3), one can take an optimal leisure path $\{l_t^*\}$ as given and set $g_t(x_t, x_{t+1}) = [f_t(x_t, 1 - l_t^*) - x_{t+1}]e(l_t^*)$. The model then reduces to (2.1) again.

We have shown that maximization problems of the form (4.1) reduce to (2.1) in typical parametric cases if an optimal leisure path is taken as given. There is no problem in taking an optimal leisure path as given, since necessary optimality conditions such as Euler equations and TVCs are expressed taking an optimal path as (implicitly) given anyway. The discussions here along with our main results indicate that there is practically no issue about necessity of the STVC for standard parametric stochastic growth models, whether labor supply is exogenous or endogenous.

5 Concluding Remarks

In this paper we showed the necessity of the standard transversality condition (STVC) for a multi-sector growth model with a single consumption good and constant-relative-risk-aversion (CRRA) utility under general conditions. In particular, it was shown that the STVC is necessary in the case of logarithmic utility. It was also shown that the STVC is necessary in the case of non-logarithmic CRRA utility as long as lifetime utility is finite at the optimum. These results apply to various stochastic growth models, including RBC models with endogenous labor supply.

Our results indicate that there is practically no issue about necessity of the STVC for stochastic models with CRRA utility. We believe that these results help eliminate the concern about necessity of the STVC that has annoyed the profession for decades.

Appendix A Preliminary Lemmas

The proofs of our propositions use a result shown in Kamihigashi (2003) for a general reduced-form model. We state it without proof after introducing some notation. After stating the result, we show two further lemmas. All the assumptions stated in Section 2 are maintained here though not all of them are used.

For $t \in \mathbb{Z}_+, (y, z) \in X_t$, and $\omega \in \Omega$, define

$$v_t(y(\omega), z(\omega), \omega) = \beta^t u(g_t(y(\omega), z(\omega), \omega)). \tag{A.1}$$

As for $g_t(x_t, x_{t+1})$, $v_t(x_t, x_{t+1})$ denotes the random variable $v_t(x_t(\cdot), x_{t+1}(\cdot), \cdot) : \Omega \to \mathbb{R} \cup \{-\infty\}$ for $(x_t, x_{t+1}) \in X_t$. For $t \in \mathbb{N}$ and $\lambda < 1$ with $(\lambda x_t^*, \lambda x_{t+1}^*) \in X_t$, define

$$w_t(\lambda) = \frac{Ev_t(x_t^*, x_{t+1}^*) - Ev_t(\lambda x_t^*, \lambda x_{t+1}^*)}{1 - \lambda}, \tag{A.2}$$

$$\hat{w}_t(\lambda) = \sup_{z \in [\lambda, 1)} w_t(z), \tag{A.3}$$

where $\hat{w}_t(\lambda)$ is defined for $\lambda < 1$ such that $\forall z \in [\lambda, 1), (zx_t^*, zx_{t+1}^*) \in X_t$.

Lemma A.1. *Suppose*

$$\exists \{b_t\}_{t=1}^{\infty} \subset \mathbb{R}, \exists \lambda \in [\underline{\lambda}_0, 1), \forall t \in \mathbb{N}, \quad \hat{w}_t(\lambda) \leq b_t, \tag{A.4}$$

$$\sum_{t=1}^{\infty} b_t \text{ exists in } [-\infty, \infty), \tag{A.5}$$

where $\underline{\lambda}_0$ is given by Assumption 2.6. Then

$$\overline{\lim_{t \uparrow \infty}} Ev_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) \leq 0, \tag{A.6}$$

where $v_{t,2}$ is defined as in (2.4).

Proof. See Kamihigashi (2003, Theorem 2.2). □

For simplicity, we use the following notations for $t \in \mathbb{Z}_+$:

$$g_t^* = g_t(x_t^*, x_{t+1}^*), \quad g_t(\lambda) = g_t(\lambda x_t^*, \lambda x_{t+1}^*). \tag{A.7}$$

Lemma A.2. *We have*

$$\forall t \in \mathbb{N}, \forall \lambda \in [\underline{\lambda}_0, 1), \quad g_t(\lambda) \geq m(\lambda) g_t^*, \tag{A.8}$$

where $\underline{\lambda}_0$ is given by Assumption 2.6 and

$$m(\lambda) = \frac{\lambda - \underline{\lambda}_0}{1 - \underline{\lambda}_0}. \tag{A.9}$$

Proof. It is easy to see that

$$m(\lambda) + [1 - m(\lambda)]\underline{\lambda}_0 = \lambda. \tag{A.10}$$

Let $t \in \mathbb{N}$. Since $g_t(\lambda)$ is concave in $\lambda \in [\underline{\lambda}_0, 1]$ by Assumption 2.7, for $\lambda \in [\underline{\lambda}_0, 1]$,

$$g_t(\lambda) \geq m(\lambda)g_t^* + [1 - m(\lambda)]g_t(\underline{\lambda}_0) \tag{A.11}$$
$$\geq m(\lambda)g_t^*, \tag{A.12}$$

where the last inequality holds since $g_t(\underline{\lambda}_0) \geq 0$ by Assumptions 2.6 and 2.3. Now (A.8) follows. □

Lemma A.3. *TVC (A.6) is equivalent to TVC (2.6).*

Proof. Since Assumption 2.8 implies $v_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) \geq 0$, TVC (A.6) is equivalent to

$$\lim_{t\uparrow\infty} Ev_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) = 0. \tag{A.13}$$

Thus it suffices to verify the following for $t \in \mathbb{Z}_+$:

$$v_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*) = \beta^t u'(g_t^*)g_{t,2}(x_t^*, x_{t+1}^*; -x_{t+1}^*). \tag{A.14}$$

Let $t \in \mathbb{Z}_+$. Let $\epsilon > 0$ be such that $(1-\epsilon) \in [\underline{\lambda}_0, 1)$. Fix $\omega \in \Omega$ (though it is omitted in what follows). Note that

$$v_t(x_t^*, x_{t+1}^* - \epsilon x_{t+1}^*) - v_t(x_t^*, x_{t+1}^*) \tag{A.15}$$
$$= \beta^t[u(g_t(x_t^*, x_{t+1}^* - \epsilon x_{t+1}^*)) - u(g_t^*)] \tag{A.16}$$
$$= \beta^t u'(\tilde{g})[g_t(x_t^*, x_{t+1}^* - \epsilon x_{t+1}^*) - g_t^*], \tag{A.17}$$

where (A.17) holds by the mean value theorem for some

$$\tilde{g} \in [g_t^*, g_t(x_t^*, (1-\epsilon)x_{t+1}^*)]. \tag{A.18}$$

Since $g_t(x_t^*, (1-\epsilon)x_{t+1}^*)$ is continuous in $\epsilon \in [0, 1 - \underline{\lambda}_t)$ by Assumption 2.8, $\tilde{g}$ converges to g_t^* as $\epsilon \downarrow 0$. Thus dividing (A.15) and (A.17) by ϵ and applying $\underline{\lim}_{\epsilon\downarrow 0}$ yields (A.14). □

Appendix B Proofs

B.1 Proof of Proposition 3.1

By Lemma A.3, to conclude TVC (2.6) from Lemma A.1, it suffices to verify (A.4) and (A.5). Let $\underline{\lambda} \in (\underline{\lambda}_0, 1), \lambda \in [\underline{\lambda}, 1)$, and $t \in \mathbb{N}$. We have

$$(1-\lambda)\beta^{-t} w_t(\lambda) = E[\ln g_t^* - \ln g_t(\lambda)] \tag{B.1}$$
$$\leq E\left[\ln g_t^* - \ln\left(m(\lambda) g_t^*\right)\right] \tag{B.2}$$
$$= -\ln m(\lambda), \tag{B.3}$$

where (B.2) uses Lemma A.2. It follows that

$$\beta^{-t} w_t(\lambda) \leq \frac{-\ln m(\lambda)}{1-\lambda} \leq A, \tag{B.4}$$

where

$$A = \sup_{\lambda \in [\underline{\lambda}, 1)} \frac{-\ln m(\lambda)}{1-\lambda}. \tag{B.5}$$

Note that A is finite since the supremand is continuous on $[\underline{\lambda}, 1)$ and converges to $1/(1-\underline{\lambda}_0)$ as $\lambda \uparrow 1$. Now (A.4) and (A.5) hold with $b_t = \beta^t A$.

B.2 Proof of Proposition 3.2

By Lemma A.3, to conclude TVC (2.6) from Lemma A.1, it suffices to verify (A.4) and (A.5). Let $\underline{\lambda} \in (\underline{\lambda}_0, 1), \lambda \in [\underline{\lambda}, 1)$, and $t \in \mathbb{N}$. We have

$$(1-\lambda)\beta^{-t} w_t(\lambda) = E[u(g_t^*) - u(g_t(\lambda))] \tag{B.6}$$
$$\leq E[u(g_t^*) - u(m(\lambda) g_t^*)] \tag{B.7}$$
$$= [1 - (m(\lambda))^\alpha] E u(g_t^*), \tag{B.8}$$

where (B.7) holds by Lemma A.2, and (B.8) holds by (3.2). It follows that

$$\beta^{-t} w_t(\lambda) \leq \frac{1-(m(\lambda))^\alpha}{(1-\lambda)\alpha} \alpha E u(g_t^*) \tag{B.9}$$
$$\leq A \alpha E u(g_t^*), \tag{B.10}$$

where

(B.11) $$A = \sup_{\lambda \in [\underline{\lambda}, 1)} \frac{1 - (m(\lambda))^\alpha}{(1 - \lambda)\alpha}.$$

Note that A is finite since the supremand is continuous in $\lambda \in [\underline{\lambda}, 1)$ and converges to $1/(1 - \underline{\lambda}_0)$ as $\lambda \uparrow 1$. Now (B.9), (B.10), and (3.3) imply (A.4) and (A.5) with $b_t = \beta^t A \alpha E u(g_t^*)$.

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