A Di¤erential Game Model of Tari¤ War

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Abstract

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We present a simple two(-country) by two(-good) di¤erential game model of international trade in which the governments of the two countries play a tari¤-setting game. We explicitly derive a unilateral optimum tari¤ rate and then a Markov-perfect equilibrium pair of tari¤ strategies (bilateral optimum tari¤ strategies) and compare the welfare level of each country among autarchic, free- trade, unilateral and bilateral optimum-tari¤ equilibria.

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1. Introduction

While there is a huge literature on optimum tari¤s and tari¤ wars, most studies of those topics are set in a static framework and intertemporal analyses are few. Moreover, the intertemporal models are mainly applications of repeated games containing no inherent dynamics: the stocks of assets or productive factors remain the same over time. To our knowledge, there is no paper which formulates a dynamic general equilibrium model of international trade in which the governments play a tari¤-setting game, and which derives a feedback-Nash equilibrium pair of tari¤ strategies and characterizes it. We aim to do this job in this paper.

As it will become evident, our di¤erential-game model relies on rather speci…c assumptions. However, at the same time, we believe that it has several virtues. First, although there are two state variables, one in each country, we are able to simplify the di¤erential-game model in such a way that e¤ectively one state variable can be eliminated. Therefore, we can avoid the di¢culty that would arise from having to deal with Markov-perfect tari¤ strategy as a function of two state variables[1]. Second, due to the model speci…city, we can explicitly derive the feedback-Nash equilibrium pair of tari¤ strategies and easily analyze how it depends on the economic fundamentals of the model. Third, the model is very simple and its tractability may facilitate applications to other problems concerning optimum tari¤s and tari¤ war.

Section 2 presents the model. Section 3 simpli…es it for tractability. Section 4 explicitly derives the optimum tari¤ set by one of the government under the assumption that the other government is committed to either free trade or a constant tari¤ rate. Section 5 formulates a tari¤-setting game and explicitly derives a Markov-perfect pair of strategies to characterize. Section 6 compare the welfare level of each country among autarchic, free-trade, unilateral and bilateral optimum-tari¤ equilibria. Section 7 concludes. The appendices show

[1] As we show in the Appendix, Markov-perfect strategies are obtained by solving the Hamilton-Jacobi-Bellman (HJB) equation which is a di¤erential equation. If there are multiple state variables, the HJB equation becomes a partial di¤erential equation in which the solution is a function of multiple state variables. It is well known that partial di¤erential equations are, in general, very di¤cult to solve.

an alternative method of arriving at the same solution.

2. The Model

Our dynamic model is based on the one developed by Shimomura (1993) which investigated the implications of time preference for the trade pattern. There are two countries, the home and the foreign, both producing two tradable goods, rice and cars. Rice is non-durable, while cars are durable. Both goods are pure consumption goods, that is, neither is a factor of production. Each car provides one unit of service flow per unit of time. Newly produced cars are tradable, but existing cars are internationally immobile, just like the investment good in Oniki and Uzawa (1965). There exists a domestic rental market for existing cars in each country. All markets are perfectly competitive.

The (infinitely-lived and perfectly farsighted) representative household in the home country maximizes the discounted sum of utility

$$\int_0^\infty u(c; B)\exp(-\rho t)dt \tag{1}$$

subject to

$$\dot{A} = Y(pT) + r(A - B) - pTc - \delta A + \Delta \tag{2}$$

where c is the consumption of rice, B is the service flow from the cars that the household uses, A is the stock of cars owned by the household (implying that $A - B$ is the quantity of cars the household leases to other households), r is the rental rate of a car, p is the world price of rice in terms of car, pT is the corresponding internal (i.e., domestic) price, ρ is the rate of time preference, δ is the rate of depreciation of cars, $Y(pT)$ is the GNP function, and Δ is the lumpsum transfer of the tariff revenue to each household. The felicity function $u(c; B)$ is twice-differentiable, strictly increasing, and concave in c and B. We define $T \equiv 1 + \tau$, where τ is the ad valorem tariff on rice imposed by the home government.

There is a large number of households, each treating Δ, T; p and r, as known and given functions of time. (Strictly speaking, we have a continuum of identical households, and the measure of this continuum is one). Clearly, in equilibrium, since all households in the home countries are identical, we have $A - B = 0$ and

$$\Delta = p(T - 1)(c - Y_P(pT))$$

where $Y_P(pT) \equiv \frac{d}{d(pT)} Y(pT)$ is the output of rice in the home country.

Notice that $Y(pT) + r(A - B) + \Delta - pTc$, being the excess of the household's income over its expenditure on rice, represents its purchase of new cars. (It is implicit in this formulation is that there is no international borrowing; thus trade balance is zero at each point of time.)

We now derive the necessary conditions for the household's optimization problem.

Define the Hamiltonian as

$$H = u(c; B) + \lambda[Y(pT) + r(A - B) - pTc - \delta A + \Delta] \tag{3}$$

where λ is the co-state variable. The optimality conditions are

$$\frac{\partial H}{\partial c} = u_c(c; B) - \lambda pT = 0 \tag{4}$$

$$\frac{\partial H}{\partial B} = u_B(c; B) - \lambda r = 0 \tag{5}$$

$$\dot{\lambda} = (\rho + \delta - r)\lambda \tag{6}$$

and the transversality condition is

$$\lim_{t \to \infty} A(t)\lambda(t)\exp(-\rho t) = 0 \tag{7}$$

A parallel formulation applies to the foreign country. Indicating the variables and functions belonging to the foreign country by means of an asterisk (*), we have

$$\dot{A}^* = Y^*(pT^*) + r^*(A^* - B^*) - pT^*c^* - \delta^* A^* + \Delta^*; \tag{8}$$

where $\Delta^* \equiv p(T^* - 1)(c - Y_P^*(pT^*)) = p(1 - T^*)(Y_P^*(pT^*) - c)$ can be interpreted as the foreign country's revenue from taxing its rice exports (see the remark below).

Remark 1: The following observations concerning the tariffs may be useful. To fix ideas, suppose the home country imports rice and exports cars. The world price of rice in terms of cars (i.e., the inverse of the home country's terms of trade) is p. The home country imposes an ad valorem tariff rate $\tau > 0$ on rice. Hence the home country's internal price ratio is $p_{int} = p(1+\tau) > p$. This causes home consumers to substitute away from rice. The foreign country, which imports cars, can impose an ad valorem tariff on cars. But, by Lerner's symmetry theorem[2], this is equivalent to the foreign country taxing its exports of rice. Since p is the international price of rice (in terms of cars), an export tax on rice at the rate $\varepsilon > 0$ will result in an internal price ratio (in the foreign country) of $p^*_{int} = p(1-\varepsilon) \equiv pT^* < p$. (For example, if $p = 100$ and $\varepsilon = 20\%$, then rice producers in the foreign country will get a net revenue of 80 for each unit of rice they export; if they sell in the internal market, they also get $p^*_{int} = 80$. In what follows, we will refer to T and T^* as the home and foreign tariff factors.

The necessary conditions for the optimization problem of the rep-

[2] According to Lerner's symmetry, an ad valorem export tax at the rate ε on the world price of the exported good is equivalent to an ad valorem tariff at the rate $\Omega = \varepsilon/(1-\varepsilon)$ on the world price of the imported good. See, for example, Vousden (1991). For example, let the nominal world prices of rice and cars be $\Pi^R_W = \$100$ and $\Pi^C_W = \$10$. The foreign country exports rice, and imposes an export tax of 20% (i.e. $\varepsilon = .2$). The internal price of rice is then $\Pi^R_{int} = (1-.20)\Pi^R_W = \$80$; and the internal price of cars is $\Pi^C_{int} = \$10$. The internal relative price of rice is $p^*_{int} = (1-\varepsilon)\Pi^R_W/\Pi^C_W = 8$. Alternatively, the foreign country can impose a tariff Ω on imported cars. The internal price of car is then $\Pi^C_{int} = (1+\Omega)\Pi^C_W$ and the internal price of car is $\Pi^R_{int} = \Pi^R_W = \$100$. The internal relative price of rice is $p^*_{int} = \Pi^R_W/[\Pi^C_W(1+\Omega)] = 8$ if and only if $(1-\varepsilon) = 1/(1+\Omega)$, i.e., $\Omega = \varepsilon/(1-\varepsilon) = 0.25$.

resentative foreign household are

$$\frac{\partial H^*}{\partial c^*} = u^*_c(c^*; B^*) - \lambda^* p T^* = 0 \tag{9}$$

$$\frac{\partial H^*}{\partial B^*} = u^*_B(c^*; B^*) - \lambda^* r^* = 0 \tag{10}$$

$$\dot{\lambda}^* = (\rho^* + \delta^* - r^*)\lambda^* \tag{11}$$

$$\lim_{t \to \infty} A^*(t)\lambda^*(t)\exp(-\rho^* t) = 0 \tag{12}$$

Finally, the market-clearing conditions are

$$c + c^* = Y_P(pT) + Y_P(pT^*) \tag{13}$$

$$A - B = 0 \tag{14}$$

$$A^* - B^* = 0 \tag{15}$$

Note that we assume that there is no world rental market for existing cars. The system, (2) and (4)-(15), constitutes the model of this paper. Equation (9) says that the world market for rice is in equilibrium. Since the trade balance is zero at each point of time, equilibrium in the market for rice implies equilibrium in the market for new cars, by Walras' law.

3. Simplifying the Model

The model just presented is quite complicated. To obtain sharp results, we must make a number of simplifying assumptions. First, we assume that the outputs of rice and cars are constant in both countries, and we denote them by $\alpha > 0$ (resp. $\alpha^* > 0$) and $\beta > 0$ (resp. $\beta^* > 0$) The GNP functions are thus $Y(p) = \alpha p + \beta$ and $Y^*(p) = \alpha^* p + \beta^*$. This is the case if each good is produced by industry-specific factors alone. Second, there is no international difference in the rate of depreciation, i.e., $\delta = \delta^* > 0$. Third, the felicity function of the representative agent in each country is assumed to be quasi-linear

$$u(c; B) = \eta \ln c + B \tag{16}$$

$$u^*(c^*; B^*) = \eta^* \ln c^* + B^* \tag{17}$$

From these assumptions, the model can be rewritten as follows

$$\dot{A} = pT(\alpha - c) + \beta - \delta A + \Delta \tag{18}$$

$$\dot{A}^* = pT^*(\alpha^* - c^*) + \beta^* - \delta A^* + \Delta^* \tag{19}$$

$$\dot{\lambda} = (\rho + \delta)\lambda - 1 \tag{20}$$

$$\dot{\lambda}^* = (\rho^* + \delta)\lambda^* - 1 \tag{21}$$

$$\frac{\eta}{c} - \lambda p = 0 \tag{22}$$

$$\frac{\eta^*}{c^*} - \lambda^* p = 0 \tag{23}$$

$$c + c^* = \alpha + \alpha^* \tag{24}$$

The dynamic system can be simplified further. Let $S(t) = A(t) + A^*(t)$ denote the sum of the two stocks of cars. First, from (17), (18) and (23), we have

$$\dot{S}(t) = \frac{d(A(t) + A^*(t))}{dt} = (\beta + \beta^*) - \delta S(t), \tag{25}$$

then the time path of $S(t)$ is independent of tariff policies and of preferences:

$$S(t) = S_\infty + (S_0 - S_\infty)e^{-\delta t}; \quad S_0 \equiv A_0 + A_0^*; \quad S_\infty \equiv \frac{\beta + \beta^*}{\delta} \tag{26}$$

Without loss, we can concentrate on the time profile of $A(t)$, because once $A(t)$ is known, we can obtain $A^*(t) = S_1 + (S_0 - S_1)e^{-\delta t} - A(t)$.

Second, (20) means that if $\lambda(0)$ and $\lambda^*(0)$ are chosen as $\frac{1}{\rho+\delta}$ and $\frac{1}{\rho^*+\delta}$, respectively, $\lambda(t) = \frac{1}{\rho+\delta}$ and $\lambda^*(t) = \frac{1}{\rho^*+\delta}$ for all $t > 0$. Since the Hamiltonian associated with the dynamic optimization by the representative household in the foreign country is concave with respect to c^*, B^* and A^* under (16) and (18), the standard sufficiency conditions[3] ensures us that the representative household behaves optimally under these choices of $\lambda(0)$ and $\lambda^*(0)$, respectively, provided $A(t)$ is bounded.

Let us therefore choose $\lambda(0) = \frac{1}{\rho+\delta}$ and $\lambda^*(0) = \frac{1}{\rho^*+\delta}$. Not only does this choice simplify the mathematical manipulations, it also makes much economic sense. If we think of $A(0)$ as the stock of wealth of the household, then $\lambda(0)$ measures the marginal contribution of wealth to the welfare of the household. That is, it measures how much additional welfare the household would gain if we give it an additional car. Since an additional car given at time 0 yields a service flow at the rate $e^{-\delta t}$ at time t, $(0 \le t \le \infty)$, and since one unit of service flow at t yields $e^{-\rho t}$ units of felicity, the present value of the stream of felicity obtained from the additional car is $\frac{1}{\rho+\delta}$.

With these choices, we derive from (12) the home and foreign demand functions for rice, $c(t) = \frac{(\rho+\delta)\eta}{p(t)T(t)} \equiv \frac{x(t)}{p(t)}$ and $c^*(t) = \frac{(\rho^*+\delta)\eta^*}{p(t)T^*(t)} \equiv \frac{x^*(t)}{p(t)}$, respectively[4], where $x \equiv \frac{\eta(\rho+\delta)}{T}$ and $x^* \equiv \frac{\eta^*(\rho^*+\delta)}{T^*}$. Substituting these expressions into the market clearing condition (23), we obtain

$$p(t) = \frac{x(t) + x^*(t)}{\alpha + \alpha^*}$$

This equation shows how the equilibrium terms of trade depend on (i) the tariff rates τ and τ^*, (ii) the rates of time preferences ρ and ρ^*, (iii) the depreciation rate δ, and (iv) the parameters η and η^* which are measures of the substitutability of rice for cars. Notice that the

[3] See, for example, Leonard and Long (1992, Chapter 9).

[4] Hence, the elasticities of demand for rice are unity in each country.

demand function for rice is "almost" static, it depends only on the current terms of trade $p(t)$, and the current tariff rates, and the rates of time preference and depreciation (the only intertemporal elements). Initial wealth does not appear in the function, nor does the output of cars. This, of course, is due to the assumption of a quasi-linear felicity function[5].

After substitution for p, c and Δ in (17), the differential equation for $A(t)$ becomes:

$$\dot{A} = \beta - \delta A - \frac{\alpha^* x}{\alpha + \alpha^*} + \frac{\alpha x^*}{\alpha + \alpha^*} \tag{27}$$

Let us assume that each government seeks to maximize the welfare of each household in its country. Then, the objective functionals of the governments are

$$\int_0^\infty \left[\eta \ln\left(\frac{(\alpha+\alpha^*)x}{x+x^*}\right) + A\right] e^{-\rho t}dt =$$
$$\int_0^\infty \left[\eta \ln\left(\frac{x}{x+x^*}\right) + A\right] e^{-\rho t}dt + \frac{\eta \ln(\alpha+\alpha^*)}{\rho} \tag{28}$$

and

$$\int_0^\infty \left[\eta^* \ln\left(\frac{(\alpha+\alpha^*)x^*}{x+x^*}\right) + A^*\right] e^{-\rho^* t}dt =$$
$$\int_0^\infty \left[\eta^* \ln\left(\frac{x^*}{x+x^*}\right) - A\right] e^{-\rho^* t}dt + \frac{\eta^* \ln(\alpha+\alpha^*)}{\rho^*} + \int_0^\infty S(t)e^{-\rho^* t}dt \tag{29}$$

where $S(t)$ is an exogenous time path, given by (24b).

[5] In a model of the optimal tariff on exhaustible resources, Kemp and Long (1980), using a quasi-linear utility function, obtain a similar result: the tariff rate is a constant. However, they show that, because of the exhaustibility of the resource, such a constant tariff rate is feasible only with precommitment, without which the time-inconsistency problem will arise. In the present model, there is no time-inconsistency.

4. The Optimum Tariff

Let us assume for the moment that the foreign government does not change the tariff rate τ^*. We can then easily obtain the optimum tariff rate of the home country. For a given x^*, the home optimum tariff rate can be obtained by solving the following simple dynamic optimization problem.

$$\max \int_0^\infty \left[\theta \ln \left(\frac{x}{x + x^*} \right) + A \right] e^{-\rho t} dt \tag{30}$$

subject to (25). Associated with this problem is the Hamiltonian

$$H^g = \theta \ln \left(\frac{x}{x + x^*} \right) + A + \mu \left[\beta - \delta A - \frac{\alpha^* x}{\alpha + \alpha^*} + \frac{\alpha x^*}{\alpha + \alpha^*} \right]$$

The first-order conditions and the transversality conditions are

$$\frac{\partial H^g}{\partial x} = \frac{\theta}{x} - \frac{\theta}{x + x^*} - \frac{\alpha^* \mu}{\alpha + \alpha^*} = 0 \tag{31}$$

$$\dot{\mu} = (\rho + \delta)\mu - 1 \tag{32}$$

$$\lim_{t \to \infty} A(t) e^{-\rho t} \mu(t) = 0 \tag{33}$$

Since

$$\frac{\partial^2 H^g}{\partial x^2} = -\frac{\theta(2x + x^*)}{x(x + x^*)^2} < 0;$$

the Hamiltonian H^g is strictly concave in x and concave in x and A. Therefore, sufficiency conditions for optimality are assured, and we have the following lemma.

Lemma 1: The optimal solution to the above problem is as follows. First, $\mu(t)$ is constant over time,

$$\mu(t) = \frac{1}{\rho + \delta} \quad \text{for any } t \geq 0 \tag{34}$$

Second, $x(t)$ is also constant overtime and solves the equation

$$\frac{\theta}{x} - \frac{\theta}{x + x^*} - \frac{\alpha^{*}1}{\alpha + \alpha^*} = \frac{\theta}{x} - \frac{\theta}{x + x^*} - \frac{\alpha^*}{(\alpha + \alpha^*)(\rho + \delta)} = 0 \quad (35)$$

so that

$$x = \frac{1}{2}\left[-x^* + \sqrt{(x^*)^2 + \frac{4x^*\theta(\rho + \delta)(\alpha + \alpha^*)}{\alpha^*}} \right] \quad (36)$$

Third, for any initial condition $A(0)$, $A(t)$ monotonely converges to

$$A_\infty = \frac{1}{\delta}\left[\beta - \frac{\alpha^* x}{\alpha + \alpha^*} + \frac{\alpha x^*}{\alpha + \alpha^*} \right] \quad (37)$$

Based on the lemma, we derive the optimum tariff rate.

Remark 2: Strictly speaking, we must require that A_∞ be non-negative. This means that β must be sufficiently great.

Proposition 1: For a given foreign tariff rate T^*, the home optimum tariff rate is

$$T^{op} = \frac{1 + \sqrt{1 + 2(1 + D)CT^*}}{2(1 + D)} \quad (38)$$

where $C \equiv \frac{2\theta(\rho+\delta)}{\theta^*(\rho^*+\delta)}$ and $D \equiv \frac{\alpha}{\alpha^*}$. It follows from (35) that the home optimum tariff rate rises if C and T^* increase and/or D decreases.

Proof: Omitted.

Remark 3: Formula (35) makes sense. For example, if the two countries are identical in preferences ($\theta = \theta^*$ and $\rho = \rho^*$ so that $C = 2$) and if the foreign country adopts free trade ($T^* = 1$); then the home country's tariff on rice will be positive if $\alpha < \alpha^*$: If α increases toward α^*, the tariff will fall steadily to zero (i.e., T falls toward 1). If the home rate of time preference rises, then the tariff on rice will also rise. More generally, we can interpret (35) as the home country's "reaction function". Recall that $T = (1 + \tau)$ and $T^* = (1 - \varepsilon)$ where ε is the foreign country's export tax on rice (or, in view of Lerner's symmetry, $\varepsilon/(1 - \varepsilon)$ is the foreign tariff on cars). As ε increases, the optimal τ decreases; thus the reaction function $\tau = \tau(\epsilon)$ is downward sloping. The absolute value of this slope is $\left\|\frac{d\tau}{d\epsilon}\right\| = \frac{C}{2\sqrt{1+2(1+D)CT^*}} < 1$; if $C < 2\sqrt{1+2(1+D)CT^*}$.

5. A Markov-perfect Equilibrium Pair of Tariff Strategies

Now let us assume that both governments play a tariff game. The strategy space from which each country can choose its strategy is restricted to the set of functions which map the current values of the state variables $(A(t); A^*(t))$ to the set of positive real numbers, which are the tariff factors $T(t)$ and $T^*(t)$. This restriction means that we do not allow punishments that are based on the full history of the game. In particular, trigger strategies[6] (which say that if "you deviate from cooperative behaviour, I will punish you, and if you continue to cooperate, I will cooperate") are not permitted. Restricting strategies this way amounts to assuming that the only manner in which history can affect the present is through the "summary statistics" $(A(t); A^*(t))$. Basically, we do not allow strategies that condition current actions on the full history because we know, from the folk theorem on infinitely repeated games, that, if the rate of discounts are sufficiently small, almost any outcome (including full cooperation) can be sustained as Nash equilibria. We think that this is quite unrealistic, in view of the observed absence of completely free trade in the world.

First, consider the home government. The maximization problem it faces is

$$\max \int_0^\infty \left[\theta \ln\left(\frac{x}{x+x^*(A)}\right) + A\right] e^{-\rho t} dt \tag{39}$$

subject to

$$\dot{A} = \beta - \delta A - \frac{\alpha^* x}{\alpha+\alpha^*} + \frac{\alpha x^*(A)}{\alpha+\alpha^*} \tag{40}$$

[6] For a treatment of these strategies in differential games, see Dockner et al. (2000).

where $x^*(A)$ is the foreign tariff strategy. The Hamiltonian is

$$\overline{H}^g = \eta \ln\left(\frac{x}{x + x^*(A)}\right) + A + \nu\left[\beta - \delta A - \frac{\alpha^* x}{\alpha + \alpha^*} + \frac{\alpha x^*(A)}{\alpha + \alpha^*}\right] \quad (41)$$

The first-order conditions are

$$\frac{\eta}{x} - \frac{\eta}{x + x^*(A)} - \frac{\alpha^* \nu}{\alpha + \alpha^*} = 0 \quad (42)$$

and

$$\dot{\nu} = (\rho + \delta)\nu - 1 + \frac{\eta}{x + x^*(A)}\frac{d}{dA}x^*(A) - \frac{\nu\alpha}{\alpha + \alpha^*}\frac{d}{dA}x^*(A) \quad (43)$$

We can construct a parallel argument for the foreign country and derive

$$\frac{\eta^*}{x^*} - \frac{\eta^*}{x(A) + x^*} + \frac{\alpha \nu^*}{\alpha + \alpha^*} = 0 \quad (44)$$

$$\dot{\nu}^* = (\rho + \delta)\nu^* + 1 + \frac{\eta}{x + x^*(A)}\frac{d}{dA}x^*(A) + \frac{\nu^*\alpha^*}{\alpha + \alpha^*}\frac{d}{dA}x(A) \quad (45)$$

Thus, a Markov-perfect equilibrium strategy pair $(x(A), x^*(A))$ is that of constant strategies, $x(A) \equiv \overline{x}$ and $x^*(A) \equiv \overline{x}^*$, where $\overline{x}$ and $\overline{x}^*$ are positive, constant over time and independent of A. Substituting them into the above equations, we can derive $\overline{x}$ and $\overline{x}^*$ as the solutions to

$$\frac{1}{\overline{x}} - \frac{1}{\overline{x} + \overline{x}^*} = \frac{\alpha^* \nu}{\alpha + \alpha^*} = \frac{\alpha^*}{(\alpha + \alpha^*)\eta(\rho + \delta)} \quad (46)$$

$$\frac{1}{\overline{x}^*} - \frac{1}{\overline{x} + \overline{x}^*} = \frac{-\alpha \nu^*}{\alpha + \alpha^*} = \frac{\alpha}{(\alpha + \alpha^*)\eta^*(\rho^* + \delta)} \quad (47)$$

Note that if $\nu(0)$ and $\nu^*(0)$ are chosen as $1/(\rho + \delta)$ and $-1/(\rho^* + \delta)$, so are $\nu(t)$ and $\nu^*(t)$. Considering the definitions of $\overline{x}$ and $\overline{x}^*$, we have from (43) and (44)

$$\overline{T}^2 = \frac{1}{1 + D}\left[\frac{C}{2}\overline{T}^* + \overline{T}\right] \quad (48)$$

$$(\bar{T}^*)^2 = \frac{D}{1+D}\left[\frac{2}{C}\bar{T} + \bar{T}^*\right] \tag{49}$$

Solving for $\bar{T}$, we obtain

$$\bar{T} = \frac{1 + \sqrt{CD/2}}{1+D} \tag{50}$$

Proposition 2: A Markov-perfect equilibrium tariff pair is

$$(\bar{T}, \bar{T}^*) = \left(\frac{1 + \sqrt{\frac{\eta(\rho+\delta)\alpha}{\eta^*(\rho^*+\delta)\alpha^*}}}{1 + \frac{\alpha}{\alpha^*}}, \frac{1 + \sqrt{\frac{\eta^*(\rho^*+\delta)\alpha^*}{\eta(\rho+\delta)\alpha}}}{1 + \frac{\alpha^*}{\alpha}}\right)$$

Note that $\bar{T}$ increases if $\frac{\eta(\rho+\delta)}{\eta^*(\rho^*+\delta)}$ increases or if $\frac{\alpha}{\alpha^*}$ decreases.

Remark 4: The equilibrium tariff pair depends on the two rates of time preference, the rate of depreciation, the ratio of rice productions, and the parameters of substitution in consumption. Small changes in the parameters β and β^* do not affect the equilibrium tariff rates. (The role played by β and β^* is that they lie in a certain subset of R^2 to ensure that $A(t)$ and $A^*(t)$ are non-negative; they do affect the welfare levels, as can be seen from the value functions in the Appendix.)

6. The Welfare Effects of Equilibrium Tariffs

Based on the foregoing argument, we check how the imposition of tariff would affect the welfare level of each country.

First, assume that x and x^* are positive and constant. Solving the differential equation (25) with the initial condition $A(0)$ and substituting the solution $A(t; A(0))$ into (26) and (27), we can derive the home and foreign welfare levels as functions of x and x^*;

$$W(x; x^*) \equiv \frac{\eta}{\rho}\ln\left[\frac{(\alpha+\alpha^*)x}{x+x^*}\right] - \frac{1}{\rho(\rho+\delta)}\left(\frac{\alpha^* x - \alpha x^*}{\alpha+\alpha^*}\right) + \frac{\beta}{\rho(\rho+\delta)} + \frac{A(0)}{\rho+\delta} \tag{51}$$

and

$$W^*(x; x^*) \equiv \frac{\eta^*}{\rho^*}\ln\left[\frac{(\alpha+\alpha^*)x^*}{x+x^*}\right] - \frac{1}{\rho^*(\rho^*+\delta)}\left(\frac{\alpha x^* - \alpha^* x}{\alpha+\alpha^*}\right) + \frac{\beta^*}{\rho^*(\rho^*+\delta)} + \frac{A^*(0)}{\rho^*+\delta} \tag{52}$$

Figure 1 shows the indifference map of $W(x; x^*)$[7]. Note that the locus $W_x(x; x^*) \equiv \frac{\partial}{\partial x} W(x; x^*) = 0$ is the home reaction curve and that the intersection of the two loci, $W_x(x; x^*) = 0$ and $W_{x^*}(x; x^*) \equiv \frac{\partial}{\partial x^*} W(x; x^*) = 0$, is a saddle point. Thus, there are two indifference curves both of which cross the intersection of $W_x(x; x^*) = 0$ and $W_{x^*}(x; x^*) = 0$, say Q. Since for any $x^* > 0$ $W(\frac{\alpha}{\alpha^*} x^*; x^*) = W_A$; one of the two indifference curves is $x = \frac{\alpha}{\alpha^*} x^*$: The other indifference curve is depicted as mm': A parallel observation can be made for the foreign welfare function $W^*(x; x^*)$.

Second, let us consider the autarchic welfare level of each country. Since $\alpha = c$ and $\Delta = 0$, (1) and (2) are simplified as

$$W^A \equiv \int_0^\infty [\eta \ln(\alpha) + A] \exp(-\rho t) dt \qquad (53)$$

and

$$\dot{A} = \beta - \delta A \qquad (54)$$

We can derive

$$W_A = \frac{\eta}{\rho} \ln \alpha + \frac{\beta}{\rho(\rho + \delta)} + \frac{A(0)}{\rho + \delta}$$

as the autarchic welfare level of the home country. Similarly, we derive

$$W_A^* = \frac{\eta^*}{\rho^*} \ln \alpha^* + \frac{\beta^*}{\rho^*(\rho^* + \delta)} + \frac{A^*(0)}{\rho^* + \delta}$$

as the autarchic welfare level of the foreign country. One can verify that $W(x; x^*) = W_A$ and $W^*(x; x^*) = W_A^*$ for any point on the line $\alpha^* x = \alpha x^*$ and mm' in Figure 1.

Making use of $W(x; x^*)$; $W^*(x; x^*)$; W_A and W_A^*, we shall compare the welfare level of each country among the following four cases.

[7] To save space, we do not explain how to depict the figures that appear in this section, which is available from the authors on request.

(i) Autarchy

(ii) Free trade: $T = T^* = 1$, which mean that $x = \eta(\rho + \delta)$ and $x^* = \eta^*(\rho^* + \delta)$

(iii) Unilateral optimum tariff: $T^* = 1$ and $T = T^{op}$, which mean that $x = \eta(\rho + \delta)/T^{op}$ and $x^* = \eta^*(\rho^* + \delta)$

(iv) Bilateral optimum tariffs (a Markov-perfect equilibrium pair of tariff strategies): $T = \bar{T}$ and $T^* = \bar{T}^*$, which mean that $x = \eta(\rho + \delta)/\bar{T}$ and $x^* = \eta^*(\rho^* + \delta)/\bar{T}^*$

6.1. The symmetric case: $\alpha = \alpha^*$, $\rho = \rho^*$ and $\eta = \eta^*$

First of all, let us consider the symmetric world in which $\alpha = \alpha^*$, $\rho = \rho^*$ and $\eta = \eta^*$: (48) becomes

$$W(x; x^*) \equiv \frac{\eta}{\rho}\ln\left[\frac{2\alpha x}{x + x^*}\right] - \frac{x - x^*}{2\rho(\rho + \delta)} + \frac{\beta}{\rho(\rho + \delta)} + \frac{A(0)}{\rho + \delta} \qquad (55)$$

and

$$W^*(x; x^*) \equiv \frac{\eta}{\rho}\ln\left[\frac{2\alpha x^*}{x + x^*}\right] - \frac{x^* - x}{2\rho(\rho + \delta)} + \frac{\beta^*}{\rho(\rho + \delta)} + \frac{A^*(0)}{\rho + \delta} \qquad (56)$$

Since $\alpha = \alpha^*$, the 45°-line $x = x^*$ in the $(x; x^*)$- plane is an indifference curve the welfare level of which is equal to the autarchic level $W_A (= W_A^*)$. And since $\eta(\rho + \delta) = \eta^*(\rho^* + \delta)$, and since Proposition 1 and Proposition 2 ensure us that $T^{op} = 1$, and $\bar{T} = \bar{T}^* = 1$ in the symmetric world, we see that the welfare level of each country is the same among the four cases (i)-(iv).

6.2. The case in which the two countries differ only in α and α^*

If $\alpha \neq \alpha^*$ while $\eta = \eta^*$ and $\rho = \rho^*$, i.e., if only the output of rice differs between the two countries, (48) can be rewritten as

$$W(x; x^*) \equiv \frac{\eta}{\rho}\ln\left[\frac{(\alpha + \alpha^*)x}{x + x^*}\right] - \frac{1}{\rho(\rho + \delta)}\left(\frac{\alpha^* x - \alpha x^*}{\alpha + \alpha^*}\right) + \frac{\beta}{\rho(\rho + \delta)} + \frac{A(0)}{\rho + \delta} \qquad (57)$$

and

$$W^*(x;x^*) \equiv \frac{\eta}{\rho}\ln[\frac{(\alpha+\alpha^*)x^*}{x+x^*}] - \frac{1}{\rho(\rho+\delta)}\left(\frac{\alpha x^* - \alpha^* x}{\alpha+\alpha^*}\right) + \frac{—^*}{\rho(\rho+\delta)} + \frac{A^*(0)}{\rho+\delta} \quad (58)$$

Figure 2A depicts the reaction curves of the two countries under $\alpha > \alpha^*$. Inspection of the figure exhibits that

1_A The free-trade equilibrium (point E_A) and tariff-ridden equilibrium points (unilateral optimum tariff points F_A and F_A^* and bilateral optimum tariff point G_A) are Pareto-superior to the autarchic equilibrium for each country. Thus, trade gains are confirmed, irrespective of free trade or tariff-ridden trade[8].

2_A The unilateral optimum tariff point F_A (resp. F_A^*) makes the home (resp. foreign) country better off and the foreign (resp. home) country worse off, compared with the free trade and bilateral tariff points E_A and G_A.

3_A The free-trade equilibrium point E_A and all the three tariff-ridden equilibrium points G_A, F_A and F_A^* are below the line $\alpha^* x = \alpha x^*$, i.e., $\alpha^*/\alpha < x^*/x = \frac{\eta(\rho+\delta)/\sqrt{T^*}}{\eta(\rho+\delta)/\sqrt{T}} = c^*/c$; which means that the home country exports rice and imports cars[9]. x is greater than $\eta(\rho+\delta)$ at points G_A and F_A, while x^* is smaller than $\eta(\rho+\delta)$ at points G_A and F_A^*, which implies that $T < 1$ at G_A and E_A and $T^* > 1$ at G_A and E_A^*. Thus, the home and foreign countries imposes import tariffs on cars and rice, respectively.

What remains is the welfare comparison between the free-trade equilibrium (point E_A) and the bilateral optimum tariff equilibrium

[8] Inspection of Figure 1 reveals that any indifference curve in the area PQm' has a higher value of W than the indifference curves mm' and OP. We can make a parallel argument for the comparison of the level between W^*.and W_A^*:

[9] The world market-clearing condition, $c + c^* = \alpha + \alpha^*$, can be rewritten as $(1+\frac{c^*}{c})/(1+\frac{\alpha^*}{\alpha}) = \alpha/c$. Hence $\alpha^*/\alpha < c^*/c$ is equivalent to $\alpha - c > 0$.

(point G_A). Let us consider the sign of

$$
\begin{aligned}
&\frac{\rho}{\eta}\left[W(\eta(\rho+\delta);\eta(\rho+\delta)) - W(\bar{x};\bar{x}^*)\right]\\
=\;& \ln\left[\frac{1}{2}\left(1+\sqrt{\frac{\alpha^*}{\alpha}}\right)\right] - \frac{\sqrt{\alpha}-\sqrt{\alpha^*}}{\sqrt{\alpha}+\sqrt{\alpha^*}} + \frac{\alpha-\alpha^*}{\alpha+\alpha^*}\\
\equiv\;& G^A(\alpha;\alpha^*)
\end{aligned}
$$

For any given $\alpha^* > 0$, we observe that $G^A(0;\alpha^*) = \infty$; $G^A(\alpha^*;\alpha^*) = 0$ and $G^A(\infty;\alpha^*) = -\ln 2 < 0$. The partial differentiation of $G^A(\alpha;\alpha^*)$ with respect to α yields

$$
G^A_\alpha(\alpha;\alpha^*) \equiv \frac{\partial}{\partial\alpha}G^A(\alpha;\alpha^*) = -\frac{1}{2}\frac{\alpha^*\left(\sqrt{\alpha}-\sqrt{\alpha^*}\right)R^A(\sqrt{\alpha};\sqrt{\alpha^*})}{\alpha^3\left(\sqrt{\alpha}+\sqrt{\alpha^*}\right)^2(\alpha+\alpha^*)^2}
$$

where

$$
R^A(\sqrt{\alpha};\sqrt{\alpha^*}) \equiv -\left(\sqrt{\alpha^*}\right)^4 - 4\left(\sqrt{\alpha^*}\right)^3\left(\sqrt{\alpha}\right) - 2\left(\sqrt{\alpha^*}\right)^2\left(\sqrt{\alpha}\right)^2 + 3\left(\sqrt{\alpha}\right)^4
$$

Thus,

$$
G^A_\alpha(\alpha^*;\alpha^*) = 0
$$

Moreover,

$$
\left.\frac{\partial^2}{\partial\alpha^2}G^A(\alpha;\alpha^*)\right|_{\alpha=\alpha^*} = \frac{1}{4\alpha^*} > 0
$$

Next, let us focus on $R^A(\sqrt{\alpha};\sqrt{\alpha^*})$. First, $R^A(0;\sqrt{\alpha^*}) = -\left(\sqrt{\alpha^*}\right)^4$ and $R^A(\sqrt{\alpha^*};\sqrt{\alpha^*}) = -4\left(\sqrt{\alpha^*}\right)^4$. Second, considering Descartes' rule of sign, it follows that there exists a unique positive real number $\sqrt{\Theta} > \sqrt{\alpha^*}$ such that

$$
R^A(\xi;\sqrt{\alpha^*}) \begin{pmatrix} < \\ = \\ > \end{pmatrix} 0 \quad \text{for any } \xi \begin{pmatrix} \in (0;\sqrt{\Theta}) \\ = \sqrt{\Theta} \\ \in (\sqrt{\Theta};\infty) \end{pmatrix}
$$

Based on the foregoing analysis, we can depict the graph of $G^A(\alpha;\alpha^*)$ like Figure 3. There exists a unique $\hat{\alpha} > \alpha^*$ such that

$$G^A(\alpha;\alpha^*) \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \text{ for any } \alpha \begin{pmatrix} \in (0;\hat{\alpha}) - \{\alpha^*\} \\ = \alpha^* \text{ and } \hat{\alpha} \\ \in (\hat{\alpha};\infty) \end{pmatrix}$$

Thus we derive the following result.

4_A There is a threshold for the parameter value $\hat{\alpha} > \alpha^*$ such that if α is not equal to α^* and smaller (resp. larger) than $\hat{\alpha}$; the free-trade equilibrium is better (worse) than the bilateral optimum-tariff equilibrium for the home country. A parallel statement can be made for the foreign country. That is, free trade is better than tariff-ridden trade to each country unless the output of rice in the country is sufficiently greater than that of the other country.

6.3. The case in which the two countries differ only in ρ and ρ^*

If $\rho \neq \rho^*$ while $\alpha = \alpha^*$and $\eta = \eta^*$, i.e., if only the rate of time preference differs between the two countries, (48) becomes

$$W(x;x^*) \equiv \frac{\eta}{\rho}\ln[\frac{2\alpha x}{x+x^*}] - \frac{x-x^*}{2\rho(\rho+\delta)} + \frac{\beta}{\rho(\rho+\delta)} + \frac{A(0)}{\rho+\delta} \quad (59)$$

and

$$W^*(x;x^*) \equiv \frac{\eta}{\rho}\ln[\frac{(\alpha+\alpha^*)x^*}{x+x^*}] - \frac{x^*-x}{2\rho(\rho+\delta)} + \frac{\beta^*}{\rho(\rho+\delta)} + \frac{A^*(0)}{\rho+\delta} \quad (60)$$

Figure 2B depicts the reaction curves of the two countries under $\rho > \rho^*$. Inspection of the figure reveals that

1_B The free-trade equilibrium (point E_B) as well as the tariff-ridden equilibrium points (unilateral optimum tariff points F_B and F_B^* and bilateral optimum tariff point G_B) are Pareto-superior to the autarchic equilibrium at which the welfare levels are $W(x, x)$ and $W^*(x^*, x^*)$ for the respective countries. Trade gains are confirmed, both for free trade and for tariff-ridden trade.

2. The unilateral optimum tariff point F_0 (resp. F_0^*) makes the home (resp. foreign) country better off and the foreign (resp. home) country worse off, compared with the free trade and bilateral tariff points E_0 and G_0.

3. All the three tariff-ridden equilibrium points and free-trade equilibrium points are above the line $x = x^*$, which means that the home country (=the more impatient country) exports cars and imports rice. x is smaller than $\eta(\rho+\delta)$ at points G_0 and F_0, while x^* is greater than $\eta(\rho+\delta)$ at points G_0 and F_0^*, which implies that $T > 1$ at G_0 and F_0 and $T^* < 1$ at G_0 and F_0^*. Thus, the home and foreign countries impose import tariffs on rice and cars, respectively.

What remains is the welfare comparison between the free-trade equilibrium (point E_0) and the bilateral optimum tariff equilibrium (point G_0). Considering that $(x, x^*) = (\eta(\rho+\delta), \eta(\rho^*+\delta))$ at point E_0, and recalling the definition of $(\bar{T}, \bar{T}^*)$,

$$(\bar{x}, \bar{x}^*) = \left(\frac{2\eta(\rho+\delta)\sqrt{\rho^*+\delta}}{\sqrt{\rho+\delta}+\sqrt{\rho^*+\delta}}, \frac{2\eta(\rho^*+\delta)\sqrt{\rho+\delta}}{\sqrt{\rho+\delta}+\sqrt{\rho^*+\delta}} \right),$$

we obtain

$$\begin{aligned} & \frac{\rho}{\eta}[W(\eta(\rho+\delta), \eta(\rho^*+\delta)) - W(\bar{x}, \bar{x}^*)] \\ = \; & \ln \left(\frac{\sqrt{\rho+\delta}(\sqrt{\rho+\delta}+\sqrt{\rho^*+\delta})}{(\rho+\delta)+(\rho^*+\delta)} \right) + \frac{\{(\rho+\delta)+(\rho^*+\delta)\}(\sqrt{\rho^*+\delta}-\sqrt{\rho+\delta})}{(\rho+\delta)(\sqrt{\rho^*+\delta}+\sqrt{\rho+\delta})} \\ \equiv \; & G^B(\rho+\delta, \rho^*+\delta) \end{aligned}$$

Making virtually the same argument as we did for $G^A(\rho+\delta;\rho^*+\delta)$, we can show that there exist a parameter value $\hat{\rho}$ $(>\rho^*)$ such that

$$G^B(\rho+\delta;\rho^*+\delta)\left(\begin{array}{c} > \\ = \\ < \end{array}\right)0 \text{ for any } \rho\left(\begin{array}{c} \in(0,\hat{\rho})-\{\rho^*\} \\ =\rho^* \text{ and } \hat{\rho} \\ \in(\hat{\rho},\infty) \end{array}\right)$$

We arrive at the following result

4_B There is a threshold for the parameter value $\hat{\rho}>\rho^*+\delta$ such that if ρ is not equal to ρ^* and is smaller (resp. larger) than $\hat{\rho}$, the free-trade equilibrium is better (worse) than the bilateral optimum-tariff equilibrium for the home country. A parallel statement can be made for the foreign country. That is, free trade is better than tariff-ridden trade to each country unless the output of rice in the country is sufficiently greater than that of the other country.

Finally, we note that the case in which the two countries differ only in η and η^* is parallel to the present case in the sense that there exists $\hat{\eta}$ that plays the same role as $\hat{\rho}$. The exercise is left to the reader.

7. Concluding Remarks

We have been able to derive explicitly a bilateral optimum-tariff equilibrium pair of strategies for a dynamic game model of tariff-war in a two(-country) by two(-good) dynamic general equilibrium model where assets are accumulated.

The model is subject to a number of restrictive assumptions and specifications. In particular, the assumption of quasi-linear felicity functions is crucial for obtaining the equilibrium pair of strategies. However, to our knowledge, this paper is the first attempt to derive explicitly a feedback-Nash equilibrium solution to a dynamic tariff-setting game and therefore we believe that it makes some contribution to the trade literature on tariff in the sense that it opens a door to a more general dynamic analysis of tariff war.

We would also like to emphasize that the rates of time preference of both countries play an important role not only in the determination of the equilibrium tariff pair but also in the welfare comparison between the free-trade equilibrium and the bilateral optimum-tariff equilibrium for each country. These are in sharp contrast with static models.

APPENDIX

Appendix A1: Optimal Tariff (The HJB approach)

Assume that the foreign tariff is fixed, so that x^* is given. The home country's task is to find the time path $x(t)$ which maximizes

$$\int_0^\infty [\eta \ln(x) + \eta \ln(x + x^*) + A] e^{-\rho t} dt$$

subject to

$$\dot{A} = \beta - \delta A - \gamma x + \theta x^* \qquad (61)$$

where

$$\gamma \equiv \frac{\alpha^*}{\alpha + \alpha^*} \text{ and } \theta \equiv \frac{\alpha}{\alpha + \alpha^*}$$

and

$$A(0) = A_0; \qquad A(t) \geq 0.$$

The Hamilton-Jacobi-Bellman (HJB) equation for this problem is

$$\rho V(A) = \max_x [\eta \ln(x) + \eta \ln(x + x^*) + A + V'(A)(\beta - \delta A - \gamma x + \theta x^*)]$$

Let us try the functional form $V(A) = QA + W$, where Q and W are constants to be determined. Then the HJB equation becomes

$$\rho QA + \rho W = \max_x [\eta \ln(x) + \eta \ln(x + x^*) + A + Q(\beta - \delta A - \gamma x + \theta x^*)] \qquad (62)$$

This gives the first-order condition

$$\frac{1}{x} - \frac{1}{x + x^*} = \frac{\gamma Q}{\eta} \equiv \frac{1}{z} \tag{63}$$

Solving (A3), taking the positive root,

$$x = \frac{1}{2}\left[-x^* + \sqrt{(x^*)^2 + 4zx^*} \right] \equiv \hat{x} \tag{64}$$

Substituting (4) into (2),

$$[(\rho + \delta)Q - 1]\,A + \rho W = \eta \ln\left(\frac{\hat{x}}{\hat{x} + x^*}\right) + Q(\beta - \gamma\hat{x} + \theta x^*)$$

Since this equation must hold for all $A \geq 0$, it follows that

$$Q = \frac{1}{\rho + \delta} \tag{65}$$

and

$$W = \frac{1}{\rho}\left[\eta \ln\left(\frac{\hat{x}}{\hat{x} + x^*}\right) + \frac{1}{\rho + \delta}(\beta - \gamma\hat{x} + \theta x^*)\right]$$

Furthermore,

$$\hat{x} = \frac{1}{2}\left[-x^* + \sqrt{(x^*)^2 + \frac{4(\delta + \rho)(\alpha + \alpha^*)\eta}{\alpha^*}} \right] \tag{66}$$

$A(t)$ converges to

$$A_\infty \equiv \frac{1}{\delta}[\beta - \gamma\hat{x} + \theta x^*]$$

which is positive if β is sufficiently great. (We assume this.) The transversality condition is satisfied:

$$\lim_{t \to \infty} e^{-\rho t} V(A(t)) = 0.$$

Appendix A2: Markov-Perfect Nash Equilibrium (The HJB approach)

Now we look at the optimization problem of the foreign country. It seeks the time path $x^*(t)$ which maximizes

$$\int_0^\infty [\eta^* \ln(x^*) + \eta^* \ln(x + x^*) + A^*] e^{-\rho^* t} dt \tag{67}$$

where

$$A^*(t) = S(t) - A(t) \tag{68}$$

and $S(t)$ is the sum of the two stocks of cars, its time path given by

$$S(t) = S_\infty + (S_0 - S_\infty)e^{-\delta t}; \qquad S_0 = A_0 + A_0^*; \qquad S_\infty = \frac{\beta + \beta^*}{\delta} \tag{69}$$

The maximization is subject to

$$\dot{A} = \beta - \delta A - \gamma x(A) + \theta x^*$$

Substituting (8) and (9) into (7), the foreign objective function reduces to

$$\int_0^\infty [\eta^* \ln(x^*) + \eta^* \ln(x + x^*) - A] e^{-\rho^* t} dt + \Psi$$

where Ψ is a constant:

$$\Psi = \int_0^\infty \left[\frac{\beta + \beta^*}{\delta} + \left(A_0 + A_0^* - \frac{\beta + \beta^*}{\delta}\right) e^{-\delta t} \right] e^{-\rho^* t} dt$$

Since Ψ is a constant, it can be ignored, and we can write the HJB equation for the foreign country as

$$\rho^* V^*(A) =$$

$$\max_{x^*} \left[\eta^* \ln(x^*/(x(A) + x^*)) - A + V^{*\prime}(A)(\beta - \delta A - \gamma x(A) + \theta x^*) \right] \tag{70}$$

Similarly, the HJB equation for the home country is

$$\rho V(A) =$$

$$\max_x \left[\eta \ln(x/(x + x^*(A))) - \eta \ln + A + V'(A)(\beta - \delta A - \gamma x + \theta x^*(A))\right] \quad (71)$$

The equations (10) and (11) define the differential game. To solve this game, we conjecture that both countries choose a stock-independent strategy:

$$x(A) = \bar{x}\,; \qquad x^*(A) = \bar{x}^* \quad (72)$$

and we conjecture that

$$V(A) = QA + W; \qquad V^*(A) = Q^*A + W^*$$

Proceeding as in appendix A1, we find, for the home country

$$\bar{x} = \frac{1}{2}\left[-\bar{x}^* + \sqrt{(\bar{x}^*)^2 + \frac{4(\delta + \rho)(\alpha + \alpha^*)\eta}{\alpha^*}}\right] \quad (73)$$

and that, for the foreign country

$$\bar{x}^* = \frac{1}{2}\left[-\bar{x} + \sqrt{(\bar{x})^2 + \frac{4(\delta + \rho^*)(\alpha + \alpha^*)\eta^*}{\alpha}}\right] \quad (74)$$

The two equations simultaneously determine $\bar{x}$ and $\bar{x}^*$.

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