Discussion Paper Series

RIEB

Kobe University

DP2024-05

The Impact of Individual Loss Aversion on Market Risk-Return Trade-off: A Non-linear Approach*

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March 1, 2024

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The Impact of Individual Loss Aversion on Market Risk-Return Trade-off:

A Non-linear Approach[1]

Shoka Hayaki[2]

ABSTRACT

Traditional frameworks often fail to adequately explain the observed procyclical nature of the risk-return trade-off associated with aggregate risk aversion in recent years. This study introduces a simple model incorporating the concepts of loss aversion and state-dependent preferences. The model suggests an initial positive adjustment to the risk-return trade-off when the shock occurs, followed by a negative adjustment once the shock fully manifests. Essentially, the risk-return trade-off temporarily becomes procyclical as the shock spreads. In this study, the nonlinear structure of the risk-return trade-off is approximated using natural cubic splines with several constraints. Estimation results based on market excess returns in the United States indicate that a nonlinear risk-return trade-off, consistent with the model, offers valuable insights for pricing.

[1] I would like to thank Takashi Hatakeda, Naoya Mori, Takashi Yamasaki, Kohei Aono, Masaru Chiba, Kenya Fujiwara, Tokuo Iwaisako, Takahiro Miura, Kazuhiko Ohashi, Katsuhiko Okada, Kosuke Oya, Ryuta Sakemoto, Hidenori Takahashi, and Hirofumi Uchida for useful discussions. This work was supported by JSPS KAKENHI Grant Number JP22K20166.

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1. INTRODUCTION

Aggregate risk aversion, often assessed via the risk-return trade-off in the stock market, has traditionally been believed to display a countercyclical pattern in various asset pricing models. This notion is exemplified in models featuring state-dependent preferences, such as the external habit formation model proposed by Campbell and Cochrane (1999). Aggregate risk aversion can fluctuate in response to changes in the risk aversion levels of individuals within the economy, as well as shifts in the distribution of individual wealth (Chan and Kogan, 2002; Xiouros and Zapatero, 2010). Recent empirical evidence has supported this concept, particularly through studies such as those by Cohn et al. (2015) and Guiso et al. (2018).

However, recent empirical findings regarding the relationship between the risk-return trade-off and business cycles remain inconclusive. For instance, studies by Frazier and Liu (2016) and Alemany et al. (2023) emphasize the procyclicality of the risk-return trade-off. Additionally, research by Liu (2017) and Adrian et al. (2019) suggests that the association between the risk-return trade-off and business cycles is not simply binary but may exhibit a more complex non-linear structure. Despite this, there is a limited body of research addressing the mechanisms underlying variations in the risk-return trade-off under different economic conditions, including its nonlinear nature. This study aims to fill this gap in the literature.

This study focuses on loss aversion, a significant component of prospect theory (Kahneman and Tversky, 1979; Tversky and Kahneman, 1992), as a novel approach to understanding the variability of aggregate risk aversion within the state-dependent preference framework. Several consumption-based asset pricing models, incorporating reference dependence and loss aversion, have already been investigated. These include models with reference-dependent preferences (Kőszegi and Rabin, 2006, 2007) and those considering disappointment aversion (Gul, 1991; Routledge and Zin, 2010). Agents with state-dependent preferences focus their attention not on absolute consumption but on relative consumption, using exogenously given states (or habits) as a reference point. Essentially, reference dependence is inherently embedded in a state-dependent preference framework. However, the integration of loss aversion into this framework remains underexplored.

This study specifically addresses the optimization problem encountered by agents exhibiting pseudo-loss aversion. Upon identifying a decline in the state of the economy, these agents transition their utility function from a standard (gain-type) form to a loss-type utility function characterized by a relatively low utility level and high marginal utility of consumption. It is noteworthy that the state prompting the recognition of an economic downturn may differ among agents. When a shock occurs and a substantial number of agents newly recognize a deterioration in the state of the economy, their utility function shifts to a loss-type utility function with a relatively high marginal utility of consumption. This leads to a further increase in the aggregate marginal utility of the economy, resulting in a significant rise in the level of aggregate risk aversion. In essence, similar to numerous asset pricing models, this proposition suggests countercyclical variations in aggregate risk aversion.

The new model presented in this study explains not only the countercyclicality of aggregate risk aversion but also its procyclicality under specific conditions. In this model, the shift to a loss-type utility function essentially occurs only once. Thus, agents transitioning to a loss-type utility function at the onset of an economic downturn maintain this utility function even in the event of further deterioration in the state of the economy. With the worsening of the economy, the number of agents newly transitioning to a loss-type utility function will plateau and eventually start declining. This implies that the marginal utility increases at a slower rate or decreases at a faster rate as the economy deteriorates, potentially leading to a decline in aggregate risk aversion. In other words, aggregate risk aversion, or the risk-return trade-off, may exhibit procyclical variations.

This model's insights can be summarized into three hypotheses. The first hypothesis suggests that the risk-return trade-off follows a pattern of increase, decrease, and subsequent increase with the decrease in relative consumption. When shocks occur, and many agents acknowledge the "loss" stemming from a decline in relative consumption, there tends to be a positive correction in aggregate risk aversion. In contrast, after the shocks have fully propagated, there is potential for a negative correction in aggregate risk aversion. The second hypothesis suggests that the risk-return trade-off stabilizes under states of high relative consumption levels, where no one acknowledges losses, or low relative consumption levels, where everyone recognizes losses. In such extreme states, the mechanism that influences the risk-return trade-off doesn't function adequately. The third hypothesis suggests that under such extreme conditions, the risk-return trade-off levels should be almost the same. If the risk aversion of individual agents remains constant and homogeneous, regardless of the variations in relative consumption, the risk-return trade-off is anticipated to converge to a uniform level at both the right- and left-hand limits.

The empirical findings indicate robust support for the first hypothesis, partial support for the second, and a complete lack of support for the third hypothesis, spanning from January 1959 to

December 2019. Specifically, the non-linear relationship between relative consumption and the risk-return trade-off is specified using natural cubic splines and estimated using Hansen's (1982) generalized method of moments (GMM). To conserve parameters and reduce the number of moment conditions, and to test the characteristics of the risk-return trade-off under extremely high and extremely low relative consumption levels, several models with constraints are also examined. Models selected based on out-of-sample root mean squared error (RMSE) consistently exhibit behavior aligning with the first hypothesis of the risk-return trade-off under any imposed constraints. Subsequently, the risk-return trade-off tends to somewhat stabilize under low relative consumption levels, lending partial support to the second hypothesis. However, the characteristics of the risk-return trade-off under high relative consumption levels are unclear, indicating the potential oversight of critical variables associated with pricing. Moreover, the levels of the risk-return trade-off differ substantially between high and low relative consumption, strongly contradicting the third hypothesis, possibly because the risk aversion of individual agents varies or is heterogeneous. Alternatively, individual agents might have grown accustomed to acknowledging losses due to the continual low levels of relative consumption, considering it as the new norm. In this scenario, with a further decrease in relative consumption, individual agents recognizing additional losses could potentially result in further positive adjustments to aggregate risk aversion. Qualitatively similar results are obtained even when considering the consumption-wealth ratio (CAY) proposed by Lettau and Ludvigson (2001) as a robust variable to capture fluctuations in investment opportunities. Overall, it can be concluded that relative consumption plays a significant role in pricing the market portfolio, and its nonlinear structure is largely consistent with the model proposed in this paper.

The structure of this paper is outlined as follows. Section 2 introduces a simple model that integrates the concept of loss aversion into the framework of state-dependent preferences, presenting three hypotheses derived from its insights. Section 3 presents the methodology for testing these hypotheses alongside an overview of the data. Section 4 offers empirical results and discusses the outcomes of the three hypotheses. Finally, Section 5 concludes the paper.

2. MODEL AND HYPOTHESES

This section introduces a simple model that incorporates the idea of loss aversion into the state-dependent preference framework. With this model, it becomes feasible to elucidate not only scenarios where the risk-return trade-off varies countercyclically, as observed in prior research, but also situations where it fluctuates procyclically contingent upon the stage of the shock.

Assuming a pure exchange economy, the only uncertainty in the economy is the aggregate endowment (aggregate consumption) Y_t. The aggregate endowment is assumed to follow a geometric Brownian motion.

$$\mathrm{d}Y_t = Y_t(\mu \mathrm{d}t + \sigma \mathrm{d}B_t), \qquad \forall t \geq 0, \tag{1}$$

where $\mathrm{d}B_t$ is a standard Brownian motion, $\mu > \sigma^2/2$ and $\sigma > 0$. In this economy, two financial securities are traded in the market. One is a single risk asset with a net supply of 1 that pays the aggregate endowment Y_t as dividends. The other is an instantaneous risk-free asset, with a net supply of zero. Consider r_t as the instantaneous rate of return for a risky asset, and σ_t as its instantaneous volatility. Additionally, let r_t^f denote the instantaneous risk-free interest rate for a risk-free asset.

In many models with state-dependent preferences, the relative consumption $\omega = \ln Y - \ln X$, which measures the deviation between the past consumption benchmark X and current consumption Y, serves as a variable that captures the state of the economy. Following an approach similar to Chan and Kogan (2002), the consumption benchmark $x_t = \ln X_t$ is specified as the weighted average of past values of the logarithmic aggregate endowment $y_t = \ln Y_t$.

$$x_t = e^{-(1-\kappa)t} x_0 + (1-\kappa) \int_0^t e^{-(1-\kappa)(t-s)} y_s \mathrm{d}s \,, \tag{2}$$

$$\mathrm{d}x_t = (1-\kappa)(y_t - x_t)\mathrm{d}t. \tag{3}$$

where the parameter κ governs the degree of dependence on past values of the consumption benchmark. This is linked to the autocorrelation of price-dividend ratios within the state-dependent preference frameworks, as noted in works such as Campbell and Cochrane (1999), Chan and Kogan (2002), and Xiouros and Zapatero (2010). Intuitively, relative consumption, as defined in this manner, rises when consumption expands over an extended period and declines when consumption contracts over a prolonged duration. In other words, this can be considered to reflect the state of the economy.

This study analyzes the effects of disparities in the timing at which individual agents recognize the worsening economic conditions and adjust their marginal utility accordingly. For this purpose, the utility function of an agent is expressed below, where the agent identifies a *loss*, or a deterioration in the economic state, when the given relative consumption ω_t falls below a threshold ζ.

$$u(C_t, X_t, \omega_t; \zeta) = \frac{\Lambda(\omega_t; \zeta)}{1-\gamma} \left(\frac{C_t}{X_t}\right)^{1-\gamma} \quad \text{for } \gamma > 1, \tag{4}$$

$$\text{where } \Lambda(\omega_t;\zeta) = \begin{cases} 1 & \text{if } \omega_t \geq \zeta \\ \lambda & \text{if } \omega_t < \zeta \end{cases} \quad \text{for } \lambda \geq 1. \tag{5}$$

C_t is the agent's consumption, γ is the relative risk aversion, and λ is the pseudo-loss aversion. Such an agent has a standard (profit-type) utility function $(1-\gamma)^{-1}(C_t/X_t)^{1-\gamma}$ in the range $\omega_t \geq \zeta$ and has a loss-type utility function $\lambda(1-\gamma)^{-1}(C_t/X_t)^{1-\gamma}$ in the range $\omega_t < \zeta$. In this study, it is assumed that γ is uniform across all agents to mitigate the influence of the mechanism of varying aggregate risk aversion stemming from heterogeneity in individual risk aversion, as proposed by Chan and Kogan (2002). Furthermore, for simplifying the analysis, it is assumed that the pseudo-loss aversion λ is uniform across all agents.

Expressing the societal weights allocated to each type ζ as $f(\zeta)$, individual decision-making problems can be described as a central planner problem. The problem faced by the central planner is represented by the following, with the aggregate endowment Y_t and consumption benchmark X_t given.

$$\sup_{\{C_t(Y_t,X_t,\omega_t;\zeta);t\geq 0\}} \left\{ E\left[\int_0^\infty e^{-\rho t} \int f(\zeta)\frac{\Lambda(\omega_t;\zeta)}{1-\gamma}\left(\frac{C_t(Y_t,X_t,\omega_t;\zeta)}{X_t}\right)^{1-\gamma} \mathrm{d}\zeta\, \mathrm{d}t\right], \right. \\ \left. \text{s.t.} \quad \int C_t(Y_t,X_t,\omega_t;\zeta)\mathrm{d}\zeta \leq Y_t \quad \forall t \geq 0 \right\}, \tag{6}$$

where ρ is a parameter representing time preference rate. The utility function specified in eq. (4) has a time-additive structure, thus reducing this problem to a series of period-by-period optimization problems. In the following, subscripts related to time points are omitted for brevity. By solving the optimality conditions for each time point in eq. (6), the optimal consumption share $c^* = C^*/Y$ can be determined as follows:[3]

$$c^*(\omega;\zeta) = f(\zeta)^{1/\gamma}\Lambda(\omega;\zeta)^{1/\gamma}L(\omega)^{-1}, \tag{7}$$

[3] See Appendix A for the proof of the results presented in this section.

$$\text{where}\quad L(\omega) = \int f(\zeta)^{1/\gamma}\Lambda(\omega;\zeta)^{1/\gamma}\mathrm{d}\zeta\,. \tag{8}$$

From eq. (5), eq. (8) can be rewritten as $L(\omega) = \int_{\zeta<\omega} f(\zeta)^{1/\gamma}\mathrm{d}\zeta + \lambda^{1/\gamma}\int_{\zeta>\omega} f(\zeta)^{1/\gamma}\mathrm{d}\zeta$. When the distribution $f(\zeta)$ representing the timing of recognizing the economic conditions' deterioration follows a bell-shaped continuous distribution similar to a normal distribution, the following properties regarding L can be demonstrated.

$$L > 0, \qquad L' < 0, \qquad f' \lesseqgtr 0 \;\Rightarrow\; L'' \gtreqless 0, \tag{9}$$

$$\lim_{\omega\to\pm\infty} L'(\omega) = 0, \qquad \lim_{\omega\to\pm\infty} L''(\omega) = 0. \tag{10}$$

Under the optimal consumption share, the aggregate utility $U = \int f(\zeta)u(C^*, X, \omega;\zeta)\mathrm{d}\zeta$ is expressed as a function of relative consumption.

$$U(\omega) = e^{(1-\gamma)\omega}L(\omega)^{\gamma}. \tag{11}$$

Under the settings, the instantaneous expected return of the risky asset at time t is expressed as follows:

$$\mu_t - r_t^f = -\frac{Y_t U_{YY}}{U_Y}\sigma_Y\sigma_t, \tag{12}$$

where $U_Y = \partial U/\partial Y$ and $U_{YY} = \partial^2 U/\partial Y^2$.

Under the optimal consumption share, the aggregate risk aversion $\gamma^a = -YU_{YY}/U_Y = 1 - U''/U'$ is expressed as a function of relative consumption as follows:

$$\gamma^a(\omega) = \gamma + \left(\frac{1}{\gamma - 1 - \gamma\frac{L'(\omega)}{L(\omega)}}\frac{L'(\omega)}{L(\omega)} - 1\right)\frac{\gamma L'(\omega)}{L(\omega)} + \frac{1}{\gamma - 1 - \gamma\frac{L'(\omega)}{L(\omega)}}\frac{\gamma L''(\omega)}{L(\omega)}. \tag{13}$$

The first term represents the uniform relative risk aversion across all agents, while the second and third terms depict the deviations from it. If there are no agents whose utility functions change with the deterioration of economic conditions (i.e., $\lambda = 1$), the second and third terms become zero, and $\gamma^a(\omega) = \gamma$. When $\lambda > 1$, as $L > 0$ and $L' < 0$, the second term is always positive. However, the sign of the third term depends on the sign of L''. From eq. (9), if $f' < 0$, then $L'' > 0$, meaning the third term is positive, resulting in a positive correction to $\gamma^a(\omega)$. The condition $f' \leq 0$ implies an increasing number of agents shifting towards loss-type utility functions as the economic conditions deteriorate. This suggests a stage where the impact of the shock has not yet fully permeated throughout the economy. Conversely, if $f' > 0$, then $L'' < 0$, meaning the third term is negative, resulting in a potential negative correction to $\gamma^a(\omega)$. The condition $f' > 0$ implies a decreasing number of agents shifting towards loss-type utility functions as the economic conditions deteriorate, suggesting a stage where the impact of the shock is increasingly permeating throughout the economy. Furthermore, according to (10), when the economic conditions are extremely good or bad, $\gamma^a(\omega)$ can be approximated by γ, and there is little to no correction applied.

To enhance clarity and provide substantive insights, examining numerical illustrations that are pertinent to the discussion is relevant. The necessary parameters are those that characterize a homogeneous relative risk aversion γ, homogeneous pseudo-loss aversion λ, and the distribution $f(\zeta)$ governing the timing of recognizing the deterioration of economic conditions. The homogeneous relative risk aversion γ is set to the median value of 6.3, as investigated by Kimball et al. (2008) for risk aversion among Americans. The homogeneous pseudo-loss aversion λ is set to the median value of 1.7, as surveyed by Wang et al. (2017) for loss aversion among Americans. The distribution $f(\zeta)$ is assumed to follow a normal distribution with mean μ_N and standard deviation σ_N, and the parameters are chosen to satisfy the following conditions under the setting of relative consumption from Chan and Kogan (2002) (with a mean of 0.32 and standard deviation of 0.12).

$$\int_{0.32}^{\infty} f(\zeta;\mu_N,\sigma_N)\mathrm{d}\zeta = 0.001 \quad \text{and} \quad \int_{0.32-0.12}^{\infty} f(\zeta;\mu_N,\sigma_N)\mathrm{d}\zeta = 0.999$$

This indicates a scenario where agents recognizing the deterioration of economic conditions under average state $\bar{\omega}$ represent only about 0.1% of the total, while under extremely adverse conditions (mean minus one standard deviation), approximately 99.9% of agents recognize the deterioration. To fulfill these two conditions, μ_N is set to 0.26, and σ_N to 0.019.

Figure 1 depicts the relationship between the state of the economy (relative consumption) and aggregate risk aversion under such numerical examples, providing three key insights. First, under appropriate numerical examples, as the economy deteriorates, initially a positive correction is applied, but as a sufficient number of agents recognize the deterioration, a negative correction is introduced. This implies that, following the propagation of shocks, aggregate risk aversion initially fluctuates countercyclically, then switches to procyclical fluctuations, and eventually returns to countercyclical fluctuations. Second, even under the average state where there are relatively few agents recognizing the deterioration of the economy (i.e., $\omega = 0.32$), the level of aggregate risk aversion becomes notably higher compared to the better state. In the numerical example of this paper, the state where aggregate risk aversion is maximized generally corresponds to the average state of the economy.

(Figure 1)

Based on the insights above, the following three hypotheses regarding the nature of the risk-return trade-off corresponding to aggregate risk aversion are considered.

Hypothesis 1: As relative consumption decreases, the risk-return trade-off increases, decreases, and then increases again.

Hypothesis 2: In states where almost all agents do not recognize the deterioration of relative consumption levels, or conversely, where almost all agents recognize the deterioration of relative consumption levels, the risk-return trade-off stabilizes.

Hypothesis 3: In states where almost all agents do not recognize the deterioration of relative consumption levels, and conversely, where almost all agents recognize the deterioration of relative consumption levels, the risk-return trade-off reaches nearly the same level.

3. DATA AND METHODOLOGY

In this section, the methodology for verifying the three hypotheses constructed in the previous section will be discussed. In much of the previous research, risk for the market portfolio is proxied by conditional volatility, and typically, the following relationship between risk and return is of interest.

$$E_{t-1}[r_t^e] = g_{t|t-1} \operatorname{Var}_{t-1}[r_t^e]^{q/2} , \tag{14}$$

where r_t^e represents the excess return of the market portfolio and $g_{t|t-1}$ is the risk-return trade-off. When risk is captured by conditional variance, $q = 2$, and when it is captured by conditional volatility, $q = 1$.

In this paper, the market excess return is specified as follows, considering its correspondence with eq. (12).

$$r_t^e = g(\omega_{t-1})\sigma_{r,t|t-1} + \epsilon_t, \qquad E[\epsilon_t | \mathcal{I}_{t-1}] = 0, \tag{15}$$

where $g(\omega_{t-1})$ is the risk-return trade-off associated with relative consumption ω_{t-1}, $\sigma_{r,t|t-1}$ is the conditional volatility of the market excess return, ϵ_t is the error term, and $\mathcal{I}_{t-1}$ is the set of information variables. It is worth noting that the emphasis is on the relationship between $g(\omega_{t-1})$ and aggregate risk aversion, and therefore, a constant term is not included in the model. From an empirical perspective, Lanne and Saikkonen (2007) highlight that including unnecessary constant terms not suggested by the model notably reduces the power of standard Wald tests. The model is estimated using the generalized method of moments (GMM), as proposed by Hansen (1982), following the approach outlined by Guo et al. (2013).[4]

Relative consumption is specified in the following form, which corresponds to eq. (2) in discrete time.

$$\omega_t = \ln Y_t / Y_{t-1} + \kappa\omega_{t-1}, \qquad \omega_0 = \frac{\frac{1}{T}\sum_{t=1}^{T} \ln Y_t / Y_{t-1}}{1-\kappa} \tag{16}$$

Real personal consumption expenditures for the United States from January 1959 to December 2019 are utilized for Y. Following previous studies, κ is set to $0.914^{1/12}$ based on the lag-12

[4] Bolleslev et al. (1992) noted that estimation using the ARCH-M (autoregressive conditional heteroskedasticity in mean) model proposed by Engle et al. (1987) can lead to significant bias when there are specification errors in the model.

autocorrelation of monthly price-dividend ratios over the same period.[5] The market excess return r_t^e is defined as the difference between the value-weighted return of all CRSP firms incorporated in the US and one-month Treasury bill rate, while the conditional volatility $\sigma_{r,t|t-1}$ is determined using the realized volatility of period $t-1$.[6] The information variables encompass relative consumption, the logarithm of price-dividend ratio (PD), term spread (TERM), credit spread (DEF), and stochastically detrended risk-free rate (RREL).[7] However, it has been noted that the GMM estimation may suffer from serious bias when the number of moment conditions is excessive (Newey and Smith, 2004). To mitigate this issue and economize on the number of moment conditions, only the first principal component of these five variables is employed as additional information variables.[8]

Figure 2 illustrates the evolution of relative consumption and the information variables over the analysis period. The gray areas indicate recession periods as defined by the National Bureau of Economic Research (NBER). It can be observed that relative consumption sharply declines during recessions. However, the relative consumption level during periods of economic expansion

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[5] The annual autocorrelation of relative consumption in this paper, $\kappa^{12} = 0.914$, is close to 0.91 as found in Xiouros and Zapatero (2010). Additionally, Campbell and Cochrane (1999) set it to 0.87, while Chan and Kogan (2002) set it to 0.94. Referring to the lag-12 autocorrelation is aimed at eliminating the influence of seasonality in price-dividend ratios.

[6] For realized volatility, the sum of squared daily excess returns within the corresponding month is utilized. Moreover, as a proxy for conditional volatility, the predicted volatility from an ARFIMA (autoregressive fractionally integrated moving average) model with the realized volatility of period t as the dependent variable yielded qualitatively similar results. Additionally, to align the estimated $g(\omega)$ with the annual Sharpe ratio, r_t^e is multiplied by 12, and $\sigma_{r,t|t-1}$ is multiplied by $\sqrt{12}$ for annualization.

[7] PD represents the logarithm of the ratio of the current S&P500 price to the total dividends accumulated over the past 12 months. TS stands for the difference between the 10-year Treasury bond yield and the 3-month T-bill rate, DEF represents the spread between Moody's Baa and Aaa corporate bond yields, and RREL is the difference between the current risk-free rate and risk-free rate over the past 12 months. The data was sourced from the FRB of St. Louis, K. French's website, and R. Shiller's website.

[8] Substituting all five variables instead of the first principal component as information variables did not qualitatively alter the subsequent results.

does not necessarily surpass that during economic downturns. However, using relative consumption as a proxy for the state of the economy is significant. For instance, the economic downturn of the 1970s can be interpreted as relatively mild compared to that of the 1980s from the perspective of relative consumption levels. During the former, there were few agents recognizing losses, whereas during the latter, there might have been a shift of many agents towards loss-type utility functions. Furthermore, since the global financial crisis of 2008, relative consumption levels have remained low, indicating a clear regime shift. Considering the economic growth in the US during the 2010s, it is unlikely that this period can be classified as a "bad state." However, in the model presented in this paper, the 2010s could also be interpreted as a "stable state" where agents have become accustomed to the decline in relative consumption levels. Additionally, considering that the first principal component (PC1) effectively captures the variability of the underlying five information variables and consistently exhibits a significant upward trend during recessionary periods, it is deemed to possess sufficient information crucial for pricing in the market portfolio.

(Figure 2)

Moving forward, the specification of the risk-return trade-off function $g(\omega)$ will be examined. If Hypothesis 1 constructed in the previous section holds true, the risk-return trade-off should be a nonlinear function of relative consumption. To incorporate the nonlinear structure, the following formulation is specified, employing natural cubic spline basis functions $b_k(\omega)$ for $k = 1, \ldots, K$:

$$g(\omega) = \sum_{k=1}^{K} \delta_k b_k(\omega)\,. \tag{17}$$

The basis functions $b_k(\omega)$ are defined as follows:

$$b_1(\omega) = 1, \qquad b_2(\omega) = \omega, \qquad b_{2+k} = d_k(\omega) - d_{K-1}(\omega) \text{ for } k = 1,2,\dots,K-2,$$

$$d_k(\omega) = \frac{\{\max(0, \omega - \xi_k)\}^3 - \{\max(0, \omega - \xi_K)\}^3}{\xi_K - \xi_k}, \tag{18}$$

where ξ_k represents the $(k-1)/(K-1) \times 100$th percentile of the sample of ω. The natural cubic spline basis is specified to ensure $g''(\omega) = 0$ outside the data range ($\omega < \xi_1, \omega > \xi_K$). As the number of K increases, it can capture a more complex nonlinear structure, but it also leads to the problem of overfitting. A typical approach to address overfitting is to set a larger K and smooth $g(\omega)$ by penalizing large $g''(\omega)$. However, in the GMM estimation, minimizing the number of moment conditions is essential to eliminate bias. Thus, using the smallest possible K is preferable. Therefore, this paper explores the optimal model by comparing the out-of-sample RMSE across models with different K.

Furthermore, based on Hypothesis 2 from the previous section, it is also considered to impose additional constraints on $g(\omega)$. If Hypothesis 2 is correct, then two states exist, away from shocks associated with changes in utility functions due to deteriorations in economic conditions: one where the economic situation is extremely favorable and another where the economic situation has deteriorated significantly. In these states, the risk-return trade-off should stabilize. To reflect this property, constraints are imposed such that $g'(\omega) = 0$ outside the data range, i.e., $\omega < \xi_1$ or $\omega > \xi_K$. To incorporate this constraint, eq. (17) is rewritten as follows:[9]

$$g(\omega) = \delta_1 + \delta_2 \left(\omega - \frac{b_K(\omega)}{3(\xi_{K-1} - \xi_{K-2})}\right) + \sum_{k=1}^{K-3} \delta_{k+2} b_{2+k}^{adj}(\omega) + \delta_K^{adj} b_K(\omega),$$

[9] See Appendix B for the proof of the results presented in this section.

$$\text{where} \quad b_{2+k}^{adj}(\omega) = b_{2+k}(\omega) - \frac{\xi_{K-1} - \xi_k}{\xi_{K-1} - \xi_{K-2}} b_K(\omega) \quad \text{for } k = 1,2, \ldots, K-3,$$

$$\delta_K^{adj} = \frac{\delta_2 + 3\sum_{k=1}^{K-2} \delta_{k+2}(\xi_{K-1} - \xi_k)}{3(\xi_{K-1} - \xi_{K-2})}. \tag{19}$$

Using this specification, $\delta_2 = 0$ implies that the risk-return trade-off stabilizes when the economy is at its worst, while $\delta_K^{adj} = 0$ suggests stability in the risk-return trade-off during periods of exceptionally strong economic conditions. By adding constraints, reducing the number of basis functions is possible, thereby conserving parameters and moment conditions while still investigating more complex structures.

To test Hypothesis 3 from the previous section, constraints are added to ensure that the risk-return trade-off remains nearly the same when the economy is either extremely favorable or in deep distress. For this purpose, constraints are added such that $g(\omega) = \delta_0$ outside the data range, that is, when $\omega < \xi_1$ or $\omega > \xi_K$. Under these constraints, $g(\omega)$ aligns as follows.

$$g(\omega) = \delta_1 + \sum_{k=1}^{K-4} \delta_{k+2} b_{k+2}^{level}(\omega),$$

$$\text{where} \quad b_{k+2}^{level}(\omega) = b_{k+2}^{adj}(\omega) - \frac{\Xi_k - \Xi_{K-2}}{\Xi_{K-3} - \Xi_{K-2}} b_{K-1}^{adj}(\omega),$$

$$\Xi_k = \frac{(\xi_K + \xi_k)\xi_k - (\xi_K + \xi_{K-1})\xi_{K-1}}{\xi_{K-1} - \xi_k}. \tag{20}$$

Finally, the addition of variables that might influence the pricing of the market portfolio in eq. (15) is under consideration. Initially, the framework of Section 2 will be revised to incorporate additional state variables F_t besides X_t, for instance, variables that capture fluctuations in investment opportunities. F_t is assumed to follow the following stochastic process:

$$\mathrm{d}F_t = \mu_{Ft}\mathrm{d}t + \sigma_{Ft}\mathrm{d}B_t, \qquad \forall t \geq 0. \tag{21}$$

If aggregate utility is represented by $U(Y_t, X_t, F_t)$, then instantaneous returns are updated as follows.

$$\mu_t - r_t^f = -\frac{Y_t U_{YY}}{U_Y}\sigma_Y \sigma_t + \frac{U_{YF}}{U_Y}\sigma_{Ft}\sigma_t, \tag{22}$$

where $U_{YF} = \partial^2 U/(\partial Y \partial F)$. To simplify, assuming that F_t does not influence aggregate relative risk aversion, the conditional expected market excess return can be expressed as follows:

$$E_{t-1}[r_t^e] = g(\omega_{t-1})\sigma_{r,t|t-1} + \beta_{Ft}\sigma_{F,t|t-1},$$
$$\text{where } \beta_{Ft} = \frac{U_{YF}}{U_Y}\frac{\text{Cov}_{t-1}[F_t, r_t^e]}{\sigma_{F,t|t-1}}. \tag{23}$$

If the risk exposure of the investment opportunity β_{Ft} can be approximated by a constant value β_F, then the following can be obtained.

$$E_{t-1}[r_t^e] \approx g(\omega_{t-1})\sigma_{r,t|t-1} + \beta_F \sigma_{F,t|t-1}. \tag{24}$$

F_t is not directly observable and thus requires the use of proxy variables. One commonly used proxy is the consumption-wealth ratio (CAY) proposed by Lettau and Ludvigson (2001). What is crucial to note is that, as demonstrated by Guo et al. (2013), the CAY is influenced not only by the risk factor $\sigma_{F,t|t-1}$ associated with hedging portfolios but also by the market factor $\sigma_{r,t|t-1}$. For the sake of simplicity, assuming $CAY_{t-1} = a_0 + a_1\sigma_{t|t-1} + a_2\sigma_{F,t|t-1}$, then eq. (23) is updated as follows.

$$E_{t-1}[r_t^e] \approx \left\{g(\omega_{t-1}) - \frac{a_1}{a_2}\right\}\sigma_{t|t-1} - \beta_F\frac{a_0}{a_2} + \frac{\beta_F}{a_2}CAY_{t-1}. \tag{25}$$

This way, by incorporating variables such as the CAY to capture changes in investment opportunities as control variables, a better model can potentially be constructed. Considering the above, the following model is estimated.[10]

$$r_t^e = g^{ctrl}(\omega_{t-1})\sigma_{r,t|t-1} + \theta_0 + \theta_1 CAY_{t-1} + \epsilon_t, \qquad E[\epsilon_t | \mathcal{I}_{t-1}] = 0, \tag{26}$$

where $g^{ctrl}(\omega_{t-1})$ represents risk-return trade-off and is specified in the same way as $g(\omega_{t-1})$. It is worth noting that the risk-return trade-off estimated here, denoted as $g(\omega_{t-1}) - a_1/a_2$ in eq. (25), should not be interpreted as aggregate risk aversion.[11] However, as the estimated $g^{ctrl}(\omega_{t-1})$ is derived from several simplifications as mentioned above, which effectively shift the aggregate risk aversion up or down, its variability could be preserved. Therefore, it is crucial to verify whether this variability, rather than its level, satisfies Hypotheses 1 to 3.

4. EMPIRICAL RESULTS

In this section, Hypotheses 1 to 3, as discussed in Section 2, will be validated. Model selection relies on the root mean squared error (RMSE) from predictions. The first half of the analysis period, from January 1959 to June 1989, is designated as the in-sample period, while the latter half, from July 1989 to December 2019, serves as the out-of-sample period. Additionally, consideration is given to the RMSE based on updating predictions using data up to one period before the forecast period within the same out-of-sample period.

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[10] The data for the CAY was obtained from M. Lettau's website. However, because it is quarterly data, the value of CAY_{t-1} uses the value from the quarter preceding the one that includes period $t-1$.

[11] Especially when $a_1/a_2 > 0$, the estimated $g^{ctrl}(\omega_{t-1})$ may take on negative values even with positive aggregate risk aversion. Furthermore, when using β_{Ft} instead of β_F, the interpretation of the risk-return trade-off becomes even more intricate due to the inclusion of $\sigma_{r,t|t-1}$.

Table 1 displays the estimated results of eq. (15) for $K = 1, 2, \ldots, 10$. Eq. (15) represents a model without a constant term. As a baseline, a model including a constant term and assuming a constant risk-return trade-off is employed to verify if improvements in RMSE can be achieved. In the baseline model, a significant positive constant term is indicated. However, the results of the overidentification test (J-stat) are rejected at the 5% level, suggesting potential misspecification in the model. The second column presents the results of tests (z-tests or F-tests) where $g(\omega) = 0$, or $\delta_1 = \delta_2 = \cdots = \delta_K = 0$, is the null hypothesis, indicating that no significant risk-return trade-off is observed in the baseline model.

(Table 1)

The results in Table 1 strongly suggest that relative consumption plays a crucial role in pricing the market portfolio, indicating a nonlinear structure for $g(\omega)$. The model with $K = 2$ (i.e., $g(\omega) = \delta_1 + \delta_2\omega$) improves the out-of-sample RMSE by 0.425% compared to the baseline model assuming a constant risk-return trade-off. Moreover, unlike the baseline model, the overidentification test is not rejected. These strongly imply the importance of relative consumption in forecasting r_t^e. However, as the F-test fails to reject $g(\omega) = 0$ as the null hypothesis in this model, it cannot be definitively stated that volatility holds crucial information in price formation. However, the nonlinear model with $K = 5$, which exhibits the most improvement in out-of-sample RMSE, achieves a 1.610% improvement rate, and the F-test rejects $g(\omega) = 0$ as the null hypothesis. This suggests that assuming a nonlinear structure for $g(\omega)$ could lead to improvements in the model and its predictive power.

The top left portion of Figure 2 illustrates the risk-return trade-off $g(\omega; \hat{\boldsymbol{\delta}})$ with the parameter vector $\hat{\boldsymbol{\delta}}$ estimated by the model with $K = 5$, which yields the most significant improvement in RMSE. Except for periods characterized by high relative consumption levels, the trade-off shows a pattern of increasing, decreasing, and then increasing again with decreasing relative consumption, mirroring Hypothesis 1. The average relative consumption during the analysis period is 0.375, roughly aligning with the peak of the curve. As anticipated, even under average state, aggregate risk aversion appears to reach high levels. While these findings broadly support Hypothesis 1, the observation that $g(\omega)$ occasionally takes negative values and demonstrates unexpected procyclical fluctuations at high ω contradicts expectations. Additionally, the significant increase in $g(\omega)$ at low ω is inconsistent with Hypothesis 2.

(Figure 2)

To gain deeper insights, particularly regarding Hypotheses 2 and 3, four constrained models are examined. The first model, labeled as the lower constraint model, imposes constraints to stabilize the movement of $g(\omega)$ at low ω. This model sets the constraint $\delta_2 = 0$ in eq. (19). The second model, labeled as the upper constraint model, imposes constraints to stabilize the movement of $g(\omega)$ at high ω. This model sets the constraint $\delta_K^{adj} = 0$ in eq. (19). The third model, labeled as the two-sided constraint model, imposes both constraints. These are valuable for testing Hypothesis 2. The fourth model, designated as the level constraint model, incorporates additional constraints to ensure that $g(\omega)$ remains nearly identical at both extremely high and extremely low values of ω, alongside all the other constraints. This is specified in eq. (20) and is valuable for testing Hypothesis 3.

Table 2 displays the estimation results for both the unconstrained model and the four constrained models. For the sake of brevity, only the model with the most significant improvement in RMSE for each $K = 1,2,\dots,10$ is provided.[12] In an exceptional case, within the level constraint model, the overidentification test for the best RMSE model is rejected. Consequently, the model with the best RMSE among those where the overidentification test was not rejected was selected. Exceptionally, in the level constraint model, the overidentification test for the best RMSE model is rejected, resulting in the selection of the model with the best RMSE among those where the overidentification test was not rejected.

(Table 2)

Table 2 presents partial evidence supporting Hypothesis 2 and contradicting Hypothesis 3. Across all five selected models, the F-tests consistently reject the null hypothesis of $g(\omega) = 0$, implying that volatility holds crucial information for pricing, unlike the baseline model. The Diebold-Mariano test results indicate rejection for both the lower constraint model and the two-sided constraint model, implying that predictive power equivalent to the baseline model is not observed in either case. This suggests that employing these models leads to a significant improvement in predictive capability compared to the baseline. The two-sided constraint model

[12] Regardless of whether selected based on out-of-sample or updating predictions, the results remain consistent.

demonstrates the most substantial enhancement in RMSE, surpassing a 2% improvement rate. From a predictive perspective, this provides evidence supporting Hypothesis 2, suggesting stability in $g(\omega)$ at both high and low ω. However, considering the slight decline in predictive performance of the upper constraint model relative to the unconstraint model, the significance of the upper constraint may be somewhat diminished. On another note, the level constraint model exhibits the poorest predictive performance, with scant evidence supporting Hypothesis 3, indicating little convergence between $g(\omega)$ at extremely high ω and $g(\omega)$ at extremely low ω.

The rest of Figure 2 illustrates the risk-return trade-off $g\left(\omega; \hat{\boldsymbol{\delta}}\right)$ estimated under the four constraint models, offering crucial insights into the three hypotheses. Outside of extremely high and extremely low ω, the behavior of the four constraint models appears to be similar, with each model generally showing an initial increase, followed by a decrease, and then another increase, all in response to the decline in relative consumption. The summit of the central peak broadly corresponds to the average relative consumption of 0.375. This evidence provides robust support for Hypothesis 1. Furthermore, in the two-sided constraint model, which exhibits the most significant improvement in RMSE, the stabilization of $g(\omega)$ at both high and low ω is evident, aligning well with Hypothesis 2. Conversely, the substantial difference between $g(\omega)$ at high ω and $g(\omega)$ at low ω strongly refutes Hypothesis 3. Furthermore, in all models except the low-performing level constraint model, $g(\omega)$ turns negative at high ω. This poses a challenge when linking the risk-return trade-off $g(\omega)$ with aggregate risk aversion. It implies the potential oversight of some factor that may significantly impact the pricing of the market portfolio, particularly during periods of exceptionally high relative consumption levels.

From this point forward, the examination will focus on how the discussion evolves when additional variables that could potentially influence the pricing of the market portfolio are introduced. Particularly, the consumption-wealth ratio (CAY) proposed by Lettau and Ludvigson (2001) is known to be useful in capturing fluctuations in investment opportunities. Eq. (26) is estimated incorporating this effect. However, the risk-return trade-off estimated here does not correspond to the aggregate risk aversion, at least not in levels. Nevertheless, it accommodates the variability in aggregate risk aversion under certain simplifying assumptions.

Table 3 illustrates the estimation results of eq. (26) for $K = 1,2,\ldots,10$, alongside the unconstrained risk-return trade-off outlined in eq. (17). Given the lack of necessity to exclude the constant term, the model with $K = 1$ is deemed the baseline. In this model, neither the CAY nor volatility proves to be statistically significant, and the overidentification test is rejected as well. The $K = 5$ model stands out as the one that brings the most significant improvement in RMSE from the baseline model, with an enhancement rate of 1.911%. The results of the Diebold-Mariano test indicate a substantial enhancement in predictive accuracy compared to the baseline model. In this framework, both the CAY and volatility exert significant influence on pricing. Moreover, the non-rejection of the overidentification test suggests a minimal chance of misspecification in the model. Consistent with the results excluding the CAY, it is indicated that relative consumption holds crucial information, with the risk-return trade-off likely represented as its nonlinear function.

(Table 3)

The top left panel of Figure 3 illustrates the risk-return trade-off $g^{ctrl}\left(\omega;\hat{\boldsymbol{\delta}}\right)$ under the estimated parameter vector $\hat{\boldsymbol{\delta}}$ for the $K=5$ model, which demonstrates the most significant improvement in RMSE. Comparing it to the result without CAY (top left panel of Figure 1), a nearly identical shape is observed, albeit shifted downward. This downward adjustment at this level is attributed to the correction of $-a_1/a_2$ as depicted in eq. (25). The crucial part here is not merely the level but its variability. Similar to the result without CAY, except for high ω, there is a pattern of rise, fall, and rise in relative consumption, which aligns with Hypothesis 1.

(Figure 3)

Table 4 displays the estimation outcomes of both the unconstrained model and various constrained models. While each model ranging from K=1 to 10 is estimated, for the sake of brevity, only the model that yields the most significant improvement in RMSE is provided.[13] However, in both the lower constraint model and the two-sided constraint model, the overidentification test is rejected for the best RMSE model. Thus, the model with the best RMSE among those where the overidentification test is not rejected is selected. The level constraint model has omitted the results as the overidentification test was rejected for all models with $K \leq 10$. In all four selected models, the F-test, with the null hypothesis of $g(\omega)=0$, is rejected, indicating that volatility holds significant information for pricing. Moreover, in every model, there is a significant enhancement in predictive capability compared to the baseline model. The lower constraint model

[13] As with the case without the CAY, regardless of whether selected based on out-of-sample or updating predictions, the results remain consistent.

shows the most substantial improvement in RMSE (with an improvement rate of 2.122%), while the two-sided constraint model is comparable (2.117%). Under low ω, results consistent with Hypothesis 2 seem apparent. Additionally, the complete failure of the level constraint model strongly contradicts Hypothesis 3.

(Figure 4)

The rest of Figure 3 depicts the risk-return trade-off $g^{ctrl}\left(\omega; \hat{\boldsymbol{\delta}}\right)$ under the estimated parameter vector $\hat{\boldsymbol{\delta}}$ for three different types of constrained models. The results closely resemble the risk-return trade-off estimated by the model excluding the CAY, thus strengthening implications for the three hypotheses. The behavior of the three types of constrained models is largely similar, except for extremely high and extremely low ω, showing a consistent pattern of rise, fall, and rise corresponding to a decrease in relative consumption. Moreover, the peak of the central mountain typically coincides with an average relative consumption of 0.375. These findings strongly endorse Hypothesis 1. Moreover, in all models, $g^{ctrl}(\omega)$ significantly differs between extremely high and extremely small ω, providing strong evidence against Hypothesis 3. Hypothesis 2 seems unclear in the results. Both the lower constraint model and the two-sided constraint model, which significantly improve RMSE, exhibit a tendency for $g^{ctrl}(\omega)$ to stabilize somewhat at low omega. However, their behavior differs at high omega. In the former model, $g^{ctrl}(\omega)$ fluctuates sharply, whereas in the latter model, $g^{ctrl}(\omega)$ remains relatively stable. Given that both models have nearly equal predictive power, it suggests that the contribution of relative consumption to pricing may be minimal during periods of high ω.

5. CONCLUSION

The initial half of the paper introduces a simple model, which integrates the concept of loss aversion into the state-dependent preference framework. This model posits that when the relative consumption ω, used to gauge the state of the economy, falls below a certain threshold, individuals' utility functions shift to a loss-type utility function, characterized by high marginal utility and low utility levels. When shocks occur that decrease ω, many agents undergo these shifts successively, causing the marginal utility at the aggregate level to increase rapidly and introduce a positive correction to aggregate risk aversion. Conversely, when shocks propagate widely enough, there might be few agents remaining to undergo further shifts. This could lead to a decrease in the rate of increase of marginal utility at the aggregate level and potentially result in a negative correction to aggregate risk aversion. These characteristics suggest that as ω decreases, aggregate risk aversion initially rises, then declines, and finally rises again. This forms the basis of Hypothesis 1. Moreover, when ω is extremely high or extremely low, there are few agents left to undergo such shifts. Consequently, aggregate risk aversion is expected to remain stable, which is hypothesized as Hypothesis 2. Furthermore, when individual risk aversion remains constant and homogeneous, the level of aggregate risk aversion is expected to be the same in situations where hardly anyone shifts their utility functions and situations where almost everyone shifts their utility functions. This is hypothesized as Hypothesis 3.

The latter part of the paper examines three hypotheses derived from the model. Irrespective of whether control is exercised using the consumption-wealth ratio (CAY) as a proxy variable for investment opportunities, the variability of the risk-return trade-off $g(\omega)$ corresponding to aggregate risk aversion confirms Hypothesis 1, provides partial support for Hypothesis 2, and does not support Hypothesis 3. In all models, excluding extremely high and extremely low ω, the risk-return trade-off undergoes a pattern of rise, fall, and rise as relative consumption decreases. Moreover, there is a tendency for the risk-return trade-off to peak at average levels of relative consumption. These findings support Hypothesis 1. Hypothesis 2 appears to generally hold true, at least under low ω, but becomes ambiguous under high ω. Despite using the influential variable the CAY to capture investment opportunities, Hypothesis 2 remains unclear under high ω, indicating a potential oversight of certain variables. Hypothesis 3 cannot be considered valid, given the significant decline in predictive power caused by the level constraint, as well as the notable disparity in the estimated values of the risk-return trade-off between high and low ω in other models.

Hypothesis 3 may not hold for two potential reasons. First, the variability and heterogeneity in individual risk aversion, not considered in this study's model, may be significant. These elements contribute to the countercyclical variation in aggregate risk aversion, thus explaining the low $g(\omega)$ at extremely high ω and the high $g(\omega)$ at extremely low ω. Second, multiple shifts in utility functions may occur with the decline in relative consumption. To explain this, attention is drawn to the period since the 2008 global financial crisis, during which relative consumption has remained at low levels. It is likely that most agents shifted towards loss-type utility functions in response to the decrease in relative consumption during the 2008 global financial crisis. In other recessions, relative consumption tends to recover afterward, leading to a shift back to standard gain-type utility functions. However, in scenarios similar to the 2010s, where relative consumption fails to bounce back, such a situation may have become the new normal. Consequently, there could be agents whose utility functions shift to ones with even higher marginal utility of consumption than loss-type utility functions as relative consumption continues to decrease. In the presence of such agents, aggregate risk aversion may start to rise again as relative consumption declines, eventually leading to a decrease.

APPENDIX A: PROPERTIES OF OPTIMAL CONSUMPTION SHARE

This section elaborates on the characteristics of the optimal consumption share model introduced in Section 2. Initially, to solve optimization problem (6), introduce $\mathcal{L}$ defined as follows.

$$\mathcal{L} = \int f(\zeta)\frac{\Lambda(\omega;\zeta)}{1-\gamma}\left(c(Y,X,\omega;\zeta)\frac{Y}{X}\right)^{1-\gamma}\mathrm{d}\zeta + H\left(1-\int c(Y,X,\omega;\zeta)\mathrm{d}\zeta\right), \tag{A1}$$

where $H > 0$. Solving the first-order conditions leads to the following equation for the optimal consumption share.

$$c^*(\omega;\zeta) = f(\zeta)^{1/\gamma}\Lambda(\omega;\zeta)^{1/\gamma}H^{-1/\gamma}e^{\frac{1-\gamma}{\gamma}\omega} \quad \text{for all } \zeta, \tag{A2}$$

$$\int c^*(\omega;\zeta)\mathrm{d}\zeta = 1. \tag{A3}$$

Substituting eq. (A2) into eq. (A3) gives us the equation for H.

$$H = e^{(1-\gamma)\omega}L(\omega)^{\gamma} \tag{A4}$$

where $L(\omega) \equiv \int f(\zeta)^{1/\gamma}\Lambda(\omega;\zeta)^{1/\gamma}\mathrm{d}\zeta$. Substituting this into eq. (A2) leads to the derivation of eqs. (7) and (8).

Next, the characteristics of $L(\omega)$ when $f(\zeta)$ follows a bell-shaped continuous distribution, such as a normal distribution, will be discussed. $L(\omega)$ can be rewritten as follows.

$$L(\omega) = \lambda^{1/\gamma} \int_{\zeta > \omega} f(\zeta)^{1/\gamma} \mathrm{d}\zeta + \int_{\zeta \leq \omega} f(\zeta)^{1/\gamma} \mathrm{d}\zeta \tag{A5}$$

Using a certain distribution function $\phi(\zeta)$, without loss of generality, $f(\zeta)$ can be expressed as $f(\zeta) = \big(\phi(\zeta)\big)^{\gamma} F^{-1}$, where $F = \int_{\zeta} \big(\phi(\zeta)\big)^{\gamma} \mathrm{d}\zeta$. Subsequently, the following equation is derived.

$$\int_{\zeta \leq \omega} f(\zeta)^{1/\gamma} \mathrm{d}\zeta = F^{-1/\gamma} \Phi(\omega), \tag{A6}$$

where $\Phi(\omega) = \int_{\zeta \leq \omega} \phi(\zeta) \mathrm{d}\zeta$. Eq. (A5) can be rewritten as follows.

$$L(\omega) = F^{-1/\gamma} \lambda^{1/\gamma} - (\lambda^{1/\gamma} - 1) F^{-1/\gamma} \Phi(\omega) \tag{A7}$$

Therefore, the first and second derivatives of $L(\omega)$ are as follows.

$$L'(\omega) = -(\lambda^{1/\gamma} - 1) f(\omega)^{1/\gamma}, \tag{A8}$$

$$L''(\omega) = -(\lambda^{1/\gamma} - 1) \frac{1}{\gamma} f(\omega)^{1/\gamma - 1} f'(\omega). \tag{A9}$$

Eq. (9) follows from eqs. (A5), (A8), and (A9). When f follows a bell-shaped continuous distribution as normal distribution, $\lim_{\omega \to \pm\infty} f(\omega) = 0$, thus eq. (10) holds.

APPENDIX B: CONSTRAINED SPLINE

This section presents the derivation of the constrained spline introduced in Section 3. Let us consider adding constraints to ensure that $g'(\omega) = 0$ when $\omega < \xi_1$ in eq. (17). When $\omega < \xi_1$, the expression for $g(\omega)$ is $\delta_1 + \delta_2 \omega$, thus the necessary and sufficient condition for $g'(\omega) = 0$ is that $\delta_2 = 0$.

Next, consider adding constraints to eq. (17) such that $g'(\omega) = 0$ when $\omega > \xi_K$. Noting that $d_k(\omega) = \{(\omega - \xi_k)^3 - (\omega - \xi_K)^3\}/(\xi_K - \xi_k)$ when $\omega > \xi_K$, the following is derived.

$$g'(\omega) = \delta_2 + 3\sum_{k=1}^{K-2} \delta_{k+2}(\xi_{K-1} - \xi_k) \tag{A10}$$

Thus, the necessary and sufficient condition for $g'(\omega) = 0$ is $\delta_2 + 3\sum_{k=1}^{K-2} \delta_{k+2}(\xi_{K-1} - \xi_k) = 0$. The expression on the left-hand side of this condition represents the numerator of δ_K^{adj} as defined in eq. (19). In essence, setting $\delta_K^{adj} = 0$ becomes the necessary and sufficient condition for $g'(\omega) = 0$ when $\omega > \xi_K$. Rewriting the definition of δ_K^{adj} in terms of δ_K gives the following.

$$\delta_K = \frac{3(\xi_{K-1} - \xi_{K-2})\delta_K^{adj} - \delta_2 - 3\sum_{k=1}^{K-3} \delta_{k+2}(\xi_{K-1} - \xi_k)}{3(\xi_{K-1} - \xi_{K-2})} \tag{A11}$$

Substituting this into eq. (17) leads to the derivation of the remaining components of eq. (19).

Finally, examining the constraint where $g(\omega)$ matches for $\omega < \xi_1$ and $\omega > \xi_K$. Under this constraint, considering that all the previously mentioned conditions are satisfied, $g(\omega)$ is taken at the point where $\delta_2 = \delta_K^{adj} = 0$ as the starting point.

$$g(\omega) = \delta_1 + \sum_{k=1}^{K-3} \delta_{k+2} b_{2+k}^{adj}(\omega) \tag{A12}$$

When $\omega > \xi_K$, the condition $\sum_{k=1}^{K-3} \delta_{k+2} b_{2+k}^{adj}(\omega) = 0$ implies the following.

$$\delta_{K-1} = -\sum_{k=1}^{K-4} \delta_{k+2} \frac{(\xi_{K-1} - \xi_k)(\Xi_k - \Xi_{K-2})}{(\xi_{K-1} - \xi_{K-3})(\Xi_{K-3} - \Xi_{K-2})}$$

where $\Xi_k = \{(\xi_K + \xi_k)\xi_k - (\xi_K + \xi_{K-1})\xi_{K-1}\}/(\xi_{K-1} - \xi_k)$. By substitution and rearrangement, eq. (20) is derived.

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TABLES

TABLE 1.

				Out-of-sample prediction		Updating prediction	
	Constant	H_0: $g(\omega)=0$	J-stat	RMSE-IR [%]	DM	RMSE-IR [%]	DM
Baseline	0.118**	-1.134	4.732*	-	-	-	-
K=1	-	1.405	2.527	-0.249	-0.374	-0.243	-0.364
K=2	-	2.994	1.399	0.425	0.465	0.434	0.474
K=3	-	7.017	1.491	0.455	0.412	0.463	0.419
K=4	-	18.493**	1.454	0.797	0.928	0.802	0.933
K=5	-	26.244**	2.013	1.610	1.754	1.616	1.759
K=6	-	43.112**	3.127	1.451	1.674	1.461	1.683
K=7	-	44.622**	2.754	1.036	1.119	1.046	1.129
K=8	-	45.739**	2.847	0.847	0.851	0.856	0.859
K=9	-	47.809**	2.852	1.175	1.297	1.182	1.303
K=10	-	49.674**	2.579	1.454	1.525	1.460	1.530

Note: ** and * represent statistical significance at the 1 and 5% levels, respectively. H_0: $g(\omega)=0$ represents the z-value or F-value with $\delta_1=\delta_2=\cdots=\delta_K=0$ as the null hypothesis. J-stat means Hansen's J statistic. In out-of-sample prediction, the period from January 1959 to June 1989 is in-sample, and the period from July 1989 to December 2019 is out-of-sample. Updating prediction means a prediction from July 1989 to December 2019 using data from one period before the prediction period. RMSE-IR represents the root mean squared error improvement rate compared to the baseline model. DM means Diebold-Mariano (1995) test statistic with the null hypothesis that there is no difference in predictive power between the model with K=1.

TABLE 2.

	H_0: $g(\omega) = 0$	J-stat	Out-of-sample prediction		Updating prediction	
			RMSE-IR [%]	DM	RMSE-IR [%]	DM
Unconstraint (K=5)	26.244**	2.013	1.610	1.754	1.616	1.759
Lower constraint (K=9)	31.669**	2.961	1.960	2.140*	1.967	2.146*
Upper constraint (K=6)	42.303**	3.166	1.531	1.738	1.540	1.746
Two-sided constraint (K=9)	28.186**	3.139	2.020	2.224*	2.027	2.230*
Level constraint (K=7)	19.241**	3.260	1.077	1.253	1.088	1.264

Note: ** and * represent statistical significance at the 1 and 5% levels, respectively.

TABLE 3.

	Constant	CAY	H_0: $g(\omega) = 0$	J-stat	Out-of-sample prediction		Updating prediction	
					RMSE-IR [%]	DM	RMSE-IR [%]	DM
Baseline	0.120**	1.361	-1.292	4.932*	-	-	-	-
K=2	0.132**	1.954*	12.499**	2.821	1.151	2.167*	1.151	2.168*
K=3	0.118**	2.361*	12.503**	3.351	1.326	2.168*	1.326	2.168*
K=4	0.120**	2.159	12.612*	3.186	1.481	2.587**	1.481	2.587**
K=5	0.108**	2.209*	13.706*	3.494	1.911	2.924**	1.910	2.925**
K=6	0.082**	2.118	19.952**	4.292*	1.737	2.635**	1.739	2.639**
K=7	0.087*	2.333*	25.729**	4.024*	1.731	2.743**	1.733	2.748**
K=8	0.081**	2.254	29.583**	3.931*	1.335	2.052*	1.336	2.055*
K=9	0.079**	2.229*	30.891**	3.964*	1.502	2.348*	1.503	2.350*
K=10	0.070*	2.054	36.784**	3.649	1.684	2.296*	1.685	2.297*

Note: ** and * represent statistical significance at the 1 and 5% levels, respectively.

TABLE 4.

	Constant	CAY	H_0: $g(\omega) = 0$	Out-of-sample prediction RMSE-IR [%]	Out-of-sample prediction DM	Updating prediction RMSE-IR [%]	Updating prediction DM
Unconstraint (K=5)	0.108**	2.209*	13.706*	1.911	2.924**	1.910	2.925**
Lower constraint (K=7)	0.110**	2.355*	18.955**	2.122	2.847**	2.123	2.851**
Upper constraint (K=5)	0.112**	2.022*	13.883**	1.815	2.828**	1.815	2.830**
Two-sided constraint (K=7)	0.110**	2.445*	16.036**	2.117	2.869**	2.117	2.872**

Note: ** and * represent statistical significance at the 1 and 5% levels, respectively.

FIGURES

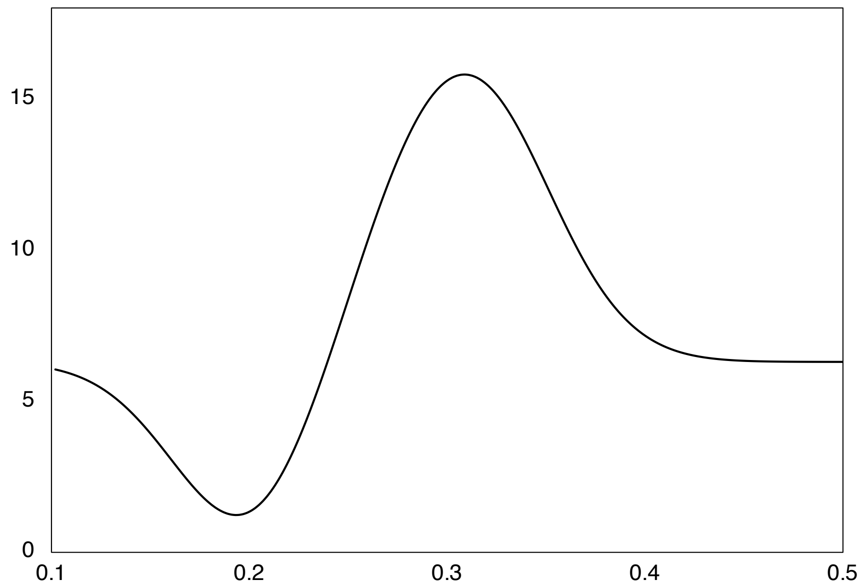

FIGURE 1.

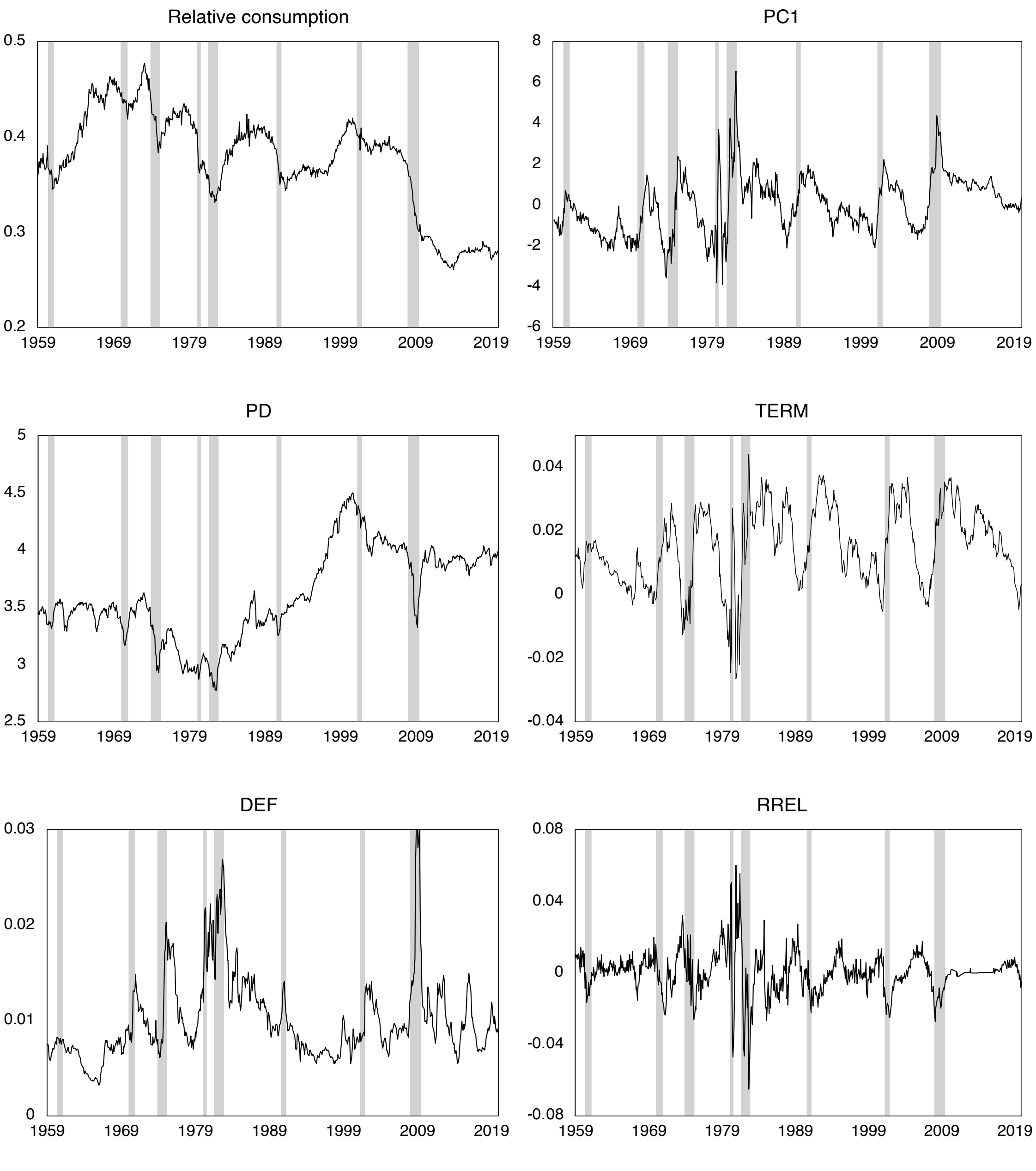
Relative consumption
PC1
PD
TERM
DEF
RREL
1959
1969
1979
1989
1999
2009
2019

FIGURE 2.

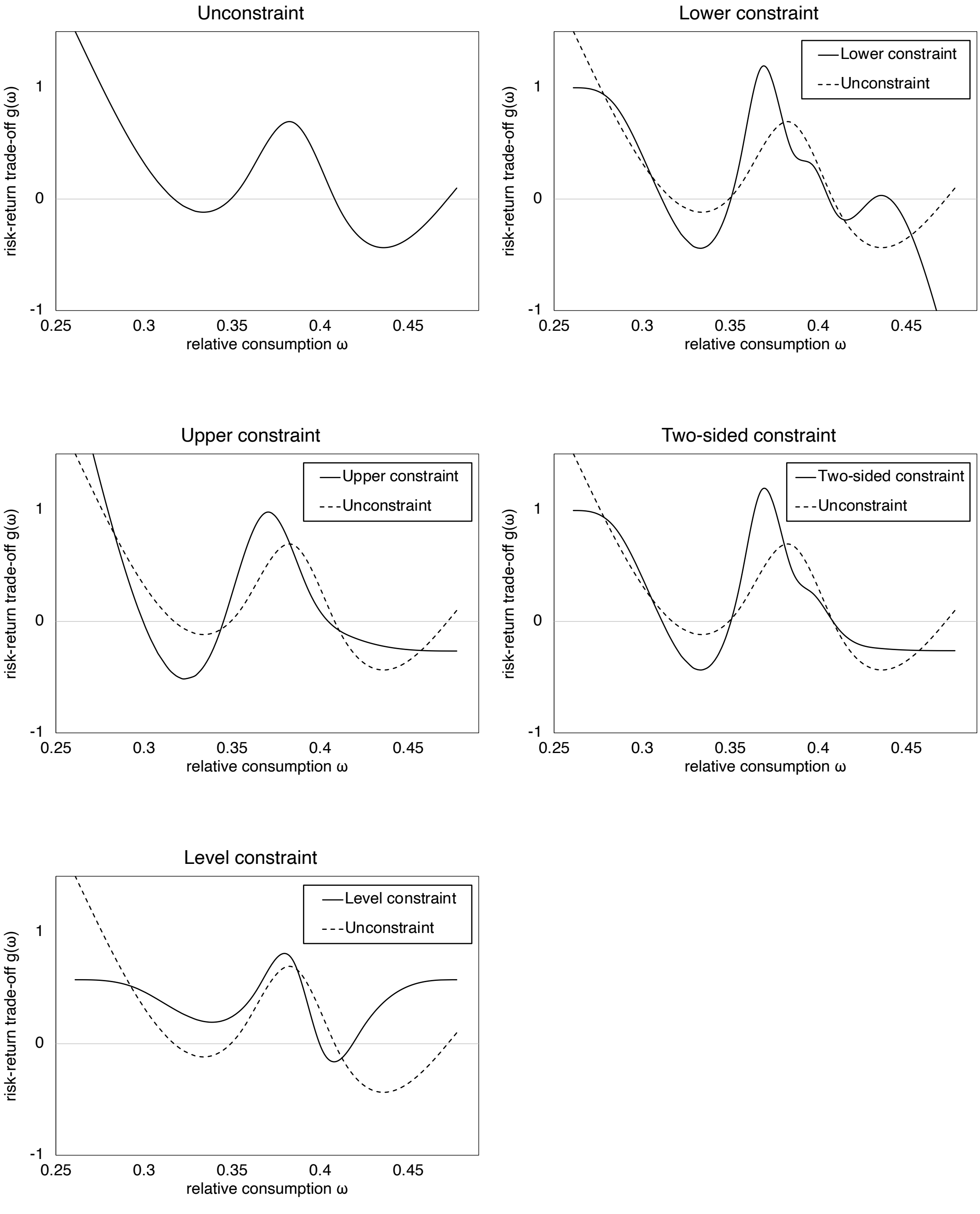
Unconstraint
Lower constraint
Lower constraint
Unconstraint
Upper constraint
Upper constraint
Unconstraint
Two-sided constraint
Two-sided constraint
Unconstraint
Level constraint
Level constraint
Unconstraint
risk-return trade-off g(ω)
relative consumption ω
1
0
-1
0.25
0.3
0.35
0.4
0.45

FIGURE 3.

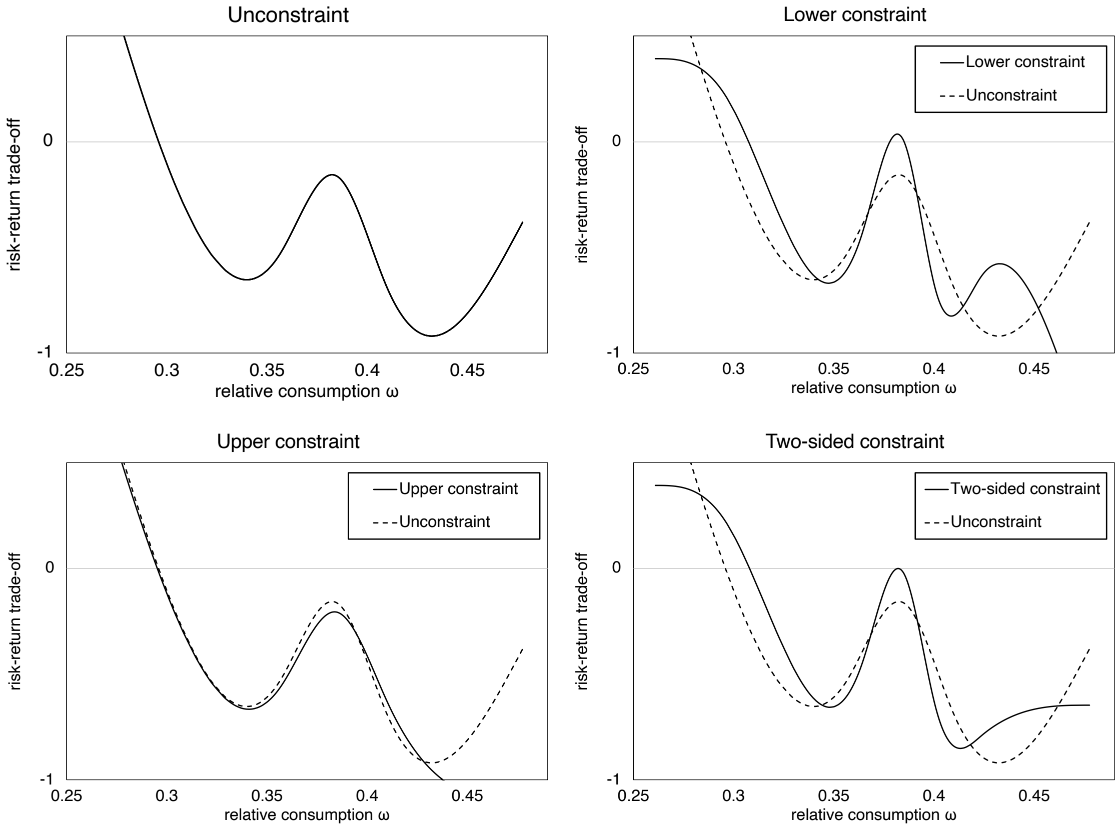
Unconstraint
Lower constraint
Lower constraint
Unconstraint
Upper constraint
Upper constraint
Unconstraint
Two-sided constraint
Two-sided constraint
Unconstraint
risk-return trade-off
relative consumption ω
0
-1
0.25
0.3
0.35
0.4
0.45

FIGURE 4.