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Endowments-swapping-proofness and Efficiency in Multiple-Type Housing Markets*

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Endowments-swapping-proofness and Efficiency in Multiple-Type Housing Markets*

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Abstract

For Shapley-Scarf housing markets (Shapley and Scarf, 1974), Fujinaka and Wakayama (2018) propose a new incentive property, *endowments-swapping-proofness*, that excludes manipulations that a pair of agents can conduct before the operation of the selected mechanism by swapping their endowments. We investigate *endowments-swapping-proofness* for Moulin (1995)'s multiple-type housing markets, which are an extension of Shapley-Scarf housing markets with multi-unit demands. Differing from Shapley-Scarf housing markets, for multiple-type housing markets, there are various ways to swap endowments. Motivated by this observation, we introduce three extensions of *endowments-swapping-proofness*: *bundle endowments-swapping-proofness*, *one type endowments-swapping-proofness*, and *flexible endowments-swapping-proofness*.

Based on the first two weaker *endowments-swapping-proofness* properties that we propose, and other well-studied properties (*individual rationality*, *strategy-proofness*, and *non-bossiness*), on several domains of preference profiles, we provide characterizations of two extensions of the top-trading-cycles (TTC) mechanism: the bundle top-trading-cycles (bTTC) mechanism and the coordinatewise top-trading-cycles (cTTC) mechanism. Moreover, we also show that the strongest possible *endowments-swapping-proofness* property (*flexible endowments-swapping-proofness*) leads to an impossibility. Furthermore, our results explicitly show that our new properties correspond to efficiency notions.

Keywords: multiple-type housing markets; *endowments-swapping-proofness*; *strategy-proofness*; *constrained efficiency*; top-trading-cycles (TTC) mechanism; market design.

JEL codes: C78; D61; D47.

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1 Introduction

Assignments of scarce resources are attracting a lot of attention in mechanism design and market design.[1] Many studies, both theoretical and empirical, focus on indivisible resources, e.g., in auctions (Myerson, 1981; Hortaçsu and McAdams, 2010), school choice (Abdulkadiroğlu and Sönmez, 2003; Kapor et al., 2020), and medical resource allocation (Pathak et al., 2021). Existing literature has often investigated such indivisible resource allocation problems with "unit-demand". Additionally, when monetary transfers are not allowed, many studies focus on the so-called housing market model (Shapley and Scarf, 1974). This model is an exchange economy in which each agent owns an indivisible object (say, a house); each agent has preferences over houses and wishes to consume exactly one house. When preferences are strict, Shapley and Scarf (1974) show that the strict core (defined by a weak blocking notion) has remarkable features: it is non-empty,[2] and can be easily calculated by the so-called top-trading-cycles (TTC) algorithm (due to David Gale). Moreover, the TTC mechanism that assigns the unique strict core allocation satisfies important incentive properties, *strategy-proofness* (Roth, 1982) and even *group strategy-proofness* (Bird, 1984). Furthermore, Ma (1994) and Svensson (1999) show that the TTC mechanism is the unique mechanism satisfying *Pareto efficiency*, *individual rationality*, and *strategy-proofness*.

Although in some important cases (e.g., kidney exchange, school choice, etc.), "unit-demand" is an appropriate assumption, in many other cases, agents may wish to receive more than one object, and indeed, many studies have analyzed such situations, e.g., Pápai (2001, 2007), Manjunath and Westkamp (2021), Biró et al. (2022a,b), and Echenique et al. (2022). However, it is known to be very difficult to extend the results from unit-demand indivisible resource allocation problems to multi-unit demands. In this paper, we consider an extension of the classical Shapley-Scarf housing markets by allowing multi-unit demands: multiple-type housing markets, to use the language of Moulin (1995).[3] In this model, objects are of different types (say, houses, cars, etc.) and agents initially own and wish to consume exactly one object of each type.

We believe that the analysis of multiple-type housing markets is relevant for three reasons. First, from a theoretical point of view, this is a simple extension of Shapley-Scarf housing markets with multi-unit demands. Therefore, the analysis of multiple-type housing markets, as a benchmark or stepping stone, is potentially useful for addressing issues for other multi-unit demand models, such as Pápai (2007) and Anno and Kurino (2016). Second, for multiple-type housing markets, agents are "balanced" in the sense that all agents have the same numbers and types of endowments and demands. This balanced structure provides some tractability and hence some hope for positive results.[4] Third, similar to Shapley-Scarf housing markets,

[1] See the Nobel prize lectures in economic sciences in 2012 and 2020 for examples.

[2] Roth and Postlewaite (1977) show that the strict core is single-valued.

[3] In Echenique et al. (2022), multiple-type housing markets are called categorical economies.

[4] We provide a discussion to compare a more general model with ours in Subsection 6.3.

multiple-type housing markets are applicable to many problems in reality: a familiar example for most readers would be students' enrollments at many universities where courses are taught in small groups and in multiple sessions (Klaus, 2008). Additionally, for term paper presentations during a course, students may want to exchange their assigned topics and dates (Mackin and Xia, 2016); hospitals may want to improve their surgery schedules for surgeons by swapping surgery staff, operating rooms, and dates (Huh et al., 2013); and in shift-reallocation, people may want to switch their shifts for personal reasons (Manjunath and Westkamp, 2021). Furthermore, nowadays several types of resources can be allocated together due to technological development, e.g., cloud computing (Ghodsi et al., 2011, 2012) and 5G network slicing (Peng et al., 2015; Bag et al., 2019; Han et al., 2019). Such situations can also be modeled as multiple-type housing markets. Thus, the analysis of multiple-type housing markets may also impact the real world.

Despite their importance and generality, there is little research on multiple-type housing markets. One main reason for this is that for multiple-type housing markets, even with additively separable preferences, the strict core may be empty, and there is no mechanism satisfying *Pareto efficiency*, *individual rationality*, and *strategy-proofness* (Konishi et al., 2001).

One could conjecture that for multiple-type housing markets, some extensions of the TTC mechanism are still desirable, even if they cannot satisfy all of the desirable properties above. For instance, when preferences are strict, one extension, the bundle TTC (bTTC) mechanism is *group strategy-proof* (Feng et al., 2022b). When preferences are also separable, the coordinatewise TTC (cTTC) mechanism, is *coalitional proof* (Wako, 2005),[5] and *second-best incentive compatible* (Klaus, 2008).[6] When preferences are also lexicographic,[7] the multi-type TTC (mTTC) mechanism, always assigns a strict core allocation (Sikdar et al., 2017).

However, the works above also lead to a new challenge: to distinguish different extensions of the TTC mechanism from a normative point of view. More precisely, how can we make a compelling argument for one extension over another? The goal of this paper is to shed light on desirable properties of different TTC extensions and, in particular, to distinguish between bTTC and cTTC mentioned above.[8]

[5]Feng et al. (2022a) refer to this property as *effective group strategy-proofness*.

[6]A mechanism is *second-best incentive compatible* if it is not Pareto dominated by another *strategy-proof* mechanism.

[7]When preferences are lexicographic, the strict core is non-empty.

[8]In this paper, we only focus on *strategy-proof* mechanisms, Thus we do not consider the mTTC mechanism as it is not *strategy-proof*.

2 Our contributions

For Shapley-Scarf housing markets, Fujinaka and Wakayama (2018) propose a new incentive property, *endowments-swapping-proofness*,[9] that excludes manipulations that a pair of agents can conduct before the operation of the selected mechanism by swapping their endowments.[10] They also show that the TTC mechanism is the only mechanism satisfying *individual rationality*, *strategy-proofness* and *endowments-swapping-proofness*.

Inspired by Fujinaka and Wakayama (2018), we also investigate this property for multiple-type housing markets. However, differing from Shapley-Scarf housing markets, for multiple-type housing markets, agents can swap their endowments in various ways. For instance, they can: swap their endowments completely; swap only one type of endowments; swap more than one (but not all) types of endowments. Motivated by this observation, we introduce three extensions of *endowments-swapping-proofness*: *bundle endowments-swapping-proofness*, *one type endowments-swapping-proofness*, and *flexible endowments-swapping-proofness*. Note that our concern with ex-ante swapping is also of interest in real life, e.g., it help us to rule out some collusion that is outside the designer's control.

We make four contributions to the literature. Our first and most direct contribution is axiomatic.[11] Based on two of our new properties (*bundle endowments-swapping-proofness* and *one type endowments-swapping-proofness*), together with other well-studied properties (*individual rationality*, *strategy-proofness*, and *non-bossiness*), we provide characterizations of bTTC and cTTC on several domains of preference profiles. Such characterizations help us to find the difference between these two mechanisms. Also, our characterization results help us to verify the trade-off between desirable properties, e.g., the (in)compatibility between incentive properties and efficiency properties.

Additionally, based on our other new property, *flexible endowments-swapping-proofness*, we discuss the cost of adding *strategy-proofness* for multiple-type housing markets by providing several impossibility results. Our findings suggest that it could even be difficult to find a reasonable stability notion (stronger than *individual rationality*), that is compatible with *strategy-proofness*.

Moreover, it is worth noting that on the full domain of strict preference profiles, i.e., when agents' preferences are strict but otherwise unrestricted, our characterizations of bTTC consti-

[9]Gale (1974) first proposes the notion of the ex-ante manipulation via endowments. Moulin (1995) proposes *endowments-swapping-proofness* in Shapley-Scarf housing markets, but Fujinaka and Wakayama (2018) formally define and study it.

[10]Apart from Fujinaka and Wakayama (2018), there are many studies that have also examined ex-ante manipulation via endowments in different settings, e.g., Postlewaite (1979); Atlamaz and Klaus (2007) in exchange economics, Sertel (1994) in public good economies, and Sertel and Özkal-Sanver (2002); Fiestras-Janeiro et al. (2004) in two-sided matching problems.

[11]An axiomatic study often identifies a particular mechanism as the only one satisfying a class of normative properties. For the meaningfulness of the axiomatic studies, see the discussion in Thomson (2001).

tute the first characterizations of an extension of the prominent TTC mechanism to multiple-type housing markets. The analysis on the domain of strict preference profiles is demanding, because it allows agents' preferences to be "complementary". Thus, our results also contribute to the literature on allocation problems with complements, e.g., Sun and Yang (2006), Rostek and Yoder (2020), and Jagadeesan and Teytelboym (2021).

Finally, the standard proof methods for Shapley-Scarf housing markets cannot be generalized to our higher dimensional model. Hence, we need to develop a different approach. In particular, similar to Feng et al. (2022a), we use a novel proof strategy to build results for lexicographic preferences and a method to extend the results from lexicographic preferences to separable preferences and strict preferences. Our extension method is potentially useful for addressing issues in markets other than multiple-type housing markets. Therefore, methodologically, we view our new approach as a separate contribution.

3 Organization

Our paper is organized as follows. In the following section, Section 4, we introduce multiple-type housing markets, mechanisms and their properties, and two extensions of the TTC mechanism, bTTC and cTTC.

We state our results in Section 5.

- In Subsection 5.1, we show several impossibility results related to *flexible endowments-swapping-proofness* (Theorems 1 and 2).
- In Subsection 5.2, we show that for lexicographic preferences, separable preferences, and strict preferences, a mechanism is *individually rational*, *strategy-proof*, *non-bossy*, and *bundle endowments-swapping-proof* if and only if it is bTTC (Theorems 3, 4, and 5).
- In Subsection 5.3, we show that for lexicographic preferences and separable preferences, a mechanism is *individually rational*, *strategy-proof*, and *one type endowments-swapping-proof* if and only if it is cTTC (Theorems 6 and 7), and on the domain of strict preference profiles, these three properties are incompatible (Theorem 8).
- Moreover, in Subsection 5.4, by providing two characterization results (Theorems 9 and 10), we discuss the relation between our new incentive properties and efficiency properties.

In Section 6, we conclude with a discussion of our results and how they relate to the literature. In Appendix A, we provide the proofs of our results that are not included in the main text. In Appendix B, we provide several examples to establish the logical independence of the properties in our characterization results.

4 Preliminaries

The basic model setup is similar to Feng and Klaus (2022).

4.1 Multiple-type housing markets

We consider a barter economy without monetary transfers formed by n agents and $n \times m$ indivisible objects. Let $N = \{1, \ldots, n\}$ be a finite *set of agents*. A nonempty subset of agents $S \subseteq N$ is a *coalition*. We assume that there exist $m \geq 1$ *(distinct) types of indivisible objects* and n *(distinct) indivisible objects of each type*. We denote the *set of types* by $T = \{1, ..., m\}$. For each $t \in T$, the set of type-t objects is $O^t = \{o_1^t, ..., o_n^t\}$, and the *set of all objects* is $O = \{o_1^1, o_1^2, \ldots, o_n^1, o_n^2, \ldots, o_n^m\}$. In particular, $|O| = n \times m$. Each agent $i \in N$ initially owns exactly one object of each type $t \in T$, denoted by e_i^t. Hence, each *agent i's endowment* is a list $e_i = (e_i^1, \ldots, e_i^m)$. Moreover, each agent $i \in N$ exactly wishes to consume one object of each type, and hence, each agent i's *(feasible) consumption set* is $\Pi_{t \in T} O^t$. An element in $\Pi_{t \in T} O^t$ is a *(consumption) bundle*. Note that for $m = 1$, our model equals the classical Shapley-Scarf housing market model (Shapley and Scarf, 1974).

An *allocation* x partitions the set of all objects O into n bundles to agents. Formally, $x = \{x_1, \ldots, x_n\}$ is such that for each $t \in T$, $\cup_{i \in N} x_i^t = O^t$ and for each pair $i \neq j$, $x_i^t \neq x_j^t$. The *set of all allocations* is denoted by X, and the *endowment allocation* is denoted by e. Given an allocation $x \in X$, for each agent $i \in N$, we say that x_i is agent i's *allotment* (at x) and for each type $t \in T$, x_i^t is i's *type-t allotment*. For simplicity, sometimes we will restate an allocation as a list $x = (x_1, \ldots, x_n) \in (\Pi_{t \in T} O^t)^N$. Given x, we define $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ to be the list of all agents' allotments, except for agent i's allotment; and $x_S = (x_i)_{i \in S}$ to be the list of allotments of members in coalition S.

First, we assume that each agent i has *complete*, *antisymmetric*, and *transitive preferences* R_i over all bundles (allotments), i.e., R_i is a linear order over $\Pi_{t \in T} O^t$.[12] For two allotments x_i and y_i, x_i is *weakly better than* y_i if $x_i \, R_i \, y_i$, and x_i is *strictly better than* y_i if $[x_i \, R_i \, y_i$ and not $y_i \, R_i \, x_i]$, denoted $x_i \, P_i \, y_i$. Finally, since preferences over allotments are strict, x_i is indifferent to y_i only if $x_i = y_i$. We denote preferences as ordered lists, e.g., $R_i : x_i, \; y_i, \; z_i$ instead of $x_i \, P_i \, y_i \, P_i \, z_i$. The *set of all preferences* is denoted by $\mathcal{R}$, which we will also refer to as the *strict preference domain*.

Furthermore, we assume that when facing an allocation x, there are no consumption externalities and each agent $i \in N$ only cares about his own allotment x_i. Hence, each agent i's preferences over allocations X are essentially equivalent to his preferences over allotments $\Pi_{t \in T} O^t$. With some abuse of notation, we use notation R_i to denote an agent i's preferences

[12] Preferences R_i are *complete* if for any two allotments x_i, y_i, $x_i \; R_i \; y_i$ or $y_i \; R_i \; x_i$; they are *antisymmetric* if $x_i \, R_i \, y_i$ and $y_i \, R_i \, x_i$ imply $x_i = y_i$; and they are *transitive* if for any three allotments x_i, y_i, z_i, $x_i \, R_i \, y_i$ and $y_i \, R_i \, z_i$ imply $x_i \; R_i \; z_i$.

over allotments as well as his preferences over allocations, i.e., for each agent $i \in N$ and for any two allocations $x, y \in X$, $x \, R_i \, y$ if and only if $x_i \, R_i \, y_i$.[13]

A *preference profile* specifies preferences for all agents and is denoted by a list $R = (R_1, \ldots, R_n) \in \mathcal{R}^N$. We use the standard notation $R_{-i} = (R_1, \ldots, R_{i-1}, R_{i+1}, \ldots, R_n)$ to denote the list of all agents' preferences, except for agent i's preferences. Furthermore, for each coalition S we define $R_S = (R_i)_{i \in S}$ and $R_{-S} = (R_i)_{i \in N \setminus S}$ to be the lists of preferences of the members in coalitions S and $N \setminus S$, respectively.

In addition to the domain of strict preferences, we consider several preference subdomains based on agents' "marginal preferences": assume that for each $i \in N$ and for each type $t \in T$, agent i has complete, antisymmetric, and transitive preferences R_i^t over the set of type-t objects O^t. We refer to R_i^t as *agent i's type-t marginal preferences*, and denote by $\mathcal{R}^t$ the *set of all type-t marginal preferences.* Then, we can define the following two preference domains.

(Strictly) Separable preferences. Agent i's preferences $R_i \in \mathcal{R}$ are *separable* if for each $t \in T$ there exist type-t marginal preferences $R_i^t \in \mathcal{R}^t$ such that for any two allotments x_i and y_i,

$$\text{if for all } t \in T, \; x_i^t \, R_i^t \, y_i^t, \; \text{then } x_i \, R_i \, y_i.$$

$\mathcal{R}_s$ denotes the *domain of separable preferences.*

We use the standard notation $R^t = (R_1^t, \ldots, R_n^t)$ to denote the list of all agents' marginal preferences of type-t, and $R^{-t} = (R^1, \ldots, R^{t-1}, R^{t+1}, \ldots, R^m)$ to denote the list of all agents' marginal preferences of all types except for type-t.

Before defining our next preference domain, we introduce some notation. We use a bijective function $\pi_i : T \to T$ to order types according to agent i's "(subjective) importance," with $\pi_i(1)$ being the most important and $\pi_i(m)$ being the least important object type. We denote π_i as an ordered list of types, e.g., by $\pi_i = (2, 3, 1)$, we mean that $\pi_i(1) = 2$, $\pi_i(2) = 3$, and $\pi_i(3) = 1$. So for each agent $i \in N$ and each allotment $x_i = (x_i^1, \ldots, x_i^m)$, by $x_i^{\pi_i} = (x_i^{\pi_i(1)}, \ldots, x_i^{\pi_i(m)})$ we denote the allotment after rearranging it with respect to the *object-type importance order* π_i.

(Separably) Lexicographic preferences. Agent i's preferences $R_i \in \mathcal{R}$ are *(separably) lexicographic* if they are separable with type-t marginal preferences $(R_i^t)_{t \in T}$ and there exists an object-type importance order $\pi_i : T \to T$ such that for any two allotments x_i and y_i,

$$\begin{aligned}
&\text{if } x_i^{\pi_i(1)} P_i^{\pi_i(1)} y_i^{\pi_i(1)} \text{ or} \\
&\text{if there exists a positive integer } k \leq m-1 \text{ such that} \\
&x_i^{\pi_i(1)} = y_i^{\pi_i(1)}, \; \ldots, \; x_i^{\pi_i(k)} = y_i^{\pi_i(k)}, \; \text{and } x_i^{\pi_i(k+1)} P_i^{\pi_i(k+1)} y_i^{\pi_i(k+1)}, \\
&\text{then } x_i \, P_i \, y_i.
\end{aligned}$$

[13]Note that when extending strict preferences over allotments to preferences over allocations without consumption externalities, strictness is lost because any two allocations where an agent gets the same allotment are indifferent to that agent.

$\mathcal{R}_l$ denotes the *domain of lexicographic preferences.*

Note that $R_i \in \mathcal{R}_l$ can be restated as a $m+1$-tuple $R_i = (R_i^1, \ldots, R_i^m, \pi_i) = ((R_i^t)_{t \in T}, \pi_i)$, or a strict ordering of all objects,[14] i.e., R_i lists first all $\pi(1)$ objects (according to $R_i^{\pi(1)}$), then all $\pi(2)$ objects (according to $R_i^{\pi(2)}$), and so on. We provide a simple illustration in Example 1.

Note that when $|T| = m > 1$,

$$\mathcal{R}_l \subsetneq \mathcal{R}_s \subsetneq \mathcal{R}.$$

A *(multiple-type housing) market* is a triple (N, e, R); as the set of agents N remains fixed throughout, we will simply denote the market (N, e, R) by a pair (e, R). Let $\mathcal{M}$ be the set of all markets with strict preferences, i.e., for each $(e, R) \in \mathcal{M}$, $R \in \mathcal{R}^N$. Similarly, let $\mathcal{M}_s$ / $\mathcal{M}_l$ be the set of all markets with separable preferences / lexicographic preferences.

Remark 1 (**Lexicographic preferences**).
There are several examples of markets with lexicographic preferences. Each professor at a university may initially have a course to teach per semester, and some professors may prefer to exchange their assigned courses. Professors may have expertise for certain courses and so lexicographically prefer some to others. For instance, the administrative office determines the class schedule for four courses, "microeconomics" and "mathematical economics" in fall, and "macroeconomics" and "econometrics" in spring. A professor who is a game theorist prefers to teach in fall and a professor who focuses on empirical asset pricing prefers to teach in spring. Similarly, students at a university wish to reschedule their courses by exchanging their assigned slots, and students may strongly prefer certain timings, e.g., a party boy does not like to attend the class on Friday.

Moreover, from a theoretical point of view, the domain of lexicographic preference profiles is a good starting point for our analysis. First, this domain is well-studied in the literature, e.g., Altuntaş et al. (2021) and Biró et al. (2022b). Thus, it is easy to compare our results with other studies on the domain of lexicographic preference profiles. Second, it is well-known that (i) for Shapley-Scarf housing markets, the top trading cycle (TTC) mechanism coincides with the strict core-stable mechanism; and (ii) for multiple-type housing markets with lexicographic preferences, the strict core is non-empty. Thus, starting from the lexicographic preference domain can help us to understand the difference between strict core-stable mechanisms and the extensions of TTC mechanisms for multiple-type housing markets. ◇

4.2 Mechanisms and their properties

Note that all following definitions for the set of markets with strict preferences ($\mathcal{M}$) can alternatively be formulated for markets with separable preferences ($\mathcal{M}_s$) or with lexicographic

[14] See Feng and Klaus (2022, Remark 1) for details.

preferences ($\mathcal{M}_l$).

A mechanism is a function $f : \mathcal{M} \to X$ that assigns to each market (e, R) an allocation $f(e, R) \in X$, and

- for each $i \in N$, $f_i(e, R)$ denotes agent i's allotment
- for each $i \in N$ and each $t \in T$, $f_i^t(e, R)$ denotes agent i's type-t allotment

under mechanism f at (e, R).

We next introduce and discuss some well-known properties for allocations and mechanisms. Let $(e, R) \in \mathcal{M}$.

Welfare lower bounds

First we consider a voluntary participation condition for an allocation x to be implementable without causing agents any harm: no agent will be worse off than at his endowment.

Definition (Individual rationality).
An allocation $x \in X$ is *individually rational* if for each agent $i \in N$, $x_i \; R_i \; o_i$. A mechanism is *individually rational* if for each market, it assigns an individually rational allocation.

Efficiency notions

Next, we consider two well-known efficiency criteria.

Definition (Pareto efficiency).
An allocation $y \in X$ *Pareto dominates* allocation $x \in X$ if for each agent $i \in N$, $y_i \; R_i \; x_i$, and for at least one agent $j \in N$, $y_j \; P_j \; x_j$. An allocation $x \in X$ is *Pareto efficient* if there is no allocation $y \in X$ that Pareto dominates it. A mechanism is *Pareto efficient* if for each market, it assigns a Pareto efficient allocation.

Definition (Unanimity).
An allocation $x \in X$ is *unanimously best* if for each agent $i \in N$ and each allocation $y \in X$, we have $x \; R_i \; y$.[15] A mechanism is *unanimous* if, for each market, it assigns only the unanimously best allocation whenever it exists.

If a *unanimously best* allocation exists for at (e, R), then that allocation is the only *Pareto efficient* allocation at (e, R). Hence, *Pareto efficiency* implies *unanimity*.

[15] Since all preferences are strict, the set of unanimously best allocations is empty or single-valued.

Incentive properties

Next, we discuss some incentive properties for mechanisms that require that no agent / group of agents can benefit from untruthful actions.

The next two properties, *strategy-proofness* and *group strategy-proofness*, are two of the properties that are most frequently used in the literature on mechanism design. They model that no agent / coalition can benefit from misrepresenting his / their preferences.

Definition (**Strategy-proofness**).
A mechanism f is *strategy-proof* if for each market $(e, R) \in \mathcal{M}$, each agent $i \in N$, and each preference relation $R'_i \in \mathcal{R}$, $f_i(e, R_i, R_{-i})\, R_i\, f_i(e, R'_i, R_{-i})$, i.e., agent i cannot manipulate mechanism f at R via R'_i.

Definition (**Group strategy-proofness**).
A mechanism f is *group strategy-proof* if for each each market $(e, R) \in \mathcal{M}$, there is no coalition $S \subseteq N$ and no preference list $R'_S = (R'_i)_{i \in S} \in \mathcal{R}^S$ such that for each $i \in S$, $f_i(e, R'_S, R_{-S})\ R_i\ f_i(e, R)$, and for some $j \in S$, $f_j(e, R'_S, R_{-S})\ P_j\ f_j(e, R)$, i.e., coalition S cannot manipulate mechanism f at R via R'_S.

Next, we consider a well-known property for mechanisms that restricts each agent's influence: no agent can change other agents' allotments without changing his own allotment.

Definition (**Non-bossiness**).
A mechanism f is *non-bossy* if for each each market $(e, R) \in \mathcal{M}$, each agent $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(e, R_i, R_{-i}) = f_i(e, R'_i, R_{-i})$ implies $f(e, R_i, R_{-i}) = f(e, R'_i, R_{-i})$. Otherwise it is *bossy*.

Remark 2 (**Non-bossiness**).
Non-bossiness is proposed by Satterthwaite and Sonnenschein (1981).[16] Normally, *non-bossiness* together with *strategy-proofness* is considered as an incentive property because the combination of them is equivalent to *group strategy-proofness*. Unfortunately, this equivalence is not valid for multiple-type housing markets (see Feng et al., 2022a, Proposition 1). However, *non-bossiness* still plays an important role as an incentive property. First, *non-bossiness* itself, can be considered as a "collusion-proofness" property (Miyagawa, 2001). Collusion is observed often in many marketplaces, e.g., auctions. In particular, *non-bossiness* is satisfied by the first-price auction and is violated by the second-price auction, and this is why the first-price auction is becoming increasingly popular (Pycia and Raghavan, 2022). For instance, in 2019, Google AdX changed its auction mechanism from a second-price auction to a first-price auction.

Moreover, Alva (2017, Proposition 1) shows the equivalence of (a) *effective pairwise strategy-proofness* and (b) the combination of *strategy-proofness* and *non-bossiness*.[17] Thus, his study

[16] See Thomson (2016) for an excellent survey.

[17] The term *effective pairwise strategy-proofness* is due to Serizawa (2006). It excludes joint misreports by two

provides an intuition of why the invariance property *non-bossiness* can be considered to be an incentive property as well.[18] ◇

Stability notions

Next, in order to introduce the standard cooperative solutions of the weak and the strict core, we introduce two blocking notions: at (e, R), an *allocation* $x \in X$ *is strictly blocked by coalition* $S \subseteq N$ if there exists an allocation $y \in X$ such that

(i) at allocation y agents in S reallocate their endowments, i.e., for each $i \in S$ and each $t \in T$, $y_i^t \in \Pi_{j \in S} e_j^t$ and

(ii) all agents in S are strictly better off, i.e., for each $i \in S$, $y_i \, P_i \, x_i$.

An *allocation* $x \in X$ *is weakly blocked by coalition* $S \subseteq N$ if there exists an allocation $y \in X$ such that (ii) and

(ii') all agents in S are weakly better off with at least one of them being strictly better off, i.e., for each $i \in S$, $y_i \, R_i \, x_i$, and for some $j \in S$, $y_j \, P_j \, x_j$.

Given the blocking notions above, we can restate *individual rationality* and *Pareto efficiency* as stability notions in the following way: an allocation is *individually rational* if it is not weakly or strictly blocked by any singleton coalition $\{i\}$ and an allocation is *Pareto efficient* if it is not weakly blocked by the set of all agents N.

We now introduce the first type of (possibly empty or multi-valued) solution to multiple-type housing markets that we will consider: core solutions.

Definition (Core allocation and core-stability).
An allocation is a *weak core allocation* if it is not strictly blocked by any coalition; the set of all weak core allocations is the *weak core*, and a mechanism is *weakly core-stable* if for any markets, it assigns only weak core allocations.

Similarly, an allocation is a *strict core allocation* if it is not weakly blocked by any coalition; the set of all strict core allocations is the *strict core*, and a mechanism is *strictly core-stable* if for any markets, it assigns only strict core allocations.

Next, we introduce a weaker condition than *core-stability* that considers the blocking with only one or two agents. Note that *pairwise weak stability* is appealing because in reality, it is considered to be difficult for large coalitions to block, while blocking by two agents is easy.

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agents without self-enforcing. In other words, it insists on the robustness of a pairwise misreporting to a further deviation by one of the misreported agents. For further studies of *effective pairwise strategy-proofness*, see Alva (2017) and Shinozaki (2022).

[18] Alva's result applies to our model as well (see Feng et al., 2022a, Lemma 1, for details).

Definition (Pairwise stability).

An allocation is a *pairwise weak core allocation* if it is not strictly blocked by any coalition $S \subseteq N$ with $|S| \leq 2$; the set of all pairwise weak core allocations is the *pairwise weak core*. A mechanism is *pairwise weakly stable* if for any markets, it assigns only pairwise weak core allocations.

Similarly, an allocation is a *pairwise strict core allocation* if it is not weakly blocked by any coalition $S \subseteq N$ with $|S| \leq 2$; the set of all pairwise strict core allocations is the *pairwise strict core*. A mechanism is *pairwise strictly stable* if for any markets, it assigns only pairwise strict / weak core allocations.

Next, we list some pertinent results based on the properties we introduced above.[19]

Result 1 (Strict core-stability for Shapley-Scarf housing markets)**.**

- A mechanism is *strictly core-stable* if and only if it is *individually rational*, *Pareto efficient*, and *strategy-proof* (Ma, 1994; Svensson, 1999).
- If a mechanism is *strictly core-stable*, then it is *group strategy-proof* (Bird, 1984).

Result 2 (Impossibility for multiple-type housing markets)**.**

- For markets with separable preferences $\mathcal{M}_s$, there is no mechanism satisfying *individual rationality*, *Pareto efficiency*, and *strategy-proofness* (Konishi et al., 2001).
- For markets with lexicographic preferences $\mathcal{M}_l$, there is no mechanism satisfying *individual rationality*, *Pareto efficiency*, and *strategy-proofness* (Sikdar et al., 2017).

4.3 *Endowments-swapping-proofness* and its extensions

In this subsection, we introduce our main incentive properties that exclude the manipulations of two agents by swapping their endowments before the operation of the selected mechanism. Our properties are motivated by *endowments-swapping-proofness* for Shapley-Scarf housing markets (Fujinaka and Wakayama, 2018).

Endowments-swapping-proofness for Shapley-Scarf housing markets

We first introduce *endowments-swapping-proofness* (Fujinaka and Wakayama, 2018) for Shapley-Scarf housing markets as a benchmark. As mentioned before, for $m = 1$ our model equals the classical Shapley-Scarf housing market model. Note that for Shapley-Scarf housing markets, each agent only owns one object.

[19] For Shapley-Scarf housing markets, when preferences are strict, but otherwise unrestricted, only the top trading cycles (TTC) mechanism is strictly core-stable (see Subsection 4.4 for a formal definition of the TTC mechanism).

For each endowment allocation e and a pair of agents $\{i, j\}$, let $\tilde{e}(i, j)$ be an endowment allocation such that agents i and j swap their endowments and all other agents keep their endowments, i.e., $\tilde{e}_i = e_j$, $\tilde{e}_j = e_i$, and for each agent $k \notin \{i, j\}$, $\tilde{e}_k = e_k$. For each market (e, R), let $(\tilde{e}(i, j), R)$ be the corresponding market according to this swapping.

Definition (Endowments-swapping-proofness).
A mechanism f is *endowments-swapping-proof* if for each Shapley-Scarf housing market (e, R), there is no pair of agents $\{i, j\} \subseteq N$ such that $f_i(\tilde{e}(i, j), R)\, P_i\, f_i(e, R)$ and $f_j(\tilde{e}(i, j), R)\, P_j\, f_j(e, R)$.

Extensions of *endowments-swapping-proofness* for multiple-type housing markets

Note that for Shapley-Scarf housing markets, endowments-swapping is easy because each agent only owns one object. However, for multiple-type housing markets, the situation becomes complicated: alternatively, agents can partially swap their endowments. For instance, they can: swap only one type of endowments; swap their endowments completely; swap more than one (but not all) types of endowments.

To model manipulations via endowments-swapping for multiple-type housing markets, we introduce three different ways of ex-ante endowments-swapping by a pair of agents before the operation of the selected mechanism.[20]

Given an endowment allocation e and a pair of agents $\{i, j\} \subseteq N$, we consider endowments-swapping in the following ways: (i) flexible swapping, (ii) (complete) bundle swapping, and (iii) one type swapping.

(i) Let $\bar{e}(i, j) = (\bar{e}_1, \ldots, \bar{e}_n)$ be be a *new endowment allocation by flexible swapping*, such that $\{i, j\}$ can swap any subsets of endowments types, i.e., (a) for each $k \in \{i, j\}$ and each $t \in T$, $\bar{e}_k^t \in \{e_i^t, e_j^t\}$, and (b) for each $k \notin \{i, j\}$, $\bar{e}_k = e_k$.

(ii) Let $e'(i, j) = (e'_1, \ldots, e'_n)$ be a *new endowment allocation by bundle swapping*, such that $\{i, j\}$ swap their endowments completely, i.e., (a) $e'_i = e_j, e'_j = e_i$, and (b) for each $k \notin \{i, j\}$, $e'_k = e_k$.

(iii) Let $\hat{e}(t, i, j) = (\hat{e}_1, \ldots, \hat{e}_n)$ be a *new endowment allocation by one type swapping of type-t*, such that $\{i, j\}$ only swap their type-t endowments, i.e., (a.1) $\hat{e}_i = (e_i^1, \ldots, e_i^{t-1}, e_j^t, e_i^{t+1}, \ldots, e_i^m)$, (a.2) $\hat{e}_j = (e_j^1, \ldots, e_j^{t-1}, e_i^t, e_j^{t+1}, \ldots, e_j^m)$, and (b) for each $k \notin \{i, j\}$, $\hat{e}_k = e_k$.

Next, we introduce three properties of immunity to the three manipulations introduced above.

Definition (Flexible endowments-swapping-proofness).
A mechanism f is *flexibly endowments-swapping-proof* if for each $(e, R) \in \mathcal{M}$, there is no pair of agents $\{i, j\} \subseteq N$ and no $(\bar{e}(i, j), R)$ such that $f_i(\bar{e}(i, j), R)\, P_i\, f_i(e, R)$ and $f_j(\bar{e}(i, j), R)\, P_j\, f_j(e, R)$.

[20]Similar to Fujinaka and Wakayama (2018, Remark 4), our incentive properties and corresponding results can be extended to any coalition. That is, the restriction for two agents can be relaxed for any numbers of agents.

Definition (Bundle endowments-swapping-proofness).
A mechanism f is *bundle endowments-swapping-proof* if for each $(e, R) \in \mathcal{M}$ there is no pair of agents $\{i, j\} \subseteq N$ such that $f_i(e'(i, j), R)\ P_i\ f_i(e, R)$ and $f_j(e'(i, j), R)\ P_j\ f_j(e, R)$.

Definition (One type endowments-swapping-proofness).
A mechanism f is *one type endowments-swapping-proof* if for each $(e, R) \in \mathcal{M}$, there is no pair of agents $\{i, j\} \subseteq N$ and no type $t \in T$, such that $f_i(\hat{e}(t, i, j), R)\, P_i\, f_i(e, R)$ and $f_j(\hat{e}(t, i, j), R)\, P_j$ $f_j(e, R)$.

Discussion of our endowments-swapping properties and their implicit constraints

Our three properties above are extensions of *endowments-swapping-proofness* for Shapley-Scarf housing markets (Fujinaka and Wakayama, 2018). When $|T| = m = 1$, all three properties are equivalent. By definition, it is easy to see that *flexible endowments-swapping-proofness* is the strongest property, i.e., *flexible endowments-swapping-proofness* implies *bundle endowments-swapping-proofness* and *one type endowments-swapping-proofness*. However, there is no logical relation between *bundle endowments-swapping-proofness* and *one type endowments-swapping-proofness*. For instance, our results in Section 5 show that bTTC is *bundle endowments-swapping-proof* but not *one type endowments-swapping-proof*, while cTTC is *one type endowments-swapping-proof* but not *bundle endowments-swapping-proof*.[21]

Similar but different to ours, there are some properties that rule out other forms of ex-ante manipulation via endowments, such as *transfer-proofness* (Atlamaz and Klaus, 2007) and *non-manipulability via pre-arranged matches* (Sönmez, 1999a; Kesten, 2012).[22] It is worth mentioning that concerns about these types of ex-ante manipulation are often present in real markets (see Kojima and Pathak, 2009, for details).

Our first property, *flexible endowments-swapping-proofness* is a natural extension because it allows agents to swap freely, i.e., there is no restriction on how to swap. However, together with other desirable properties (*individual rationality* and *strategy-proofness*), this strong property leads to an impossibility (Theorem 1). To avoid this impossibility, we may want to weaken *flexible endowments-swapping-proofness*. Thus, we propose two natural ways of restricting ex-ante swapping and two corresponding weaker properties, *bundle endowments-swapping-proofness* and *one type endowments-swapping-proofness*: for *bundle endowments-swapping-proofness*, agents can only "bundle swap", whereas for *one type endowments-swapping-proofness*, agents can only "one type swap".

One may argue that such ex-ante swapping constraints are strong. However, note that for many models with multi-unit demands, constraints on how to trade are necessary to guarantee

[21] The definitions of these two mechanisms are in Subsection 4.4.

[22] *Transfer-proofness* says that no pair of agents can be better off by transferring part of one's endowment to another. *Non-manipulability via pre-arranged matches* means that no pair of a college and a student can benefit by agreeing to match before receiving their allocations from the centralized matching mechanism.

positive results, e.g., Pápai (2007); Raghavan (2015), and Shinozaki and Serizawa (2022). On the other hand, in practice, such constraints are weaker than they may seem at first. For instance, in the context of course allocation problems (e.g., Klaus, 2008; Mackin and Xia, 2016) and emergency medicine rotation problems (e.g., Manjunath and Westkamp, 2021), swapping slots arbitrarily might be impossible due to administrative rules, i.e., to avoid systemic complexity, the administration might only allow agents to swap their full bundles of slots (or just one slot). Moreover, we also show that when we keep other desirable properties, such constraints are sufficient and necessary to preserve some efficiency properties (see Subsection 5.4 for details).

4.4 Extensions of the top trading cycles mechanism

Note that two of our endowments-swapping constraints also imply two ways of trading: agents can trade their endowments completely, or they can trade their endowments type by type, e.g., trade your house with others' houses and trade your car with others' cars. To simplify notation, we first introduce an extension of the top trading cycles (TTC) mechanism with "type-by-type" trading.

Definition (The type-t top trading cycles (TTC) algorithm).
Consider a market (N, e, R) such that $R \in \mathcal{R}_s$. For each type $t \in T$, let $(N, e^t, R^t) = (N, (e_1^t, \ldots, e_n^t), (R_1^t, \ldots, R_n^t))$ be its *associated type-t submarket*.

We define the top trading cycles (TTC) allocation for each type-t submarket as follows.

Input. A type-t submarket (N, e^t, R^t).

Step 1. Let $N_1 := N$ and $O_1^t := O^t$. We construct a directed graph with the set of nodes $N_1 \cup O_1^t$. For each agent $i \in N_1$, there is an edge from the agent to his most preferred type-t object in O_1^t according to R_i^t. For each edge (i, o) we say that agent i points to type-t object o. For each type-t object $o \in O_1^t$, there is an edge from the object to its owner.

A *trading cycle* is a directed cycle in the graph. Given the finite number of nodes, at least one trading cycle exists. We assign each agent in a trading cycle the type-t object he points to and remove all trading cycle agents and type-t objects. If there are some agents (and hence objects) left, we continue with the next step. Otherwise we stop.

Step k. Let N_k be the set of agents that remain after Step $k-1$ and O_k^t be the set of type-t objects that remain after Step $k-1$. We construct a directed graph with the set of nodes $N_k \cup O_k^t$. For each agent $i \in N_k$, there is an edge from the agent to his most preferred type-t object in O_k^t according to R_i^t. For each type-t object $o \in O_k^t$, there is an edge from the object to its owner. At least one trading cycle exists and we assign each agent in a trading cycle the type-t object he points to and remove all trading cycle agents and objects. If there are some agents (and hence objects) left, we continue with the next step. Otherwise we stop.

Output. The type-t TTC algorithm terminates when each agent in N is assigned an object in

O^t, which takes at most n steps. We denote the object in O^t that agent $i \in N$ obtains in the type-t TTC algorithm by $TTC_i^t(e^t, R^t)$ and the final type-t allocation by $TTC^t(e^t, R^t)$.

Definition (The cTTC allocation / mechanism).
The *coordinatewise top trading cycles (cTTC) allocation*, $cTTC(R)$, is the collection of all type-t TTC allocations, i.e., for each $R \in \mathcal{R}_s^N$,

$$cTTC(e,R) = \left(\left(TTC_1^1(e^1,R^1),\dots,TTC_1^m(e^m,R^m)\right),\dots,\left(TTC_n^1(e^1,R^1),\dots,TTC_n^m(e^m,R^m)\right) \right).$$

The *cTTC* mechanism (introduced by Wako, 2005) assigns each market its cTTC allocation.

Next, we consider another extension of the TTC mechanism, which only allows agents to trade their endowments completely.

Definition (The bundle top trading cycles (bTTC) algorithm / mechanism).
The *bundle top trading cycles (bTTC) mechanism* assigns to each market $(e, R) \in \mathcal{M}$ the unique top-trading allocation that results from the TTC algorithm if agents are only allowed to trade their whole endowments among each other.

Formally, for each market (e, R) and $i \in N$, let $R_i|^e$ be the restriction[23] of R_i to endowments $\{e_1, \dots, e_n\}$ and $R|^e \equiv (R_i|^e)_{i\in N}$ be the restriction profile. We then use the TTC algorithm to compute the bTTC allocation for $(e, R|^e)$. Note that the difference with the classical TTC algorithm (for Shapley-Scarf housing markets) is that instead of an object, each agent can only point to a whole endowment.

The *bTTC* mechanism assigns the bTTC allocation above to each market.

Remark 3. Note that bTTC is also well-defined on the domain of strict preference profiles as well as lexicographic preference profiles and separable preference profiles. Meanwhile, cTTC is only well-defined for lexicographic preferences and separable preferences. Also note that for $m = 1$, cTTC and bTTC reduce to the standard TTC mechanism. ◇

5 Results

We now focus on the multiple-type extension of the Shapley-Scarf housing market model as introduced by Moulin (1995), where $|N| = n \geq 2$ and $|T| = m \geq 2$.[24]

[23]That is, for each $i \in N$, $R_i|^e$ are preferences over $\{e_1, \dots, e_n\}$ such that for each $e_j, e_k \in \{e_1, \dots, e_n\}$, $e_j \; R_i|^e \; e_k$ if and only if $e_j \; R_i \; e_k$.

[24]One agent multiple-type housing market problems are rather trivial since no trade occurs, and for just one object type we are back to the Shapley-Scarf housing market model.

5.1 *Flexible endowments-swapping-proofness*

In this subsection, we focus on the results related to *flexible endowments-swapping-proofness*. Note that our results in this subsection are also valid for separable preferences and strict preferences.

The next result shows that *flexible endowments-swapping-proofness* is strong: it is incompatible with *individual rationality* and *strategy-proofness*.

Theorem 1. *For multiple-type housing markets with lexicographic preferences $\mathcal{M}_l$, there is no mechanism satisfying individual rationality, strategy-proofness, and flexible endowments-swapping-proofness.*

We prove Theorem 1 in Appendix A.2 by a counterexample. Note that the independence of *strategy-proofness* from the other properties in Theorem 1 is an open problem.

Inspired by Theorem 1, we also find an incompatibility between *pairwise weak stability* and *strategy-proofness*.[25]

Theorem 2. *For multiple-type housing markets with lexicographic preferences $\mathcal{M}_l$, there is no mechanism satisfying pairwise weak stability and strategy-proofness.*

We prove Theorem 2 in Appendix A.2 by a counterexample.

Remark 4 (A trade off between *strategy-proofness* and *stability* notions). The (in)compatibility between *strategy-proofness* and *core-stability* has received considerable attention in economic theory. In a seminal paper, Sönmez (1999b) discusses the link between *strategy-proofness* and the single-valuedness strict core. More precisely, he shows that a *strategy-proofness* mechanism always chooses a strict core allocation only if the strict core is (essentially) single-valued.[26] However, for multiple-type housing markets, usually the strict core (if it is no-empty) is not single-valued (Konishi et al., 2001; Sikdar et al., 2017). Also, note that Sönmez (1999b)'s result relies on his preference domain richness conditions. Without Sönmez (1999b)'s preference domain richness conditions, Feng and Klaus (2022) show that *strict core-stability* and *strategy-proofness* are still incompatible. In the presence of *strategy-proofness*, to avoid this impossibility, it is natural to weaken *core-stability*. A possibility is weakening *core-stability* to the combination of *individual rationality* and *Pareto efficiency*. However, this does not work, i.e., there is no mechanism satisfying *individual rationality*, *Pareto efficiency*, and *strategy-proofness* (see Result 2). Here, we consider another way, weakening *core-stability* to *pairwise weak stability*. Theorem 2 shows that this does not work either. Therefore, Theorem 2 can be considered as a complement to Result 2. Furthermore, Result 2 and Theorem 2 together

[25]We discuss the relation between *flexible endowments-swapping-proofness* and *pairwise weak stability* in Subsection 6.1.

[26]The strict core is *(essentially) single-valued* if all agents are indifferent between all strict core allocations.

indicate that it could be difficult to find a reasonable stability notion (stronger than *individual rationality*),[27] that is compatible with *strategy-proofness*.

Overall, together with Result 2, our result (Theorem 2) explores the cost of *strategy-proofness*. If the mechanism designer chooses an *individually rational* and *strategy-proof* mechanism, then he has to accept that, the corresponding allocation might be inefficient and unstable: some agents might realize that they could be better off by pairwise trade and the market might be thin. ◇

5.2 *Bundle endowments-swapping-proofness*

In this subsection, we focus on the results related to *bundle endowments-swapping-proofness*.

5.2.1 Characterizing bTTC for lexicographic preferences

We first characterize bTTC for lexicographic preferences.

Theorem 3. *For markets with lexicographic preferences $\mathcal{M}_l$, bTTC is the only mechanism satisfying individual rationality, strategy-proofness, non-bossiness, and bundle endowments-swapping-proofness.*

We prove Theorem 3 in Appendix A.5. Here we only explain the intuition of the uniqueness part of the proof. Consider a top trading cycle that forms at the first step of the bTTC algorithm for (e, R). We first show that, by *individual rationality*, *strategy-proofness*, and *bundle endowments-swapping-proofness*, agents in this top trading cycle receive their most preferred object (Lemma 2). Next, we show that by *individual rationality*, *strategy-proofness*, and *non-bossiness*, agents in this top trading cycle receive their bTTC allotments (Lemma 3). Once we have shown that agents who trade at the first step of the bTTC algorithm always receive their bTTC allotments, we can consider agents who trade at the second step of the bTTC algorithm by following the same proof arguments as for first step trading cycles, and so on.

Note that since bTTC is *group strategy-proof* (Feng et al., 2022b), we also have the following result.

Corollary 1. *For markets with lexicographic preferences $\mathcal{M}_l$, bTTC is the only mechanism satisfying individual rationality, group strategy-proofness, and bundle endowments-swapping-proofness.*

5.2.2 Characterizing bTTC for separable preferences and strict preferences

Next, we use the result for lexicographic preferences (Theorem 3) as a "stepping stone" to obtain corresponding results for separable preferences strict preferences.

[27] As we mentioned earlier, *Pareto efficiency* can also be considered as a stability notion via a weak blocking notion.

Theorem 4. *For markets with separable preferences $\mathcal{M}_s$, bTTC is the only mechanism satisfying individual rationality, strategy-proofness, non-bossiness, and bundle endowments-swapping-proofness.*

Theorem 5. *For markets with strict preferences $\mathcal{M}$, bTTC is the only mechanism satisfying individual rationality, strategy-proofness, non-bossiness, and bundle endowments-swapping-proofness.*

We prove Theorems 4 and 5 in Appendix A.5. The proof of Theorem 4 works as follows. We first consider a preference profile such that only one agent has separable and non-lexicographic preferences. We show that for this agent, if he (mis)reports lexicographic preferences without changing his marginal preferences, then he must receive the same allotment. According to Theorem 3, the allotment (in fact, the whole allocation) then equals the bTTC allotment (allocation). Hence, f assigns the bTTC allocation if all but one agent have lexicographic preferences. By applying this preference replacement argument, one by one, for all other agents, we conclude that f equals bTTC on the domain of separable preference profiles. Also, Theorem 5 can be proven by exactly the same way above.

Note that Theorems 4 and 5 imply that Corollary 1 is valid for separable preferences and strict preferences as well.

Corollary 2. *For markets with separable preferences $\mathcal{M}_s$, bTTC is the only mechanism satisfying individual rationality, group strategy-proofness, and bundle endowments-swapping-proofness.*

Corollary 3. *For markets with strict preferences $\mathcal{M}$, bTTC is the only mechanism satisfying individual rationality, group strategy-proofness, and bundle endowments-swapping-proofness.*

5.3 *One type endowments-swapping-proofness*

In this subsection, we focus on the results related to *one type endowments-swapping-proofness*.

5.3.1 Characterizing cTTC for lexicographic preferences

We first characterize cTTC for lexicographic preferences.

Theorem 6. *For markets with lexicographic preferences $\mathcal{M}_l$, cTTC is the only mechanism satisfying individual rationality, strategy-proofness, and one type endowments-swapping-proofness.*

We prove Theorem 6 in Appendix A.6. The uniqueness part of the proof consists of two steps. We first consider a restricted domain of preference profiles such that all agents share the same importance order. We show that f equals cTTC on the restricted domain (Proposition 1). Then we extend this result to the full domain of lexicographic preference profiles by showing that changes in agents' importance orders do not affect the allocation.

5.3.2 Characterizing cTTC for separable preferences

Next, we show that Theorem 6 also holds on the domain of separable preference profiles.

Theorem 7. *For markets with separable preferences $\mathcal{M}_s$, cTTC is the only mechanism satisfying individual rationality, strategy-proofness, and one type endowments-swapping-proofness.*

The intuition of the proof is similar to Theorem 4: by replacing agents' preferences, one by one, from the domain of lexicographic preference profiles to the domain of separable preference profiles, we extend Theorem 6 to the domain of separable preference profiles.

5.3.3 An impossibility for strict preferences

Note that cTTC is not well-defined for strict preferences. Then, a natural question is if there exists an extension of cTTC for strict preferences that satisfies our properties. Our answer is negative: there is no extension of cTTC to the domain of strict preference profiles that satisfies our properties.

Theorem 8. *For markets with strict preferences $\mathcal{M}$, there is no mechanism satisfying individual rationality, strategy-proofness, and one type endowments-swapping-proofness.*

We prove Theorem 8 in Appendix A.6 by a counterexample. Note that the independence of *strategy-proofness* from the other properties in Theorem 8 is an open problem.

5.4 Relation to efficiency

In this subsection, we further discuss the relation between two of our incentive properties (*bundle endowments-swapping-proofness* and *one type endowments-swapping-proofness*) and two efficiency properties.

Recall that bTTC and cTTC, both are *individually rational* and *strategy-proof*, but not *Pareto efficient*. Moreover, the example below shows that none of them is more efficient than the other.

Example 1. Consider a market with two agents and two types, i.e., $N = \{1, 2\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, C_1, C_2\}$, and where each agent i's endowment is $e_i = (H_i, C_i)$. The preference profile $R \in \mathcal{R}_l^N$ is as follows:

$$R_1 : H_2, \boldsymbol{H_1}, \boldsymbol{C_1}, C_2,$$

$$R_2 : C_1, \boldsymbol{C_2}, \boldsymbol{H_2}, H_1.$$

Thus, agent 1, who primarily cares for houses, would like to trade houses but not cars and agent 2, who primarily cares about cars, would like to trade cars but not houses. One easily verifies that $cTTC(e,R) = e = ((H_1, C_1), (H_2, C_2))$ and $bTTC(e,R) = x = ((H_2, C_2), (H_1, C_1))$. Since preferences are lexicographic, both agents prefer the *bTTC* allocation to the *cTTC* allocation at R.

Furthermore, consider another preference profile $R' \in \mathcal{R}_l^N$ as follows:

$$R'_1 : \boldsymbol{H_1}, H_2, C_2, \boldsymbol{C_1},$$

$$R'_2 : \boldsymbol{H_2}, H_1, C_1, \boldsymbol{C_2}.$$

Thus, both agents primarily care for houses, would like to trade cars but not houses. One easily verifies that $cTTC(e,R') = y = ((H_1, C_2), (H_2, C_1))$, $bTTC(e,R') = e = ((H_1, C_1), (H_2, C_2))$, and both agents prefer the *cTTC* allocation to the *bTTC* allocation at R'. ◇

To avoid Result 2, one may wonder whether the *bTTC* mechanism and *cTTC* mechanism satisfy some efficiency properties that are weaker than *Pareto efficiency*. To answer this question, we consider the following two efficiency properties.

The first one, *pairwise efficiency*, introduced by Ekici (2022),[28] rules out efficiency improvements by pairwise reallocation.

Definition (Pairwise efficiency)**.**
An allocation $x \in X$ is *pairwise efficient* if there is no pair of agents $\{i, j\} \subseteq N$ such that $x_j \, P_i \, x_i$ and $x_i \, P_j \, x_j$. A mechanism is *pairwise efficient* if for each market, it assigns a pairwise efficient allocation.

Next, we consider a natural modification of *Pareto efficiency* for multiple-type housing markets, *coordinatewise efficiency*, that rules out coordinatewise Pareto-improvements for any type. Note that *coordinatewise efficiency* is only well-defined for lexicographic preferences and separable preferences.

Definition (Coordinatewise efficiency)**.**
A mechanism $f : \mathcal{M}_s \to X$ is *type-t-efficient* if for each market with separable preferences, $(e, R) \in \mathcal{M}_s$, $f^t(e, R)$ is Pareto efficient at R^t. Furthermore, f is *coordinatewise-efficient* if for each type $t \in T$, f is *type-t-efficient*.

Recall that the above definitions are also valid for lexicographic preference profile domains. For $|T| = m = 1$, *coordinatewise efficiency* coincides with *Pareto efficiency*. Moreover, since the TTC mechanism is *Pareto efficient* for Shapley-Scarf housing markets, it is easy to see that for each type $t \in T$, cTTC is *type-t-efficient*, and hence is *coordinatewise efficient*.

One easily verifies that *Pareto efficiency* implies *pairwise efficiency* and *coordinatewise efficiency*, and *coordinatewise efficiency* implies *unanimity*. However, there is no logical relation between *pairwise efficiency* and *coordinatewise efficiency*.

[28] Ekici (2022) originally refers to this property as *pair efficiency*.

Next, we show that together with other desirable properties, the *bTTC* mechanism can be characterized by *pairwise efficiency*.

Theorem 9.

- *For markets with lexicographic preferences $\mathcal{M}_l$, bTTC is the only mechanism satisfying individual rationality, strategy-proofness, non-bossiness, and pairwise efficiency.*
- *For markets with separable preferences $\mathcal{M}_s$, bTTC is the only mechanism satisfying individual rationality, strategy-proofness, non-bossiness, and pairwise efficiency.*
- *For markets with strict preferences $\mathcal{M}$, bTTC is the only mechanism satisfying individual rationality, strategy-proofness, non-bossiness, and pairwise efficiency.*

We prove Theorem 9 in Appendix A.7. The proof sketch is similar to Theorem 3. That is, by *individual rationality*, *strategy-proofness*, *non-bossiness*, and *pairwise efficiency*, we show that agents who trade at the first step of the bTTC algorithm receive their bTTC allotments (Lemmas 6 and 7). Then, we use the same arguments to show that agents who at the second step, and so on.

Finally, we show that together with other desirable properties, the *cTTC* mechanism can be characterized by *coordinatewise efficiency*.

Theorem 10.

- *For markets with lexicographic preferences $\mathcal{M}_l$, cTTC is the only mechanism satisfying individual rationality, strategy-proofness, and coordinatewise efficiency.*
- *For markets with separable preferences $\mathcal{M}_s$, cTTC is the only mechanism satisfying individual rationality, strategy-proofness, and coordinatewise efficiency.*

This result can be proven similarly to Theorems 6 and 7: the only difference is that instead of *one type endowments-swapping-proofness*, we use *coordinatewise efficiency* to show that top trading cycles are executed. Thus, we omit the proof.

Note that *coordinatewise efficiency* is not well-defined on the domain of strict preference profiles, thus Theorem 10 cannot be extended to markets with strict preferences $\mathcal{M}$.

We next discuss the relation between our incentive properties and efficiency properties. It is not hard to see that our incentive properties, *bundle endowments-swapping-proofness*, and *one type endowments-swapping-proofness*, rule out some inefficient mechanisms, such as the no-trade mechanism.[29] Unfortunately, there is no direct link between any efficiency properties and our incentive properties. That is, none of our incentive properties, itself implies any efficiency

[29]The *no-trade* mechanism is a mechanism that always assigns the endowment allocation to all markets.

properties. However, together with our previous characterization results, Theorems 9 and 10 show that (i) together with *individual rationality*, *strategy-proofness*, and *non-bossiness*, *bundle endowments-swapping-proofness* is equivalent to *pairwise efficiency*, and (ii) together with *individual rationality* and *strategy-proofness*, *one type endowments-swapping-proofness* is equivalent to *coordinatewise efficiency*. In other words, in the presence of other properties we mentioned above, *bundle endowments-swapping-proofness* and *one type endowments-swapping-proofness*, each corresponds to an efficiency property.

It is worth mentioning that *bundle endowments-swapping-proofness* (*one type endowments-swapping-proofness*) and *pairwise efficiency* (*coordinatewise efficiency*) are logically independent. Ekici (2022) presents two mechanisms for Shapley-Scarf housing markets to show that *endowments-swapping-proofness* and *pairwise efficiency* are logically independent (see Ekici, 2022, Section 4.3, for details). These two mechanisms can be extended to multiple-type housing markets by only trading agents' endowment bundles. Similarly, Fujinaka and Wakayama (2018) present several mechanisms for Shapley-Scarf housing markets to show that *endowments-swapping-proofness* and *Pareto efficiency* are logically independent (see Fujinaka and Wakayama, 2018, Section 3.3, for details). These mechanisms can be extended to multiple-type housing markets by applying them coordinatewise.

Remark 5 (Constraints and efficiency).
As we mentioned earlier in Subsection 4.3, to avoid negative results, such as Result 2, it is natural to make constraints to restrict trades. For instance, Kalai et al. (1978) impose restrictions regarding trading among certain agents and Pápai (2007) restricts the set of feasible trades. However, this also induces a new problem for the mechanism designer: what constraints to enforce? In other words, if any constraint is admissible, which constraint is better? Our results partially answer this problem: if we still care about efficiency, then, without of loss other properties, allowing agents to trade their endowments completely (type by type) is sufficient and necessary to achieve *pairwise efficiency* (*coordinatewise efficiency*). ◇

Remark 6 (Second-best incentive compatibility).
Klaus (2008) weakens *Pareto efficiency* to another efficiency property, *second-best incentive compatibility*. To be precise, she shows that cTTC is *second-best incentive compatible*, i.e., there exists no other strategy-proof mechanism that Pareto dominates cTTC. However, she also shows that there exists another mechanism that is *individually rational*, *strategy-proofness*, and *second-best incentive compatible*. In a follow-up work, Anno and Kurino (2016) consider *second-best incentive compatibility* for *independent* mechanisms.[30] They also show that cTTC is not the unique *independent* mechanism that satisfies these properties. Thus, Theorem 10 can be considered as a complement to Klaus (2008) and Anno and Kurino (2016)'s work: by strengthening *second-best incentive compatibility* to *coordinatewise efficiency*, we find that cTTC is

[30] An *independent* mechanism treats each submarket independently and separately. That is, the selected allocation of each type only depends on agents' marginal preferences of each type.

the only *independent* mechanism that satisfies *individual rationality*, *strategy-proofness*, and *coordinatewise efficiency*. ◇

6 Discussion

6.1 A discussion of our new properties in a normative view

In this subsection, we further discuss the role of our new properties. We first review the role of *endowments-swapping-proofness* for Shapley-Scarf housing markets.

Result 3 (Endowments-swapping-proof mechanisms (Fujinaka and Wakayama, 2018))**.**

(a) If an *individually rational* and *strategy-proof* mechanism is *endowments-swapping-proof*, then it is *group strategy-proof*.

(b) A mechanism is *strict core-stable* if and only if it is *individually rational*, *strategy-proof*, and *endowments-swapping-proof*.

(c) If an *individually rational* mechanism is *endowments-swapping-proof*, then it is *pairwise weakly stable*.

Note that in Result 3, we always require the presence of *individual rationality*. The reason for this is simple: if we ignore property rights that are established via the endowments, then *endowments-swapping-proofness* will be very weak. For instance, any constant mechanisms and serial dictatorships are *endowments-swapping-proof*.[31]

Result 3 suggests that the role of *endowments-swapping-proofness* for Shapley-Scarf housing markets can also be divided into two parts as follows.[32]

(i) *endowments-swapping-proofness* as an incentive property

By definition, it is easy to see that *endowments-swapping-proofness* is an incentive property since it excludes the possibility that two agents within a pair may gain by swapping their endowments. Moreover, Result 3 (a) shows that, in the presence of *individual rationality* and *strategy-proofness*, *endowments-swapping-proofness* also implies a stronger incentive property, *group strategy-proofness*. Thus, together with *individual rationality* and *strategy-proofness*, *endowments-swapping-proofness* also rules out beneficial joint misreporting.

[31]The *constant* mechanism is one that always assigns a constant allocation to all markets. The *serial dictatorship* mechanism is determined by the following procedure: agents move sequentially; the first mover obtains his most preferred allotment in O; the second mover obtains his most preferred allotment from the remaining set of objects, and so on.

[32]Together with other properties, *endowments-swapping-proofness* also contains some efficiency. And in Subsection 5.4, we already show that our new properties inherit this feature.

(ii) *endowments-swapping-proofness* as a stability notion

Fujinaka and Wakayama (2018) discuss the relation between *endowments-swapping-proofness* and stability notions. In particular, Result 3 (b) shows that *strict core-stability* implies *endowments-swapping-proofness*, and (c) shows that together with *individual rationality*, *endowments-swapping-proofness* implies *pairwise weak stability*. Thus, one can consider *endowments-swapping-proofness* as a stability notion.

The role of our new properties

Our results show that the new properties that we have drawn from *endowments-swapping-proofness*, inherits these two feature of *endowments-swapping-proofness* as well. To be more precise, together with other desirable properties, (i) *bundle endowments-swapping-proofness* can be considered as an incentive property to induce *group strategy-proofness* (Corollaries 1, 2, and 3), and (ii) *flexible endowments-swapping-proofness* might be considered as a stability notion (see Statement 1 in Appendix A.1).

It is worth mentioning that the last relation is not clear yet. In Statement 1, we show that (a) *strict core-stability* does not imply *endowments-swapping-proofness* and (b) together with *individual rationality*, *endowments-swapping-proofness* implies *pairwise weak stability*. However, Statement 1 (b) is a vacuous truth. That is, the existence of *individually rational* and *endowments-swapping-proof* mechanisms is an open problem.

6.2 Comparison of bTTC and cTTC

In this subsection, by comparing bTTC with cTTC, we emphasize the meaning of our characterization results for these two extensions of the TTC mechanism.

First, note that both extensions satisfy several desirable properties: *individual rationality*, *strategy-proofness*, and *non-bossiness*. Thus, it is hard to make a compelling argument for one extension over the other. However, our analysis provides some justifications for distinguishing between these two extensions of the TTC mechanism.

Generalizability

As we mentioned earlier in Remark 3, bTTC can also be used for strict preferences while cTTC cannot. Moreover, Theorem 8 shows that no extension / modification that inherits properties from cTTC can be used on the domain of strict preference profiles. Thus, we conclude that bTTC performs better than cTTC in terms of generalizability (with respect to more complex preference profiles).

It is worth mentioning that the domain of separable preference profiles is the maximal domain for cTTC, i.e., cTTC cannot be used if any agents' preferences are not separable, and separability allows substitutes preferences only. However, sometimes with multi-unit demands, agents' preferences might be (strongly) complementary, e.g., Leszczyc and Häubl (2010). Thus, cTTC cannot be used for such a situation. bTTC allows for that but at the cost of limiting outcomes by only allowing whole endowment bundle trading.

Strategic robustness

Note that each TTC extensions only satisfies one of our new properties. That is, (a) only bTTC is *bundle endowments-swapping-proof*, (b) only cTTC is *one type endowments-swapping-proof*, and (c) none of them is *flexibly endowments-swapping-proof*. Thus, we cannot say that one of them is better in terms of strategic robustness via endowments-swapping. However, since bTTC is also *group strategy-proof* (Corollary 1,Corollary 2, and Corollary 3), while cTTC is not (Feng et al., 2022a, Proposition 1), we conclude that bTTC is better if we also take into account agents' joint misreports.

6.3 Relation to other literature

In this subsection, we provide a discussion of our results and how they relate to the literature.

Another characterization result of cTTC

Here, we discuss the relation between our characterization results of cTTC (Theorem 6 and Theorem 7) and Feng et al. (2022a, Theorem 1 and Theorem 2), which characterizes cTTC on the basis of *individual rationality*, *strategy-proofness*, *non-bossiness*, and *unanimity*. To distinguish between Feng et al. (2022a)'s results and ours, note that their results are established by weakening *Pareto efficiency* and strengthening *strategy-proofness*, whereas ours are based on another approach. We ignore all efficiency notions but consider an additional incentive property that excludes ex-ante manipulations via endowments-swapping. It is worthwhile to note that we cannot invoke their results to establish ours. Therefore, while there is a close connection between our results, there is no direct logical relation between Feng et al. (2022a)'s and ours.

Another characterization result of bTTC

Here, we discuss the relation between our characterization results of bTTC (Theorem 3,Theorem 4, and Theorem 5) and Feng et al. (2022b)'s, which essentially characterize bTTC on the basis of *individual rationality*, *group strategy-proofness* and *anonymity*.[33] [34] Since (a) *group strategy-proofness* is stronger than the combination of *strategy-proofness* and *non-bossiness* (see Remark 2), and (b) there is no link between *bundle endowments-swapping-proofness* and *anonymity*,[35] our results are logically independent.

[33] *Anonymity* says that the mechanism is defined independently of the names of the agents.

[34] They show that only a class of hybrid mechanisms between bTTC and the no-trade mechanism, satisfies all of their properties.

[35] See Subsection 4.2 in Fujinaka and Wakayama (2018) for details.

Object allocation problems with multi-demands and with ownership

Finally, we compare our results to Altuntaş et al. (2021) and Biró et al. (2022a). Both papers consider a more general model for allocating objects to the set of agents who can consume any set of objects. Each object is owned by an agent, but now each agent has strict preferences over all objects, and his preferences over sets of objects are monotonically responsive to these "objects-preferences". In our model, we impose more structure by assuming that (i) the set of objects is partitioned into sets of exogenously given types and (ii) each agent owns and wishes to consume one object of each type.

Altuntaş et al. (2021) consider another extension of the TTC mechanism: the generalized top trading cycles (gTTC) mechanism, which satisfies *individual rationality* and *Pareto efficiency* but violates *strategy-proofness*.[36] By strengthening *individual rationality* and weakening *strategy-proofness*, they provide a characterization result of the gTTC mechanism (only for lexicographic preferences). Thus, their results complement ours: if we exclude *strategy-proofness*, then there exists another extension of the TTC mechanism, which performs better than our mechanisms, in terms of efficiency and stability.

Biró et al. (2022a) extend bTTC from multiple type housing markets to their more general model. They show that this extension is neither *Pareto efficient* nor *strategy-proof*. Thus, their results show the limitation of bTTC: without the structure in our model, bTTC (which is *group strategy-proof* in our model) and its extensions might be manipulable.

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[36] In fact, the gTTC mechanism is *strictly core-stable* and *bossy*.

A Appendix: proofs

A.1 The link between ***flexibly endowments-swapping-proofness*** and stability

Statment 1. For multiple-type housing markets with lexicographic preferences $\mathcal{M}_l$,[37]

(a) if an *individually rational* mechanism is *flexibly endowments-swapping-proof*, then it is *pairwise weakly stable*.

(b) a *strictly core-stable* mechanism is not necessarily *flexibly endowments-swapping-proof*.

Proof. Part (a). Suppose not. Let f be a mechanism that is *individually rational* and *flexibly endowments-swapping-proof*, but not *pairwise weakly stable*. Thus, there exists a market (e, R), a pair of agents $\{i, j\} \subseteq N$, and an allocation $y \in X$ such that $\{i, j\}$ strictly pairwise blocks $f(e, R)$ via y. By the definition of pairwise blocking, (y_i, y_j) is a reallocation among $\{i, j\}$ themselves, i.e., for each $k \in \{i, j\}$ and $t \in T$, $y_k^t \in \{e_i^t, e_j^t\}$. Let $\bar{e}(i, j)$ be such that $\bar{e}_i = y_i$ and $\bar{e}_j = y_j$. Let $z \equiv f(\bar{e}(i, j), R)$. By *individual rationality* of f, $z_i \, R_i \, y_i$ and $z_j \, R_j \, y_j$. Since $\{i, j\}$ strictly blocks $f(e, R)$ via y, $z_i \, R_i \, y_i \, P_i \, f_i(e, R)$ and $z_j \, R_j \, y_j \, P_j \, f_j(e, R)$, which contradicts *flexible endowments-swapping-proofness* of f.

Part (b). Consider markets with three agents and two types, i.e., $N = \{1, 2, 3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$.[38] Consider a market $(e, R) \in \mathcal{M}_l$ as follows.

Each agent i's endowment is (H_i, C_i) and preference profile $R \in \mathcal{R}_l^N$ is as follows:

$$\boldsymbol{R_1} : C_2, \boldsymbol{C_1}, C_3, H_3, H_2, \boldsymbol{H_1};$$

$$\boldsymbol{R_2} : H_1, \boldsymbol{H_2}, H_3, C_1, \boldsymbol{C_2}, C_3;$$

$$\boldsymbol{R_3} : \boldsymbol{C_3}, C_2, C_1, H_1, \boldsymbol{H_3}, H_2.$$

Note that at (e, R), there are two strict core allocations $x \equiv (x_1, x_2, x_3) = ((H_2, C_2), (H_1, C_1), (H_3, C_3))$ and $y \equiv (y_1, y_2, y_3) = ((H_3, C_2), (H_2, C_1), (H_1, C_3))$.

Let f be a *strictly core-stable* mechanism such that $f(e, R) = x$.

Let $\bar{e}(1, 3) \equiv ((H_3, C_1), (H_2, C_2), (H_1, C_3))$. Note that at $(\bar{e}(1, 3), R)$, there is only one strict core allocation y. Thus, $f(\bar{e}(1, 3), R) = y$. Since $y_1 \; P_1 \; x_1$ and $y_3 \; P_3 \; x_3$, we find that $\{1, 3\}$ can be strictly better off by swapping their endowments via $\bar{e}(1, 3)$. Thus, f is not *flexibly endowments-swapping-proof*.

□

[37]Note that for multiple-type housing markets with lexicographic preferences, the strict core is non-empty (Sikdar et al., 2017), hence *strictly core-stable* mechanisms exist.

[38]Note that for two agent cases, i.e., $|N| = 2$, *flexible endowments-swapping-proofness*, *pairwise weak (or strict) stability*, and *weak (or strict) core-stability* are equivalent. Thus, our counterexample is made with at least three agents.

A.2 Proof of Theorem 1 and Theorem 2

Consider markets with two types, i.e., $T = \{1, 2\}$. Suppose that there is a mechanism f that is *individually rational*, *strategy-proof*, and *flexible endowments-swapping-proof* (or *pairwise weakly stable*).

Let $(e, R) \in \mathcal{M}_l$ be defined as follows.

At R_1, agent 1's type order is $\pi_1 : 1, 2$, and the only object that he prefers to one of his endowments is the type-1 endowment of agent 2, i.e.,

$$R_1^1 : e_2^1, e_1^1, \ldots$$
$$R_1^2 : e_1^2, \ldots.$$

At R_2, agent 2 only prefers the type-1 and type-2 endowments of agent 1 to some of his endowments, i.e.,

$$R_2^1 : e_1^1, e_2^1, \ldots$$
$$R_2^2 : e_1^2, e_2^2, \ldots.$$

For each $i = 3, \ldots, n$, agent i prefers his full endowment to all other allotments, i.e.,

$$R_i^1 : e_i^1, \ldots$$
$$R_i^2 : e_i^2, \ldots.$$

Let $x, y \in X$ be such that at x agents 1 and 2 swap their endowments of both types, i.e.,

$$x_1 = (e_2^1, e_2^2),$$
$$x_2 = (e_1^1, e_1^2),$$
$$\text{and for each } i = 3, \ldots, n, \quad x_i = e_i$$

and at y agents 1 and 2 swap their endowments of type-1, i.e.,

$$y_1 = (e_2^1, e_1^2),$$
$$y_2 = (e_1^1, e_2^2),$$
$$\text{and for each } i = 3, \ldots, n, \quad y_i = e_i.$$

Obviously, $x \neq y$. Also note that agent 1 prefers y to x while agent 2 prefers x to y, i.e., $y_1 \, P_1 \, x_1$ and $x_2 \, P_2 \, y_2$.

By *individually rationality* of f, $f(e, R) \in \{x, y, e\}$. By *flexible endowments-swapping-proofness* (or *pairwise weak stability*) of f, $f(e, R) \in \{x, y\}$.

Next, we define two preferences $R_1', R_2' \in \mathcal{R}_l$ for agents 1 and 2.

At R_1', agent 1's type order is $\pi_1' : 2, 1$, and the marginal preferences of R_1' are the same as R_1, i.e., $R_1'^1 = R_1^1, R_1'^2 = R_1^2$, and $R_1' : e_1^2, \ldots, e_2^1, e_1^1, \ldots$

At R'_2, agent 2's type order is $\pi'_2 : 2, 1$, and the only object that he prefers to one of his endowments is the type-2 endowment of agent 1, i.e.,

$$R'^1_2 : e^1_2, e^1_1, \ldots$$
$$R'^2_2 : e^2_1, e^2_2, \ldots.$$

and

$$R'_2 : e^1_2, e^1_1, \ldots, e^2_1, e^2_2, \ldots.$$

Suppose that $f(e, R) = x$. By *individually rationality* of f, $f(e, R'_1, R_{-1}) \in \{y, e\}$. By *strategy-proofness* of f, $f_1(e, R'_1, R_{-1}) \neq y_1$. Thus, $f_1(e, R'_1, R_{-1}) = e_1$ and hence $f(e, R'_1, R_{-1}) = e$. However, this violates *flexible endowments-swapping-proofness* (or *pairwise weak stability*) of f, because $\{1, 2\}$ can be strictly better off by swapping their endowments of type-1 (or strictly block e via y).

Suppose that $f(e, R) = y$. By *individually rationality* of f, $f(e, R'_2, R_{-2}) \in \{x, e\}$. By *strategy-proofness* of f, $f_2(e, R'_2, R_{-2}) \neq x_2$. Thus, $f_2(e, R'_2, R_{-2}) = e_2$ and hence $f(e, R'_2, R_{-2}) = e$. However, this violates *flexible endowments-swapping-proofness* (or *pairwise weak stability*) of f, because $\{1, 2\}$ can be strictly better off by swapping their endowments of both types (or strictly block e via x).

A.3 Auxiliary properties and results

We introduce the well-known property of *(Maskin) monotonicity*, which requires that if an allocation is chosen, then that allocation will still be chosen if each agent shifts it up in his preferences. We formulate *monotonicity* as well as our first auxiliary result for markets with strict preferences $\mathcal{M}$; however, we could use markets with separable preference $\mathcal{M}_s$ and with lexicographic preferences $\mathcal{M}_l$ instead.

Let $i \in N$. Given preferences $R_i \in \mathcal{R}$ and an allotment x_i, let $L(x_i, R_i) = \{y_i \in \Pi_{t \in T} O^t \mid x_i R_i y_i\}$ be the *lower contour set of* R_i *at* x_i. Preference relation R'_i is a *monotonic transformation of* R_i *at* x_i if $L(x_i, R_i) \subseteq L(x_i, R'_i)$. Similarly, given a preference profile $R \in \mathcal{R}^N$ and an allocation x, a preference profile $R' \in \mathcal{R}^N$ is a *monotonic transformation of* R *at* x if for each $i \in N$, R'_i is a monotonic transformation of R_i at x_i.

Definition (**Monotonicity**).
A mechanism f is *monotonic* if for each $(e, R) \in \mathcal{M}$ and for each monotonic transformation $R' \in \mathcal{R}^N$ of R at $f(e, R)$, we have $f(e, R') = f(e, R)$.

Strategy-proofness and *non-bossiness* imply *monotonicity*.

Lemma 1 (Lemma 3 in Feng et al. (2022a)).
For markets with strict preferences $\mathcal{M}$ ($\mathcal{M}_s$ / $\mathcal{M}_l$, respectively), if a mechanism is strategy-proof and non-bossy, then it is monotonic.

Next, we list some useful results based on *strategy-proofness*, *non-bossiness*, and *monotonicity*.

Fact 1 (Fact 1 in Feng et al. (2022a))**.**
Let x_i be an allotment. Let $R_i, \hat{R}_i$ be lexicographic preferences such that (1) $\pi_i = \hat{\pi}_i$ and (2) for each $t \in T$, $\hat{R}_i^t$ is a monotonic transformation of R_i^t at x_i^t. Then, $\hat{R}_i$ is a monotonic transformation of R_i at x_i.

Fact 2. Let f be a *strategy-proof* and *non-bossy* mechanism. Let $R \in \mathcal{R}_l^N$, $z \equiv f(e, R)$, $i \in N$, and $R_i^* \in \mathcal{R}_l$ be preferences that only differ with R_i in the marginal preference of the most important type (type-t), i.e., (1) $\pi_i = \pi_i^*$ where $\pi_i^*(t) = 1$, and (2) for each $\tau \neq t$, $R_i^\tau = R_i^{*\tau}$.

If $f_i^t(e, R_i^*, R_{-i}) = z_i^t$, then $f(e, R_i^*, R_{-i}) = z$.

Proof. It is without loss of generality to assume that $t = 1$ and $\pi_i : 1, \ldots, m$. Let $y \equiv f(e, R_i^*, R_{-i})$ and assume $y_i^1 = z_i^1$. By strategy-proofness of f, $z_i \; R_i \; y_i$ and $y_i \; R_i^* \; z_i$. Since R_i are lexicographic preferences, $z_i \; R_i \; y_i$ implies $z_i^2 \; R_i^2 \; y_i^2$. Similarly, since R_i^* are lexicographic preferences, $y_i \; R_i^* \; z_i$ implies $y_i^2 \; R_i^{*2} \; z_i^2$. Since $R_i^2 = R_i^{*2}$, we find that $z_i^2 = y_i^2$. Applying the same argument sequentially for type-τ marginal preferences with $\tau = 3, \ldots, m$ yields $z_i = y_i$. By non-bossiness of f, $z = y$. □

A.4 Alternative definition of bTTC and results

We restate bTTC for lexicographic preferences by adjusting the multiple-type top trading cycles (mTTC) algorithm from Feng and Klaus (2022).

The bundle top trading cycles (bTTC) algorithm / mechanism.

Input. A multiple-type housing market $(e, R) \in \mathcal{M}_l$.

Step 1. ***Building step.*** Let $N(1) = N$ and $U(1) = O$. We construct a directed graph $G(1)$ with the set of nodes $N(1) \cup U(1)$. For each $o \in U(1)$, we add an edge from the object to its owner and for each $i \in N(1)$, we add an edge from the agent to his most preferred object in O (according to the linear representation of R_i). For each edge $(i, o) \in N \times O$ we say that agent i points to object o.

Implementation step. A *trading cycle* is a directed cycle in graph $G(1)$. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent i in a trading cycle the object that he pointed to, and denote the object assigned to him in this step by $a_i(1)$. Moreover, let $e_i(1)$ be the whole endowment of object $a_i(1)$'s owner, and assign the allotment $x_i(1) = \{e_i(1)\}$ to agent i. If agent $i \in N$ was not part of a trading cycle, then $x_i(1) = \emptyset$.

Removal step. We remove all agents and objects that were assigned in the implementation step, let $N(2)$ and $U(2)$ be the remaining agents and objects, respectively. Go to Step 2.

In general, at Step q (≥ 2) we have the following:

Step *q*. If $U(q)$ (or equivalently $N(q)$) is empty, then stop; otherwise do the following.

Building step. We construct a directed graph $G(q)$ with the set of nodes $N(q) \cup U(q)$. For each $o \in U(q)$, we add an edge from the object to its owner and for each $i \in N$, we add an edge from the agent to his most preferred feasible continuation object in $U_i(q)$ (according to the linear representation of R_i).

Implementation step. A *trading cycle* is a directed cycle in graph $G(q)$. Given the finite number of nodes, at least one trading cycle exists. We assign to each agent i in a trading cycle the object that he pointed to, and denote the object assigned to him in this step by $a_i(q)$. Moreover, let $e_i(q)$ be the whole endowment of object $a_i(q)$'s owner, and assign the allotment $x_i(q) = \{e_i(q)\}$ to agent i. If agent $i \in N$ was not part of a trading cycle, then $x_i(q) = \emptyset$.

Removal step. We remove all agents and objects that were assigned in the implementation step, let $N(q+1)$ and $U(q+1)$ be the remaining agents and objects, respectively. Go to Step $q+1$.

Output. The bTTC algorithm terminates when all objects in O are assigned (it takes at most n steps). Assume that the final step is Step q^*. Then, the final allocation is $x(q^*) = \{x_1(q^*), \ldots, x_n(q^*)\}$.

The *bundle top trading cycles (bTTC) mechanism*, *bTTC*, assigns to each market $(e, R) \in \mathcal{M}_l$ the allocation $x(q^*)$ obtained by the bTTC algorithm.

Example 2 (bTTC).
Consider $(e, R) \in \mathcal{M}_l$ with $N = \{1, 2, 3\}$, $T = \{H(ouse), C(ar)\}$, $O = \{H_1, H_2, H_3, C_1, C_2, C_3\}$, each agent i's endowment is $e_i = (H_i, C_i)$, and

$$\boldsymbol{R_1} : H_2, H_3, \boldsymbol{H_1}, C_3, C_2, \boldsymbol{C_1},$$

$$\boldsymbol{R_2} : C_1, \boldsymbol{C_2}, C_3, H_3, \boldsymbol{H_2}, H_1,$$

$$\boldsymbol{R_3} : H_2, H_1, \boldsymbol{H_3}, C_1, \boldsymbol{C_3}, C_2.$$

The bTTC allocation at (e, R) is obtained as follows.

Step 1. ***Building step.*** $G(1) = (N \cup O, E(1))$ with set of directed edges $E(1) = \{(H_1, 1), (H_2, 2), (H_3, 3), (C_1, 1), (C_2, 2), (C_3, 3), (1, H_2), (2, C_1), (3, H_2)\}$.

Implementation step. The trading cycle $1 \to H_2 \to 2 \to C_1 \to 1$ forms. Then, $a_1(1) = H_2$, $a_2(1) = C_1$, and $e_1(1) = \{H_2, C_2\}$, $e_2(1) = \{H_1, C_1\}$; thus, $x_1(1) = \{H_2, C_2\}$, $x_2(1) = \{H_1, C_1\}$, and $x_3(1) = \emptyset$.

Removal step. $N(2) = 3$, $U(2) = \{H_3, C_3\}$.

Step 2. ***Building step.*** $G(2) = (N(2) \cup U(2), E(2))$ with set of directed edges $E(2) = \{(H_3, 3), (C_3, 3), (3, H_3)\}$.

Implementation step. The trading cycle $3 \to H_3 \to 3$ forms. Then, $a_3(2) = H_3$ and $e_3(2) = \{H_3, C_3\}$; $x_1(2) = \{H_2, C_2\}$, $x_2(2) = \{H_1, C_1\}$, and $x_3(2) = \{H_3, C_3\}$.

Removal step. $N(3) = \emptyset$ and $U(3) = \emptyset$.

Thus, the bTTC algorithm computes the allocation $x = ((H_2, C_2), (H_1, C_1), (H_3, C_3))$. ◇

Our next lemma states that Fujinaka and Wakayama (2018, Lemma 1) partially extends to multiple-type housing markets with lexicographic preferences $\mathcal{M}_l$.

Given a market $(e, R) \in \mathcal{M}_l$, let $\mathcal{C}(e, R)$ be a set of top trading cycles that are obtained at step 1 of the re-defined bTTC mechanism above at (e, R). We say that a trading cycle C is a *first step top trading cycle* if $C \in \mathcal{C}(e, R)$. For each first step top trading cycle C, let $S_C \subseteq N$ be the set of agents who are involved in C, and for each $i \in S_C$, let c_i be the object that agent i points at in C, and t_i be the type of object c_i, i.e., $c_i \in O^{t_i}$. We say that a trading cycle C is *executed* under f at (e, R) if for each $i \in S_C$, agent i receives c_i at $f(e, R)$.

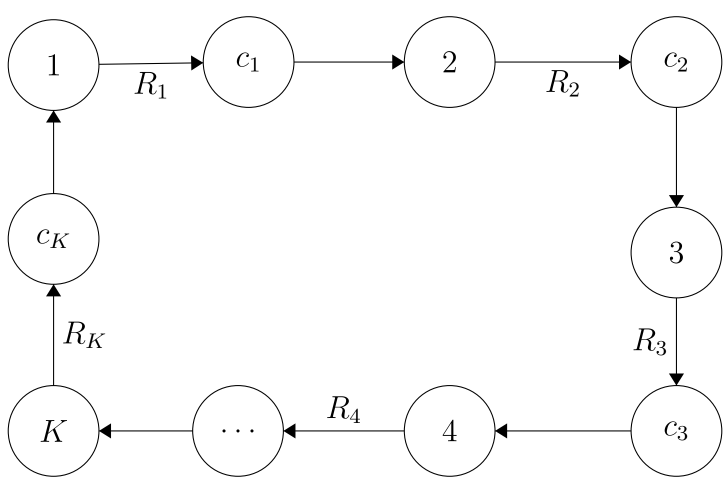
Lemma 2. *If a mechanism $f : \mathcal{M}_l \to X$ is individually rational, strategy-proof, and bundle endowments-swapping proof, then for each $(e, R) \in \mathcal{M}_l$, each first step top trading cycle $C(\in \mathcal{C}(e, R))$ is executed under f at (e, R).*

Proof. The proof is a straightforward extension of Fujinaka and Wakayama (2018, Lemma 1). Let $C \in \mathcal{C}(e, R)$ be a first step top trading cycle that consists of agents $S_C \subseteq N$. We prove this lemma by induction on $|S_C|$.

Induction basis. $|S_C| = 1$. In this case, agent $i \in S_C$ points to one of his endowed object, i.e., $c_i \in e_i$ and hence $C = (i \to c_i(= e_i^{t_i}) \to i)$. Since preferences are lexicographic, i.e., $R_i \in \mathcal{R}_l$, i will be strictly worse off if he receives any other type-t_i objects. Thus, C must be executed by individual rationality of f.

Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that C is executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and $C = (1 \to c_1 \to 2 \to c_2 \to \ldots \to K \to c_K \to 1)$. See the figure below for the graphical explanation.



By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C$ who does not receive c_i, i.e., $f_i^{t_i}(e, R) \neq c_i$. Let $\hat{R}_i$ be such that i only wants to receive type-t_i object c_i and no other objects, i.e.,

$$\hat{R}_i^{t_i} : c_i, e_i^{t_i}, \ldots,$$

$$\text{for each } t \in T \setminus \{t_i\} : \hat{R}_i^t : e_i^t, \ldots, \text{ and}$$

$$\hat{\pi}_i(t_i) = 1.$$

Note that at $\hat{R}_i$, if i does not receive c_i, then from individual rationality of f, he must receive his full endowment e_i.

Let $\hat{R} \equiv (\hat{R}_i, R_{-i})$ and $\hat{M}_i \equiv (e, (\hat{R}_i, R_{-i}))$. We proceed in two steps.

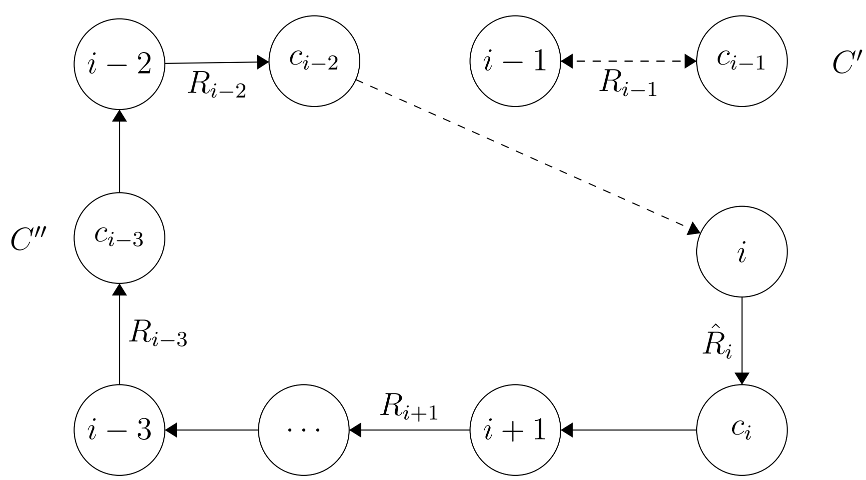
Step 1. We show that agent i receives c_i under f at $\hat{M}_i$, i.e., $f_i^{t_i}(\hat{M}_i) = c_i$.

By *individual rationality* of f, $f_i^{t_i}(\hat{M}_i) \in \{c_i, e_i^{t_i}\}$. By *strategy-proofness* of f, $f_i^{t_i}(e, R) \neq c_i$ implies that $f_i^{t_i}(\hat{R}_i) \neq c_i$, otherwise instead of R_i, agent i has an incentive to misreport $\hat{R}_i$ at (e, R). Thus, $f_i^{t_i}(\hat{M}_i) = e_i^{t_i}$. Then, by *individual rationality* of f, $f_i(\hat{M}_i) = e_i$. Thus, agent $i-1$ cannot receive c_{i-1} from agent i because it is assigned to agent i. Overall, we find that

$$f_i^{t_i}(\hat{M}_i) = e_i^{t_i} \neq c_i \text{ and } f_{i-1}^{t_{i-1}}(\hat{M}_i) \neq c_{i-1} (\in e_i). \tag{1}$$

Let agents $i-1$ and i swap their endowments completely. i.e., $e'(i-1, i)$. Note that at $e'(i-1, i)$, $i-1$ owns $e_i (= e'_{i-1})$ and i owns $e_{i-1} (= e'_i)$. Correspondingly, $i-1$ owns his most preferred object $c_{i-1} (\in e_i)$, and i owns $c_{i-2} (\in e_{i-1})$, recall that c_{i-2} is agent $i-2$'s most preferred object.

Consider $\hat{M}'_i \equiv (e'(i-1, i), (\hat{R}_i, R_{-i}))$. at $\hat{M}'_i$, there are two first step top trading cycles $C' = (i-1 \to c_{i-1} \to i-1)$ and $C'' = (1 \to c_1 \to 2 \to \ldots \to i-2 \to c_{i-2} \to i \to c_i \to i+1 \ldots \to K \to c_K \to 1)$ at $\hat{M}'_i$, i.e., $C', C'' \in \mathcal{C}(\hat{M}'_i)$. See the figure below for the graphical explanation.



Note that $|S_{C'}| = 1$ and $|S_{C''}| = K-1$. Thus, by the induction hypothesis, these two cycles are executed under f at M'_i. Hence, agents $i-1$ and i receive their most preferred object at $f(\hat{M}'_i)$. Thus, together with (1), we find that $f_{i-1}^{t_{i-1}}(\hat{M}'_i) = c_{i-1}\ P_{i-1}^{t_{i-1}}\ f_{i-1}^{t_{i-1}}(\hat{M}_i)$ and $f_i^{t_i}(\hat{M}'_i) = c_i\ P_i^{t_i}\ f_i^{t_i}(\hat{M}_i)$. Therefore, $f_{i-1}(\hat{M}'_i)\ P_{i-1}\ f_{i-1}(\hat{M}_i)$ and $f_i(\hat{M}'_i)\ P_i\ f_i(\hat{M}_i)$. However, this implies that f is not *bundle endowments-swapping proof*, a contradiction.

Step 2. We show that agent i receives c_i under f at (e, R), i.e., $f_i^{t_i}(e, R) = c_i$.

Note that c_i is agent i's most preferred object at R_i. By strategy-proofness of f, $f_i(e, R)\ R_i\ f_i(\hat{M}_i)$. Hence, $f_i^{t_i}(e, R)\ R_i^{t_i}\ f_i^{t_i}(\hat{M}_i) = c_i$, which implies that $f_i^{t_i}(e, R) = c_i$. □

A.5 Proof of Theorems 3, 4, and 5

Proof of Theorem 3

It is well-known that the TTC mechanism satisfies *individual rationality*, *strategy-proofness*, *non-bossiness*, and *endowments-swapping-proofness* for Shapley-Scarf housing markets. By using similar arguments we also obtain that bTTC inherits *individual rationality*, *strategy-proofness*, *non-bossiness*, and *bundle endowments-swapping-proofness* from the underlying top trading cycles algorithm for the restricted market $(e, R|^e)$. So we only show that bTTC is the only mechanism satisfying all our properties above.

Next, we show that all agents who are involved in top trading cycles at step 1 of bTTC, have to receive the corresponding bTTC allocation. Note that in the proof below, we do not use *bundle endowments-swapping-proofness*. That is, we only use *individual rationality*, *strategy-proofness*, *non-bossiness*, and *monotonicity*. Thus, from now and until the end of the proof, we will denote a market simply by its preference profile, e.g., $f(R)$ instead of $f(e, R)$, when the rest in the proof is clear. We will show that for each $R \in \mathcal{R}_l^N$, $f(R) = bTTC(R)$.

Let $f : \mathcal{R}_l^N \to X$ be a mechanism satisfying our all properties above. By Lemma 1, f is *monotonic*.

For each market $R \in \mathcal{R}_l^N$, each first step top trading cycle $C \in \mathcal{C}(R)$, each $i \in S_C$ and $c_i \in O$, let i' be the owner of c_i. Since i and i' are involved in C, $i' \in S_C$. We say that a trading cycle C is *fully executed* under f at (e, R) if for each $i \in S_C$, agent i receives $e_{i'}$ at $f(e, R)$, i.e., $f_i(e, R) = e_{i'}$.

Lemma 3. *For each $R \in \mathcal{R}_l^N$, each $C \in \mathcal{C}(R)$, and each $i \in S_C$, $f_i(R) = bTTC_i(R)$.*

Proof. Let $C \in \mathcal{C}(R)$ be a first step trading cycle that consists of agents $S_C \subsetneq N$. We prove this lemma by induction on $|S_C|$.

Induction basis. $|S_C| = 1$. In this case, agent $i \in S_C$ points to one of his endowed object, i.e., $c_i \in e_i$. Without loss of generality, assume that $S_C = \{1\}$ and $\pi_1 : t_1, \ldots$. Thus, $C = (1 \to e_1^{t_1} \to 1)$.

Let $y \equiv f(R)$. By contradiction, suppose that $y_1 \neq e_1$. Note that by Lemma 2, C is executed, and hence $y_1^{t_1} = e_1^{t_1}$. Let $t \in T \setminus \{t_i\}$ be such that $y_1^t \neq e_1^t$. Without loss of generality, assume that agent 1 receives agent 2's endowment of type-t at y, i.e., $y_1^t = e_2^t$.

Let $\hat{R} \in \mathcal{R}_l^N$ be such that each agent j positions y_j at the top and changes his importance order as π_1, i.e., (i) for each agent $j \in N$, $\hat{\pi}_j = \pi_1 : t_1, \ldots$; and (ii) for each $j \in N$ and each $\tau \in T$, $\hat{R}_j^\tau : y_j^\tau, \ldots$

By monotonicity of f, $f(\hat{R}) = y$.

Let $\bar{R}_2$ be such that

$$\bar{\pi}_2 = \hat{\pi}_2 (= \pi_1),$$

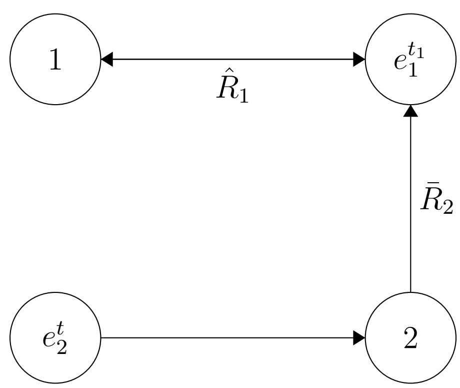
$$\bar{R}_2^{t_1} : e_1^{t_1}, y_2^{t_1}, \ldots, \text{ and}$$

$$\text{For each } \tau \in T \setminus \{t_1\}, \bar{R}_2^\tau = \hat{R}_2^\tau.$$

Let

$$\bar{R} \equiv (\bar{R}_2, \hat{R}_{-2}).$$

Note that by strategy-proofness of f, agent 2 either receives $e_1^{t_1}$ or $y_2^{t_1}$; otherwise he has an incentive to misreport $\hat{R}_2$ at $\bar{R}$. Moreover, C is still a first top trading cycle at $\bar{R}$, i.e., $C \in \mathcal{C}(\bar{R})$. Thus, by Lemma 2, C is executed and hence agent 1 receives $e_1^{t_1}$ at $f(\bar{R})$. See the figure below for the graphical explanation.



Thus, agent 2 still receives $y_2^{t_1}$, and hence by Fact 2,

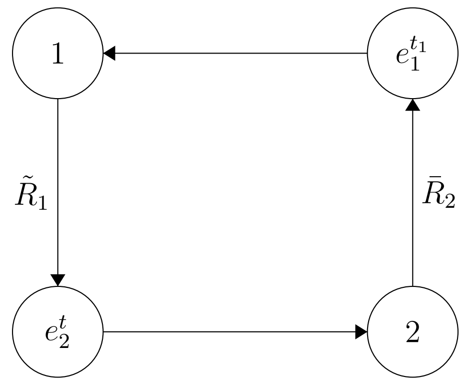
$$f(\bar{R}) = f(\hat{R}) = y \text{ and particularly, } f_1^{t_1}(\tilde{R}) = y_1^{t_1} = e_1^{t_1}. \tag{2}$$

Let $\tilde{R}_1$ be such that agent 1 only changes his importance order as t is the most important, i.e., $\tilde{\pi}_1 : t, \ldots$ and $\tilde{R}_1 = (\hat{R}_1^1, \ldots, \hat{R}_1^m, \tilde{\pi}_1)$.

Let

$$\tilde{R} \equiv (\tilde{R}_1, \bar{R}_{-1}).$$

By monotonicity of f, $f(\tilde{R}) = f(\bar{R}) = y$. However, at $\tilde{R}$, there is a first step top trading cycle $C' \in \mathcal{C}(\tilde{R})$ consisting of agents 1 and 2, i.e., $C' = (1 \to e_2^t (= y_1^t) \to 2 \to e_1^{t_1} \to 1)$. See the figure below for the graphical explanation.



By Lemma 2, C' is executed under f at $\tilde{R}$. Thus, $f_2^{t_1}(\tilde{R}) = e_1^{t_1}$, which contradicts with the fact that $f_1^{t_1}(\tilde{R}) = y_1^{t_1} = e_1^{t_1}$ (see (2)).

Induction hypothesis. Let $K \in \{2, \dots, n\}$. Suppose C is fully executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \dots, K\}$ and $C = (1 \to e_2^{t_1} \to 2 \to \cdots \to K \to e_1^{t_K} \to 1)$. We will prove that for each $i \in S_C$, $f_i(R) = bTTC_i(R) = e_{i+1}$ (mod K).

Let $x \equiv bTTC(R)$, $y \equiv f(R)$, and by contradiction, suppose that there is an agent $i \in S_C$ such that $y_i \neq x_i$. Without loss of generality, let $i = 1$. By Lemma 2, C is executed under f at R. In particular,

$$y_1^{t_1} = e_2^{t_1} = x_1^{t_1} \text{ and } y_K^{t_K} = e_1^{t_K} = x_K^{t_K}. \tag{3}$$

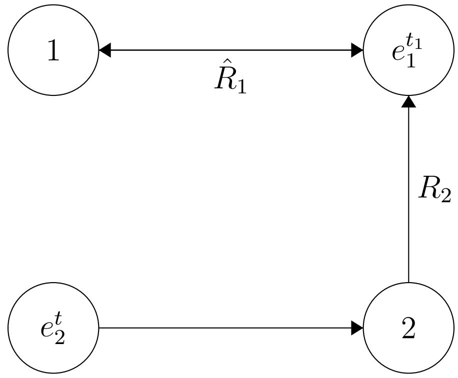
Since $y_1 \neq x_1 (= e_2)$, there is a type $t \in T \setminus \{t_1\}$ and an agent $j \neq 2$ such that $y_1^t = e_j^t$. There are two cases.

Case 1: $j \in S_C$. Let $\hat{R}_1$ be such that agent 1 positions y_1 at the top and moves t to the most important, i.e., (i) $\hat{\pi}_1 : t, \dots$; and (ii) for each $\tau \in T$, $\hat{R}_1^\tau : y_1^\tau, \dots$. Since $\hat{R}_1$ is a monotonic transformation of R_1 at y, we have

$$f(\hat{R}_1, R_{-1}) = f(R) = y \text{ and particularly, } f_1^{t_1}(\hat{R}_1, R_{-1}) = e_2^{t_1}. \tag{4}$$

Note that there is a first step top trading cycle $C' \equiv (1 \to e_j^t \to j \to e_{j+1}^{t_j} \to j+1 \to \cdots \to K \to e_1^{t_K} \to 1)$ at $(\hat{R}_1, R_{-1})$. i.e., $C' \in \mathcal{C}(\hat{R}_1, R_{-1})$. Since $j \neq 2$, cycle C' contains less than K agents. There are two sub-cases.

Sub-case one: $K = 2$. $j \in S_C$ implies That $j = 1$ and $C' = (1 \to e_1^t \to 1)$, then we are back to the situation in the induction basis, see the figure below for the graphical explanation.



Thus, by the the induction basis, $f_1(\hat{R}_1, R_{-1}) = e_1$ and hence $f_1^{t_1}(\hat{R}_1, R_{-1}) = e_1^{t_1}$, which contradicts with the fact that $f_1^{t_1}(\hat{R}_1, R_{-1}) = e_2^{t_1}$ (see (4)).

Sub-case two: $K > 2$. By the induction hypothesis, cycle C' is fully executed at $f(\hat{R}_1, R_{-1})$. Therefore, $f_1(\hat{R}_1, R_{-1}) = e_j$ and hence $f_1^{t_1}(\hat{R}_1, R_{-1}) = e_j^{t_1}$, which contradicts with the fact that $f_1^{t_1}(\hat{R}_1, R_{-1}) = e_2^{t_1}$ (see (4)).

Case 2: $j \notin S_C$. Let $\hat{R}_j$ be such that (i) $\hat{\pi}_j : t_K, \ldots$; (ii) $\hat{R}_j^{t_K} : e_1^{t_K}, y_j^{t_K}, \ldots$; and (iii) for each $\tau \in T \backslash \{t_K\}$, $\hat{R}_j^{\tau} : y_j^{\tau}, \ldots$.

Let

$$\hat{R} \equiv (\hat{R}_j, R_{-j}).$$

Note that C is still a first step top trading cycle at $\hat{R}$, and hence, by Lemma 2, C is executed. In particular, with (3), we have

$$f_K^{t_K}(\hat{R}) = y_K^{t_K} = e_1^{t_K}. \tag{5}$$

So, agent j does not receive $e_1^{t_K}$ at $f(\hat{R})$. So, by strategy-proofness of f, $f_j(\hat{R}) = y_j$; otherwise he has an incentive to misreport R_j at $\hat{R}$. So, by non-bossiness of f, $f(\hat{R}) = y$.

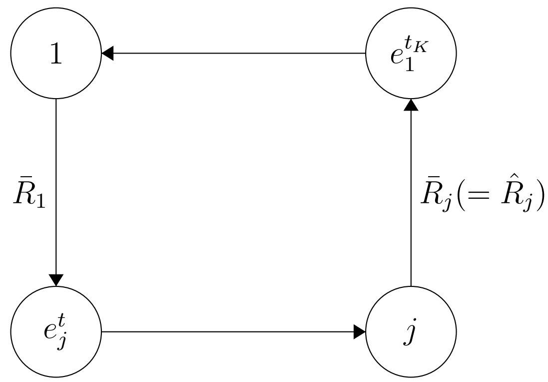
Let $\bar{R}_1$ be such that agent 1 positions y_1 at the top and moves t to the most important, i.e., (i) $\bar{\pi}_1 : t, \ldots$; and (ii) for each $\tau \in T$, $\bar{R}_1^{\tau} : y_1^{\tau}, \ldots$. Let

$$\bar{R} \equiv (\bar{R}_1, \hat{R}_{-1}).$$

Since $\bar{R}_1$ is a monotonic transformation of $\hat{R}_1 (= R_1)$ at y, we have

$$f(\bar{R}) = y \text{ and particularly, } f_K^{t_K}(\bar{R}) = e_1^{t_K}. \tag{6}$$

Note that $C' \equiv (1 \to y_1^t (= e_j^t) \to j \to e_1^{t_K} \to 1)$ is a first step top trading cycle at $\bar{R}$, i.e., $C' \in \mathcal{C}(\bar{R})$. See the figure below for the graphical explanation.



Thus, by Lemma 2, cycle C' is executed at $f(\bar{R})$. Therefore, $f_j^{t_K}(\bar{R}) = e_1^{t_K}$. Since $j \notin S_C$, this contradicts with the fact that $f_K^{t_K}(\hat{R}) = e_1^{t_K}$ (see (6)). □

By Lemma 3, we have shown that agents who trade at step 1 of the bTTC algorithm always receive their bTTC allotments under f. Next, we can consider agents who trade at step 2 of the bTTC algorithm by following the same proof arguments as for first step trading cycles, and so on. Thus, the proof of Theorem 3 is completed.

Proof of Theorem 4

Proof. Let $f : \mathcal{M}_s \to X$ be *individually rational*, *strategy-proof*, *non-bossy*, and *bundle endowments-swapping proof*. Note that by Lemma 1, f is *monotonic*.

Let $S \subseteq N$ and $(e, R) \in \mathcal{M}_s$ be such that only agents in S do no have lexicographic preferences, i.e., $R_S \notin \mathcal{R}_l^S$ and $R_{-S} \in \mathcal{R}_l^{-S}$. We show that $f(e, R) = bTTC(e, R)$ by induction on $|S|$.

We first consider $S = \{i\}$, i.e., $|S| = 1$ as the induction basis.

Let $x \equiv f(e, R)$ and $y \equiv bTTC(e, R)$.

Let $\hat{R}_i \in \mathcal{R}_l$ be such that for each $t \in T$, $\hat{R}_i^t : x_i^t, \ldots$

By *monotonicity* of f, $f(e, (\hat{R}_i, R_{-i})) = x$. Note that $(\hat{R}_i, R_{-i}) \in \mathcal{R}_l^N$. Thus, by Theorem 3, f coincides with bTTC, i.e., $bTTC(e, (\hat{R}_i, R_{-i})) = f(e, (\hat{R}_i, R_{-i})) = x$.

Let $\bar{R}_i \in \mathcal{R}_l$ be such that (a) $\bar{\pi}_i = \hat{\pi}_i$; and (b) for each $t \in T$, $\hat{R}_i^t : y_i^t, \ldots$

By *monotonicity* of $bTTC$, $bTTC(e, (\bar{R}_i, R_{-i})) = y$. Note that $(\bar{R}_i, R_{-i}) \in \mathcal{R}_l^N$. Thus, again by Theorem 3, $f(e, (\bar{R}_i, R_{-i})) = bTTC(e, (\bar{R}_i, R_{-i})) = y$.

By *strategy-proofness* of $bTTC$, $y_i = bTTC_i(e, R)\ R_i\ bTTC_i(e, (\hat{R}_i, R_{-i})) = x_i$; by *strategy-proofness* of f, $x_i = f_i(e, R)\ R_i\ f_i(e, (\bar{R}_i, R_{-i})) = y_i$. Thus, $x_i = y_i$. Subsequently, by *non-bossiness* of $bTTC$, $x = bTTC(e, (\hat{R}_i, R_{-i})) = bTTC(e, (\bar{R}_i, R_{-i})) = y$.

We can apply repeatedly the same argument to obtain that for $|S| \in \{2, \ldots, n\}$, and for each profile $R \in \mathcal{R}_s^N$ where exactly $|S|$ agents have non-lexicographic preferences, $f(e, R) = bTTC(e, R)$. Thus, for each $(e, R) \in \mathcal{M}_s$, $f(e, R) = bTTC(e, R)$. □

Theorem 5 can be proven by exactly the same way to Theorem 4 and hence we omit it.

A.6 Proof of Theorems 6, 7, and 8

Feng et al. (2022a) show that cTTC is *individual rationality* and *strategy-proofness*. Thus, here we only show that cTTC is *one-type endowments-swapping-proof*.

By contradiction, suppose that cTTC is not *one-type endowments-swapping-proof*. Thus, there exists a market (e, R), a type t, a pair of agents $\{i, j\}$, and a new endowment allocation by one type swapping $\hat{e}(t, i, j)$ such that $cTTC_i(\hat{e}(t, i, j), R) P_i cTTC_i(e, R)$ and $cTTC_j(\hat{e}(t, i, j), R) P_i$ $cTTC_j(e, R)$. Let $x \equiv cTTC(e, R)$ and $y \equiv cTTC(\hat{e}(t, i, j), R)$. By the definition of cTTC, we know that

(i) for each $\tau \neq t$, $x_i^\tau = TTC_i^\tau(e^\tau, R) = y_i^\tau$ and $x_j^\tau = TTC_j^\tau(e^\tau, R) = y_j^\tau$.

(ii) $x_i^t = TTC_i^t(e, R^t)$ and $x_j^t = TTC_j^t(e, R^t)$.

(iii) $y_i^t = TTC_i^t(\hat{e}^t(t, i, j), R^t)$ and $y_j^t = TTC_j^t(\hat{e}^t(t, i, j), R^t)$.

Since the TTC mechanism is *endowments-swapping-proof* (Fujinaka and Wakayama, 2018), together with (ii) and (iii), we find that for one agent $k \in \{i, j\}$, $x_k^t \ R_k \ y_k^t$. However, by (i), $y_i \ P_i \ x_i$ and $y_j \ P_j \ x_j$ implies that $y_i^t \ P_i^t \ x_i^t$ and $y_j^t \ P_j^t \ x_j^t$, a contradiction.

Next, we show that cTTC is the only mechanism satisfying all our properties above. Let $f : \mathcal{M}_l \to X$ be a mechanism that is *individual rationality*, *strategy-proofness*, and *one-type endowments-swapping-proof*.

A result for restricted preferences

We first consider a restricted domain $\mathcal{R}_\pi^N$ such that all agents share the same importance order π. It is without loss of generality to assume that $\pi : 1, \ldots, m$.

Proposition 1. *For each market (e, R) with restricted preference profile $R \in \mathcal{R}_\pi^N$, $f(e, R) = cTTC(e, R)$.*

The proof of Proposition 1 consists of three claims.

First, we show that for each market with restricted preferences, f assigns the cTTC allocation of type-1.

Claim 1. For each (e, R) with $R \in \mathcal{R}_\pi^N$, $f^1(e, R) = cTTC^1(e, R)$.

Proof. Note that the proof is similar to Lemma 2.

Let C be a first step top trading cycle under TTC^1 at (e, R) that consists of a set of agents $S_C \subseteq N$. We first show that C is executed at $f(e, R)$ by induction on $|S_C|$.

Induction basis. $|S_C| = 1$. In this case, agent $i \in S_C$ points to his type-1 endowed object, i.e., $C = (1 \to e_1^1 \to 1)$. Since preferences are lexicographic, agent 1 will be strictly worse off if he receives any other type-1 object. Thus, by individual rationality of f, C must be executed.

Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that C is executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and $C = (1 \to e_2^1 \to 2 \to \ldots \to K \to e_1^1 \to 1)$.

By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C$ who does not receive e_{i+1}^1, i.e., $f_i^1(e, R) \neq e_{i+1}^1$.

Let $\hat{R}_i \in \mathcal{R}_\pi$ be such that i only wants to receive one type-1 object e_{i+1}^1 than his type-1 endowment, i.e.,

$$\hat{R}_i^1 : e_{i+1}^1, e_i^1, \ldots,$$

$$\text{for each } t \in T \setminus \{1\} : \hat{R}_i^t = R_i^t, \text{ and}$$

$$\hat{\pi}_1 = \pi : 1, \ldots, m.$$

Note that at $\hat{R}_i$, if i does not receive e_{i+1}^1, then by individual rationality of f, he must receive his type-1 endowment e_i^1.

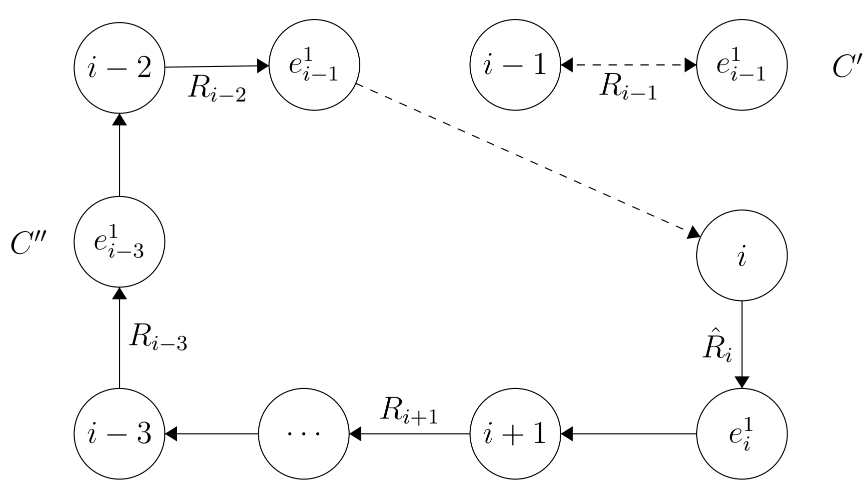
Let $\hat{R} \equiv (\hat{R}_i, R_{-i})$ and $\hat{M}_i \equiv (e, (\hat{R}_i, R_{-i}))$. We proceed in two steps.

Step 1. We show that agent i receives e_{i+1}^1 under f at $\hat{M}_i$, i.e., $f_i^1(\hat{M}_i) = e_{i+1}^1$.

Since $\hat{R}_i \in \mathcal{R}_l$ and f is *individually rational*, $f_i^1(\hat{M}_i) \in \{e_{i+1}^1, e_i^1\}$. By *strategy-proofness* of f, $f_i^1(e, R) \neq e_{i+1}^1$ implies that $f_i^1(\hat{M}_i) \neq e_{i+1}^1$, otherwise instead of R_i, agent i has an incentive to misreport $\hat{R}_i$ at (e, R). Thus, $f_i^1(\hat{M}_i) = e_i^1$. Thus, agent $i-1$ cannot receive e_i^1 from agent i because it is assigned to agent i. Overall, we find that

$$f_i^1(\hat{M}_i) = e_i^1 \neq e_{i+1}^1 \text{ and } f_{i-1}^1(\hat{M}_i) \neq e_i^1. \tag{7}$$

Consider that agents $i-1$ and i swap their type-1 endowments, i.e., $\hat{e}(t=1, i-1, i)$. Note that at this new endowment allocation, $i-1$ owns his most preferred object e_i^1, and i owns e_{i-1}^1, and recall that e_{i-1}^1 is agent $(i-2)$'s most preferred type-1 object. Consider that $\hat{M}_i' \equiv (\hat{e}(t = 1, i-1, i), (\hat{R}_i, R_{-i}))$. At $\hat{M}_i'$, there are two first step top trading cycles $C' = (i-1 \to e_i^1 \to i-1)$ and $C'' = (1 \to e_2^1 \to 2 \to \ldots \to i-2 \to e_{i-1}^1 \to i \to e_{1+1}^1 \to i+1 \ldots \to K \to e_1^1 \to 1)$ at $\hat{M}_i'$. See the figure below for the graphical explanation.



Note that $|S_{C'}| = 1$ and $|S_{C''}| = K - 1$. Thus, by the induction hypothesis, these two cycles are executed under f at $\hat{M}'_i$.

Hence, agents $i-1$ and i receive their most preferred object at $f(\hat{M}'_i)$. By (7), $f^1_{i-1}(\hat{M}'_i) = e^1_i \; P^1_{i-1} \; f^1_{i-1}(\hat{M}_i)$ and $f^1_i(\hat{M}'_i) = e^1_{i+1} \; P^1_i \; f^1_i(\hat{M}_i)$. Therefore, $f_{i-1}(\hat{M}'_i) \; P_{i-1} \; f_{i-1}(\hat{M}_i)$ and $f_i(\hat{M}'_i) \; P_i \; f_i(\hat{M}_i)$. However, this implies that f is not *one type endowments-swapping-proof*, a contradiction.

Step 2. We show that agent i receives e^1_{i+1} under f at (e, R), i.e., $f^1_i(e, R) = e^1_{i+1}$.

Note that e^1_{i+1} is agent i's most preferred type-1 object at R_i. By *strategy-proofness* of f, $f_i(e, R) \; R_i \; f_i(\hat{M}_i)$. Hence, $f^1_i(e, R) \; R^1_i \; f^1_i(\hat{M}_i) = e^1_{i+1}$, which implies that $f^1_i(e, R) = e^1_{i+1}$.

It suffices to show that C is executed at $f(e, R)$ because once we have shown that agents who trade at the first step of the TTC algorithm (of type-1) always receive their TTC allotments of type-1 under f, we can consider agents who trade at the second step of the TTC (of type-1) by following the same proof arguments as for first step trading cycles, and so on. Thus, the proof of Claim 1 is completed. □

Note that at step 1 and step 2 of the proof of Claim 1, we only require that agents in S_C have restricted preferences in $\mathcal{R}_\pi$, i.e., if $R_{S_C} \in \mathcal{R}^{S_C}_\pi$ then for any $R_{-S_C} \in \mathcal{R}^{-S_C}_l$, $f^1_{S_C}(R_{S_C}, R_{-S_C}) = cTTC^1_{S_C}(R_{S_C}, R_{-S_C})$. Therefore, Claim 1 implies the following fact.

Fact 3 (Restricted preferences)**.**
For each (e, R) with $R \in \mathcal{R}^N_\pi$, let $\mathbb{C} \equiv \{C_1, C_2, \ldots, C_I\}$ be the set of top trading cycles that are obtained via the TTC algorithm of type-1 at (e^1, R^1). Moreover, for each top trading cycle $C_i \in \mathbb{C}$, assume that C_i is executed at step s_i, and without loss of generality, assume that if $i < i'$ then $s_i \leq s_{i'}$.

For each $C_i \in \mathbb{C}$, if all agents in $S_{C_1}, S_{C_2}, \ldots, S_{C_{i-1}}, S_{C_i}$ have restricted preferences, then $C_1, \ldots, C_{i-1}, C_i$ are executed, regardless of the preferences of other agents in $C_{i+1}, \ldots, C_I$.

Formally, for each $C_i \in \mathbb{C}$, let $S' \equiv \cup^i_{k=1} S_{C_k}$. If (e, R) is such that for each $j \in S'$, $R_j \in \mathcal{R}_\pi$, then $f^1_{S'}(e, R) = cTTC^1_{S'}(e, R)$.

Next, we show that for each agent, his type-2 allotment is weakly better than his type-2 endowment.

Claim 2. For each (e, R) with $R \in \mathcal{R}^N_\pi$ and each $i \in N$, $f^2_i(e, R) \; R^2_i \; e^2_i$.

Proof. By contradiction, assume that there exists a market (e, R) with $R \in \mathcal{R}^N_\pi$ and an agent $i \in N$ such that $e^2_i \; P^2_i \; f^2_i(e, R)$.

Let $y \equiv f(e, R)$. Recall that by Claim 1, $y^1 = cTTC^1(e, R) = TTC^1(e^1, R^1)$. It is without loss of generality to assume that $i = 1$. Since $\hat{R}_1 \in \mathcal{R}_l$ and f is *individually rational*,

$$y^1_1 \neq e^1_1. \tag{8}$$

Let $\hat{R}_1 \in \mathcal{R}_\pi$ be such that

$$\hat{R}_1^2 : e_1^2, y_1^2, \ldots,$$
$$\text{for each } t \in T \setminus \{2\}, \hat{R}_1^t : y_1^t, e_1^t, \ldots, \text{ and}$$
$$\hat{\pi}_1 = \pi : 1, \ldots, m.$$

By *strategy-proofness* of f, $f_1(e, \hat{R}_1, R_{-1}) = y_1$. Note that $\hat{\pi}_1 = \pi : 1, \ldots, m$ and $(\hat{R}_1, R_{-1}) \in \mathcal{R}_\pi^N$. By Claim 1, $f^1(e, \hat{R}_1, R_{-1}) = cTTC^1(e, \hat{R}_1, R_{-1}) = y^1$.

Let $\bar{R}_1 \in \mathcal{R}_l$ be such that $\bar{R}_1$ and $\hat{R}_1$ only differ in the importance order, where the orders of type-1 and type-2 are switched, i.e., $\bar{\pi}_1 : 2, 1, 3, \ldots, m$ and $\bar{R}_1 = (\hat{R}_1^1, \ldots, \hat{R}_1^m, \bar{\pi}_1)$.

By *individual rationality* of f, $f_1^2(e, \bar{R}_1, R_{-1}) = e_1^2$. Therefore, by *individual rationality* of f, $f_1^1(e, \bar{R}_1, R_{-1}) \in \{y_1^1, e_1^1\}$.

By *strategy-proofness* of f, $f_1^1(e, \bar{R}_1, R_{-1}) \neq f_1^1(e, \hat{R}_1, R_{-1}) = y_1^1$; otherwise agent 1 has an incentive to misreport $\bar{R}_1$ at $(\hat{R}_1, R_{-1})$. Thus,

$$f_1^1(e, \bar{R}_1, R_{-1}) = e_1^1. \tag{9}$$

Next, we show that (9) contradicts with *one type endowments-swapping-proofness* of f.

let ℓ be the step of the TTC algorithm at which agent 1 receives his type-1 object y_1^1. Let C be the corresponding top trading cycle that involves agent 1, i.e., $1 \in S_C$.

Note that by Claim 1 and Fact 3, all top trading cycles that are obtained before step ℓ are executed at $f(e, \bar{R}_1, R_{-1})$. Thus, by the definition of TTC, we know that for each agent in S_C, the object that he pointed at in C is his most preferred type-1 object among the unassigned type-1 objects, i.e., for each $i \in S_C$, all better type-1 objects for him, are assigned to someone else via top trading cycles that are obtained before step ℓ.

Since $y_1^1 \neq e_1^1$ (see (8)), $|S_C| > 1$. We show a contradiction by induction on $|S_C|$.

Induction basis. $|S_C| = 2$. Without loss of generality, let $C = (1 \to e_2^1 \to 2 \to e_1^1 \to 1)$. Since $f_1^1(e, \bar{R}_1, R_{-1}) = e_1^1$ (see (9)), agent 2 does not receive his most (feasible) preferred object e_1^1.

Consider that agents 1 and 2 swap their type-1 endowments, i.e., $\hat{e}(t = 1, 1, 2)$. Note that at this new endowment allocation, agent 1 owns e_2^1 and agent 2 owns e_1^1. Consider that $M' \equiv (\hat{e}(t = 1, 1, 2), \bar{R}_1, R_{-1})$.

Recall that $\bar{R}_1^1 : y_1^1 (= e_2^1 = \hat{e}_1^1), e_1^1 (= \hat{e}_2^1), \ldots$, and for agent 2, all type-1 objects that are better than $y_2^1 (= e_1^1 = \hat{e}_2^1)$ are already assigned. Thus, By *individual rationality* of f, $f_1^1(M') = \hat{e}_1^1 = e_2^1$ and $f_2^1(M') = \hat{e}_2^1 = e_1^1$. This implies that agents 1 and 2 are strictly better off by swapping their type-1 endowments, which is in contradiction with *one type endowments-swapping-proofness* of f.

The following induction arguments for $K > 2$ are similar to the induction basis part.

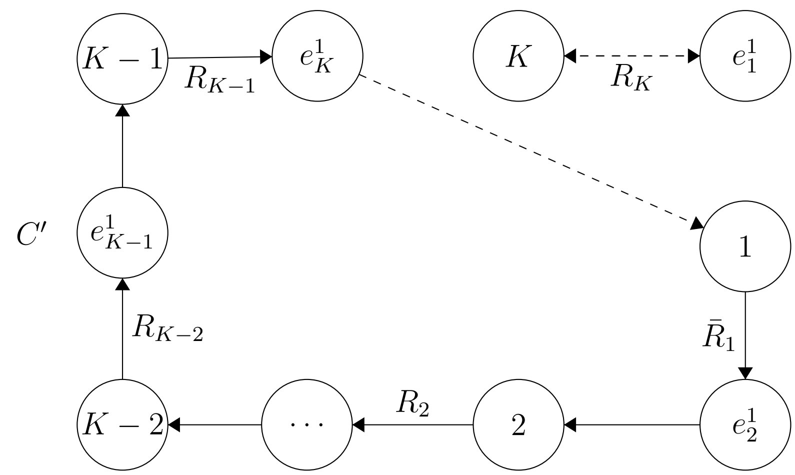
Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that C is executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and hence $C = (1 \to e_2^1 \to 2 \to \ldots \to K-1 \to e_K^1 \to K \to e_1^1 \to 1)$.

Recall that by (9), agent 1 receives his type-1 endowment and hence agent K does not receive his most (feasible) preferred object e_1^1.

Consider that agents 1 and K swap their type-1 endowments, i.e., $\hat{e}(t = 1, 1, K)$. Note that at this new endowment allocation, agent 1 owns e_K^1 and agent K owns e_1^1. Consider that $M' \equiv (\hat{e}(t = 1, 1, K), \bar{R}_1, R_{-1})$.

Recall that for agent K, all type-1 objects that are better than $e_1^1 (= \hat{e}_K^1)$ are already assigned. Thus, By *individual rationality* of f, $f_K^1(M') = \hat{e}_K^1 = e_1^1$. Moreover, at M', agent 1 is involved in a (step ℓ) top trading cycle $C' = (1 \to e_2^1 \to 2 \to \ldots \to K-1 \to e_K^1 \to 1)$. See the figure below for the graphical explanation.



Since $|S_{C'}| = K - 1$, by the induction hypothesis, C' is executed. Thus, $f_1^1(M') = e_2^1$. This implies that agents 1 and K are strictly better off by swapping their type-1 endowments, which is in contradiction with *one type endowments-swapping-proofness* of f. □

Next, we show that f also assigns the cTTC allocation of type-2.

Claim 3. For each (e, R) with $R \in \mathcal{R}_\pi^N$, $f^2(e, R) = cTTC^2(e, R)$.

Proof. The proof is similar to Claim 1, the main difference is that instead of *individual rationality*, we use Claim 2.

Let C be a first step top trading cycle under TTC^2 at (e, R) which consists of a set of agents $S_C \subseteq N$.

Similar to Claim 1, we first show that C is executed at $f(e, R)$ by induction on $|S_C|$.

Induction basis. $|S_C| = 1$. In this case, agent $i \in S_C$ points to his type-1 endowed object, i.e., $C = (i \to e_i^2 \to i)$. Thus, by Claim 2, C must be executed.

Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that C is executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and $C = (1 \to e_2^2 \to 2 \to \ldots \to K \to e_1^2 \to 1)$.

By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C$ who does not receive e_{i+1}^2, i.e., $f_i^2(e, R) \neq e_{i+1}^2$.

Let $\hat{R}_i \in \mathcal{R}_\pi$ be such that i only wants to receive one type-2 object e_{i+1}^2 rather than his type-2 endowment, i.e.,

$$\hat{R}_i^2 : e_{i+1}^2, e_i^2, \ldots,$$

$$\text{for each } t \in T \setminus \{2\} : \hat{R}_i^t = R_i^t, \text{ and}$$

$$\hat{\pi}_1 = \pi : 1, \ldots, m.$$

Note that at $\hat{R}_i$, if i does not receive e_{i+1}^2, then by Claim 2, he must receive his type-2 endowment e_i^2.

Let $\hat{R} \equiv (\hat{R}_i, R_{-i})$ and $\hat{M}_i \equiv (e, (\hat{R}_i, R_{-i}))$. We proceed in two steps. Note that by Claim 1, $f^1(\hat{M}_i) = cTTC^1(\hat{M}_i) = cTTC^1(e, R) = f^1(e, R)$.

Step 1. We show that agent i receives e_{1+1}^2 under f at $\hat{M}_i$, i.e., $f_i^2(\hat{M}_i) = e_{i+1}^2$.

By Claim 2, $f_i^2(\hat{M}_i) \in \{e_{i+1}^2, e_i^2\}$. By *strategy-proofness* of f, $f_i^2(e, R) \neq e_{i+1}^2$ implies that $f_i^2(\hat{M}_i) \neq e_{i+1}^2$, otherwise instead of R_i, agent i has an incentive to misreport $\hat{R}_i$ at (e, R). Thus, $f_i^2(M_i) = e_i^2$. Thus, agent $i-1$ cannot receive e_i^2 from agent i because it is assigned to agent i.

Overall, we find that

$$f_i^2(\hat{M}_i) = e_i^2 \neq e_{i+1}^2 \text{ and } f_{i-1}^2(\hat{M}_i) \neq e_i^2. \tag{10}$$

Consider that agents $i-1$ and i swap their type-2 endowments, i.e., $\hat{e}(t = 2, i-1, i)$. Note that at this new endowment allocation, $i-1$ owns his most preferred type-2 object e_i^2, and i owns e_{i-1}^2, and recall that e_{i-1}^2 is agent $(i-2)$'s most preferred type-2 object. Consider that $\hat{M}_i' \equiv (\hat{e}(t = 2, i-1, i), (\hat{R}_i, R_{-i}))$.

At $\hat{M}_i'$, for type-2, there are two first step top trading cycles $C' = (i - 1 \to e_i^2 \to i - 1)$ and $C'' = (1 \to e_2^2 \to 2 \to \ldots \to i-2 \to e_{i-1}^2 \to i \to e_{1+1}^2 \to i+1 \ldots \to K \to e_1^2 \to 1)$ at $\hat{M}_i'$. The graphical explanation is similar to the figure in Claim 1 and hence we omit it (the only difference is that now the superscript is type-2).

Note that $|S_{C'}| = 1$ and $|S_{C''}| = K - 1$. Thus, by the induction hypothesis, these two cycles are executed under f at $\hat{M}_i'$.

Hence, agents $i-1$ and i receive their most preferred type-2 objects at $f(\hat{M}_i')$. By (10), we find that $f_{i-1}^2(\hat{M}_i') = e_i^2 P_{i-1}^2 f_{i-1}^2(\hat{M}_i)$ and $f_i^2(\hat{M}_i') = e_{i+1}^2 P_i^2 f_i^2(\hat{M}_i)$. Moreover, by Claim 1, agents $i-$ and i still receive their cTTC allocation of type-1 at $f(\hat{M}_i')$. Therefore, $f_{i-1}(\hat{M}_i') \, P_{i-1} \, f_{i-1}(\hat{M}_i)$ and $f_i(\hat{M}_i') \, P_i \, f_i(\hat{M}_i)$. However, this implies that f is not *one type endowments-swapping proof*, a contradiction.

Step 2. We show that agent i receives e_{i+1}^2 under f at (e, R), i.e., $f_i^2(e, R) = e_{1+1}^2$.

Note that e^2_{1+1} is agent i's most preferred type-2 object at R_i. By *strategy-proofness* of f, $f_i(e,R)\ R_i\ f_i(\hat{M}_i)$. By Claim 1, $f^1_i(e,R) = f^1_i(\hat{M}_i)$. Hence, $f^2_i(e,R)\ R^2_i\ f^2_i(\hat{M}_i) = e^2_{i+1}$, which implies that $f^2_i(e,R) = e^2_{i+1}$.

It suffices to show that C is executed at $f(e,R)$ because once we have shown that agents who trade at the first step of the TTC algorithm (of type-2) always receive their TTC allotments of type-2 under f, we can consider agents who trade at the second step of the TTC (of type-2) by following the same proof arguments as for first step trading cycles, and so on. Thus, the proof of Claim 3 is completed. □

By Claim 1 and Claim 3, we know that for each (e,R) with $R \in \mathcal{R}^N_\pi$, the allocations of type-1 and type-2 under f are the same as *cTTC* allocation, i.e., $f^1(e,R) = cTTC^1(e,R)$ and $f^2(e,R) = cTTC^2(e,R)$. By applying similar arguments, we can also show that $f^3(e,R) = cTTC^3(e,R)$ and so on. Thus, we conclude that for each (e,R) with $R \in \mathcal{R}^N_\pi$, and each $t \in T$, $f^t(e,R) = cTTC^t(e,R)$, which completes the proof of Proposition 1.

Proof of Theorem 6 and 7

The proof of Theorem 6 will be shown by extending Proposition 1 to the full domain of lexicographic preference profiles.

Let $\bar{S} \subseteq N$ and (e,R) be such that only agents in $\bar{S}$ do no have restricted (but lexicographic) preferences, i.e., $R_{-\bar{S}} \in \mathcal{R}^{-\bar{S}}_\pi$ and for each $i \in \bar{S}$, $R_i \notin \mathcal{R}_\pi$. We show that $f(e,R) = cTTC(e,R)$ by induction on $|\bar{S}|$.

We first consider the case that only agent i does no have restricted (but lexicographic) preferences, i.e., $\bar{S} = \{i\}$. We will show that f still assigns the cTTC allocation.

Lemma 4. *For each $(e,R) \in \mathcal{M}_l$, each $i \in N$ with $R_i \notin \mathcal{R}_\pi$ and $R_{-i} \in \mathcal{R}^{-i}_\pi$, $f(e,R) = cTTC(e,R)$.*

The proof of Lemma 4 consists of four claims.

It is without loss of generality to assume that $i = 1$. Thus, $R_1 \in \mathcal{R}_l$ and $\pi_1 \neq \pi$. Let $y \equiv f(e,R)$ and $x \equiv cTTC(e,R)$.

We first show that agent 1 still receives his cTTC allocation at R, i.e., $y_1 = x_1$.

Claim 4. $y_1 = x_1$.

***Proof*.** Suppose not.

Let $\bar{R}_1 \in \mathcal{R}_\pi$ be such that $\bar{R}_1$ and R_1 are only different in the importance order, i.e., for each $t \in T$, $\bar{R}^t_1 = R^t_1$, and $\bar{\pi}_1 = \pi : 1, \ldots, m$. Note that $(\bar{R}_1, R_{-1}) \in \mathcal{R}^N_\pi$ and hence by Proposition 1, $f(e, \bar{R}_1, R_{-1}) = cTTC(e,R) = x$.

Note that if for each $t \in T$, $x^t_1\ R^t_1\ y^t_1$, then agent 1 has an incentive to misreport $\bar{R}_1$ at R. Thus, by *strategy-proofness* of f, there exists one type $\tau \in T$ such that $y^\tau_1\ P^\tau_1\ x^\tau_1$.

Note that by the definition of $cTTC$, $x_1^\tau \; R_1^\tau \; e_1^\tau$ and hence $y_1^\tau \neq e_1^t$. Overall, we have

$$y_1^\tau \; P_1^\tau \; x_1^\tau \; R_1^\tau \; e_1^\tau. \tag{11}$$

Let $\hat{R}_1 \in \mathcal{R}_\pi$ be such that for each type $t \in T$, agent 1 positions y_1^t first and e_1^t second, i.e.,

$$\text{for each } t \in T : \hat{R}_1^t : y_1^t, e_1^t, \ldots;$$

$$\hat{\pi}_1 = \pi : 1, \ldots, m$$

Let

$$\hat{R} \equiv (\hat{R}_1, R_{-1}).$$

By *strategy-proofness* of f, $f_1(e, \hat{R}) = f_1(e, R) = y_1$. Also note that $\hat{R} \in \mathcal{R}_\pi^N$. Hence, by Proposition 1, $f(e, \hat{R}) = cTTC(e, \hat{R})$. In particular, $y_1^\tau = cTTC_1^\tau(e, \hat{R})$.

Next, we show that $cTTC_1^\tau(e, \hat{R}) = e_1^\tau$. By the definition of $cTTC$,

$$cTTC_1^\tau(e, \hat{R}) = TTC_1^\tau(e^1, \hat{R}^\tau) \in \{y_1^\tau, e_1^\tau\}. \tag{12}$$

Recall that $cTTC^\tau(e, R) = TTC^\tau(e^\tau, R^\tau) = x^\tau$ and $y_1^\tau \; P_1^\tau \; x_1^\tau \; R_1^\tau \; e_1^\tau$ (see (11)). Thus, by *strategy-proofness* of TTC, $x_1^\tau = TTC_1^\tau(e, R^\tau) \, R_1^\tau \, TTC_1^\tau(e, \hat{R}^\tau)$. Together with (12), we conclude that $TTC_1^\tau(e, \hat{R}^\tau) = e_1^\tau$. It implies that $cTTC_1^\tau(e, \hat{R}_1, R_{-1}) = e_1^\tau \neq y_1^\tau$. □

Note that Claim 4 implies that for each (e, R) with $(R_1, R_{-1}) \in (\mathcal{R}_l \backslash \mathcal{R}_\pi) \times \mathcal{R}_\pi^{N \backslash \{1\}}$, $f_1(e, R) = cTTC_1(e, R)$.

Next, we show that $y = x$ by applying similar arguments in Claims 1, 2, and 3.

Claim 5. For each (e, R) with $(R_1, R_{-1}) \in (\mathcal{R}_l \setminus \mathcal{R}_\pi) \times \mathcal{R}_\pi^{N \backslash \{1\}}$, $f^1(e, R) = cTTC^1(e, R)$.

Proof. Let ℓ be the step of the TTC algorithm at which agent 1 receives type-1 object $y_1^1 (= x_1^1 = cTTC_1^1(e, R))$. Let C be the corresponding top trading cycle that involves agent 1, i.e., $1 \in S_C$.

Note that by Claim 1 and Fact 3, all top trading cycles that are obtained before step ℓ are executed at $f(e, R)$. Moreover, if C is executed, then again by Claim 1 and Fact 3, all remaining top trading cycles are also executed. Thus, it suffices to show that C is executed, i.e., for each $i \in S_C$, $f_i^1(e, R) = cTTC_i^1(e, R)$.

Since all top trading cycles that are obtained before step ℓ are executed, by the definition of TTC, we know that for each agent in S_C, the object that he pointed at in C is his most preferred type-1 object among the unassigned type-1 objects, i.e., for each $i \in S_C$, all better type-1 objects for him, are assigned to someone else via top trading cycles that are obtained before step ℓ.

Similar to Claim 1, we show that C is executed at $f(e, R)$ by induction on $|S_C|$.

Induction basis. $|S_C| = 1$. In this case, $S_C = \{1\}$. By Claim 4, $f_1^1(e, R) = cTTC_1^1(e, R)$.

Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that C is executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and $C = (1 \to e_2^1 \to 2 \to \ldots \to K \to e_1^1 \to 1)$.

By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C \setminus \{1\}$ who does not receive e_{i+1}^1, i.e., $f_i^1(e,R) \neq e_{i+1}^1$.

Let $\hat{R}_i \in \mathcal{R}_\pi$ be such that i only wants to receive one type-1 object e_{i+1}^1 than his type-1 endowment, i.e.,

$$\hat{R}_i^1 : e_{i+1}^1, e_i^1, \ldots,$$

$$\text{for each } t \in T \setminus \{1\} : \hat{R}_i^t = R_i^t, \text{ and}$$

$$\hat{\pi}_1 = \pi : 1, \ldots, m.$$

Note that at $\hat{R}_i$, if i does not receive e_{i+1}^1, then by *individual rationality* of f, he must receive his type-1 endowment e_i^1.

Let $\hat{R} \equiv (\hat{R}_i, R_{-i})$ and $\hat{M}_i \equiv (e, (\hat{R}_i, R_{-i}))$. We proceed in two steps.

Step 1. We show that agent i receives e_{i+1}^1 under f at $\hat{M}_i$, i.e., $f_i^1(\hat{M}_i) = e_{i+1}^1$.

Since $\hat{R}_i \in \mathcal{R}_l$ and f is *individually rational*, $f_i^1(\hat{M}_i) \in \{e_{i+1}^1, e_i^1\}$. By *strategy-proofness* of f, $f_i^1(e,R) \neq e_{i+1}^1$ implies that $f_i^1(\hat{M}_i) \neq e_{i+1}^1$, otherwise instead of R_i, agent i has an incentive to misreport $\hat{R}_i$ at (e,R). Thus, $f_i^1(\hat{M}_i) = e_i^1$. Thus, agent $i-1$ cannot receive e_i^1 from agent i because it is assigned to agent i.

Consider that agents $i-1$ and i swap their type-1 endowments, i.e., $\hat{e}(t=1, i-1, i)$. Note that at this new endowment allocation, $i-1$ owns his most (feasible) preferred type-1 object e_i^1, and i owns e_{i-1}^1, and recall that e_{i-1}^1 is agent $(i-2)$'s most (feasible) preferred type-1 object. Consider that $\hat{M}_i' \equiv (\hat{e}(t=1, i-1, i), (\hat{R}_i, R_{-i}))$.

At $\hat{M}_i'$, there are two first step top trading cycles $C' = (i-1 \to e_i^1 \to i-1)$ and $C'' = (1 \to e_2^1 \to 2 \to \ldots \to i-2 \to e_{i-1}^1 \to i \to e_{1+1}^1 \to i+1 \ldots \to K \to e_1^1 \to 1)$ at $\hat{M}_i'$. The graphical explanation is exactly the same as the figure in Claim 1.

Note that $|S_{C'}| = 1$ and $|S_{C''}| = K-1$. Thus, by the induction hypothesis, these two cycles are executed under f at $\hat{M}_i'$.

Hence, agents $i-1$ and i receive their most (feasible) preferred type-1 objects at $f(\hat{M}_i')$. Thus, $f_{i-1}^1(\hat{M}_i') = e_i^1 \, P_{i-1}^1 \, f_{i-1}^1(\hat{M}_i)$ and $f_i^1(\hat{M}_i') = e_{i+1}^1 \, P_i^1 \, f_i^1(\hat{M}_i)$. Therefore, $f_i(\hat{M}_i') \, P_i \, f_i(\hat{M}_i)$ and $f_{i-1}(\hat{M}_i') \, P_{i-1} \, f_{i-1}(\hat{M}_i)$.[39] However, this implies that f is not *one type endowments-swapping proof*, a contradiction.

Step 2. We show that agent i receives e_{i+1}^1 under f at (e,R), i.e., $f_i^1(e,R) = e_{i+1}^1$.

Note that e_{i+1}^1 is agent i's most (feasible) preferred type-1 object at R_i. By *strategy-proofness* of f, $f_i(e,R) \, R_i \, f_i(\hat{M}_i)$. Hence, $f_i^1(e,R) \, R_i^1 \, f_i^1(\hat{M}_i) = e_{i+1}^1$, which implies that $f_i^1(e,R) = e_{i+1}^1$. □

[39] Note that this is also true for $i-1=1$. Because by Claim 4, agent 1 always receives his cTTC allocation of other types, i.e., for each $t \neq 1$, $cTTC_1^t(e,R) = f_1^t(e,R) = f_1^t(\hat{M}_i) = f_1^t(\hat{M}_i')$.

The next two claims, Claims 6 and 7, can be proven by a similar way to Claim 2 and 3, respectively. Thus, we omit the proofs. Note that the key point is that since agent 1 still receives his cTTC allocation, we only need to show that agents who still have restricted preferences, will also receive their cTTC allocation. Thus the proofs of Claim 2 and 3 are still valid for the case where only agent 1 does not have restricted preferences.

Claim 6. For each (e, R) with $(R_1, R_{-1}) \in (\mathcal{R}_l \setminus \mathcal{R}_\pi) \times \mathcal{R}_\pi^{N\setminus\{1\}}$, and each $i \in N$, $f_i^2(e, R)\, R_i^2\, e_i^2$.

Claim 7. For each (e, R) with $(R_1, R_{-1}) \in (\mathcal{R}_l \setminus \mathcal{R}_\pi) \times \mathcal{R}_\pi^{N\setminus\{1\}}$, $f^2(e, R) = cTTC^2(e, R)$.

Thus, similar to the proof of Proposition 1, by Claims 4, 5, 6, and 7, we conclude that for each $(e, R) \in \mathcal{M}_l$, and each $\bar{S} \subseteq N$, such that $|\bar{S}| = 1$ and $R_{-\bar{S}} \in \mathcal{R}_\pi^{-\bar{S}}$, $f(e, R) = cTTC(e, R)$. Thus, the proof of Lemma 4 is completed.

Now, we are ready to prove Theorem 6. Let $(e, R) \in \mathcal{M}_l$ and $\bar{S} \subseteq N$ be such that exactly only agents in $\bar{S}$ have non restricted (but lexicographic) preferences, we show that $f(e, R) = cTTC(e, R)$.

Lemma 5. *For each $(e, R) \in \mathcal{M}_l$ and each $\bar{S} \subseteq N$ such that $R_{-\bar{S}} \in \mathcal{R}_\pi^{-\bar{S}}$, $f(e, R) = cTTC(e, R)$.*

The proof of Lemma 5 is showing by induction on $|\bar{S}|$.

Induction basis. $|\bar{S}| = 1$. This is done by Lemma 4.

Induction hypothesis. Let $K \in \{2, \ldots, n\}$. Suppose that $f(e, R) = cTTC(e, R)$ when $|\bar{S}| < K$.

Induction step. Let $|\bar{S}| = K$. Similar to Lemma 4, the proof of this part consists of four claims.

We first show that agents in $\bar{S}$ still receive their cTTC allotments.

Claim 8. For each (e, R) with $(R_{\bar{S}}, R_{-\bar{S}}) \in (\mathcal{R}_l \setminus \mathcal{R}_\pi)^{\bar{S}} \times \mathcal{R}_\pi^{-\bar{S}}$, and each $i \in \bar{S}$, $f_i(e, R) = cTTC_i(e, R)$.

Proof. Let $y \equiv f(e, R)$ and $x = cTTC(e, R)$. By contradiction, assume that there is an agent $i \in \bar{S}$ who does not receive his cTTC allotment x_i. Without loss of generality, assume that $i = 1$.

Let $\bar{R}_1 \in \mathcal{R}_\pi$ be such that $\bar{R}_1$ and R_1 are only different in the importance order, i.e., for each $t \in T$, $\bar{R}_1^t = R_1^t$, and and $\bar{\pi}_1 = \pi : 1, \ldots, m$. Let $\bar{R} \equiv (\bar{R}_1, R_{-1})$.

Note that at $\bar{R}$, there are only $K - 1$ agents (in $\bar{S} \setminus \{1\}$) who have non restricted (but lexicographic) preferences. Thus, by the induction hypothesis and the definition of *cTTC*, $f(e, \bar{R}) = cTTC(e, \bar{R}) = cTTC(e, R) = x$. Then, the remaining proof is exactly the same as the proof of Claim 4.

Note that if for each $t \in T$, $x_1^t\, R_1^t\, y_1^t$, then agent 1 has an incentive to misreport $\bar{R}_1$ at R. Thus, by *strategy-proofness* of f, there exists one type $\tau \in T$ such that $y_1^\tau\, P_1^\tau\, x_1^\tau$.

Note that by the definition of $cTTC$, $x_1^\tau \, R_1^\tau \, e_1^\tau$ and hence $y_1^\tau \neq e_1^t$. Overall, we have

$$y_1^\tau \, P_1^\tau \, x_1^\tau \, R_1^\tau \, e_1^\tau. \tag{13}$$

Let $\hat{R}_1 \in \mathcal{R}_\pi$ be such that for each type $t \in T$, agent 1 positions y_1^t first and e_1^t second, i.e.,

$$\text{for each } t \in T : \hat{R}_1^t : y_1^t, e_1^t, \ldots;$$

$$\hat{\pi}_1 = \pi : 1, \ldots, m$$

Let

$$\hat{R} \equiv (\hat{R}_1, R_{-1}).$$

Note that $\hat{R} \in \mathcal{R}_\pi^N$.

By *strategy-proofness* of f, $f_1(e, \hat{R}) = f_1(e, R) = y_1$. Hence, by Proposition 1, $f(e, \hat{R}) = cTTC(e, \hat{R})$. In particular, $y_1^\tau = cTTC_1^\tau(e, \hat{R})$.

Next, we show that $cTTC_1^\tau(e, \hat{R}) = e_1^\tau$. By the definition of $cTTC$,

$$cTTC_1^\tau(e, \hat{R}) = TTC_1^\tau(e^1, \hat{R}^\tau) \in \{y_1^\tau, e_1^\tau\}. \tag{14}$$

Recall that $cTTC^\tau(e, R) = TTC^\tau(e^\tau, R^\tau) = x^\tau$ and $y_1^\tau \, P_1^\tau \, x_1^\tau \, R_1^\tau \, e_1^\tau$ (see (13)). Thus, by *strategy-proofness* of TTC, $x_1^\tau = TTC_1^\tau(e, R^\tau) \, R_1^\tau \, TTC_1^\tau(e, \hat{R}^\tau)$. Together with (14), we conclude that $TTC_1^\tau(e, \hat{R}^\tau) = e_1^\tau$. It implies that $cTTC_1^\tau(e, \hat{R}_1, R_{-1}) = e_1^\tau \neq y_1^\tau$. □

The following three claims can be proven by a similar way to 5, 6, and 7, respectively. Thus, we omit the proofs. Note that the key point is that since agents in $\bar{S}$ still receive their cTTC allocation, we only need to show that agents who still have restricted preferences, will also receive their cTTC allocation. Thus the proofs of Claim 6 and 7 are still valid for the case where only agents in $\bar{S}$ do not have restricted preferences.

Claim 9. For each (e, R) with $(R_{\bar{S}}, R_{-\bar{S}}) \in (\mathcal{R}_l \setminus \mathcal{R}_\pi)^{\bar{S}} \times \mathcal{R}_\pi^{-\bar{S}}$, $f^1(e, R) = cTTC^1(e, R)$.

Claim 10. For each (e, R) with $(R_{\bar{S}}, R_{-\bar{S}}) \in (\mathcal{R}_l \setminus \mathcal{R}_\pi)^{\bar{S}} \times \mathcal{R}_\pi^{-\bar{S}}$, and each $i \in N$, $f_i^2(e, R) \, R_i^2 \, e_i^2$.

Claim 11. For each (e, R) with $(R_{\bar{S}}, R_{-\bar{S}}) \in (\mathcal{R}_l \setminus \mathcal{R}_\pi)^{\bar{S}} \times \mathcal{R}_\pi^{-\bar{S}}$, $f^2(e, R) = cTTC^2(e, R)$.

Hence, we conclude that for each $(e, R) \in \mathcal{M}_l$, and each $\bar{S} \subseteq N$, such that $|\bar{S}| = K$ and $R_{-\bar{S}} \in \mathcal{R}_\pi^{-\bar{S}}$, $f(e, R) = cTTC(e, R)$. Thus, the proof of Lemma 5 is completed. Therefore, Theorem 6 is proven by applying Lemma 5 with $\bar{S} = N$.

Theorem 7 can be proven by exactly the same way to Theorem 6. That is, we first show that f assigns the cTTC allocation if only one agent does not have lexicographic (but separable) preferences. Then, by applying this preference replacement argument, one by one, for all other agents, we conclude that f equals cTTC on the domain of separable preference profiles. Note that in the proof of Lemmas 4 and 5, for each agent in $\bar{S}$, we only use the characterization of his marginal preferences. Thus the proof of Lemmas 4 and 5 are still valid for the case where only agents in $\bar{S}$, do not have lexicographic (but separable) preferences.

Proof of Theorem 8

Proof. Consider markets with two types, i.e., $T = \{1, 2\}$.

Suppose that there is a mechanism $f : \mathcal{R} \to X$ that is *one type endowments-swapping-proof*, *individually rational*, and *strategy-proof*.

Let $x, y \in X \backslash \{e\}$ be such that at x agents 1 and 2 swap their endowments of type 2, i.e.,

$$\begin{aligned} & x_1 = (e_1^1, e_2^2), \\ & x_2 = (e_2^1, e_1^2), \\ \text{and for each } i = 3, \ldots, n, \quad & x_i = e_i \end{aligned}$$

and at y agents 1 and 2 swap their endowments of type 1, i.e.,

$$\begin{aligned} & y_1 = (e_2^1, e_1^2), \\ & y_2 = (e_1^1, e_2^2), \\ \text{and for each } i = 3, \ldots, n, \quad & y_i = e_i. \end{aligned}$$

Obviously, $x \neq y$.

Let $R \in \mathcal{R}^N$ be such that agents 1 and 2 prefer only their allotments at x and y to their endowments, they disagree on which allocation is the better one, and each other agent ranks her endowments highest, i.e.,

$$\begin{aligned} & R_1 : x_1, y_1, e_1, \ldots, \\ & R_2 : y_2, x_2, e_2, \ldots, \\ \text{and for each } i = 3, \ldots, n, \quad & R_i : e_i, \ldots. \end{aligned}$$

Note that $R \in \mathcal{R}^N \setminus \mathcal{R}_s^N$. There are only three *individually rational* allocations at (e, R): x, y, and e. Also note that $\hat{e}(t = 1, \{1, 2\}) = y$ and $\hat{e}(t = 2, \{1, 2\}) = x$. By *individually rationality* of f, $f(e, R) \in \{x, y, e\}$ and

$$f(\hat{e}(t = 1, \{1, 2\}), R) = y \text{ and } f(\hat{e}(t = 2, \{1, 2\}), R) = x. \tag{15}$$

Let

- $R_1' : x_1, e_1, \ldots$, and
- $R_2' : y_2, e_2, \ldots$.

Note that by *individually rationality* of f and (15),

$$f(\hat{e}(t = 1, \{1, 2\}), R_2', R_{-2}) = y \text{ and } f(\hat{e}(t = 2, \{1, 2\}), R_1', R_{-1}) = x. \tag{16}$$

Suppose that $f(e,R) = e$. Then, agents $\{1,2\}$ can be strictly better off by swapping their endowments of type-1 (see (15)), which contradicts *one type endowments-swapping-proofness* of f. Therefore, $f(e,R) \in \{x,y\}$.

Suppose that $f(e,R) = y$. Then, by *strategy-proofness* of f, $f_1(e,R_1',R_{-1}) \neq x_1$ and hence, by *individual rationality* of f, $f(e,R_1',R_{-1}) = e$. However, this violates *one type endowments-swapping-proofness* of f because agents $\{1,2\}$ can be strictly better off by swapping their endowments of type-2 (see (16)).

Suppose that $f(e,R) = x$. Then, by *strategy-proofness* of f, $f_2(e,R_2',R_{-2}) \neq y_2$ and hence by *individual rationality* of f, $f(e,R_2',R_{-2}) = e$. However, this violates *one type endowments-swapping-proofness* of f because agents $\{1,2\}$ can be strictly better off by swapping their endowments of type-1 (see (16)). □

A.7 Proof of Theorem 9

Ekici (2022) shows that the TTC mechanism satisfies *pairwise efficiency* for Shapley-Scarf housing markets. By using similar arguments we also obtain that bTTC inherits *pairwise efficiency* from the underlying top trading cycles algorithm for the restricted market $(e, R|^e)$. So we only show that bTTC is the only mechanism satisfying all our properties in Theorem 9.

Note that in the proof below, we do not consider any endowments-swapping. Thus, from now and until the end of the proof, we will denote a market simply by its preference profile. Moreover, we only show the first part of Theorem 9, i.e., the characterization for lexicographic preferences, as the extensions to separable preferences and strict preferences can be proven by exactly the same way to Theorems 4 and 5.

Let $f : \mathcal{M}_l \to X$ be *individually rational*, *strategy-proof*, *non-bossy*, and *pair-efficient*. Note that by Lemma 1, f is *monotonic*.

Similar to the proof of Theorem 3, we show that all first step top trading cycles are fully executed.

First, we show that if a first step top trading cycle is formed by only one or two agents, then it is fully executed.

Lemma 6. *If a mechanism $f : \mathcal{M}_l \to X$ is individually rational, strategy-proof, non-bossy, and pair-efficient, then for each $R \in \mathcal{R}_l^N$, each first step top trading cycle $C(\in \mathcal{C}(R)$ with $|S_C| \leq 2$, C is fully executed under f at R.*

Proof. Let $C \in \mathcal{C}(R)$ be a first step top trading cycle that consists of agents S_C with $|S_C| \leq 2$.

We show it by two steps. First, we show that C is executed.

Claim 12. C is executed.

Proof. When $|S_C| = 1$. In this case, agent $i \in S_C$ points to one of his endowed object, i.e., $c_i = e_i^{t_i}$ and hence $C = (i \to c_i \to i)$. Since preferences are lexicographic, i.e., $R_i \in \mathcal{R}_l$, agent i

will be strictly worse off if he receives any other type-t_i objects. Thus, C must be executed by *individual rationality* of f.

When $|S_C| = 2$. Without loss of generality, assume that $S_C = \{1, 2\}$. By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C$ does not receive his most preferred object c_i. Without loss of generality, let $i = 2$.

Let $\hat{R}_2$ be such that agent 2 only wants to receive type-t_2 object c_2 and no other objects, i.e.,

$$\hat{R}_2^{t_2} : c_2(= e_1^{t_2}), e_2^{t_2}, \dots,$$

$$\text{for each } t \in T \setminus \{t_2\} : \hat{R}_2^t : e_2^t, \dots, \text{ and}$$

$$\hat{\pi}_2 = \pi_2 : t_2, \dots.$$

Note that at $\hat{R}_2$, if 2 does not receive c_2, then from individual rationality of f, he must receive his full endowment e_2. Let

$$\hat{R} \equiv (\hat{R}_2, R_{-2}).$$

By *individual rationality* of f, $f_2^{t_2}(\hat{R}) \in \{c_2, e_2^{t_2}\}$. By *strategy-proofness* of f, $f_2^{t_2}(R) \neq c_2$ implies that $f_2^{t_2}(\hat{R}) \neq c_2$, otherwise instead of R_2, agent 2 has an incentive to misreport $\hat{R}_2$ at R. Thus, $f_2^{t_2}(\hat{R}) = e_2^{t_2}$. Then, by *individual rationality* of f, $f_2(\hat{R}) = e_2$. Thus, agent 1 cannot receive $c_1(\in e_2)$ from agent 2 because it is assigned to agent 2.

Let $\bar{R}_1$ be such that be such that agent 1 only wants to receive type-t_1 object c_1 and no other objects, i.e.,

$$\bar{R}_1^{t_1} : c_1(= e_2^{t_1}), e_1^{t_1}, \dots,$$

$$\text{for each } t \in T \setminus \{t_1\} : \bar{R}_1^t : e_1^t, \dots \text{ and}$$

$$\bar{\pi}_1 = \hat{\pi}_1 = \pi_1 : t_1, \dots.$$

Note that at $\bar{R}_1$, if agent 1 does not receive c_1, then from individual rationality of f, he must receive his full endowment e_1. Let

$$\bar{R} \equiv (\bar{R}_1, \hat{R}_2, \hat{R}_3, \dots, \hat{R}_n) = (\bar{R}_1, \hat{R}_2, R_3, \dots, R_n).$$

To obtain the contradiction, we show that at $\bar{R}$, agent 1 receives c_1 and agent 2 receives c_2, i.e., $f_1^{t_1}(\bar{R}) = c_1 = e_2^{t_1}$ and $f_2^{t_2}(\bar{R}) = c_2 = e_1^{t_2}$.

By *individual rationality* of f, $f_1^{t_1}(\bar{R}) \in \{c_1, e_1^{t_1}\}$. By *strategy-proofness* of f, $f_1^{t_1}(\hat{R}) \neq c_1$ implies that $f_1^{t_1}(\bar{R}) \neq c_1$, otherwise instead of $\hat{R}_1$, agent 1 has an incentive to misreport $\bar{R}_1$ at $\hat{R}$. Thus, $f_1^{t_1}(\bar{R}) = e_1^{t_1}$. Then, by *individual rationality* of f, $f_1(\bar{R}) = e_1$, and in particular, $f_1^{t_2}(\bar{R}) = e_1^{t_2} = c_2$. Moreover, by *individual rationality* of f, $f_2(\bar{R}) = e_2$, and in particular, $f_2^{t_1}(\bar{R}) = e_2^{t_1} = c_1$. This implies that $f_2^{t_1}(\bar{R})\, P_1^{t_1}\, f_1^{t_1}(\bar{R})$ and $f_1^{t_2}(\bar{R})\, P_2^{t_2}\, f_2^{t_2}(\bar{R})$ and hence $f_2(\bar{R})\, P_1$ $f_1(\bar{R})$ and $f_1(\bar{R})\, P_2\, f_2(\bar{R})$, in which contradicts with *pair-efficiency* of f.

Overall, by contradiction we show that at $\bar{R}$, agent 1 receives c_1. Thus, by *strategy-proofness* of f, he also receives c_1 at $\hat{R}$; otherwise he has an incentive to misreport $\bar{R}_1$ at $\hat{R}$. Together with *individual rationality* of f, it implies that agent 2 receives c_2 at $\hat{R}$. Therefore, by *strategy-proofness* of f, agent 2 also receives c_2 at R; otherwise he has an incentive to misreport $\hat{R}_1$ at R. □

Next, we show that C is fully executed. There are two cases.

Case 1. $|S_C| = 1$. In this case, agent $i \in S_C$ points to one of his endowed object, i.e., $c_i \in e_i$. Without loss of generality, assume that $S_C = \{1\}$ and $\pi_1 : t_1, \ldots$. Thus, $C = (1 \to e_1^{t_1} \to 1)$.

Let $y \equiv f(R)$. By contradiction, suppose that $y_1 \neq e_1$. Note that by Lemma 2, $y_1^{t_1} = e_1^{t_1}$. Let $t \in T \setminus \{t_i\}$ be such that $y_1^t \neq e_1^t$. Without loss of generality, assume that agent 1 receives agent 2's endowment of type-t at y, i.e., $y_1^t = e_2^t$.

Let $\hat{R} \in \mathcal{R}_l^N$ be such that each agent j positions y_j at the top and changes his importance order as π_1, i.e., for each agent $j \in N$, (i) $\hat{\pi}_j = \pi_1 : t_1, \ldots$, and (ii) for each $\tau \in T$, $\hat{R}_j^\tau : y_j^\tau, \ldots$

By monotonicity of f, $f(\hat{R}) = y$.

Let $\bar{R}_2$ be such that

$$\bar{\pi}_2 = \hat{\pi}_2 (= \pi_1),$$

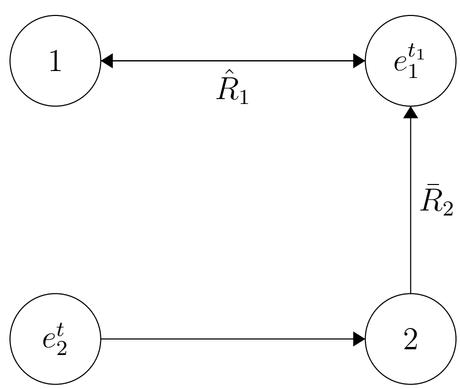
$$\bar{R}_2^{t_1} : e_1^{t_1}, y_2^{t_1}, \ldots, \text{ and}$$

$$\text{For each } \tau \in T \setminus \{t_1\}, \bar{R}_2^\tau = \hat{R}_2^\tau.$$

Let

$$\bar{R} \equiv (\bar{R}_2, \hat{R}_{-2}).$$

Note that by strategy-proofness of f, for type-t_1, agent 2 either receives $e_1^{t_1}$ or $y_2^{t_1}$; otherwise he has an incentive to misreport $\hat{R}_2$ at $\bar{R}$. Moreover, C is still a first top trading cycle at $\bar{R}$, i.e., $C \in \mathcal{C}(\bar{R})$. Thus, by Claim 12, C is executed and hence agent 1 receives $e_1^{t_1}$ at $f(\bar{R})$. See the figure below for the graphical explanation.



Thus, agent 2 still receives $y_2^{t_1}$, and hence by Fact 2,

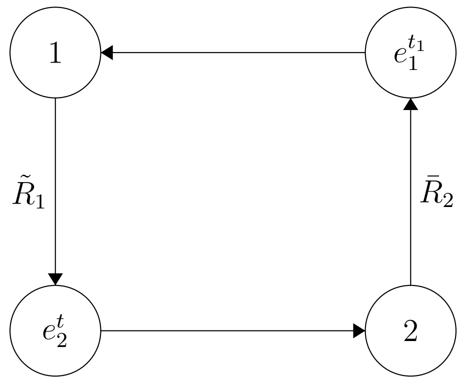
$$f(\bar{R}) = f(\hat{R}) = y \text{ and particularly, } f_1^{t_1}(\tilde{R}) = y_1^{t_1} = e_1^{t_1}. \tag{17}$$

Let $\tilde{R}_1$ be such that agent 1 only changes his importance order as t is the most important, i.e., $\tilde{\pi}_1 : t, \dots$ and $\tilde{R}_1 = (\hat{R}_1^1, \dots, \hat{R}_1^m, \tilde{\pi}_1)$.

Let

$$\tilde{R} \equiv (\tilde{R}_1, \bar{R}_{-1}).$$

By monotonicity of f, $f(\tilde{R}) = f(\bar{R}) = y$. However, at $\tilde{R}$, there is a first step top trading cycle $C' \in \mathcal{C}(\tilde{R})$ consisting of agents 1 and 2, i.e., $C' = (1 \to e_2^t(= y_1^t) \to 2 \to e_1^{t_1} \to 1)$. See the figure below for the graphical explanation.



By Claim 12, C' is executed under f at $\tilde{R}$. Thus, $f_2^{t_1}(\tilde{R}) = e_1^{t_1}$, which contradicts with the fact that $f_1^{t_1}(\tilde{R}) = y_1^{t_1} = e_1^{t_1}$ (see (17)).

Case 2. $|S_C| = 2$. Without loss of generality, assume that $S_C = \{1, 2\}$. Thus, $C = (1 \to c_1(= e_2^{t_1}) \to 2 \to c_2(= e_1^{t_2}) \to 1)$. By contradiction, assume that C is not executed. Without loss of generality, assume that agent 1 does not receive agent 2's full endowments, i.e., $f_1(R) \neq e_2$. Note that by Claim 12, $f_1^{t_1}(R) = c_1 = e_2^{t_1}$. Thus, there is a type $t \in T \setminus \{t_1\}$ such that $f_1^t(R) \neq e_2^t$. Without loss of generality, assume that agent 1 receives agent i's endowment of type-t, i.e., $f_1^t(R) = e_i^t$. Let $y \equiv f(R)$. There are two sub-cases.

Sub-case 1. $i = 1$. Let $\hat{R}_1$ be such that

$$\text{for each} t \in T, \hat{R}_1^t : y_1^t, \dots, \text{ and}$$

$$\hat{\pi}_1(t) = 1.$$

By *monotonicity* of f, $f(\hat{R}_1, R_{-1}) = f(R) = y$. Then, we are back to Case 1.

Sub-case 2. $i \neq 1$. Without loss of generality, assume that $i = 3$. Thus, $y_2^{t_2} = e_1^{t_2}$, $y_1^{t_1} = e_2^{t_1}$, and $y_2^t = e_3^t$. We will obtain a contradiction to complete the proof of this sub-case. Let $\hat{R}_3$ be such that

$$R_3^{t_2} : e_1^{t_2}, y_3^{t_2}, \dots$$

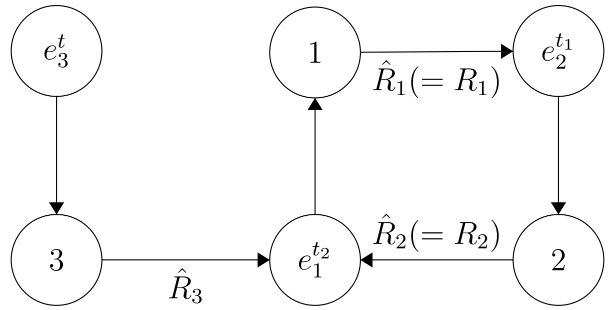
$$\text{for each} t \in T \setminus \{t_2\}, \hat{R}_3^t : y_3^t, \dots, \text{ and}$$

$$\hat{\pi}_3(t) = t_2.$$

Let

$$\hat{R} \equiv (\hat{R}_3, R_{-3}).$$

Note that C is still a first step top trading cycle at $\hat{R}$, i.e., $C \in \mathcal{C}(\hat{R})$. Thus, by Claim 12, C is executed. See the figure below for the graphical explanation.



Hence, agent 3 cannot receive $e_1^{t_2}(= c_2)$. Thus, by *strategy-proofness* of f, he still receives y_3, i.e., $f_3(\hat{R}) = y_3$. Therefore, by *non-bossiness* of f, $f(\hat{R}) = y$.

Let $\bar{R}_1$ be such that

$$\text{for each } t \in T, \bar{R}_1^t : y_1^t, \ldots, \text{ and}$$

$$\bar{\pi}_1(t) = 1.$$

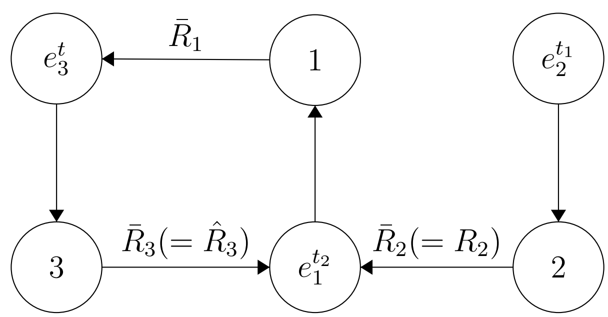
Let

$$\bar{R} \equiv (\bar{R}_1, \hat{R}_{-1}).$$

Then, since $\bar{R}_1$ is a monotonic transformation of $\hat{R}_1$ at y, $f(\bar{R}) = y$. In particular,

$$f_2^{t_1}(\bar{R}) = e_1^{t_2}(= c_2). \tag{18}$$

Note that at $\bar{R}$, there is a first step top trading cycle $C' = (1 \to y_1^t(= e_3^t) \to 3 \to e_1^{t_2} \to 1)$ that involves two agents. Thus, by Claim 12, C' is executed. See the figure below for the graphical explanation.



It implies that $f_3^{t_2}(\bar{R}) = e_1^{t_2}$, which contradicts with $f(\bar{R}) = y = f(\hat{R})$ and (18). □

Now we are ready to show that all first step top trading cycles are fully executed.

Lemma 7. *If a mechanism $f : \mathcal{M}_l \to X$ is individually rational, strategy-proof, non-bossy, and pair-efficient, then for each $R \in \mathcal{R}_l^N$, each first step top trading cycle $C(\in \mathcal{C}(R))$ is fully executed under f at R.*

Proof. Let $C \in \mathcal{C}(R)$ be a first step top trading cycle that consists of agents $S_C \subseteq N$. We prove this lemma by induction on S_C.

Induction Basis. $|S_C| \leq 2$. This is done by Lemma 6.

Induction hypothesis. Let $K \in \{3, \ldots, n\}$. Suppose that C is fully executed when $|S_C| < K$.

Induction step. Let $|S_C| = K$. Without loss of generality, assume that $S_C = \{1, \ldots, K\}$ and $C = (1 \to c_1 \to 2 \to c_2 \to \ldots \to K \to c_K \to 1)$.

Similar to Lemma 6, we first show that C is executed.

Claim 13. C is executed.

Proof. By contradiction, assume that C is not executed. Thus, there is an agent $i \in S_C$ who does not receive c_i, i.e., $f_i^{t_i}(e, R) \neq c_i$. Without loss of generality, let $i = 2$.

Let $\hat{R}_2$ be such that agent 2 only wants to receive type-t_2 object c_2 and no other objects, i.e.,

$$\hat{R}_2^{t_2} : c_2(= e_3^{t_2}), e_2^{t_2}, \ldots,$$

$$\text{for each } t \in T \setminus \{t_2\} : \hat{R}_2^t : e_2^t, \ldots, \text{ and}$$

$$\hat{\pi}_2 = \pi_2 : t_2, \ldots .$$

Note that at $\hat{R}_2$, if agent 2 does not receive c_2, then from individual rationality of f, he must receive his full endowment e_2.

Let $\hat{R} \equiv (\hat{R}_2, R_{-2})$. We proceed in two steps.

Step 1. We show that agent 2 receives c_2 under f at $\hat{R}$, i.e., $f_2^{t_2}(\hat{R}) = c_2$.

By *individual rationality* of f, $f_2^{t_2}(\hat{R}) \in \{c_2, e_2^{t_2}\}$. By *strategy-proofness* of f, $f_2^{t_2}(R) \neq c_2$ implies that $f_2^{t_2}(\hat{R}) \neq c_2$, otherwise instead of R_2, agent 2 has an incentive to misreport $\hat{R}_2$ at R. Thus, $f_2^{t_2}(\hat{R}) = e_2^{t_2}$. Then, by *individual rationality* of f, $f_2(\hat{R}) = e_2$. Thus, agent 1 cannot receive $c_1(\in e_2)$ from agent 2 because it is assigned to agent 2.

Let $y \equiv f(\hat{R})$. Overall, we find that

$$y_2 = e_2 \text{ and } y_1^{t_1} \neq c_1(= e_2^{t_1}). \tag{19}$$

Let $\bar{R}_1$ be such that

$$\bar{R}_1^{t_1} : c_1(= e_2^{t_1}), e_3^{t_1}, e_1^{t_1}, \ldots,$$

$$\text{for each } t \in T \setminus \{t_1\} : \bar{R}_1^t := \hat{R}_1^t(= R_1^t) \text{ and}$$

$$\bar{\pi}_1 = \hat{\pi}_1 = \pi_1.$$

Note that $\bar{R}_1$ and $\hat{R}_1$ only differ in type-t_1 marginal preferences. Let

$$\bar{R} \equiv (\bar{R}_1, \hat{R}_2, \hat{R}_3, \ldots, \hat{R}_n).$$

To obtain the contradiction, we want to show that at $\bar{R}$, agent 1 receives c_1 and agent 2 receives c_2, i.e., $f_1^{t_1}(\bar{R}) = c_1 = e_2^{t_1}$ and $f_2^{t_2}(\bar{R}) = c_2 = e_3^{t_2}$.

By *individual rationality* of f, $f_1^{t_1}(\bar{R}) \in \{e_2^{t_1}, e_3^{t_1}, e_1^{t_1}\}$. By *strategy-proofness* of f, $f_1^{t_1}(\hat{R}) \neq e_2^{t_1}$ implies that $f_1^{t_1}(\bar{R}) \neq e_2^{t_1}$, otherwise instead of $\hat{R}_1$, agent 1 has an incentive to misreport $\bar{R}_1$ at $\hat{R}$.

Thus, $f_1^{t_1}(\bar{R}) \in \{e_3^{t_1}, e_1^{t_1}\}$. Next, we show that $f_1^{t_1}(\bar{R}) = e_3^{t_1} = c_2$ and $f_1(\bar{R}) = e_3$.

Let $\tilde{R}_1$ be such that

$$\tilde{R}_1^{t_1} : e_3^{t_1}, e_1^{t_1}, \ldots,$$

$$\text{for each } t \in T \setminus \{t_1\} : \tilde{R}_1^t := \bar{R}_1^t (= R_1^t) \text{ and}$$

$$\tilde{\pi}_1 = \bar{\pi}_1 = \hat{\pi}_1 = \pi_1.$$

Since $f_1^{t_1}(\bar{R}) \neq c_1$, $\tilde{R}_1$ is a monotonic transformation of $\bar{R}_1$ at $f(\bar{R})$. Thus,

$$f(\bar{R}) = f(\tilde{R}_1, \bar{R}_{-1}). \tag{20}$$

Note that at $(\tilde{R}_1, \bar{R}_{-1})$, there is a first step top trading cycle $C' = (1 \to e_3^{t_1} \to 3 \to c_3 \to \ldots \to K \to c_K \to 1)$. Since $C' \in \mathcal{C}(\tilde{R}_1, \bar{R}_{-1})$ and $|S_{C'}| = K - 1$, by the Induction hypothesis, C' is fully executed. Thus, $f_1(\tilde{R}_1, \bar{R}_{-1}) = e_3$, and in particular, $f_1^{t_2}(\tilde{R}_1, \bar{R}_{-1}) = e_3^{t_2} = c_2$. Together with (20), we conclude that $f_1^{t_1}(\bar{R}) = e_3^{t_1} = c_2$ and $f_1(\bar{R}) = e_3$. Therefore, $f_2^{t_2}(\bar{R}) \neq e_3^{t_1} = c_2$. Hence, by *individual rationality* of f, $f_2(\bar{R}) = e_2$, and in particular, $f_2^{t_1}(\bar{R}) = e_2^{t_1} = c_1$.

This implies that $c_1 = f_2^{t_1}(\bar{R}) \, \bar{P}_1^{t_1} \, f_1^{t_1}(\bar{R})$ and $c_2 = f_1^{t_2}(\bar{R}) \, \bar{P}_2^{t_2} \, f_2^{t_2}(\bar{R})$. Hence, $f_2(\bar{R}) \, \bar{P}_1 \, f_1(\bar{R})$ and $f_1(\bar{R}) \, \bar{P}_2 \, f_2(\bar{R})$, in which contradicts with *pair-efficiency* of f.

Overall, by contradiction we show that at $\bar{R}$, agent 1 receives c_1. Together with *individual rationality* of f, it implies that agent 2 receives c_2 at $\bar{R}$. Subsequently, by *strategy-proofness* of f, agent 1 also receives c_1 at $\hat{R}$; otherwise he has an incentive to misreport $\bar{R}_1$ at $\hat{R}$. Again, together with *individual rationality* of f, it implies that agent 2 receives c_2 at $\hat{R}$.

Step 2. We show that agent 2 receives c_2 under f at R, i.e., $f_2^{t_2}(R) = c_2$.

Note that c_2 is agent 2's most preferred type-t_2 object at R_2. By *strategy-proofness* of f, $f_2(R) \, R_2 \, f_2(\hat{R})$. Hence, $f_2^{t_2}(R) \, R_2^{t_2} \, f_2^{t_2}(\hat{R})$, which implies that $f_2^{t_2}(R) = c_2$. □

Next, we show that C is fully executed. The proof is similar to the induction step part of Lemma 3.

Let $x \equiv bTTC(R)$, $y \equiv f(R)$. Note that if C is fully executed, then for each $i \in S_C$, $y_i = f_i(R) = x_i$.

By contradiction, suppose that there is an agent $i \in S_C$ such that $y_i \neq x_i$. Without loss of generality, let $i = 1$. By Claim 13, C is executed under f at R. In particular,

$$y_1^{t_1} = e_2^{t_1} = x_1^{t_1} \text{ and } y_K^{t_K} = e_1^{t_K} = x_K^{t_K}. \tag{21}$$

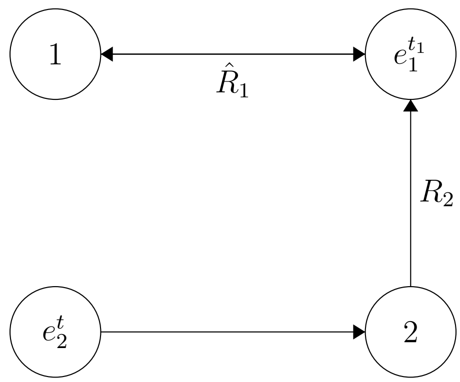
Since $y_1 \neq x_1 (= e_2)$, there is a type $t \in T \setminus \{t_1\}$ and an agent $j \neq 2$ such that $y_1^t = e_j^t$. There are two cases.

Case 1: $j \in S_C$. Let $\hat{R}_1$ such that agent 1 positions y_1 at the top and moves t to the most important, i.e., (i) $\hat{\pi}_1 : t, \ldots$; and (ii) for each $\tau \in T$, $\hat{R}_1^\tau : y_1^\tau, \ldots$. Since $\hat{R}_1$ is a monotonic transformation of R_1 at y, we have

$$f(\hat{R}_1, R_{-1}) = f(R) = y \text{ and particularly, } f_1^{t_1}(\hat{R}_1, R_{-1}) = e_2^{t_1}. \tag{22}$$

Note that there is a first step top trading cycle $C' \equiv (1 \to e_j^t \to j \to e_{j+1}^{t_j} \to j+1 \to \cdots \to K \to e_1^{t_K} \to 1)$ at $(\hat{R}_1, R_{-1})$. i.e., $C' \in \mathcal{C}(\hat{R}_1, R_{-1})$. Since $j \neq 2$, cycle C' contains less than K agents. There are two sub-cases.

Sub-case one: $K = 2$. $j \in S_C$ implies That $j = 1$ and $C' = (1 \to e_1^t \to 1)$, then we are back to the situation in the induction basis, see the figure below for the graphical explanation.



Thus, by the the induction basis, $f_1(\hat{R}_1, R_{-1}) = e_1$ and hence $f_1^{t_1}(\hat{R}_1, R_{-1}) = e_1^{t_1}$, which contradicts with the fact that $f_1^{t_1}(\hat{R}_1, R_{-1}) = e_2^{t_1}$ (see (22)).

Sub-case two: $K > 2$. By the induction hypothesis, cycle C' is fully executed at $f(\hat{R}_1, R_{-1})$. Therefore, $f_1(\hat{R}_1, R_{-1}) = e_j$ and hence $f_1^{t_1}(\hat{R}_1, R_{-1}) = e_j^{t_1}$, which contradicts with the fact that $f_1^{t_1}(\hat{R}_1, R_{-1}) = e_2^{t_1}$ (see (22)).

Case 2: $j \notin S_C$. Let $\hat{R}_j$ be such that (i) $\hat{\pi}_j : t_K, \ldots$; (ii) $\hat{R}_j^{t_K} : e_1^{t_K}, y_j^{t_K}, \ldots$; and (iii) for each $\tau \in T \backslash \{t_K\}$, $\hat{R}_j^\tau : y_j^\tau, \ldots$.

Let

$$\hat{R} \equiv (\hat{R}_j, R_{-j}).$$

Note that C is still a first step top trading cycle at $\hat{R}$, and hence, by Claim 13, C is executed. In particular, with (21), we have

$$f_K^{t_K}(\hat{R}) = y_K^{t_K} = e_1^{t_K}. \tag{23}$$

So, agent j does not receive $e_1^{t_K}$ at $f(\hat{R})$. So, by strategy-proofness of f, $f_j(\hat{R}) = y_j$; otherwise he has an incentive to misreport R_j at $\hat{R}$. So, by non-bossiness of f, $f(\hat{R}) = y$.

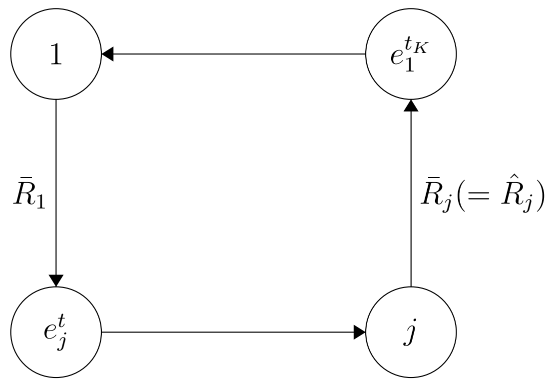
Let $\bar{R}_1$ be such that agent 1 positions y_1 at the top and moves t to the most important, i.e., (i) $\bar{\pi}_1 : t, \ldots$; and (ii) for each $\tau \in T$, $\bar{R}_1^\tau : y_1^\tau, \ldots$. Let

$$\bar{R} \equiv (\bar{R}_1, \hat{R}_{-1}).$$

Since $\bar{R}_1$ is a monotonic transformation of $\hat{R}_1 (= R_1)$ at y, we have

$$f(\bar{R}) = y \text{ and particularly, } f_K^{t_K}(\bar{R}) = e_1^{t_K}. \tag{24}$$

Note that $C' \equiv (1 \to y_1^t (= e_j^t) \to j \to e_1^{t_K} \to 1)$ is a first step top trading cycle at $\bar{R}$, i.e., $C' \in \mathcal{C}(\bar{R})$. See the figure below for the graphical explanation.



Thus, by Claim 13, cycle C' is executed at $f(\bar{R})$. Therefore, $f_j^{t_K}(\bar{R}) = e_1^{t_K}$. Since $j \notin S_C$, this contradicts with the fact that $f_K^{t_K}(\hat{R}) = e_1^{t_K}$ (see (24)). □

By Lemma 7, we have shown that agents who trade at step 1 of the bTTC algorithm always receive their bTTC allotments under f. Next, we can consider agents who trade at step 2 of the bTTC algorithm by following the same proof arguments as for first step trading cycles, and so on. Thus, the proof of Theorem 9 is completed.

B Appendix: independence of the properties

We provide several examples to establish the logical independence of the properties in our characterization results. We label examples by the property/properties that is/are not satisfied.

Mechanisms	no-trade	SD	bTTC	cTTC	Example 9	Example 10	Example 13
IR	+	−	+	+	+	+	+
SP	+	+	+	+	−	+	−
NB	+	+	+	+	+	−	+
FESP	−	+	−	−	−	−	−
BESP	−	+	+	−	+	+	−
OESP	−	+	−	+	−	−	+
PE	−	+	−	−	−	−	−
pE	−	+	+	−	+	+	−
CE	−	+	−	+	−	+	+

Satisfaction of properties of mechanisms. SD denotes the serial dictatorship mechanism. The notation "+" ("−") in a cell means that the property is satisfied (violated) by the corresponding mechanism.

Abbreviations in the first column respectively refer to *individual rationality*, *strategy-proofness*, *non-bossiness*, *flex endowments-swapping-proofness*, *bundle endowments-swapping-proofness*, *one type endowments-swapping-proofness*, *Pareto efficiency*, *pairwise efficiency*, and *coordinatewise efficiency*.

B.1 Theorem 1

The following examples establish the logical independence of the properties in Theorem 1.

Example 3 (*Flexible endowments-swapping-proofness*).
The no-trade mechanism that always assigns the endowment allocation to each market is *individually rational*, *group strategy-proof* (and hence *strategy-proof*, and *non-bossy*), but not *flexibly endowments-swapping-proof*. ◇

Example 4 (*Individual rationality*).
By ignoring property rights that are established via the endowments, we can easily adjust the well-known mechanism of serial dictatorship to our setting: based on an ordering of agents, we let agents sequentially choose their allotments. Serial dictatorship mechanisms have been shown in various resource allocation models to satisfy *Pareto efficiency*, *group strategy-proofness* (and hence *strategy-proofness*, and *non-bossiness*), and *flexible endowments-swapping-proofness* (and hence *bundle endowments-swapping-proofness* and *one type endowments-swapping-proofness*); since property rights are ignored, they violate *individual rationality*. ◇

As we mentioned earlier in the main text, the independence of *strategy-proofness* from the other properties in Theorem 1 is an open problem.

B.2 Theorem 2

The following examples establish the logical independence of the properties in Theorem 2.

Example 5 (***Strategy-proofness***).
Sikdar et al. (2017) prove that there is an extension of the TTC mechanism, the mTTC mechanism, is *strictly core-stable* (and hence *pairwise weakly stable*) but not *Strategy-proof*. ◇

Example 6 (***Pairwise weak stability***).
Same as example 3, the no-trade mechanism is and *individually rational* and *Strategy-proof*, but not *pairwise weakly stable*. ◇

B.3 Theorems 3, 4, and 5

The following examples establish the logical independence of the properties in Theorem 3.

Example 7 (***Bundle endowments-swapping-proofness***).
Same as example 3, the no-trade mechanism is *individually rational*, *Strategy-proof*, and *non-bossy*, but not *bundle endowments-swapping-proof*. ◇

Example 8 (***Individual rationality***).
Same as example 4, serial dictatorship mechanisms are *strategy-proof*, *non-bossy*, and *bundle endowments-swapping-proof*, but not *individually rational*. ◇

Example 9 (***Strategy-proofness***).
We adapt (Fujinaka and Wakayama, 2018, Example 9) to our multiple-type housing markets. We first introduce their original example for Shapley-Scarf housing markets.

Let $N = \{1, 2, 3\}$ and consider a set of markets $\hat{\mathcal{M}} \subsetneq \mathcal{M}$ such that for each $(e, R) \in \hat{\mathcal{M}}$

$$R_i : e_j, \ldots;$$

$$R_j : e_i, e_k, e_j;$$

$$R_k : e_i, e_j, e_k.$$

Let f be a mechanism such that for each $(e, R) \in \hat{\mathcal{M}}$,

$$f_i(e, R) = e_j;$$

$$f_j(e, R) = e_k;$$

$$f_k(e, R) = e_i;$$

otherwise $f(e, R) = TTC(e, R)$.

Fujinaka and Wakayama (2018) show that this mechanism satisfies *individual rationality*, *Pareto efficiency* (and hence *pairwise efficiency*), *non-bossiness*, and *endowments-swapping-proofness* but violates *strategy-proofness*.

We extend f from Shapley-Scarf housing markets to multiple-type housing markets. Let h be the mechanism that stipulates completely bundle trade through f in all types. Since f satisfy *individual rationality* and *non-bossiness*, h satisfies these two properties as well. Since f is *endowments-swapping-proof* but not strategy-proof, h is *bundle endowments-swapping-proof* but not strategy-proof. ◇

Example 10 (*Non-bossiness*).
Consider markets with three agents and two types, i.e., $N = \{1, 2, 3\}$ and $T = \{1, 2\}$.

Let $\hat{\mathcal{M}} \subsetneq \mathcal{M}_l$ be a set of markets such that for each $(e, R) \in \hat{\mathcal{M}}$, $R_1|^e$ is such that there is an agent $i \in \{2, 3\}$ and agent 1 positions agent i's full endowment at the top, i.e., for each $t \in T$, $R_1^t : e_i^t, \ldots$.

Let $y \in X$ be such that (i) $y_1 = e_i$, (ii) $y_i = (e_1^1, e_j^2)$ and $y_j = (e_j^1, e_1^2)$, where $\{i, j\} = \{2, 3\}$.

Let f be such that

$$f(e,R) = \begin{cases} y, & (e,R) \in \hat{\mathcal{M}} \text{ and } y \text{ Pareto dominates } bTTC(e,R), \\ bTTC(e,R), & \text{otherwise.} \end{cases}$$

Note that for $(e, R) \in \hat{\mathcal{M}}$, if $f_1(e, R) \neq e_i$, then there is an agent $k \in \{2, 3\}$ (possibly $k = i$) who receives e_i and prefers e_i to y_k, i.e., $f_k(e, R) = bTTC_k(e, R) = e_i \; P_k \; y_k$.

It is easy to see that f is *individually rational* and *bossy*. We show that f is *strategy-proof* and *bundle endowments-swapping-proof*.

Strategy-proofness.

We first show that agent 1 has no incentive to misreport. For $(e, R) \notin \hat{\mathcal{M}}$, agent 1 positions his full endowment at the top. Thus, $f_1(e, R) = bTTC_1(e, R) = e_1$. Clearly, for any misreport $R_1' \neq R_1$, $e_1 \; R_1 \; f_1(e, (R_1', R_2, R_3))$.

For $(e, R) \in \hat{\mathcal{M}}$, by the definition of f, e_i is agent 1's most preferred allotment among $\{e_1, e_2, e_3\}$, and $f_1(e, R) \in \{e_1, e_i\}$. If $f_1(e, R) = e_i$ then clearly $f_1(e, R) = e_i \; R_1 \; bTTC_1(e, R)$.

If $f_1(e, R) = e_1 (= bTTC_1(e, R))$ then there is an agent $k \in \{2, 3\}$ such that $bTTC_k(e, R) \, P_k \, y_k$ and $f_k(e, R) = bTTC_k(e, R) \in \{e_2, e_3\}$. Let $R_1' \neq R_1$ be a misreporting. By the definition of $bTTC$, $bTTC(e, R_1', R_{-1}) = bTTC(e, R)$. Thus, $bTTC_k(e, R_1', R_{-1}) \; P_k \; y_k$ and hence, $f(e, R_1', R_{-1}) = bTTC(e, R_1', R_{-1})$, which implies that $f_1(e, R_1', R_{-1}) = e_1$. Therefore, if $(e, R) \in \hat{\mathcal{M}}$, then $f_1(e, R) \; R_1 \; bTTC_1(e, R)$.

Next, we show that agents 2 and 3 have no incentive to misreport.

For $(e, R) \notin \hat{\mathcal{M}}$, $f_2(e, R) = bTTC_2(e, R)$ and $f_3(e, R) = bTTC_3(e, R)$. Since $bTTC$ is *strategy-proof*, agents 2 and 3 have no incentive to misreport.

For $(e, R) \in \hat{\mathcal{M}}$, there are two cases.

Case 1. $f(e,R) = bTTC(e,R)$. By the definition of f, there is an agent $k \in \{2,3\}$ such that $bTTC_k(e,R) \; P_k \; y_k$. Let $R'_k \neq R_k$ be a misreporting. Then, $f_k(e, R'_k, R_{-k}) \in \{bTTC_k(e, R'_k, R_{-k}), y_k\}$. Since $bTTC$ is *strategy-proof*, $bTTC_k(e,R) \, R_k \, bTTC_k(e, R'_k, R_{-k})$, and hence $bTTC_k(e,R) = f_k(e,R) \; R_k \; f_k(e, R'_k, R_{-k})$.

Case 2. $f(e,R) = y$. By the definition of f, $y_2 \; R_2 \; bTTC_2(e,R)$ and $y_3 \; R_3 \; bTTC_3(e,R)$. Let $k \in \{2,3\}$ and $R'_k \neq R_k$ be a misreporting. Since $bTTC$ is *strategy-proof*, $y_k \; R_k \; bTTC_k(e,R) \; R_k \; bTTC_k(e, R'_k, R_{-k})$. By the definition of f, $f_k(e, R'_k, R_{-k}) \in \{bTTC_k(e, R'_k, R_{-k}), y_k\}$. Thus, $f_k(e,R) = y_k \; R_k \; f_k(e, R'_k, R_{-k})$.

Bundle endowments-swapping-proofness.

For $(e,R) \notin \hat{\mathcal{M}}$, $f(e,R) = bTTC(e,R)$. Since $bTTC$ is *bundle endowments-swapping-proof*, no pair of agents can be strictly better off by swapping their full endowments.

For $(e,R) \in \hat{\mathcal{M}}$, there are three cases.

Case 1. $f(e,R) = y$. Let $\{i', j'\} \subsetneq N$. Let they swap their endowments completely, i.e., $e'(i',j')$. Let $M' \equiv (e'(i',j'), R)$. Since $bTTC$ is *bundle endowments-swapping-proof*, there is an agent $k \in \{i', j'\}$ such that $y_k \; R_k \; bTTC_k(e,R) \; R_k \; bTTC_k(M')$. By the definition of f, $f_k(M') \in \{y_k, bTTC_k(M')\}$. Thus, $f_k(e,R) = y_k \; R_k \; f_k(M')$, which means that agent k cannot be strictly better off by swapping his full endowments e_k.

Case 2. $f(e,R) = bTTC(e,R)$ and $f_1(e,R) = e_i$. In this case, is is easy to see that agent 1 cannot be strictly better off by swapping his full endowments e_1, because he already receives his best allotment. Moreover, $f(e,R) = bTTC(e,R)$ implies that there is an agent $k \in \{2,3\}$ who strictly prefers $bTTC_k(e,R)$ to y_k. Let agents $\{2,3\}$ swap their endowments completely, i.e., $e'(2,3)$. Let $M' \equiv (e'(2,3), R)$. By the definition of f, $f_k(M') \in \{y_k, bTTC_k(M')\}$. If $f_k(M') = y_k$. then agent k is worse off after the swapping. If $f_k(M') = bTTC_k(M')$, then $f(M') = bTTC(M')$. By *bundle endowments-swapping-proofness* of $bTTC$, there is an agent $\ell \in \{2,3\}$ such that $f_\ell(e,R) = bTTC_\ell(e,R) \; R_\ell \; bTTC_\ell(M') = f_\ell(M')$, which implies that agent ℓ cannot be strictly better off by swapping his full endowments e_ℓ.

Case 3. $f(e,R) = bTTC(e,R)$ and $f_1(e,R) \neq e_i$. By the definition of $bTTC$, we know that there is an agent $k \in \{2,3\}$ (possibly $k = i$) who receives e_i and strictly prefers e_i to y_k. Let $k' \neq k$ (possibly $k' = 1$) .

First, let agents k and k' swap their endowments completely, i.e., $e'(k,k')$. Let $M' \equiv (e'(k,k'), R)$. If $f(M') = y$, then agent k is worse off after the swapping. If $f(M') = bTTC(M')$, then by *bundle endowments-swapping-proofness* of $bTTC$, there is an agent in $\ell \in \{k, k'\}$ such that$f_\ell(e,R) = bTTC_\ell(e,R) R_\ell bTTC_\ell(M') = f_\ell(M')$, which implies that agent ℓ cannot be strictly better off by swapping his full endowments e_ℓ. This implies that agent k has no incentive to swap.

Second, let agents $\{1, k''\} \equiv N \setminus \{k\}$ swap their endowments completely, i.e., $e'(1,k'')$. Let $M'' \equiv (e'(1,k''), R)$. If $k = i$, then by the definition of f, $f_i(M'') \in \{bTTC_i(M''), y_i\}$. Since $bTTC_i(M'') = e_i \; P_i \; y_i$, $f_i(M'') = bTTC_i(M'')$ and hence $f(M'') = bTTC(M'')$. By *bundle* *endowments-swapping-proofness* of $bTTC$, we conclude that agents 1 and k'' cannot be strictly better off by swapping their full endowments. Recall that $f_1(e, R) \neq e_i$. Thus, if $k \neq i$, then $k'' = i$. This means that at M'', agent 1 positions his full endowments $e'_1(1, k'')(= e_i)$ at the top. Thus, $M'' \notin \hat{\mathcal{M}}$ and $f(M'') = bTTC(M'')$. Since $f(e, R) = bTTC(e, R)$, $f(M'') = bTTC(M'')$, and $bTTC$ is *bundle endowments-swapping-proof*, we know that agents 1 and $k''(= i)$ cannot be strictly better off by swapping their full endowments. ⋄

Examples 7, 8, 9, and 10 are well defined on the domain of separable preference (strict preference) profiles and establish the logical independence of the properties in Theorem 4 (Theorem 5).

Note that Examples 7, 8, 9, and 10 also establish the logical independence of the properties in Corollaries 1, 2, and 3 as well.

B.4 Theorems 6 and 7

The following examples establish the logical independence of the properties in Theorems 6 and 7.

Example 11 (***One type endowments-swapping-proofness***)**.**
Same as example 3, the no-trade mechanism is *individually rational*, *Strategy-proof*, and *non-bossy*, but not *one type endowments-swapping-proof*. ⋄

Example 12 (***Individual rationality***)**.**
Same as example 4, serial dictatorship mechanisms are *strategy-proof*, *non-bossy*, and *bundle endowments-swapping-proof*, but not *individually rational*. ⋄

Example 13 (***Strategy-proofness***)**.**
If we modify (Fujinaka and Wakayama, 2018, Example 9) (see details in Example 9) by applying f coordinatewise to all object types, then we obtain a mechanism which satisfies all properties excepts for *strategy-proofness*. ⋄

B.5 Theorem 8

The following examples establish the logical independence of the properties in Theorem 8.

Example 14 (***One type endowments-swapping-proofness***)**.**
Same as example 3, the no-trade mechanism is *individually rational*, and *Strategy-proof*, but not *one type endowments-swapping-proof*. ⋄

Example 15 (***Individual rationality***)**.**
Same as example 4, serial dictatorship mechanisms are *strategy-proof* and *one type endowments-swapping-proof*, but not *individually rational*. ⋄

As we mentioned earlier in the main text, the independence of *strategy-proofness* from the other properties in Theorem 8 is an open problem.

B.6 Theorem 9

The following examples establish the logical independence of the properties in Theorem 9.

Example 16 (***Pairwise efficiency***)**.**
Same as example 3, the no-trade mechanism is *individually rational*, *Strategy-proof*, and *non-bossy*, but not *pairwise efficient*. ◇

Example 17 (***Individual rationality***)**.**
Same as example 4, serial dictatorship mechanisms are *strategy-proof*, *non-bossy*, and *Pareto efficient* (and hence *pairwise efficient*), but not *individually rational*. ◇

Example 18 (***Strategy-proofness***)**.**
The mechanism in example 9 is *individually rational*, *non-bossy*, and *pairwise efficient*, but not *strategy-proof*. ◇

Example 19 (***Non-bossiness***)**.**
The mechanism in example 10 is *individually rational*, *strategy-proof*, and *pairwise efficient*, but *bossy*. ◇

B.7 Theorem 10

The following examples establish the logical independence of the properties in Theorem 10.

Example 20 (***Coordinatewise efficiency***)**.**
Same as example 3, the no-trade mechanism is *individually rational*, *Strategy-proof*, and *non-bossy*, but not *coordinatewise efficient*. ◇

Example 21 (***Individual rationality***)**.**
Same as example 4, serial dictatorship mechanisms are *strategy-proof* and *Pareto efficient* (and hence *coordinatewise efficient*), but not *individually rational*. ◇

Example 22 (***Strategy-proofness***)**.**
The mechanism in Example 13 is *individually rational* and *coordinatewise efficient*, but not *strategy-proof*. ◇

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