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A Two Country Model of Trade with International Borrowing and Lending*

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Summary.

We investigate the properties of a two-country dynamic Heckscher–Ohlin model that allows international borrowing and lending. As is well known, international trade patterns become undecidable when international borrowing and lending is allowed. To avoid this, we assume a consumable capital good to be nontradable. A key feature of our model is the existence of a continuum of steady state levels of capital stocks, which enables us to examine how the initial amount of physical capital and assets in each country affects the amount of capital and assets in the steady state.

Key words: two-country model, international borrowing and lending, continuum of steady states

JEL Classification Numbers: E13, E21, F11

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1 Introduction

Although many existing studies using dynamic two-country models have not considered international asset markets, globalization of financial markets has progressed rapidly over the past several decades, and international asset markets are becoming increasingly important. In this paper, we investigate the properties of a two-country dynamic Heckscher–Ohlin model with international borrowing and lending to examine the long-term impact of countries' physical capital and financial assets on their economic growth and production patterns.

As is well known, international trade patterns become undecidable when international borrowing and lending is allowed. To avoid this, Meng and Velasco (2004) construct a small-open economy with nontradable capital goods in which international borrowing and lending is allowed, and demonstrate the occurrence of indeterminacy, Ono and Shibata (2010) examine specialization patterns in a two-country model with international borrowing and lending by introducing an adjustment cost to the investment.

Hu and Mino (2013) consider two-country dynamic Heckscher–Ohlin models, and derive the conditions under which indeterminacy occurs. They examine two types of trade structures, one of which is a two-country version of Meng and Velasco (2004).[1]

In this study, we consider two types of consumption good, one of which is a pure consumption good and the other is a consumable capital good. Then, we assume consumable capital is nontradable as in Meng and Velasco (2004).

The key feature of our model is that there exists a continuum of steady state levels of capital stocks. So, we can examine how the initial amount of physical capital and assets in each country affects the amount of capital and assets in the steady state. It highly contrasts with results in the previous studies, where capital goods are nonconsumable and nontradable. In those models, the steady state level of capital stocks is unique, and hence both countries with relatively more and less capital initially will accumulate equal amounts of capital in the long run.

In Section 2, we present a two-sector model with international borrowing and lending. In Section 3, we derive the existence of a continuum of steady states by applying the diagrammatic analysis in Bond et al. (2012). In Section 4, we examine local stability of the steady states. Section 5 concludes the paper.

[1] They also consider the other trade structure in Nishimura and Shimomura (2002), where all goods are tradable but international borrowing and lending is not allowed, and argue that whether trade makes economic fluctuations more or less likely depends on the structure of trade.

2 The two country model of trade with international borrowing and lending

In this section, we formulate a two-country model, where there are international borrowing and lending and a nontradable good. There are two countries, home and foreign. We assume that the home and foreign countries are symmetric except for the initial capital endowment and the initial asset in each country. They have the same population normalized to be one, with each household having an endowment of labor, L. Factors of production are assumed to be mobile between sectors within a country, but immobile internationally. However, there is a market for international borrowing and lending. There are two goods using a fixed factor (labor, L) and a reproducible factor (capital, K) under conditions of perfect competition and constant returns to scale. Good 1 is a consumable capital good, and the second good is a pure consumption good. We assume good 1 is not tradable.[2] We refer to the representative country as the home country: the corresponding behavioral relations for the other (foreign) country will be denoted by a "$*$." Let p denotes the price of good 1 in home and good 2 be numeraire. Each household has a concave felicity function u defined over consumption of goods 1 and 2, C_1 and C_2, respectively.

2.1 The Production Side

On the production side, we will assume

Assumption 1: The production function in each sector is quasi-concave and linearly homogeneous. Both factors are indispensable for producing, and consumable capital good 1 is labor intensive.

The results of the static Heckscher-Ohlin model are well known, so here we provide only a brief review of properties that will be important to the dynamic model.

Letting w denote the wage rate and r the rental on capital, the technology in sector i can be characterized by the unit cost function $a_i(w,r)$, $i = 1,2$. Under incomplete specialization, the competitive profit conditions require that

$$a_1(w,r) = p, \tag{1}$$
$$a_2(w,r) = 1, \tag{2}$$

where good 2 is chosen as numeraire. Let $w(p)$ and $r(p)$ be the solution to the system of equations, (1) and (2). Then, we have

$$w(p) = \frac{pa_{2r} - a_{1r}}{\Delta}, \quad w'(p) = \frac{a_{2r}}{\Delta}, \quad r(p) = \frac{a_{1w} - pa_{2w}}{\Delta}, \quad \text{and} \quad r'(p) = \frac{-a_{2w}}{\Delta}, \tag{3}$$

[2] If we assume good 2 is not tradable, the model loses its tractability.

where a_{iw} and a_{ir} are the labor and capital input coefficients in sector i, respectively, and Assumption 1 implies that they satisfy

$$\Delta \equiv a_{1w}a_{2r} - a_{2w}a_{1r} > 0.$$

Factor market equilibrium requires that

$$L = a_{1w}Y_1 + a_{2w}Y_2, \tag{4}$$

$$K = a_{1r}Y_1 + a_{2r}Y_2, \tag{5}$$

where K is the stock of capital and Y_i is the output of good i. From the equations above, we see

$$Y_1(p, K) = w'(p)L + r'(p)K, \tag{6}$$

$$Y_2(p, K) = w(p)L + r(p)K - p[w'(p)L + r'(p)K]. \tag{7}$$

Notice that (6) and (7) yield

$$wL + rK = pY_1 + Y_2, \tag{8}$$

and hence we can use (4) and (8), instead of (4) and (5), to find outputs (Y_1, Y_2).

Since the assumptions to be made below will ensure that the economy is incompletely specialized in both the autarkic and trade steady states, we limit our presentation of the production side to the case of incomplete specialization. Our assumption on the factor intensity ranking of sectors is chosen for convenience due to emphasis on a diagrammatic presentation. We indicate below in cases where the factor intensity rankings play a role.

2.2 The Consumption Side

We analyze the optimization problem for a representative household that owns L units of labor. We will impose the following restrictions on this felicity function:

Assumption 2: The felicity function is strictly concave, with $u_{11} < 0$ and $D \equiv u_{11}u_{22} - u_{12}u_{21} > 0$ for any $(C_1, C_2) \in \{(C_1, C_2) \in \mathbb{R}^2_+ | u_i(C_1, C_2) > 0,\ i = 1, 2\}$, and satisfies $\lim_{C_i \to 0} u_i(C_1, C_2) = \infty$ $(i = 1, 2)$ for any C_j $(j \neq i)$.

The home households maximize

$$\int_0^\infty u(C_1, C_2)e^{-\rho t}dt$$

$$\text{subject to } \dot{B} = RB + rK + wL - pC_1 - C_2 - pI,$$

$$\dot{K} = I - \delta K,$$

$$\lim_{t\to\infty} \exp\left(-\int_0^t R(s)ds\right) B(t) \geq 0,$$

where B, R, and I denote the stock of bonds, the interest rate, and gross investment.

The Hamiltonian function for this optimization problem is given by

$$\mathcal{H} \equiv u(C_1, C_2) + \lambda(RB + rK + wL - pC_1 - C_2 - pI) + q(I - \delta K),$$

where λ and q denote the shadow values of bonds and capital, respectively.

Then, we have

$$\frac{\partial \mathcal{H}}{\partial C_1} = u_1(C_1, C_2) - \lambda p = 0,$$
$$\frac{\partial \mathcal{H}}{\partial C_2} = u_2(C_1, C_2) - \lambda = 0,$$
$$\frac{\partial \mathcal{H}}{\partial I} = -\lambda p + q = 0,$$
$$\frac{\partial \mathcal{H}}{\partial B} = \lambda R = \rho\lambda - \dot{\lambda},$$
$$\frac{\partial \mathcal{H}}{\partial K} = \lambda r - q\delta = \rho q - \dot{q}.$$

So, we obtain

$$C_1 = C_1(p, \lambda) \quad \text{and } C_2 = C_2(p, \lambda), \tag{9}$$
$$q = \lambda p, \tag{10}$$
$$\dot{\lambda} = \lambda(\rho - R), \tag{11}$$
$$\dot{q} = q\left(\rho + \delta - \frac{r}{p}\right), \tag{12}$$

where consumption relations $C_i(p, \lambda)$ for $i = 1, 2$ and an expenditure relation $E(p, \lambda) \equiv pC_1(p, \lambda) + C_2(p, \lambda)$ satisfy

$$\lambda C_{1\lambda} = pC_{1p} + C_{2p},$$
$$E_\lambda = pC_{1\lambda} + C_{2\lambda} < 0,$$
$$C_{1p} < 0,$$
$$E_p = C_1 + \lambda C_{1\lambda},$$

which are proved in Bond et al. (2011).

From (10), (11), and (12), we have the non-arbitrage condition between bond and capital as follows.

$$R = \frac{r}{p} - \delta + \frac{\dot{p}}{p}. \tag{13}$$

2.3 Market Equilibrium

When (13) holds, households will arbitrarily divide their surplus between financial and capital investments. So, we may conclude that the level of capital investment is determined to clear the domestic market for good 1, that is,

$$I = \dot{K} + \delta K = Y_1 - C_1$$

holds. Then, from (8) we have

$$\dot{B} = RB + Y_2(p, K) - C_2(p, \lambda).$$

Let Z_i denote the excess demand for good i $(i = 1, 2)$, where Z_2 is given by

$$Z_2(p, K, \lambda) = C_2(p, \lambda) - Y_2(p, K).$$

Notice that

$$\dot{B} + \dot{B}^* = R(B + B^*) - (Z_2 + Z_2^*)$$

holds, and hence we see that the international credit market will clear, if the international market for good 2 is cleared and $B_0 + B_0^* = 0$, where B_0 and B_0^* denote the initial stock of bond in home and foreign, respectively.

3 Steady States

A steady state will be characterized by the existence of a price $\tilde{p}$, capital stock $\tilde{K}$, and the stock of bonds $\tilde{B}$, such that $\dot{B} = \dot{K} = \dot{\lambda} = \dot{q} = 0$ and markets clear. From (12), a steady state with incomplete specialization will require that there is some $\tilde{p} > 0$ such that $\bar{r}(\tilde{p}) = \rho + \delta$ holds, where $\bar{r}(p) \equiv r(p)/p$. With Assumption 1, $\tilde{p}$ uniquely exists, but in the case where good 1 is capital intensive, some restrictions on production technologies must be assumed to guarantee the existence of $\tilde{p}$.[3]

3.1 Determination of an autarkic steady state

An autarkic steady state requires that $Z_1 = 0$ and $Z_2 = 0$ with $B = 0$. The former condition requires production of good 1 in order to sustain the steady state capital stock, and the latter condition requires production of good 2 in the steady state as a result of Assumption 2. Therefore, the autarkic steady state price must be the consistent with incomplete specialization.

The market clearing conditions in the autarkic steady state are

$$Y_1 = w'(\tilde{p})L + r'(\tilde{p})K = C_1 + \delta K \;\text{ and }\; Y_2 = C_2. \tag{14}$$

Substituting (14) into the labor market equilibrium condition, (4), we define the *steady state* Rybczynski line as follows:

$$\frac{\tilde{a}_{1w}}{\tilde{a}_{2w}} \cdot \frac{r'(\tilde{p})}{r'(\tilde{p}) - \delta} C_1 + C_2 = \left[\frac{1}{\tilde{a}_{2w}} + \frac{\tilde{a}_{1w}}{\tilde{a}_{2w}} \cdot \frac{\delta w'(\tilde{p})}{r'(\tilde{p}) - \delta} \right] L, \quad \text{for } C_1 \geq -\delta k_2(\tilde{p}) L \text{ and } C_2 \geq 0, \tag{15}$$

[3] With the Cobb-Douglas technologies, $\tilde{p}$ uniquely exists in that case.

where $\tilde{a}_{iw} \equiv a_{iw}(w(\tilde{p}), r(\tilde{p}))$ for $i = 1, 2$. The steady state Rybczynski line is the locus of steady state consumption levels that are attainable as the stock of capital is varied, given the stock of labor and relative prices, and is illustrated by the negatively sloped line in Figure 1. The steady state Rybczynski line coincides with the Rybczynski line from the static trade model if $\delta = 0$.

We obtain the following Lemma.

Lemma 1 *Let K be the steady state capital stock. Then, the outputs of two goods at the steady state, $(\hat{Y}_1(K), \hat{Y}_2(K))$, are derived from the intersection between the steady state Rybczynski line,*

$$\frac{r'(\tilde{p}) - (\rho + \delta)}{r'(\tilde{p}) - \delta} \tilde{p} C_1 + C_2 = \left[w(\tilde{p}) + \rho \frac{\tilde{p} w'(\tilde{p})}{\delta - r'(\tilde{p})} \right] L, \tag{16}$$

and the steady state resource constraint,

$$\tilde{p} C_1 + C_2 = w(\tilde{p}) L + \rho \tilde{p} K, \tag{17}$$

as $(\hat{Y}_1(K), \hat{Y}_2(K)) = (C_1 + \delta K, C_2)$.

Proof. From (3), we have

$$a_{1w} = (r - pr')\Delta, \;\; a_{1r} = (pw' - w)\Delta, \;\; a_{2w} = -r'\Delta, \;\; a_{2r} = w'\Delta, \;\text{and}\; \Delta = \frac{1}{w'r - wr'}. \tag{18}$$

Then, it is easy to see that (15) is identical to (16). On the other hand, (17) is easily derived from (8). ■

Since $r(\tilde{p}) - \delta\tilde{p} = \rho\tilde{p}$, the *steady state* resource constraint (17) also represents the budget constraint for households that have no bond ($B = 0$), but own capital stock K with investment δK. So, the intersection between the income expansion path with $\tilde{p}$ and (17) corresponds to the consumption bundle for such households.

Therefore, we may conclude that goods and bond markets will clear with $B = 0$ and $\dot{B} = \dot{K} = \dot{\lambda} = \dot{q} = 0$, when the steady state resource constraint passes through the intersection between the income expansion path and the steady state Rybczynski line.

The intersection, (C_1^A, C_2^A) in figure 1, corresponds to the autarkic steady state. Here, the steady state values of K, B, λ and q are given by

$$K^A = \frac{C_1^A - w'(\tilde{p})L}{r'(\tilde{p}) - \delta}, \;\; B^A = 0, \;\; \lambda^A = u_2(C_1^A, C_2^A) \text{ and } \; q^A = \tilde{p} u_2(C_1^A, C_2^A).$$

Hence, we have[4]

[4] Notice that the intersection must be unique when good 2 is inferior at some income levels, because the slope of the steady state Rybczynski line is steeper than that of the budget constraint (17).

Proposition 1 *An intersection between the steady state Rybczynski line and the income expansion path with the steady state price of good 1 corresponds to an autarkic steady state. Therefore, it uniquely exists as long as labor intensive good 1 is normal and preferences exhibit neither a satiation level nor a minimum subsistence level.*

In the rest of paper, we assume normality in consumption for simplicity.

Assumption 3: Both goods are normal in consumption.

3.2 Excess demand for good 2

Let $(\hat{C}_1(K,B), \hat{C}_2(K,B))$ denote the intersection between the income expansion path with $\tilde{p}$ and the steady state budget constraint,

$$\tilde{p}C_1 + C_2 = w(\tilde{p})L + \rho(\tilde{p}K + B).$$

Then, it corresponds to households' consumption bundle at the steady state where the levels of capital stock and bond are K and B, respectively, since $R = \rho$ holds at steady states.

Since good 1 is not tradable, the excess demand for good 1 must be zero. When the market for good 1 is cleared, households must have some amount of bonds at non autarkic steady states.

It is apparent from Figure 1 that for $K \in [k_1(\tilde{p})L, w'(\tilde{p})L/[\delta - r'(\tilde{p})]]$, there uniquely exists the value of B that yields market clearing for good 1 in Home. Let us denote it as $\hat{B}(K)$. Then, we have

$$\hat{Y}_1(K) - \delta K = \hat{C}_1(K, \hat{B}(K))$$

for any $K \in [k_1(\tilde{p})L, w'(\tilde{p})L/[\delta - r'(\tilde{p})]]$.

We define the steady state excess demand function as follows.

$$\hat{Z}_2(K) \equiv \hat{C}_2(K, \hat{B}(K)) - \hat{Y}_2(K),$$

which denotes the excess demand for good 2 when capital stocks in Home is K and the domestic market for good 1 is clear with $B = \hat{B}(K)$.

Under normality in consumption, excess demand will be strictly decreasing in K with $\hat{Z}_2(K^A) = 0$. Since $\hat{Y}_2(K)$ is linear in K, the shape of $\hat{Z}_2$ reflects exactly that of the income expansion path. In the case of homothetic preferences the slope of the function is constant, while the function is concave (convex) in K, when good 2 is a necessity (luxury).

3.3 The Foreign Country and World Market Equilibrium

First, we assume

Assumption 4: Both countries are identical except their initial capital stocks and asset holdings.

These assumptions ensure that the autarkic steady state prices are the same in each country, and will be the same as the trade steady state prices.[5]

Utilizing Figure 1, we will show the existence of a continuum of trade steady states.

Notice that $\hat{Z}_2(K) = \rho\hat{B}(K)$ holds for any $K \in [k_1(\tilde{p})L, w'(\tilde{p})L/[\delta - r'(\tilde{p})]]$. Therefore, the pair of capital stocks in Home and Foreign, (K^T, K^{T*}), can be a trade steady state, if and only if

$$\hat{Z}_2(K^T) + \hat{Z}_2(K^{T*}) = 0,$$

with $(K^T, K^{T*}) \in [k_1(\tilde{p})L, w'(\tilde{p})L/[\delta - r'(\tilde{p})]] \times [k_1(\tilde{p})L, w'(\tilde{p})L/[\delta - r'(\tilde{p})]]$.[6]

If

$$\hat{B}(k_1(\tilde{p})L) + \hat{B}\left(\frac{w'(\tilde{p})}{\delta - r'(\tilde{p})}L\right) < 0 \tag{19}$$

holds as in Figure 1, there is some value of K, say $\bar{K}$, that satisfies

$$\hat{B}(k_1(\tilde{p})L) + \hat{B}\left(\bar{K}\right) = 0 \ \text{ with } \bar{K} \in \left(K^A, \frac{w'(\tilde{p})}{\delta - r'(\tilde{p})}L\right).$$

On the other hand, if

$$\hat{B}(k_1(\tilde{p})L) + \hat{B}\left(\frac{w'(\tilde{p})}{\delta - r'(\tilde{p})}L\right) > 0$$

holds, then for some value of K, say $\underline{K}$, we have

$$\hat{B}(\underline{K}) + \hat{B}\left(\frac{w'(\tilde{p})}{\delta - r'(\tilde{p})}L\right) = 0 \ \text{ with } \underline{K} \in \left(k_1(\tilde{p})L, K^A\right).$$

In the rest of paper, we assume (19) holds just for simplifying the description of steady states.

Then, we obtain

Proposition 2 *For each $K \in [k_1(\tilde{p})L, \bar{K}]$, there uniquely exists a pair of stock of bond in Home and capital stock in Foreign, (B, K^*), that satisfies $B = \hat{B}(K)$ and $\hat{B}(K) + \hat{B}(K^*) = 0$. It implies that there exists a continuum of steady states under trade environment.*

Since $\hat{B}(K) = \hat{Z}_2(K)/\rho$ holds, the relationship between the home country's capital stock, K^T, and its bond holdings, B^T, can be depicted as in Figure 2, when preferences are homothetic.[7] Therefore, the capital abundant country at the steady state will be a debtor country and export capital intensive good 2 to the capital scarce country. In the case where consumable capital good

[5] Since the factor prices are equalized across two countries at any trade steady state, capital stocks in both countries satisfy

$$K \in [k_1(\tilde{p})L, k_2(\tilde{p})L] \ \text{ and } \ K^* \in [k_1(\tilde{p})L, k_2(\tilde{p})L],$$

or both countries are completely specialized with $K = K^*$, the latter case of which can not be a steady state as in the autarkic case.

[6] If $K^T = w'(\tilde{p})L/[\delta - r'(\tilde{p})]$ holds, households in Home are unable to consume good 1, and hence such a case can not be a steady state.

[7] The graph is concave (convex), when good 2 is a necessity (luxury) as stated above.

1 is capital intensive, the relationship will be reversed, and hence the capital abundant country at the steady state will be a creditor country and import labor intensive good 2 from the capital scarce country.

Also, we obtain the steady state pair of capital stocks in Home and Foreign as in Figure 3, where preferences assumed to be homothetic. Notice that the locus becomes concave (convex) to the origin, when good 2 is a necessity (luxury), because the pair (K^T, K^{T*}) must yield the pair (B^T, B^{T*}) that satisfies $B^T + B^{T*} = 0$.

In the case of homothetic preferences, $K^T + K^{T*} = 2K^A$ holds, and the pair of capital stocks in any trade steady state will be a pair of capital stocks in some steady state under free trade environment, where both goods are tradable but there is no international credit market, that is, $B = B^* = 0$ must hold. It can be derived from the fact that with homothetic preferences, the income expansion path will be linear in (C_1, C_2) space, and hence $\hat{Z}_2(K^T) + \hat{Z}_2(K^{T*}) = 0$ necessarily implies that $\hat{C}_2(K^T, 0) - \hat{Y}_2(K^T) + \hat{C}_2(K^{T*}, 0) - \hat{Y}_2(K^{T*}) = 0$.

4 Stability of Steady States

Since the discount factor is the same across two countries, we see that

$$\frac{\dot{\lambda}}{\lambda} = \frac{\dot{\lambda}^*}{\lambda^*}$$

holds, and hence we have

$$\lambda^* = m\lambda$$

for some $m > 0$.

Then, the excess demand for good 2 in the world market is given by

$$Z_2^W \equiv C_2(p, \lambda) + C_2(p^*, m\lambda) - Y_2(p, K) - Y_2(p^*, K^*). \tag{20}$$

Differentiating both sides of (20) with respect to time yields

$$\begin{aligned}\dot{Z}_2^W &= C_{2p}\dot{p} + C_{2\lambda}\dot{\lambda} + C_{2p}^*\dot{p}^* + mC_{2\lambda}^*\dot{\lambda} - Y_{2p}\dot{p} - Y_{2K}\dot{K} - Y_{2p}^*\dot{p}^* - Y_{2K}^*\dot{K}^* \\ &= pZ_{2p}(R + \delta - \bar{r}) + \lambda C_{2\lambda}(\rho - R) + p^* Z_{2p}^*(R + \delta - \bar{r}^*) + m\lambda C_{2\lambda}^*(\rho - R) \\ &\quad - Y_{2K}(Y_1 - C_1 - \delta K) - Y_{2K}^*(Y_1^* - C_1^* - \delta K^*).\end{aligned} \tag{21}$$

Notice that $\dot{Z}_2^W = 0$ yields

$$\begin{aligned}R &= R(p, p^*, K, K^*, \lambda; m) \\ &\equiv \frac{pZ_{2p}(\bar{r} - \delta) - \rho\lambda C_{2\lambda} + p^* Z_{2p}^*(\bar{r}^* - \delta) - \rho m\lambda C_{2\lambda}^* + Y_{2K}(Y_1 - C_1 - \delta K) + Y_{2K}^*(Y_1^* - C_1^* - \delta K^*)}{\Lambda},\end{aligned}$$

where $\Lambda \equiv pZ_{2p} - \lambda C_{2\lambda} + p^* Z_{2p}^* - m\lambda C_{2\lambda}^*$ is always positive, which is shown in the proof of Lemma 2 in the Appendix.

We have

Lemma 2 *At steady states,*

$$R_p = \frac{\tilde{p}\bar{r}'\lambda C_{1\lambda}}{\Lambda}, \quad R_{p^*} = \frac{\tilde{p}\bar{r}' m\lambda C^*_{1\lambda}}{\Lambda}, \quad R_K = R_{K^*} = \frac{-\tilde{p}^2\bar{r}'(r'-\delta)}{\Lambda},$$
$$\textit{and} \quad R_\lambda = \frac{\tilde{p}^2\bar{r}'(C_{1\lambda} + mC^*_{1\lambda})}{\Lambda},$$

where $\Lambda > 0$.

Proof. See the Appendix. ■

4.1 Dynamic System

Let (K^T, K^{T*}) be a pair of steady state capital stocks in home and foreign. Then, we have

$$\begin{aligned} m &= \hat{m}(K^T, K^{T*}) \\ &\equiv \frac{u_2(\hat{C}_1(K^{T*}, \hat{B}(K^{T*})), \hat{C}_2(K^{T*}, \hat{B}(K^{T*})))}{u_2(\hat{C}_1(K^T, \hat{B}(K^T)), \hat{C}_2(K^T, \hat{B}(K^T)))}. \end{aligned}$$

Since K^T and K^{T*} move in opposite directions, if K^T increases and home households' consumption bundles move toward the origin along the income expansion, then K^{T*} decreases and foreign ones move upward. Therefore, for each value of m, the steady state pair of capital stock exists and is unique, where the steady state becomes an autarkic one with $m = 1$.[8]

The dynamic general equilibrium system can be described as

$$\begin{aligned} \dot{B} &= R(p, p^*, K, K^*, \lambda; m)B - Z_2(p, K, \lambda), \\ \dot{K} &= Y_1(p, K) - C_1(p, \lambda) - \delta K, \\ \dot{K}^* &= Y_1(p^*, K^*) - C_1(p^*, m\lambda) - \delta K^*, \\ \dot{\lambda} &= \lambda\left[\rho - R(p, p^*, K, K^*, \lambda; m)\right], \\ \dot{q} &= q\left[\rho + \delta - \bar{r}(p)\right], \\ \dot{q}^* &= q^*\left[\rho + \delta - \bar{r}(p^*)\right], \\ 0 &= q - \lambda p, \\ 0 &= q^* - m\lambda p^*, \end{aligned}$$

which determines three state variables, K, K^*, B and three jump variables, λ, q, q^*. We will use this system to analyze the bond and capital accumulation on the equilibrium path, and to derive results on the dynamics in the neighborhood of the steady state with $m \in (\hat{m}(\bar{K}, k_1(\tilde{p})L), \hat{m}(k_1(\tilde{p})L, \bar{K}))$.

[8] Since K^T and K^{T*} must be in $[k_1(\tilde{p})L, \bar{K}]$, m will satisfy $\hat{m}(\bar{K}, k_1(\tilde{p})L) \leq m \leq \hat{m}(k_1(\tilde{p})L, \bar{K})$.

Differentiating the system gives the Jacobian,

$$\tilde{J} = \begin{bmatrix} \rho & BR_K - Z_{2K} & BR_{K^*} & BR_\lambda - Z_{2\lambda} & 0 & 0 & BR_p - Z_{2p} & BR_{p^*} \\ 0 & Y_{1K} - \delta & 0 & -C_{1\lambda} & 0 & 0 & Y_{1p} - C_{1p} & 0 \\ 0 & 0 & Y^*_{1K} - \delta & -mC^*_{1\lambda} & 0 & 0 & 0 & Y^*_{1p} - C^*_{1p} \\ 0 & -\lambda R_K & -\lambda R_{K^*} & -\lambda R_\lambda & 0 & 0 & -\lambda R_p & -\lambda R_{p^*} \\ 0 & 0 & 0 & 0 & 0 & 0 & -q\bar{r}' & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^*\bar{r}' \\ 0 & 0 & 0 & -\tilde{p} & 1 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -m\tilde{p} & 0 & 1 & 0 & -m\lambda \end{bmatrix},$$

where each element is evaluated at the steady state.

Then, $\tilde{J}$ yields the characteristic equation as follows.

Lemma 3 *The characteristic equation is given by*

$$J(x) = m\lambda^2 x(\rho - x)(r' - \delta - x)(\tilde{p}\bar{r}' + x)j(x),$$

where

$$j(x) = x^2 - \rho x + \frac{\tilde{p}\bar{r}'\lambda\left[(r' - \delta)(C_{2\lambda} + mC^*_{2\lambda}) + \tilde{p}^2\bar{r}'(C_{1\lambda} + mC^*_{1\lambda})\right]}{\Lambda}.$$

Then, $J(x) = 0$ *has three positive roots,* $\rho, -\tilde{p}\bar{r}'$ *and* x_1*, two negative roots,* $r' - \delta$ *and* x_2*, and one zero root, where* x_1 *and* x_2 *denote the two roots of* $j(x) = 0$.

Proof. *See the Appendix.* ■

From the Lemma above, we see that there exists two dimensional stable manifold around each of steady states under trade environment.

Based on the above, we obtain the main result of the paper as follows.

Theorem 3 *For initially given capital stocks and bond holdings in two countries, there uniquely exists the dynamic equilibrium path that converges to one of the steady states. Thus, the saddle-point stability holds under trade environment as in the case of free trade environment, where two goods are tradable but international lending and borrowing is not allowed.*

Remark 1 *The dynamic equilibrium path of this model is not Pareto efficient, despite allowing international lending and borrowing, while the path is Pareto efficient under free trade environment. This is because factor price equalization does not hold in this model due to the nontradable capital good.*

5 Concluding Remarks

We have examined the dynamic property of a two-country dynamic Heckscher–Ohlin model that allows international borrowing and lending, where the consumable capital good assumed to be nontradable. We have derived the existence of a continuum of the steady state levels of capital stocks in each country, which is a key feature of our model. Then, we have proved the saddle-point stability of steady states, which implies that the initial amount of capital and bond holdings in each country determines their steady state levels and the trade patterns in the long run.

Appendix

The proof of Lemma 2

First, totally differentiating $u_1(C_1, C_2) = \lambda p$ and $u_2(C_1, C_2) = \lambda$ yields

$$C_{1p} = \frac{\lambda u_{22}}{D}, C_{1\lambda} = \frac{u_{22}p - u_{12}}{D}, C_{2p} = \frac{-\lambda u_{12}}{D} \quad \text{and} \quad C_{2\lambda} = \frac{u_{11} - u_{12}p}{D}. \tag{22}$$

Then, we have

$$\begin{aligned}
\Lambda &= -\lambda\frac{u_{11} - u_{12}p}{D} + p\left(\frac{-\lambda u_{12}}{D} - Y_{2p}\right) - m\lambda\frac{u_{11}^* - u_{12}^*p^*}{D^*} + p^*\left(\frac{-m\lambda u_{12}^*}{D^*} - Y_{2p}^*\right) \\
&= -\frac{\lambda u_{11}}{D} - pY_{2p} - \frac{m\lambda u_{11}^*}{D^*} - p^*Y_{2p}^* \\
&> 0.
\end{aligned}$$

Since $p = p^* = \tilde{p}, R = \rho, \bar{r} = \bar{r}^* = \rho + \delta, Y_1 = C_1 + \delta K$, and $Y_1^* = C_1^* + \delta K^*$ hold at the steady states, we see from (21) that

$$\begin{aligned}
[\tilde{p}(Z_{2p} + Z_{2p}^*) - \lambda(C_{2\lambda} + mC_{2\lambda}^*)]dR =& [\bar{r}'\tilde{p}Z_{2p} + Y_{2K}(Y_{1p} - C_{1p})]dp + [\bar{r}'\tilde{p}Z_{2p}^* + Y_{2K}^*(Y_{1p}^* - C_{1p}^*)]dp^* \\
&+ Y_{2K}(Y_{1K} - \delta)dK + Y_{2K}^*(Y_{1K}^* - \delta)dK^* + (Y_{2K}C_{1\lambda} + Y_{2K}^*mC_{1\lambda}^*)d\lambda.
\end{aligned}$$

Then, we have

$$\begin{aligned}
\frac{dR}{dp} &= \frac{\bar{r}'\tilde{p}Z_{2p} + Y_{2K}(Y_{1p} - C_{1p})}{\Lambda}, \frac{dR}{dp^*} = \frac{\bar{r}'\tilde{p}Z_{2p}^* + Y_{2K}^*(Y_{1p}^* - C_{1p}^*)}{\Lambda}, \\
\frac{dR}{dK} &= \frac{Y_{2K}(Y_{1K} - \delta)}{\Lambda}, \frac{dR}{dK^*} = \frac{Y_{2K}^*(Y_{1K}^* - \delta)}{\Lambda}, \\
\text{and } \frac{dR}{d\lambda} &= \frac{Y_{2K}C_{1\lambda} + Y_{2K}^*mC_{1\lambda}^*}{\Lambda}.
\end{aligned}$$

Notice that from $p = p^*$,

$$Y_{1K} = Y_{1K}^* = r' \quad \text{and} \quad Y_{2K} = Y_{2K}^* = r - \tilde{p}r' = -\tilde{p}^2\bar{r}'$$

hold, and hence we see that from (22) and $pY_{1p}+Y_{2p}=0$,

$$\begin{aligned} & \bar{r}'\tilde{p}Z_{2p}+Y_{2K}(Y_{1p}-C_{1p})=\bar{r}'\tilde{p}(C_{2p}-Y_{2p})-\tilde{p}^2\bar{r}'(Y_{1p}-C_{1p}) \\ = \ & \bar{r}'\tilde{p}[(\tilde{p}C_{1p}+C_{2p})-(\tilde{p}Y_{1p}+Y_{2p})] \\ = \ & \bar{r}'\tilde{p}\lambda C_{1\lambda} \end{aligned}$$

and

$$\bar{r}'\tilde{p}Z^*_{2p}+Y^*_{2K}(Y^*_{1p}-C^*_{1p})=\bar{r}'\tilde{p}m\lambda C^*_{1\lambda}.$$

Thus, we have

$$R_p=\frac{\tilde{p}\bar{r}'\lambda C_{1\lambda}}{\Lambda},\quad R_{p^*}=\frac{\tilde{p}\bar{r}'m\lambda C^*_{1\lambda}}{\Lambda},\quad R_K=R_{K^*}=\frac{-\tilde{p}^2\bar{r}'(r'-\delta)}{\Lambda},$$
$$\text{and}\quad R_\lambda=\frac{\tilde{p}^2\bar{r}'(C_{1\lambda}+mC^*_{1\lambda})}{\Lambda},$$

The proof of Lemma 3

The characteristic equation $J(x)$ is given by

$$J(x)=\begin{vmatrix} \rho-x & BR_K-Z_{2K} & BR_{K^*} & BR_\lambda-Z_{2\lambda} & 0 & 0 & BR_p-Z_{2p} & BR_{p^*} \\ 0 & Y_{1K}-\delta-x & 0 & -C_{1\lambda} & 0 & 0 & Y_{1p}-C_{1p} & 0 \\ 0 & 0 & Y^*_{1K}-\delta-x & -mC^*_{1\lambda} & 0 & 0 & 0 & Y^*_{1p}-C^*_{1p} \\ 0 & -\lambda R_K & -\lambda R_{K^*} & -\lambda R_\lambda-x & 0 & 0 & -\lambda R_p & -\lambda R_{p^*} \\ 0 & 0 & 0 & 0 & -x & 0 & -q\bar{r}' & 0 \\ 0 & 0 & 0 & 0 & 0 & -x & 0 & -q^*\bar{r}' \\ 0 & 0 & 0 & -\tilde{p} & 1 & 0 & -\lambda & 0 \\ 0 & 0 & 0 & -m\tilde{p} & 0 & 1 & 0 & -m\lambda \end{vmatrix}.$$

Then,

$$
\begin{aligned}
J(x) &= (\rho - x)\begin{vmatrix}
r' - \delta - x & 0 & -C_{1\lambda} & 0 & 0 & Y_{1p} - C_{1p} & 0 \\
0 & r' - \delta - x & -mC^*_{1\lambda} & 0 & 0 & 0 & Y^*_{1p} - C^*_{1p} \\
-\lambda R_K & -\lambda R_K & -\lambda R_\lambda - x & 0 & 0 & -\lambda R_p & -\lambda R_{p^*} \\
0 & 0 & 0 & -x & 0 & -\lambda \tilde{p}\bar{r}' & 0 \\
0 & 0 & 0 & 0 & -x & 0 & -m\lambda \tilde{p}\bar{r}' \\
0 & 0 & -\tilde{p} & 1 & 0 & -\lambda & 0 \\
0 & 0 & -m\tilde{p} & 0 & 1 & 0 & -m\lambda
\end{vmatrix} \\
&= (\rho - x)\begin{vmatrix}
r' - \delta - x & 0 & -C_{1\lambda} & 0 & 0 & Y_{1p} - C_{1p} & 0 \\
-(r' - \delta - x) & r' - \delta - x & -mC^*_{1\lambda} & 0 & 0 & 0 & Y^*_{1p} - C^*_{1p} \\
0 & -\lambda R_K & -\lambda R_\lambda - x & 0 & 0 & -\lambda R_p & -\lambda R_{p^*} \\
0 & 0 & 0 & -x & 0 & -\lambda \tilde{p}\bar{r}' & 0 \\
0 & 0 & 0 & 0 & -x & 0 & -m\lambda \tilde{p}\bar{r}' \\
0 & 0 & -\tilde{p} & 1 & 0 & -\lambda & 0 \\
0 & 0 & -m\tilde{p} & 0 & 1 & 0 & -m\lambda
\end{vmatrix} \\
&= (\rho - x)(r' - \delta - x)\begin{vmatrix}
r' - \delta - x & -(C_{1\lambda} + mC^*_{1\lambda}) & 0 & 0 & Y_{1p} - C_{1p} & Y^*_{1p} - C^*_{1p} \\
-\lambda R_K & -\lambda R_\lambda - x & 0 & 0 & -\lambda R_p & -\lambda R_{p^*} \\
0 & 0 & -x & 0 & -\lambda \tilde{p}\bar{r}' & 0 \\
0 & 0 & 0 & -x & 0 & -m\lambda \tilde{p}\bar{r}' \\
0 & -\tilde{p} & 1 & 0 & -\lambda & 0 \\
0 & -m\tilde{p} & 0 & 1 & 0 & -m\lambda
\end{vmatrix}
\end{aligned}
$$

$$
= (\rho - x)(r' - \delta - x)\begin{vmatrix} r'-\delta-x & -(C_{1\lambda}+mC^*_{1\lambda}) & 0 & 0 & Y_{1p}-C_{1p} & Y^*_{1p}-C^*_{1p} \\ -\lambda R_K & -\lambda R_\lambda - x & 0 & 0 & -\lambda R_p & -\lambda R_{p^*} \\ 0 & -\tilde{p}x & -x & 0 & -\lambda(\tilde{p}\bar{r}'+x) & 0 \\ 0 & -m\tilde{p}x & 0 & -x & 0 & -m\lambda(\tilde{p}\bar{r}'+x) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{vmatrix}
$$

$$
= (\rho - x)(r' - \delta - x)\begin{vmatrix} r'-\delta-x & -(C_{1\lambda}+mC^*_{1\lambda}) & Y_{1p}-C_{1p} & Y^*_{1p}-C^*_{1p} \\ -\lambda R_K & -\lambda R_\lambda - x & -\lambda R_p & -\lambda R_{p^*} \\ 0 & -\tilde{p}x & -\lambda(\tilde{p}\bar{r}'+x) & 0 \\ 0 & 0 & m\lambda(\tilde{p}\bar{r}'+x) & -m\lambda(\tilde{p}\bar{r}'+x) \end{vmatrix}
$$

$$
= -m\lambda(\rho - x)(r' - \delta - x)(\tilde{p}\bar{r}'+x)\begin{vmatrix} r'-\delta-x & -(C_{1\lambda}+mC^*_{1\lambda}) & Y_{1p}-C_{1p}+Y^*_{1p}-C^*_{1p} \\ -\lambda R_K & -\lambda R_\lambda - x & -\lambda(R_p+R_{p^*}) \\ 0 & -\tilde{p}x & -\lambda(\tilde{p}\bar{r}'+x) \end{vmatrix}
$$

$$
= -m\lambda(\rho - x)(r' - \delta - x)(\tilde{p}\bar{r}'+x)\begin{vmatrix} r'-\delta-x+\frac{R_K}{R_\lambda}(C_{1\lambda}+mC^*_{1\lambda}) & -(C_{1\lambda}+mC^*_{1\lambda}) & Y_{1p}-C_{1p}+Y^*_{1p}-C^*_{1p} \\ -\lambda R_K+\frac{R_K}{R_\lambda}(\lambda R_\lambda+x) & -\lambda R_\lambda - x & -\lambda(R_p+R_{p^*}) \\ \frac{R_K}{R_\lambda}\tilde{p}x & -\tilde{p}x & -\lambda(\tilde{p}\bar{r}'+x) \end{vmatrix}
$$

$$
= -m\lambda(\rho - x)(r' - \delta - x)(\tilde{p}\bar{r}'+x)\begin{vmatrix} -x & -(C_{1\lambda}+mC^*_{1\lambda}) & Y_{1p}-C_{1p}+Y^*_{1p}-C^*_{1p} \\ \frac{R_K}{R_\lambda}x & -\lambda R_\lambda - x & -\lambda(R_p+R_{p^*}) \\ \frac{R_K}{R_\lambda}\tilde{p}x & -\tilde{p}x & -\lambda(\tilde{p}\bar{r}'+x) \end{vmatrix}
$$

$$
= -m\lambda x(\rho - x)(r' - \delta - x)(\tilde{p}\bar{r}'+x)\begin{vmatrix} -1 & -(C_{1\lambda}+mC^*_{1\lambda}) & Y_{1p}-C_{1p}+Y^*_{1p}-C^*_{1p} \\ 0 & -\lambda R_\lambda - x - \frac{R_K}{R_\lambda}(C_{1\lambda}+mC^*_{1\lambda}) & \begin{matrix} -\lambda(R_p+R_{p^*}) \\ +\frac{R_K}{R_\lambda}(Y_{1p}-C_{1p}+Y^*_{1p}-C^*_{1p}) \end{matrix} \\ 0 & \tilde{p}\lambda R_\lambda & -\lambda[(\tilde{p}\bar{r}'+x)-\tilde{p}(R_p+R_{p^*})] \end{vmatrix}
$$

$$
= m\lambda^2 x(\rho - x)(r' - \delta - x)(\tilde{p}\bar{r}'+x)\begin{vmatrix} -\lambda R_\lambda - x + r' - \delta & -\lambda(R_p+R_{p^*}) + \frac{R_K}{R_\lambda}(Y_{1p}-C_{1p}+Y^*_{1p}-C^*_{1p}) \\ \tilde{p}R_\lambda & -[(\tilde{p}\bar{r}'+x)-\tilde{p}(R_p+R_{p^*})] \end{vmatrix},
$$

where

$$
\begin{aligned}
&\begin{vmatrix} -\lambda R_\lambda - x + r' - \delta & -\lambda(R_p+R_{p^*}) + \frac{R_K}{R_\lambda}(Y_{1p}-C_{1p}+Y^*_{1p}-C^*_{1p}) \\ \tilde{p}R_\lambda & -[(\tilde{p}\bar{r}'+x)-\tilde{p}(R_p+R_{p^*})] \end{vmatrix} \\
=& [\lambda R_\lambda + x - r' + \delta)]\,[(\tilde{p}\bar{r}'+x)-\tilde{p}(R_p+R_{p^*})] + \tilde{p}R_\lambda\left[\lambda(R_p+R_{p^*}) + \frac{R_K}{R_\lambda}(C_{1p}-Y_{1p}+C^*_{1p}-Y^*_{1p})\right] \\
=& x^2 + [\lambda R_\lambda - r' + \delta + \tilde{p}\bar{r}' - \tilde{p}(R_p+R_{p^*})]\,x \\
&+ \tilde{p}\bar{r}'\lambda R_\lambda - (r'-\delta)[\tilde{p}\bar{r}' - \tilde{p}(R_p+R_{p^*})] + \tilde{p}R_K(C_{1p}-Y_{1p}+C^*_{1p}-Y^*_{1p}).
\end{aligned}
$$

From Lemma 2, we see that

$$
\lambda R_\lambda - r' + \delta + \tilde{p}\bar{r}' - \tilde{p}(R_p+R_{p^*}) = -r' + \delta + \tilde{p}\bar{r}' = -\rho
$$

and

$$
\begin{aligned}
& \tilde{p}\bar{r}'\lambda R_\lambda - (r'-\delta)[\tilde{p}\bar{r}' - \tilde{p}(R_p + R_{p^*})] + \tilde{p}R_K(C_{1p} - Y_{1p} + C_{1p}^* - Y_{1p}^*) \\
= \quad & \frac{1}{\Lambda}[(\tilde{p}\bar{r}' + r' - \delta)\tilde{p}^2\bar{r}'\lambda(C_{1\lambda} + mC_{1\lambda}^*) - (r'-\delta)\tilde{p}\bar{r}'(\tilde{p}Z_{2p} - \lambda C_{2\lambda} + \tilde{p}Z_{2p}^* - m\lambda C_{2\lambda}^*) \\
& -\tilde{p}^3\bar{r}'(r'-\delta)(C_{1p} - Y_{1p} + C_{1p}^* - Y_{1p}^*)] \\
= \quad & \frac{1}{\Lambda}\{(\tilde{p}\bar{r}' + r' - \delta)\tilde{p}\bar{r}'\lambda(\tilde{p}C_{1\lambda} + m\tilde{p}C_{1\lambda}^*) + (r'-\delta)\tilde{p}\bar{r}'\lambda(C_{2\lambda} + mC_{2\lambda}^*) \\
& -\tilde{p}^2\bar{r}'(r'-\delta)[\tilde{p}C_{1p} + C_{2p} - (\tilde{p}Y_{1p} + Y_{2p}) + \tilde{p}C_{1p}^* + C_{2p}^* - (\tilde{p}Y_{1p}^* + Y_{2p}^*)]\} \\
= \quad & \frac{\tilde{p}\bar{r}'\lambda\left[(r'-\delta)(C_{2\lambda} + mC_{2\lambda}^*) + \tilde{p}^2\bar{r}'(C_{1\lambda} + mC_{1\lambda}^*)\right]}{\Lambda}.
\end{aligned}
$$

Thus, we have

$$
J(x) = m\lambda^2 x(\rho - x)(r' - \delta - x)(\tilde{p}\bar{r}' + x)j(x),
$$

where

$$
j(x) = x^2 - \rho x + \frac{\tilde{p}\bar{r}'\lambda\left[(r'-\delta)(C_{2\lambda} + mC_{2\lambda}^*) + \tilde{p}^2\bar{r}'(C_{1\lambda} + mC_{1\lambda}^*)\right]}{\Lambda}.
$$

Under Assumptions 1 and 3, we obtain

$$
\frac{\tilde{p}\bar{r}'\lambda\left[(r'-\delta)(C_{2\lambda} + mC_{2\lambda}^*) + \tilde{p}^2\bar{r}'(C_{1\lambda} + mC_{1\lambda}^*)\right]}{\Lambda} < 0.
$$

References

[1] Bond, E., K. Iwasa, and K. Nishimura (2012), "The dynamic Heckscher–Ohlin model: A diagrammatic analysis," *International Journal of Economic Theory* 8, 197–211.

[2] Hu, Y., and K. Mino (2013), "Trade structure and belief-driven fluctuations in a global economy," *Journal of International Economics* 90, 414–424.

[3] Meng, Q., and A. Velasco (2004), "Market imperfections and the instability of open economies," *Journal of International Economics* 64, 503–519.

[4] Nishimura, K., and K. Shimomura (2002), "Trade and indeterminacy in a dynamic general equilibrium model," *Journal of Economic Theory* 105, 244–259.

[5] Ono, Y., and A. Shibata (2010) "Time patience and specialization patterns in the presence of asset trade," *Journal of Money, Credit and Banking* 42, 93–112.

Figure 1
The income expansion path and the steady state Rybczynski line

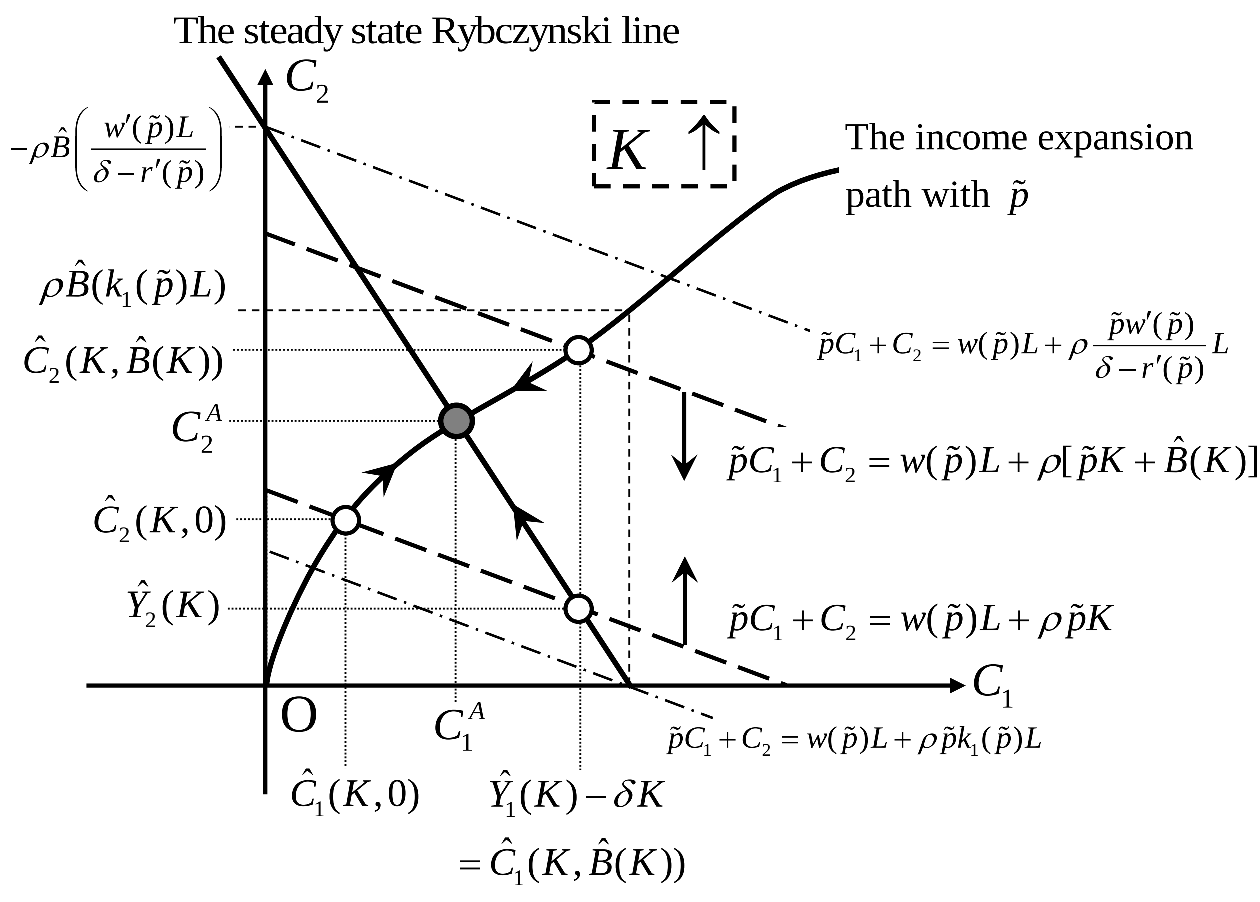

Figure 2
The relationship between capital stocks and bond holdings

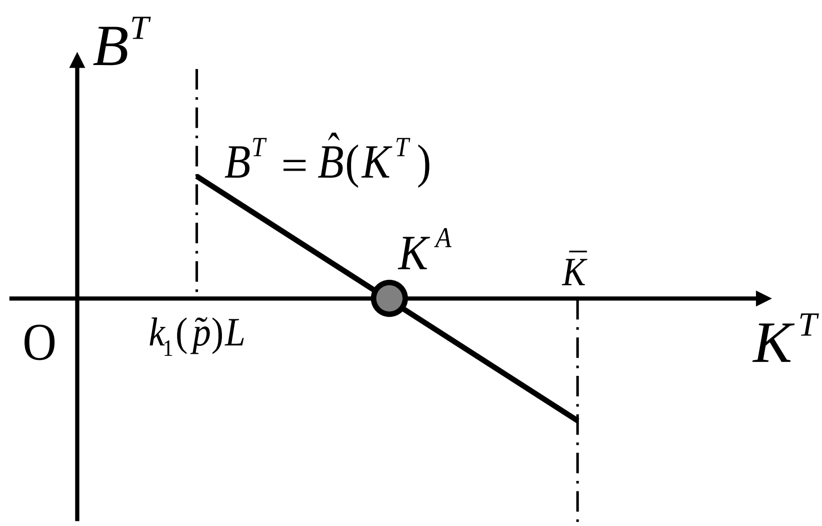

Figure 3
The locus of equilibrium pairs of K^T and K^{T*}