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Kazuya KAMIYA

So KUBOTA

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Research Institute for Economics and Business Administration

Kobe University

2-1 Rokkodai, Nada, Kobe 657-8501 JAPAN

A Search and Bargaining Model of Non-degenerate Distributions of Money Holdings*

Kazuya Kamiya[†]and So Kubota[‡]

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Abstract

We study a standard search and bargaining model of money, where goods are traded only in decentralized markets and distributions of money holdings are non-degenerate in equilibria. We assume fixed costs in each seller's production, which allows an analytical characterization of a tractable equilibrium. Each Nash bargaining solution satisfies *pay-all* property, where the buyer pays the whole amount of cash as a corner solution, and the seller produces goods as the interior solution. In the stationary equilibrium, the aggregate variables, such as total production and the number of matchings, are expressed by given parameters, i.e., determinate. On the other hand, individual-level variables are indeterminate. Distributional monetary policies are effective in both the short-run and the long-run.

1 Introduction

We propose a search and bargaining model of money and study a tractable equilibrium with non-degenerate distributions of divisible money holdings. The model is a straightforward extension of the basic search models with indivisible money such as Kiyotaki and Wright

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[†]Research Institute for Economics & Business Administration (RIEB), Kobe University (E-mail: kkamiya@rieb.kobe-u.ac.jp)

[‡]School of Political Science and Economics, Waseda University (E-mail: kubotaso@waseda.jp)

(1993), Trejos and Wright (1995), and Shi (1995). That is, divisible money and goods are traded on an environment of only random-matching and Nash bargaining. The crucial factor is fixed costs in production, which generate a tractable equilibrium with a simple structure of money holding distributions. We analytically characterize the equilibria, prove the existence, and evaluate distributional monetary policies both in the short-run and the long-run.

Originated from Kiyotaki and Wright (1989), search theory provides a solid microfoundation of money and implications for real-world economic phenomena. However, the literature has struggled to handle the problem of heterogeneous money holdings. It is a fundamental consequence of the random matching assumption, where different amounts of money are exchanged in each trade. But the characterization of dynamic equilibrium requires to keep track of the distribution of money as a state variable. To overcome the problem, the workhorse (so-called the third generation) models impose assumptions to make money holding distribution degenerate. The major approaches are large household model (Shi (1997)) and alternation between search and centralized markets (Lagos and Wright (2005)).

We consider a standard search model where agents randomly match and exchange goods and money through Nash bargaining. We construct a particular type of tractable equilibrium that satisfies *Pay-All* property. In each Nash bargaining problem, the production is decided as the interior solution. On the other hand, the cash payment is solved as a corner solution, that is, the buyer pays all the money holding. On top of that, a simple non-degenerate money holding distribution arises. There is a mass of agents at 0 money holding, and the others save different positive amounts. The transitional dynamics are also straightforward. A positive money holder pays all cash in each matching and will carry 0 in the next period. A no-cash agent takes a role as a seller and will save a positive amount. Indeed this model has a full non-degenerate distribution; however, the individual-level dynamics is as simple as the first and second-generation money search models.

The pay-all equilibrium is constructed by fixed costs in the production. The well-known difficulty in dealing with the non-degenerate money holding distribution comes from each agent's dual roles. If each agent possibly becomes both a buyer and a seller, the mapping from the current to next distribution becomes almost untraceable. In pay-all equilibrium, there is a cut-off level of money holding where each agent always becomes only a seller (buyer) if the money holding is lower (higher) than this level. This cut-off emerges by the fixed costs. A seller has no incentive to trade with a partner who has a sufficiently small amount of money because it does not cover the fixed costs. Then, agents are sorted into buyers and sellers according to their money holdings.

We find some interesting properties of the stationary equilibrium. That is, the aggregate variables are determinate, i.e., the macroeconomic measures such as total production, the number of matching, and the Benthamian welfare are uniquely determined. However, individual-level variables are indeterminate: there is a continuum of steady-state distributions of money holdings, productions, consumptions, and individual lifetime utilities. In this sense, the equilibrium is similar to the first and second-generation models in the aggregate level but parallel to Green and Zhou (1998) regarding the individual level. About the quantity theory of money, the equilibrium satisfies both neutrality and superneutrality of money if cash is injected proportionally to each agent's money holding.

We examine two types of distributional monetary policy. One is conducted on the steady-state while keeping the total money supply and the money holding distribution unchanged. In each match, the government levies a per-unit tax (subsidy) depending on the amount of goods, and distributes (collects) money depending on after-trade money holding. The tax rate is chosen so that the government budget is balanced each period. We show that this policy improves welfare, although the direction depends on the parameters. Intuitively, it modifies the intra-temporal condition of Nash bargaining depending on the bargaining power and moves the equilibrium toward Hosios condition.

Another policy is a temporal expansion of the money supply. The same amount of money is injected to all agents in one period. Then the same amount is subtracted in the next period from agents holding sufficient amount of money. It also causes a short-run improvement in relative buyer/seller allocation. Notably, there are multiple transition paths, that is, the policy's welfare effect is unpredictable. Since the individual-level allocation is indeterminate, the transition paths caused by the redistribution policy is also uncertain.

There are some models in the literature which succeed in characterizing the non-degenerate distribution of money holdings. The initial contribution is made by Green and Zhou (1998). They consider a random-matching model with divisible money where each exchange is closed by a take-it-or-leave-it offer, which is a special case of Nash bargaining. The key result is the real indeterminacy of stationary equilibrium[1]: there exists a continuum of steady-state

[1] The general framework is constructed by Kamiya and Shimizu (2006). Note that the indeterminacy emerges even if goods are divisible. See Kamiya and Shimizu (2007), Kamiya and Shimizu (2013), and Kubota (2019) for the conditions of the indeterminacy.

where each one has a different real allocation. Although the result itself is theoretically appealing, this property obstructs applied researches. Another important model is Menzio et al. (2013), which eliminates one-to-one Nash bargaining and instead assumes a competitive search environment. They construct a block-recursive structure of equilibrium, which makes a simple transitional process of agents on the non-degenerate distribution. However, the construction of equilibrium still relies on the centralized labor market. Recently, Rocheteau et al. (2018) analyze a discrete distribution caused by delayed money holding adjustment on Lagos-Wright model. Another notable approach is numerical methods conducted by Molico (2006), Chiu and Molico (2010), and Chiu and Molico (2020). Finally, Camera and Corbae (1999) consider the countable amount of money. In contrast, the novelty of our approach is (i) tractable equilibria constructed only on a random-matching market with Nash bargaining, (ii) analytical characterization and proved existence of equilibria, and (iii) analyses of distributional monetary policy.

The next section introduces the economic environment. Then, Section 3 illustrates the equilibrium and its characterization. The existence of the equilibrium is proved in Section 4. Section 5 and 6 consider distributional monetary policy in the long-run and the short-run, respectively. Section 7 concludes this paper.

2 Economic Environment

Time is discrete and time horizon is infinite, as denoted by $t = 1, 2, \cdots$. Goods are divisible. There is a continuum of agents with measure one. Each agent can produce her production goods with a cost (disutility) function

$$u^s(x) = \begin{cases} -d - cx & \text{if } x > 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where x is an amount of goods, $d > 0$ is a fixed cost, and $c > 0$ is a unit cost. Each agent cannot consume her production good for eliminating double-coincidence of wants. However, she can consume some others' production good with a temporal utility function

$$u^b(x) = \begin{cases} k + x & \text{if } x > 0 \\ 0 & \text{if } x = 0, \end{cases}$$

where $k > 0$ is a given constant. This fixed utility is interpreted as an appetizer.[2] Theoretically, k is introduced so that $v(0) > 0$ in the equilibrium. These assumptions about utility and cost functions will be discussed in Section 3.2.

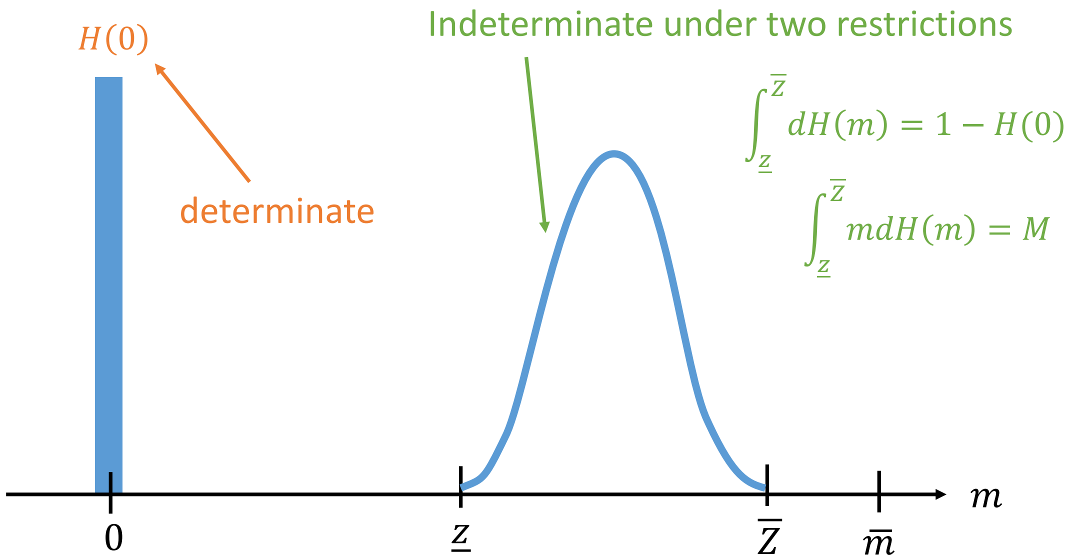
Each agent discounts her future payoff with a discount factor $\beta \in (0, 1)$. Money is perfectly divisible, and its total supply in the economy is fixed at $M > 0$. Each agent's money holding m is a real number with an upper-bound. We assume this upper-bound as 1 for normalization, that is, $m \in [0, 1]$. This assumption allows us to ignore the behavior of very rich agent who exists only off-path, and then, simplifies the proof of the existence of equilibrium (Zhou (1999)).

The timeline in each period is as follows. In the beginning of each period, agents observe the current economy-wide money holdings distribution. Then, pairwise random matching occurs with a probability $2\alpha \in (0, 1)$. In each matching, each agent observes the partner's money holding. One agent becomes a seller with probability $\frac{1}{2}$, and the other becomes a buyer. After that, bargaining over the price and the amount of goods proceeds through Nash bargaining with the buyer's bargaining power $\theta \in (0, 1)$. At the end of each period, the matching resolves and proceeds to the next period. Let H be a money holdings distribution (a Borel measure) on R_+. In a meeting of a buyer (a seller) with money holding m_b (m_s), the Nash bargaining decides the amount of goods $x(m_b, m_s, H)$ and the amount of money $p(m_b, m_s, H)$ in the trade. No trade, $x(m_b, m_s, H) = p(m_b, m_s, H) = 0$, is also a solution when the joint surplus cannot be positive. The stationary monetary equilibrium is defined as follows.

Definition 1. *Let v be a function on R_+. A pair (H, v) is called a stationary monetary equilibrium if*

[2]It might be considered that a certain fixed amount of goods, of which utility is k, is produced by the fixed cost d.

Figure 1: Money holding distribution in Pay-All Equilibrium



1. *Bellman equation is consistently constructed as*

$$v(m) = \alpha \int \left[u^b\big(x(m, m_s, H)\big) + \beta v\big(m - p(m, m_s, H)\big)\right] dH(m_s)$$
$$+ \alpha \int \left[u^s\big(x(m_b, m, H)\big) + \beta v\big(m + p(m_b, m, H)\big)\right] dH(m_b) + (1 - 2\alpha)\beta v(m),$$

2. *H is a stationary distribution of the process under the transition $p(m_b, m_s, H)$,*

3. *$x(m_b, m_s, H) \geq 0$ and $p(m_b, m_s, H) \geq 0$ solve each Nash bargaining problem.*

4. *$v(m) > 0$ for all $m \geq 0$ and $v(m)$ is strictly increasing.*

The last condition represents the individual rationality and positive equilibrium value of fiat money.

3 Pay-All Equilibrium

We consider an equilibrium with the following money holdings distribution. The support of the distribution H is $\{0\} \cup [\underline{z}, \overline{Z}]$, where $0 < \underline{z} < \overline{Z} < 1$. Let $H(0) \in [0, 1]$ be the measure of agents without money and H be a function satisfying

$$\int_{\underline{z}}^{\overline{Z}} dH = 1 - H(0) \tag{1}$$

$$\int_{\underline{z}}^{\overline{Z}} m dH = M \tag{2}$$

A Nash bargaining problem between a buyer with m_b and a seller with m_s is as follows:

- if there exists (x, p) such that $x > 0$, $0 \leq p \leq m_b$, and both the buyer and seller's surpluses are non-negative, i.e., $k + x + \beta\left[v(m_b - p) - v(m_b)\right] \geq 0$ and $-d - cx + \beta\left[v(m_s + p) - v(m_s)\right] \geq 0$, then the trade (x^*, p^*) is determined by

$$(x^*, p^*) = \arg\max_{x,p} \left\{k + x + \beta\left[v(m_b - p) - v(m_b)\right]\right\}^{\theta} \left\{-d - cx + \beta\left[v(m_s + p) - v(m_s)\right]\right\}^{1-\theta}$$
$$\text{s.t.} \quad x > 0, \quad 0 \leq p \leq m_b, \tag{3}$$

- otherwise $x^* = p^* = 0$.

This formulation assumes that the fixed utility and cost emerge only if positive amount of money and good are traded.

We construct an equilibrium where (i) on the equilibrium-path, Nash bargaining is agreed only if a buyer holds $m \in [\underline{z}, \overline{Z}]$ and a seller has $m = 0$, otherwise any trade does not cover the fixed costs for a seller. Moreover, we focus on the equilibrium where (ii) the monetary payment is binding at $p^* = m_b$, while the good production x^* is determined as the interior solution. For simplicity, let m be the buyer's money holding. Then, the first-order condition with respect to x is

$$(1 - \theta)c\{k + x^* + \beta[v(0) - v(m)]\} = \theta\{-d - cx^* + \beta[v(m) - v(0)]\}. \tag{4}$$

Note that, under the condition (i), we can write $x^* = x(m)$.

We call a stationary monetary equilibrium with (i) and (ii) a *Pay-All equilibrium*, because each buyer spends all amount of money holding. In parallel with this, (i) and (ii) are collectively called the *Pay-All property.* The Bellman equation in the pay-all equilibrium is written as follows:

$$v(m) = \alpha H(0)[k + x(m) + \beta v(0)] + [1 - \alpha H(0)]\beta v(m) \tag{5}$$

$$v(0) = \alpha \int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta v(m)]dH + \{1 - \alpha[1 - H(0)]\}\beta v(0) \tag{6}$$

3.1 Characterization

If a pay-all equilibrium exists, it satisfies the properties summarized in the following lemmas. Define the consumption elasticity of discounted value as $\xi(m) = \dfrac{x'(m)}{v'(m)}$. It must be constant in the equilibrium.

Lemma 1.

$$\xi(m) = \frac{\beta\theta(1-c)}{c} + \beta \tag{7}$$

Proof. By (4),

$$cx(m) = \theta[-d - \beta v(0)] - (1-\theta)c[k + \beta v(0)] + \theta\beta v(m) + (1-\theta)c\beta v(m),$$

$$\Rightarrow \; cx'(m) = \theta\beta v'(m) + (1-\theta)c\beta v'(m) = \beta[\theta(1-c) + c]v'(m),$$

which leads to (7). □

Next lemma derives another condition about $\xi(m)$.

Lemma 2.

$$\xi(m) = \frac{1-\beta}{\alpha H(0)} + \beta \tag{8}$$

Proof. The derivative of (5) is

$$v'(m) = \alpha H(0)x'(m) + [1 - \alpha H(0)]\beta x'(m).$$

It is rearranged to (8). □

These two equations about $\xi(m)$ determine $H(0)$. That is, the measure of non-money holders $H(0)$ is explicitly pinned down by parameters. On the other hand, we will later show that $H(m)$ for $m \in [\underline{z}, \overline{Z}]$ is indeterminate.

Lemma 3.

$$H(0) = \frac{(1-\beta)c}{\alpha\beta\theta(1-c)} \tag{9}$$

Proof. By Equation (7) and Equation (8). □

The measure of agents with positive amount of money, $1 - H(0)$, is also uniquely determined. Since they are potential buyers, the following corollary holds.

Corollary 1. *The number of matching $\alpha H(0)(1 - H(0))$ is constant.*

In the equilibrium, the aggregated discounted value of the potential buyers and sellers are $\int_{\underline{z}}^{\overline{Z}} v(m)dH$ and $H(0)v(0)$, respectively. The next lemma shows that the ratio is constant.

Lemma 4.

$$\frac{\int_{\underline{z}}^{\overline{Z}} v(m)dH}{H(0)v(0)} = \frac{\theta}{(1-\theta)c} \tag{10}$$

Proof. Appendix. □

On top of that, the discounted value of no-money holder is uniquely derived.

Lemma 5.

$$v(0) = \frac{ck - d}{\beta(1-c)} \tag{11}$$

Proof. Appendix. □

By Lemma 4, the total discounted value of buyers immediately follows.

Corollary 2.

$$\int_{\underline{z}}^{\overline{Z}} v(m)dH = \left(\frac{ck-d}{\beta(1-c)}\right)\left(\frac{\theta H(0)}{(1-\theta)c}\right) \tag{12}$$

Note that, as will be shown, the shape of $v(m)$ is indeterminate in the equilibrium. That is, although the aggregated discounted value is unique, the distribution of welfare is uncertain.

Finally, define the total welfare as $W = H(0)v(0) + \int_{\underline{z}}^{\overline{Z}} v(m)dH$. By Lemma 5 and Corollary 2, it is also constant.

Lemma 6.

$$W = \frac{\xi H(0)v(0)}{\beta(1-\theta)}$$

Proof. Appendix. □

3.2 Discussion about the model's assumptions

Both the fixed and linear terms of the utility and production functions drastically make the equilibrium tractable. The fixed costs d provides an incentive of sellers to decline trades with buyers who hold small amount of money. If the revenue is sufficiently small, it does not cover the fixed costs d. Then, the equilibrium money holding distribution $H(m)$ is divided into 0 and sufficiently large positive amount $m \geq \underline{z}$. It makes agents to play only one role, either buyer or seller, depending on money holding m. As a consequence, each agent alternates between buyer and seller, which makes a tractable transition of money holdings. The fixed utility k is assumed to overcome the fixed costs d and assure the agent's incentive to participate the market, $v(0) > 0$.

The linearity of variable costs and utility are also crucial for tractability. They make the consumption elasticity of discounted value, $\xi(m)$, constant in Lemma 1. Then, this property helps to pin down $H(0)$ in Lemma 3. This uniqueness of $H(0)$ makes Pay-All equilibrium overcome the real indeterminacy of divisible money in random-matching search models in the literature[3]. Green and Zhou (1998) derive both distributional and aggregate-level inde-

[3]The indeterminacy arises due to some identity hidden in monetary exchange. See, for the case of finite

terminacy of real allocations. This indeterminacy is one-dimensional which is characterized by the $H(0)$. The Pay-All equilibrium erases this type of indeterminacy by anchoring $H(0)$.

Although our model still has indeterminacy of money holding distribution among buyers, the determinate $H(0)$ makes the aggregated allocation determinate. The transition of agents in Pay-All equilibrium is consistent with, so called, the second generation models such as Trejos and Wright (1995) and Shi (1995), where the total money supply is $1-H(0)$. As in our model, each seller becomes a buyer with probability $\alpha[1-H(0)]$, and each buyer turns to be a seller with probability $\alpha H(0)$. Moreover, the Pay-All equilibrium decides only the quantity of goods as an interior solution of Nash bargaining as in the second generation models. Define the average discounted sum of utilities of buyers as $v^{buyer} = \frac{1}{1-H(0)} \int_{\underline{z}}^{\overline{Z}} v(m) dH(m)$. Then, Pay-All equilibrium can be interpreted as an indivisible money model with $v(0)$ for each seller and v^{buyer} for each buyer. In short, given that buyers are aggregated as a kind of one representative buyer with value v^{buyer}, Pay-All equilibrium is in line with the second generation models, where $H(0)$ is exogenously given. As Trejos and Wright (1995) and Shi (1995) do not show Green and Zhou (1998)'s indeterminacy, the allocation in Pay-All equilibrium becomes also determinate in the aggregate level.

Note that, although the aggregated allocation is indeterminate, the individual-level indeterminacy among buyers is not *nominal* but *real*. As will be shown in the proof of the existence of Pay-All equilibrium, it requires only inequality conditions about the shape of the value function. Combinations of both indeterminate $v(m)$ and $H(m)$ lead to distributional indeterminacy of each buyer's real allocation and individual welfare.

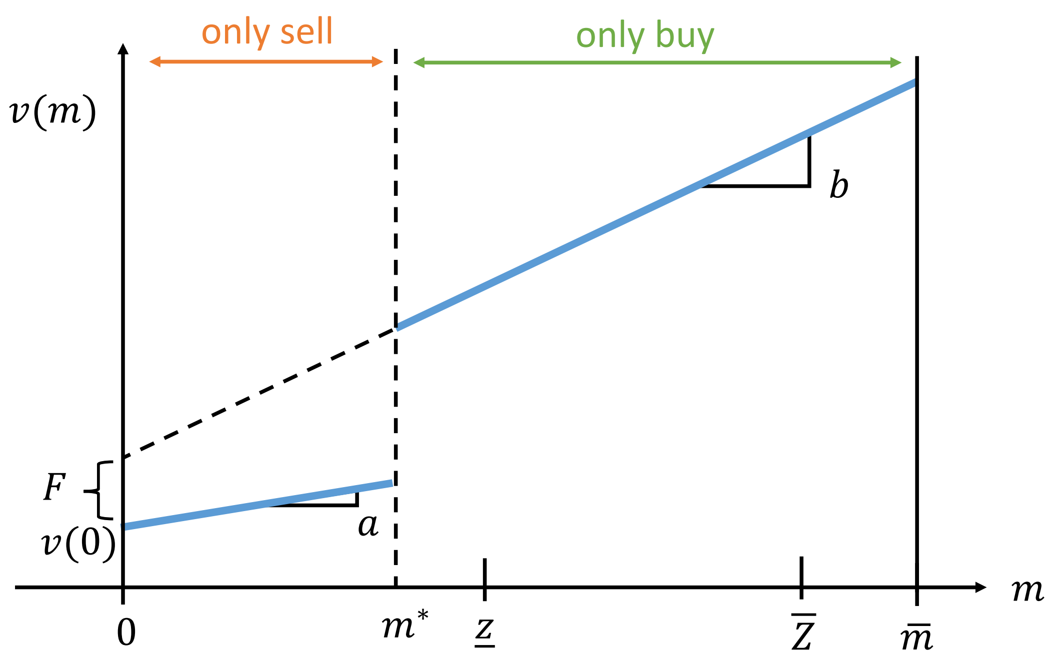
4 Existence

In this section, we show the existence of a continuum of pay-all equilibrium by construction. That is, we first present a candidate for a pay-all equilibrium (v, H) and show that it is indeed an equilibrium. The following linear function is a candidate for a value function:

$$v(m) = \begin{cases} v(0) + am & \text{for } m \in [0, m^*), \\ v(0) + F + bm, & \text{for } m \in [m^*, \infty], \end{cases} \tag{13}$$

support of money holdings distribution, Kamiya and Shimizu (2006) and Kamiya and Shimizu (2007), and for the case of infinite support, Kamiya (2019).

Figure 2: Value function



where $m^* > 0$, $b > a > 0$, and $F > 0$ will be determined later. A candidate for an equilibrium money holding distribution H satisfies that :

$$\text{the support of the distribution } H \text{ is } \{0\} \cup [\underline{z}, \overline{Z}], \tag{14}$$

where $\underline{z}$, and $\overline{Z}$ are parameters satisfying (1), (2) and

$$m^* \leq \underline{z} < \overline{Z} \leq 1 - m^*. \tag{15}$$

The above (v, H) is shown to be a pay-all equilibrium if the parameters are in some range. Note again that $H(0)$ and $v(0)$ are unique.

Proposition 1. *If the following conditions are satisfied,*

$$\frac{1-\beta}{\alpha\beta\theta} < \frac{1-c}{c}, \tag{16}$$

$$ck > d, \tag{17}$$

$$\phi \equiv \frac{\theta H(0)}{(1-\theta)[1-H(0)]c} > 1, \tag{18}$$

$$(\phi - 1)v(0) > \frac{(1-\theta)ck + \theta d}{[(1-\theta)c + \theta]\beta}, \tag{19}$$

$$\frac{\frac{1-H(0)}{M}\left[(\phi-1)\,v(0) - \frac{(1-\theta)ck+\theta d}{[(1-\theta)c+\theta]\beta}\right]}{\left[1 + \left(\frac{1-H(0)}{M}\right)\left(\frac{(1-\theta)(2-c)\chi}{2[(1-\theta)c+\theta]}\right)\right]} < 2d, \tag{20}$$

where $H(0) = \frac{(1-\beta)c}{\alpha\beta\theta(1-c)}$, $v(0) = \frac{ck-d}{\beta(1-c)}$, *and* $\chi \equiv \frac{\alpha\beta(1-\theta)[1-H(0)]}{\alpha\beta(1-\theta)[1-H(0)]+1-\beta}$, *then there exist a continuum of stationary pay-all equilibrium. Moreover, there exist parameters* $c, d, k, \alpha, \beta, \theta$, *and* M *satisfying the above conditions.*

Proof. Appendix. □

The first condition means that the unit cost c is sufficiently small and it assures $H(0) < 1$. The second conditions means that, given c and d, the fixed utility k is sufficiently large and it assures $v(0) > 0$. The third and the forth conditions assure that the slope of the value function is positive. The last condition guarantee that if both a buyer and a seller have small (large) amount of money, then they do not trade.

In the proof, we first derive conditions on (m^*, a, b, F) that Nash bargaining reaches a deal if $m_s < m^*$ and $m_b \geq m^*$, and breaks down otherwise and the pay-all property holds. Moreover, we find H and M which can be money holdings distribution and a money supply. Then we confirm that all the conditions on (m^*, a, b, F, H, M) are satisfied under the premises. Finally, we verify the consistency of (13) with (5) and (6). We also show that a continuum of (m^*, a, b, F, H, M) can be equilibria.

5 The bargaining power and welfare

The Hosios condition, a condition on the bargaining power θ for efficiency, is often discussed in the literature of search models. In this section, we discuss the impact of θ on the total welfare W, i.e., $\frac{\partial W}{\partial \theta}$.

First, note that, for given the other parameters, a change in θ may violates the conditions for the existence of a pay-all equilibrium. Namely, the pay-all property and/or $H(0) \in (0,1)$ could be violated. In what follows in this section, we assume for simplicity that a pay-all equilibrium exists for all θ given the other parameters.

Recall that $W = H(0)v(0) + \int_{\underline{z}}^{\overline{Z}} v(m)dH$. From Lemmas 3 and 5, the first term in the RHS is $\frac{(1-\beta)c}{\alpha\beta\theta(1-c)}\frac{ck-d}{\beta(1-c)}$. Thus $\frac{\partial H(0)v(0)}{\partial\theta} < 0$. From Corollary 2 and Lemma 3, the second term in the RHS is

$$\left(\frac{ck-d}{\beta(1-c)}\right)\left(\frac{\theta\frac{(1-\beta)c}{\alpha\beta\theta(1-c)}}{(1-\theta)c}\right).$$

Thus

$$\frac{\partial}{\partial\theta}\left(\int_{\underline{z}}^{\overline{Z}} v(m)dH\right) > 0. \tag{21}$$

Below, we investigate which dominates the other.

From Lemmas 1, 3, and 6, W is written as

$$W = \left(\frac{\xi v(0)}{\beta(1-\theta)}\right)H(0) = \left(\frac{v(0)(1-\beta)c}{\alpha\beta^2(1-c)}\right)\left(\frac{\theta(1-c)}{c}+1\right)\left(\frac{1}{\theta(1-\theta)}\right). \tag{22}$$

The derivative of $\left(\frac{\theta(1-c)}{c}+1\right)\left(\frac{1}{\theta(1-\theta)}\right)$ with respect to θ is

$$\frac{2\theta-1}{\theta^2(1-\theta)^2}\frac{(\theta(1-c)+c)}{c} + \frac{1-c}{c}\frac{1}{\theta(1-\theta)} = \frac{1}{c\theta^2(1-\theta)^2}((1-c)\theta^2+2c\theta-c).$$

Since

$$(1-c)\theta^2+2c\theta-c=0$$

has a negative solution and a positive solution $\theta = \frac{c}{c-\sqrt{c}}$ and $\theta = \frac{c}{c+\sqrt{c}}$. Note that $c = \frac{\theta^2}{(1-\theta)^2}$ holds at the positive solution. Thus, the following proposition holds.

Proposition 2. *$\frac{\partial W}{\partial\theta}$ is positive, if $\theta > \frac{c}{c+\sqrt{c}}$, and it is negative, if $\theta < \frac{c}{c+\sqrt{c}}$.*

Note that W is minimum at $\bar{\theta} = \frac{c}{c+\sqrt{c}}$, if a pay-all equilibrium exists at $\bar{\theta}$. Moreover, from (22), $W \to \infty$ as $\theta \to 1$, if a pay-all equilibrium exists around $\theta = 1$.

6 Distributional monetary policy in the long-run: a change in an effective bargaining power

Here, we consider a distributional monetary policy which keeps the total money supply and the money holding distribution overtime. The government imposes tax and provide subsidy so as to balance the budget. In each trade, they are transferred in the different way. Tax is collected as linear per-unit tax tx, where t is tax rate and x is the amount of good sold. Then, the collected money is injected to each seller depending on after-tax money holding. This amount is $g(m - tx)$, where g is the gross rate. Note that if $t < 0$, then it can be considered as a subsidy rate. It will be shown that the policy changes the power of the bargaining defined below as an effective bargaining power. Therefore, a Hosios-type condition on the bargaining power can be derived as in the previous section, and the policy can improve the welfare.

We assume the budget balance of the government. That is,

$$0 = \int (g-1)mdH - g\int tx(m)dH.$$

Thus

$$(g-1)M = gt\int x(m)dH. \tag{23}$$

As in the original case, the system is characterized by three equations: the first-order condition of Nash bargaining problem, buyer's value function, and seller's value function. Under pay-all equilibrium, the first one is obtained by the following problem:

$$\max_x \{k + x + \beta[v(0) - v(m)]\}^{\theta} \left\{-d - cx + \beta\left[v\big(g(m-tx)\big) - v(0)\right]\right\}^{1-\theta}.$$

Here, the seller's money holding after the trade depends on g and t. Then, the first-order condition is

$$(1-\theta)\left[c + \beta g t v'\big(g(m-tx)\big)\right]\{k+x+\beta[v(0)-v(m)]\} = \theta\left\{-d - cx + \beta\left[v\big(g(m-tx)\big) - v(0)\right]\right\}. \tag{24}$$

The Bellman equation for buyer is unchanged from (5). Whatever the policy is, each buyer pays the entire cash in pay-all equilibrium.

The value function of the seller is

$$v(0) = \alpha \int_{\underline{z}}^{\overline{Z}} \left[-d - cx(m) + \beta v\big(g(m - tx)\big)\right] dH + \{1 - \alpha[1 - H(0)]\}\beta v(0) \tag{25}$$

We assume a linear value function: $v(m) = v(0) + F + bm$ hereafter. As shown in the proof of existence, we can take such a linear value function. This assumption makes the system drastically tractable because (25) becomes identical to the non-policy case.

Lemma 7. *If the value function is linear,* (25) *coincides with* (6)

Proof. Appendix. □

From the linearity of the value function, $v'\big(g(m - tx)\big)$ is a constant b. In what follows we use v' instead of b. Therefore, (24) becomes

$$(1-\theta)\,[c + \beta g t v']\,\{k+x+\beta[v(0)-v(m)]\} = \theta\left\{-d - cx + \beta\left[v\big(g(m - tx)\big) - v(0)\right]\right\}. \tag{26}$$

Therefore, compared to the case without policy, the difference is only the first order condition of the Nash bargaining (26).

Recall that the first order condition of the Nash bargaining without policy derived in Equation (4) is

$$(1 - \theta)c\{k + x + \beta[v(0) - v(m)]\} = \theta\,\{-d - cx + \beta\,[v(m) - v(0)]\}\,.$$

Thus, the effective bargaining power of a buyer in the case of policy t, denoted by $\theta_e(t)$, is defined as follows:

$$\frac{[1 - \theta_e(t)]c}{\theta_e(t)} = \frac{(1 - \theta)[c + \beta g t v']}{\theta}.$$

Let $G = gt$. Note that $G' = g + t\frac{dg}{dt}$, and thus, at $g = 1$ and $t = 0$, $G' = g + t\frac{dg}{dt} = 1$. Then, the following proposition holds for any real number t.

Proposition 3. *Suppose $G' > 0$. Then $\frac{\partial W}{\partial t} > 0$ holds, if and only if*

$$\frac{c + \beta G v'}{c - \sqrt{c} + \beta G v'} < \theta < \frac{c + \beta G v'}{c + \sqrt{c} + \beta G v'}.$$

In particular, in the case of $t = 0$, $\frac{\partial W}{\partial t} > 0$ holds, if and only if

$$\theta < \frac{c}{c + \sqrt{c}}.$$

Proof. Appendix. □

The key implication is that, if $t = 0$, the condition becomes identical with Proposition 2. A small government intervention changes the bargaining power between buyer and seller. In this case, $\theta_e(0) = \theta$, then an increase in t leads to a decrease in $\theta_e(t)$. But, globally, it also depends on the shape of value function. Note that if the absolute value of t is not large, then $G' > 0$ holds, since $G' = 1$ at $t = 0$. Note also that, if $\frac{c+\beta G v'}{c-\sqrt{c}+\beta G v'} < 0$, then $\frac{c+\beta G v'}{c-\sqrt{c}+\beta G v'} < \theta$ can be replaced by $0 \leq \theta$. In particular, this applies in the case of $t = 0$.

7 Distributional monetary policy in the short-run

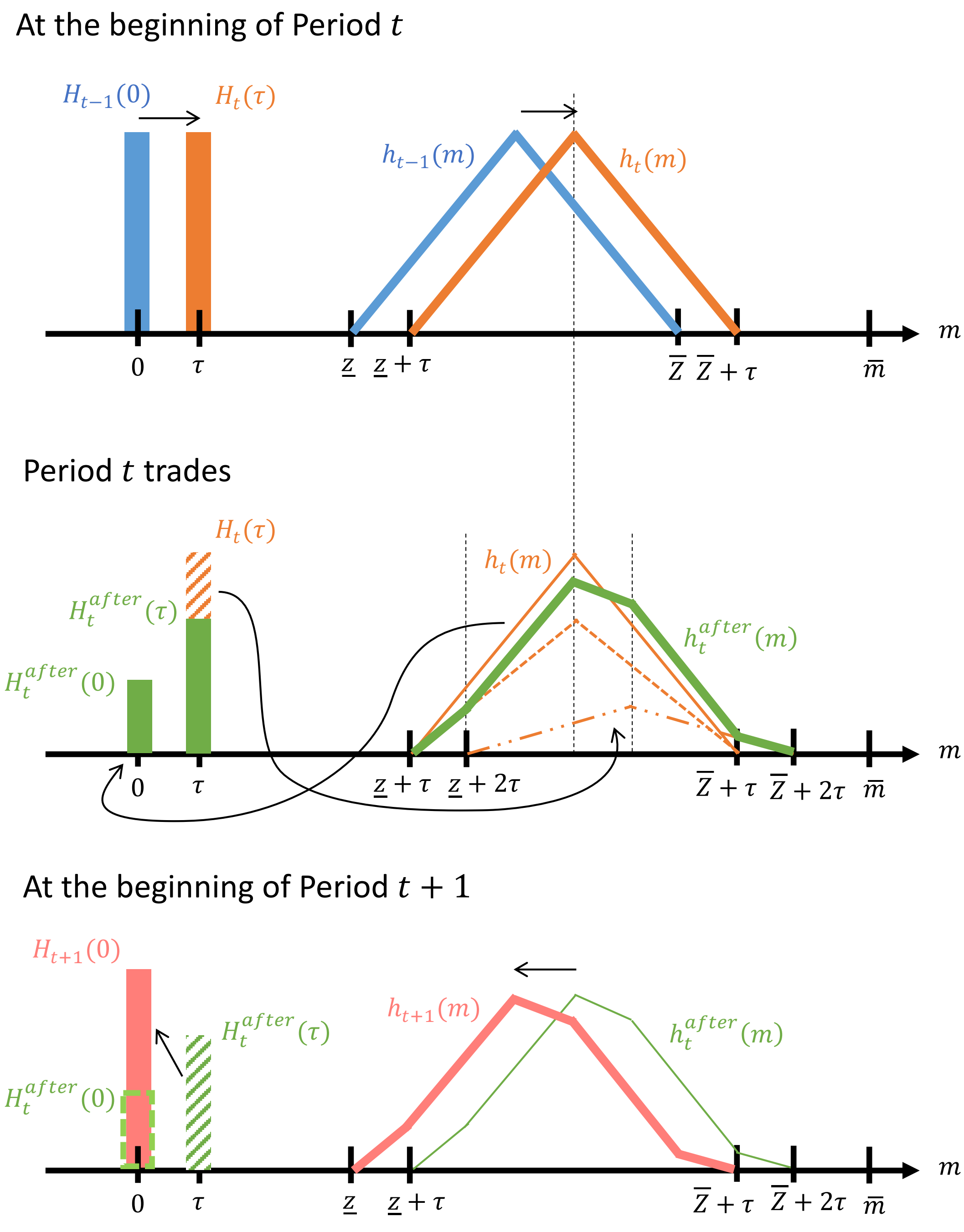
In this section, we consider one-time helicopter drop of money. At the beginning of period t, the central bank unexpectedly announces the following policy.

- The central bank injects the same τ unit of money to everybody at the beginning of period t.
- The central bank will deprive τ unit of money at period $t + 1$. If an agent's money holding is $m < \tau$, it will deprive m.

The collection of money at period $t + 1$ is for keeping Pay-All equilibrium. Otherwise, the range of money holding distribution diverges to infinity[4]. For simplicity, assume that the density function $h(m)$ associated with $H(m)$ exists for $m \in [\underline{z}, \overline{Z}]$.

[4]Suppose the maximum money holding is m_{max} at period $t - 1$. At the beginning of period t, sellers also hold τ. After the trade, the maximum amount increases to $m_{max} + \tau$. If there is no reduction in money holding, the maximum amount changes to $m_{max} + 2\tau$ at the end of period $t + 1$. It will eventually diverges and exceed the limit of money holding 1.

Figure 3: Money holding distribution



Given Pay-All equilibrium, the transition of the distribution is described as follows. Figure 3 draws an example of a triangle shape distribution.

At period $t-1$ and before, the stationary money holding distribution satisfies

- $H_{t-1}(0) = \frac{(1-\beta)c}{\alpha\beta\theta(1-c)} > 0$,
- $h_{t-1}(m) \geq 0$ for all $m \in [\underline{z}, \overline{Z}]$,

and no agents hold $m \notin \{0\} \cup [\underline{z}, \overline{Z}]$ amount of money.

At the beginning of period t, τ is injected. Then, the distribution shifts to the right by τ:

- $H_t(\tau) = H_{t-1}(0) = \frac{(1-\beta)c}{\alpha\beta\theta(1-c)} > 0$,
- $h_t(m) = h_{t-1}(m-\tau) \geq 0$ for all $m \in [\underline{z}+\tau, \overline{Z}+\tau]$.

In period t, suppose that the Pay-All property still holds in the bargaining. Then, each buyer $m \in [\underline{z}+\tau, \overline{Z}+\tau]$ pays all the amount of money. Each seller finds a buyer with probability $\alpha[1-H_t(\tau)]$, and each buyer makes a matching with $\alpha H_t(\tau)$. Let H_t^{after} be the money holding distribution after the trade in period t. Then,

- $H_t^{after}(0) = \alpha H_t(\tau)[1-H_t(\tau)] = \alpha H_{t-1}(0)[1-H_{t-1}(0)]$
- $H_t^{after}(\tau) = H_t(\tau)\{1-\alpha[1-H_t(\tau)]\} = H_{t-1}(0)\{1-\alpha[1-H_{t-1}(0)]\}$
- For $m \in [\underline{z}+\tau, \underline{z}+2\tau)$, $h_t^{after}(m) = [1-\alpha H_t(\tau)]h_t(m) = [1-\alpha H_{t-1}(0)]h_{t-1}(m-\tau)$
- For $m \in [\underline{z}+2\tau, \overline{Z}+\tau]$, $h_t^{after}(m) = \alpha H_t(\tau)h_t(m-\tau) + [1-\alpha H_t(\tau)]h_t(m) = \alpha H_{t-1}(0)h_{t-1}(m-2\tau) + [1-\alpha H_{t-1}(0)]h_{t-1}(m-\tau)$.
- For $m \in (\overline{Z}+\tau, \overline{Z}+2\tau]$, $h_t^{after}(m) = \alpha H_t(\tau)h_t(m-\tau) = [1-\alpha H_{t-1}(0)]h_{t-1}(m-2\tau)$

Finally, at the beginning of period $t+1$, money holdings of all agents except $m=0$ are declined by τ. The distribution is sustained stationary hereafter.

- $H_{t+1}(0) = H_t^{after}(0) + H_t^{after}(\tau) = \alpha H_{t-1}(0)[1-H_{t-1}(0)] + H_{t-1}(0)\{1-\alpha[1-H_t(\tau)]\} = H_{t-1}(0)$.
- For $m \in [\underline{z}, \underline{z}+\tau)$, $h_{t+1}(m) = h_t^{after}(m+\tau) = [1-\alpha H_{t-1}(0)]h_{t-1}(m)$

- For $m \in [\underline{z}+\tau, \overline{Z}]$, $h_{t+1}(m) = h_t^{after}(m+\tau) = \alpha H_{t-1}(0)h_{t-1}(m-\tau)+[1-\alpha H_{t-1}(0)]h_{t-1}(m)$.
- For $m \in (\overline{Z}, \overline{Z}+\tau]$, $h_{t+1}(m) = h_t^{after}(m+\tau) = \alpha H_{t-1}(0)h_{t-1}(m-\tau)$

The measure of non-money holder is unchanged between old and new stationary distribution: $H(0) = H_{t+1}(0) = H_{t-1}(0)$. The distribution of positive money holders spreads out. Because of the indeterminacy of the money holding distribution $h(m)$ for $m \geq \underline{z}$, the new distribution holds a stationary Pay-All Equilibrium with the same social welfare. Then, we can consider Period t allocation is on a transition path between the two steady-states. Let $\overline{v}(0) \equiv v_{t-1}(0) = v_{t+1}(0)$. Note that $v_{t-1}(m)$ and $v_{t+1}(m)$ for $m \geq \underline{z}$ are possibly different by indeterminacy. Consider the bargaining problem of a seller holding τ and a buyer holding $m_t = m_{t-1} + \tau$ at period t:

$$\max_{x_t} \{k + x_t + \beta[v_{t+1}(0) - v_{t+1}(m_{t-1})]\}^\theta \{-d - cx_t + \beta[v_{t+1}(m_{t-1}+\tau) - v_{t+1}(0)]\}^{1-\theta}$$

In the first term, the buyer holds 0 at Period $t+1$ given Pay-All property. If the bargainig fails, the buyer looses τ unit of money and hold $m_{t+1} = m_{t-1} + \tau - \tau = m_{t-1}$ unit of money, In the second term, the seller obtains $m_t = m_{t-1} + \tau$ by the trade at Period t and looses τ in the next, then $m_{t+1} = \tau + m_{t-1} + \tau - \tau = m_{t-1} + \tau$ in total at Period $t+1$. In case of no trade, the τ unit of money holding needs to be returned to the central bank, then $m_{t+1} = m_{t-1} = 0$. The first-order condition is

$$\theta\{-d - cx_t + \beta[v_{t+1}(m_{t-1}+\tau) - v_{t+1}(0)]\} = (1-\theta)c\{k + x_t + \beta[v_{t+1}(0) - v_{t+1}(m_{t-1})]\}$$

$$\Leftrightarrow \; cx_t(m_{t-1}) = \beta(1-\theta)c[v_{t+1}(m_{t-1}) - v_{t+1}(0)] + \beta\theta[v_{t+1}(m_{t-1}+\tau) - v_{t+1}(0)] - [(1-\theta)ck + \theta d]$$

$$= \beta\theta v_{t+1}(m_{t-1}+\tau) + \beta(1-\theta)cv_{t+1}(m_{t-1}) - \beta[(1-\theta)c + \theta]\overline{v}(0) - [(1-\theta)ck + \theta d]$$

Let the total production at period t as X_t.

$$X_t = \alpha H(0) \int_{\underline{z}}^{\overline{Z}} x_t(m_{t-1})h_{t-1}(m_{t-1})dm_{t-1}$$

$$= \left(\frac{\alpha\beta H(0)}{c}\right)\theta \int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1}+\tau)h_{t-1}(m_{t-1})dm_{t-1}$$

$$+\left(\frac{\alpha\beta H(0)}{c}\right)(1-\theta)c\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1})h_{t-1}(m_{t-1})dm_{t-1}$$

$$-\frac{\alpha\beta H(0)[1-H(0)]}{c}\left\{\beta[(1-\theta)c+\theta]\overline{v}(0)+[(1-\theta)ck+\theta d]\right\}$$

Now, we want to show that X_t is larger than the steady-state amount of production, $\overline{X} \equiv X_{t-1} = X_{t+1}$. By the linear utility and cost functions, it implies the welfare improvement by the short-run monetary policy.

$$\overline{X} = X_{t+1} = \frac{\alpha\beta H(0)}{c}\left\{[\theta+(1-\theta)c]\int_{\underline{z}}^{\overline{Z}+\tau} v_{t+1}(m_{t+1})h_{t+1}(m_{t+1})dm_t\right\}.$$

$$-\frac{\alpha\beta H(0)[1-H(0)]}{c}\left\{\beta[(1-\theta)c+\theta]\overline{v}(0)+[(1-\theta)ck+\theta d]\right\}$$

Then, the condition $X_t > \overline{X}$ is equivalent to

$$\theta\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1}+\tau)h_{t-1}(m_{t-1})dm_{t-1} + (1-\theta)c\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1})h_{t-1}(m_{t-1})dm_{t-1}$$

$$> [\theta+(1-\theta)c]\int_{\underline{z}}^{\overline{Z}+\tau} v_{t+1}(m_{t+1})h_{t+1}(m_{t+1})dm_{t+1}$$

By the transition of the money holding distribution,

$$\int_{\underline{z}}^{\overline{Z}+\tau} v_{t+1}(m_{t+1})h_{t+1}(m_{t+1})dm_{t+1}$$

$$= [1-\alpha H(0)]\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1})h_{t-1}(m_{t-1})dm_{t-1} + \alpha H(0)\int_{\underline{z}+\tau}^{\overline{Z}+\tau} v_{t+1}(m_{t-1})h_{t-1}(m-\tau)$$

$$= [1-\alpha H(0)]\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1})h_{t-1}(m_{t-1})dm_{t-1} + \alpha H(0)\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1}+\tau)h_{t-1}(m)$$

Therefore,

$$X_t > \overline{X} \iff$$

$$\{\theta - [\theta+(1-\theta)c]\alpha H(0)\}\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1}+\tau)h_{t-1}(m)$$

$$> \{[\theta + (1-\theta)c][1-\alpha H(0)] - (1-\theta)c\} \int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1}) h_{t-1}(m)$$

$$\Leftrightarrow \{\theta[1-\alpha H(0)] - (1-\theta)c\alpha H(0)\} \left[\int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1}+\tau) h_{t-1}(m) - \int_{\underline{z}}^{\overline{Z}} v_{t+1}(m_{t-1}) h_{t-1}(m) \right] > 0 \quad (27)$$

Since the value function is strictly increasing in Pay-All equilibrium, $v_{t-1}(m_{t-1} + \tau) > v_{t-1}(m_{t-1})$ for all m_{t-1}. Hence, the second term is positive. The sign of the production change depends on the first-term.

$$X_t > \overline{X}$$

$$\Leftrightarrow \theta[1-\alpha H(0)] > (1-\theta)c\alpha H(0)$$

$$\Leftrightarrow \theta \left[1 - \frac{(1-\beta)c}{\beta\theta(1-c)} \right] > (1-\theta) \frac{(1-\beta)c^2}{\beta\theta(1-c)}$$

$$\Leftrightarrow \theta^2\beta(1-c) - \theta(1-\beta)c > (1-\theta)(1-\beta)c^2$$

$$\Leftrightarrow \theta^2\beta(1-c) > (1-\beta)c^2 + \theta(1-\beta)c(1-c)$$

$$\Leftrightarrow \theta^2\beta(1-c) > (1-\beta)c[c + \theta(1-c)]$$

This condition holds for sufficiently large β.

Intuitively, the policy raises the production by changing each seller's future welfare. The money holding distribution will be diverged at $t+1$. Sellers at $t-1$ will hold $m \in [\underline{z} + \tau, \overline{Z} + \tau]$ at $t+1$. They are larger than the average in the steady state after $t+1$. It makes more incentive for production in Nash bargaining since the seller's threat point value $\beta[v(0) - v(m+\tau)]$ decreases.

Interestingly, this positive welfare improvement is indeterminate. The welfare improvement depends on the second term of Equation (27). Hence, the quantitative difference hinges on the shape of the value function $v_{t+1}(m)$. Therefore, the individual-level indeterminacy leads to the aggregate level indeterminacy in the welfare improvement by the distributional monetary policy.

8 Conclusion

This paper shows that a class of standard search and bargaining models of money has analytical characterizations of equilibrium aggregate variables, i.e., the welfare, the measure of non-money holders, and the other aggregate variables are uniquely expressed by given parameters. Due to the analytical characterization, the standard comparative statics can be applied and the effects of policies can be well investigated. The model has two notable features. First, it does not have centralized markets, as in Lagos and Wright (2005), so that only the effect of policies on the decentralize market can be extracted. Second, the assumption of large households, as in Shi (1997), is not made so that the effect of parameters and policies on money holdings distributions can be investigated.

A long-run and a short-run distributional monetary policies are investigated. In the long-run policy, the government levies a per-unit tax (subsidy) depending on the amount of goods, and distributes (collects) money depending on after-trade money holdings in each period, where they are chosen so that the government budget is balanced. Intuitively, it modifies the intra-temporal condition of Nash bargaining changing the effective bargaining power and moves the equilibrium toward the Hosios condition on the efficiency. In the short-run policy, the government injects the same amount of money to all agents only in one period. Then the same amount is subtracted in the next period. Intuitively, the policy raises the production by changing the future welfare of each seller, and it leads to a short-run improvement in the total welfare.

In the literature, long-term changes in the amount of money, such as the Friedman rule, are often investigated. This type of policy is not covered in this paper, because our existence proof cannot be applied to such policies. In the existence proof, we need an upper bound and a lower bound of the support of money holdings distributions. However, in such policies, they converge to infinity or to zero. Of course, an analysis of this type of policies is an important future research topic.

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Appendix

The proof of Lemma 4

By (5),

$$\left[\frac{(1-\beta)(1-\theta c)}{\alpha H(0)}\right] v(m) = (1-\theta)c\{k + x(m) + \beta[v(0) - v(m)]\}$$

$$\Rightarrow \left[\frac{(1-\beta)(1-\theta c)}{\alpha H(0)}\right] \int_{\underline{z}}^{\overline{Z}} v(m)dH = \int_{\underline{z}}^{\overline{Z}} (1-\theta)c\{k + x(m) + \beta[v(0) - v(m)]\}dH \quad \text{(A.1)}$$

By (6),

$$v(0) = \alpha \int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta v(m)]dH + \beta v(0) - \alpha \int_{\underline{z}}^{\overline{Z}} \beta v(0)dH$$

$$= \alpha \int_{\underline{z}}^{\overline{Z}} \{-d - cx(m) + \beta[v(m) - v(0])\}dH + \beta v(0)$$

$$\Leftrightarrow \left[\frac{(1-\beta)\theta}{\alpha}\right] v(0) = \theta \int_{\underline{z}}^{\overline{Z}} \{-d - cx(m) + \beta[v(m) - v(0])\}dH \quad \text{(A.2)}$$

Integrating the FOC of Nash bargaining solution, (4), derives

$$\int_{\underline{z}}^{\overline{Z}} (1-\theta)c[k + x(m) + \beta[v(0) - v(m))]dH$$
$$= \int_{\underline{z}}^{\overline{Z}} \theta\{-d - cx(m) + \beta[v(m) - v(0)]\}dH \quad \text{(A.3)}$$

Substitute Equation (A.1) and (A.2) into (A.3), then

$$\left[\frac{(1-\beta)(1-\theta)c}{\alpha H(0)}\right] \int_{\underline{z}}^{\overline{Z}} v(m)dH = \left[\frac{(1-\beta)\theta}{\alpha}\right] v(0),$$

which leads to the Lemma .

The proof of Lemma 5

By (5),

$$\left(\frac{1-\beta}{H(0)}\right) v(m) = \alpha\{k + x(m) + \beta[v(0) - v(m))]\}$$

$$\Leftrightarrow \left(\frac{(1-\beta)c}{H(0)}\right) v(m) = \alpha\{ck + cx(m) + \beta cv(0) - \beta cv(m)\}$$

$$\Leftrightarrow \alpha[ck + \beta cv(0) - \beta cv(m)] - \left(\frac{(1-\beta)c}{H(0)}\right) v(m) = \alpha[-cx(m) + \beta cv(m)]$$

$$= \alpha[d - d - cx(m) + \beta v(m) - \beta(1-c)v(m)]$$

$$= \alpha[-d - cx(m) + \beta v(m)] + \alpha[d - \beta(1-c)v(m)]$$

Then,

$$\begin{aligned} &\alpha[-d - cx(m) + \beta v(m)] \\ &= \alpha ck + \alpha\beta cv(0) - \left(\frac{(1-\beta)c}{H(0)}\right) v(m) - \alpha d + \alpha\beta(1-c)v(m)] \\ &= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right] v(m) + \alpha[ck - d + \beta cv(0)]. \end{aligned} \tag{A.4}$$

By integrating both sides of (A.4), we get

$$\alpha \int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta v(m)] dH$$

$$= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right] \int_{\underline{z}}^{\overline{Z}} v(m) dH + \alpha[1 - H(0)][ck - d + \beta cv(0)].$$

By (6),

$$\{1 - \beta + \alpha[1 - H(0)]\beta\} v(0)$$

$$= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right] \int_{\underline{z}}^{\overline{Z}} v(m) dH + \alpha[1 - H(0)][ck - d + \beta cv(0)].$$

$$\Leftrightarrow \{1 - \beta + \alpha[1 - H(0)](1-c)\beta\} v(0)$$

$$= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right] \int_{\underline{z}}^{\overline{Z}} v(m)dH + \alpha[1-H(0)](ck-d).$$

By Lemma 4,

$$\{1-\beta+\alpha[1-H(0)](1-c)\beta\}v(0)$$

$$= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right] \left[\frac{\theta H(0)v(0)}{(1-\theta)c}\right] + \alpha[1-H(0)](ck-d).$$

Then,

$$\{1-\beta+\alpha[1-H(0)](1-c)\beta\}(1-\theta)cv(0)$$

$$= \left[\alpha\beta(1-c)H(0) - (1-\beta)c\right]\theta v(0) + \alpha[1-H(0)](1-\theta)c(ck-d).$$

Thus

$$v(0) = \frac{\alpha c(1-H(0))(1-\theta)(ck-d)}{c(1-\beta)+\alpha\beta(1-\theta)(1-c) - (\alpha\beta(1-c)(c+\theta-c\theta))H(0)}.$$

Substituting $H(0) = \frac{(1-\beta)c}{\alpha\beta\theta(1-c)}$ into the denominator of the above, we obtain

$$c(1-\beta)+\alpha\beta(1-\theta)(1-c) - \frac{c}{\theta}(1-\beta)(c+\theta-c\theta) = \frac{c(1-\theta)(c\beta+\theta\alpha\beta-c-c\theta\alpha\beta)}{\theta}.$$

Since

$$1-H(0) = \frac{c\beta+\theta\alpha\beta-c-c\theta\alpha\beta}{\alpha\beta\theta(1-c)},$$

we obtain

$$v(0) = \frac{ck-d}{\beta(1-c)}.$$

The proof of Proposition 1

The steps of the proof

1. Assuming the pay-all property, we derive some properties of the Nash bargaining so-

lution.

2. We present a candidate for (v, H).

3. We derive the conditions which guarantee the pay-all property. It can be seen that a continuum of endogenously determined variables, including (v, H), satisfy the conditions.

4. We show that the conditions in Step 3 are satisfied under the assumptions in Proposition 2,

5. Finally, we show the existence of equilibria. That is, (i) (v, H) in Step 2 is an equilibrium, i,e., H is stationary and v satisfies Bellman equation, and (ii) there exist parameters satisfying the assumptions in Proposition 2,

Step 1: Some properties of the Nash bargaining solution under the pay-all property

This step derives some properties the Nash bargaining solution in the case that a meeting of a buyer holding m_b and a seller holding m_s, where the buyer pays all m_b and the amount of goods traded in the bargaining is positive. Note that in this step we do not assume the linearity of v. The results will be used in the follwoing Steps. In these steps, the off-path meetings will be also investigated, where agents hold non-equilibrium amount of money,

Lemma A.1. *Suppose the pay-all property is satisfied, i.e.,* $x(m_s, m_b, H) > 0$ *and* $q(m_s, m_b, H) = m_b$*, then*

$$cx(m_s, m_b, H) = (1-\theta)c\beta[v(m_b)-v(m_b-q)]+\theta\beta[v(m_s+q)-v(m_s)]-[(1-\theta)ck+\theta d]. \quad \text{(A.5)}$$

Moreover, the seller's surplus is

$$(1-\theta)\big\{\beta[v(m_s+q)-v(m_s)] - c\beta[v(m_b)-v(m_b-q)] + (ck-d)]\big\}. \quad \text{(A.6)}$$

and the buyer's surplus is

$$(\theta/c)\big\{\beta[v(m_s+q)-v(m_s)] - c\beta[v(m_b)-v(m_b-q)] + (ck-d)\big\} \quad \text{(A.7)}$$

Proof. The first-order condition with respect to x is

$$-(1-\theta)c[k+x+\beta(v(m_b-q)-v(m_b))]+\theta[-d-cx+\beta(v(m_s+q)-v(m_s))]=0.$$

Thus

$$cx=(1-\theta)c[\beta(v(m_b)-v(m_b-q))]+\theta\beta[v(m_s+q)-v(m_s)]-[(1-\theta)ck+\theta d].$$

Using the above, the seller's surplus is

$$-d-cx+\beta(v(m_s+q)-v(m_s))$$

$$=-d-(1-\theta)c\beta[v(m_b)-v(m_b-q)]-\theta\beta[v(m_s+q)-v(m_s)]+[(1-\theta)ck+\theta d]+\beta[v(m_s+q)-v(m_s)]$$

$$=(1-\theta)\big\{\beta[v(m_s+q)-v(m_s)]-c\beta[v(m_b)-v(m_b-q)]+(ck-d)]\big\}.$$

Similarly, the buyer's surplus is

$$k+x+\beta(v(m_b-q)-v(m_b))$$

$$=k+(1-\theta)\beta[v(m_b)-v(m_b-q)]+(\theta/c)\beta[v(m_s+q)-v(m_s)]-[(1-\theta)k+(\theta/c)d]+\beta(v(m_b-q)-v(m_b))$$

$$=(\theta/c)\beta[v(m_s+q)-v(m_s)]-\theta\beta[v(m_b)-v(m_b-q)]+\theta k-(\theta/c)d$$

$$=(\theta/c)\big\{\beta[v(m_s+q)-v(m_s)]-c\beta[v(m_b)-v(m_b-q)]+ck-d\big\}$$

□

Step 2: The candidates for an equilibrium value function and an equilibrium money holding distribution

First, note that $v(0)=\frac{ck-d}{\beta(1-c)}$ and $H(0)=\frac{(1-\beta)c}{\alpha\beta\theta(1-c)}$ are already given. The candidate for an equilibrium value function is as follows:

$$v(m) = \begin{cases} v(0) + am & \text{if } m < m^*, \\ v(0) + F + bm & \text{if } m \geq m^*, \end{cases} \tag{A.8}$$

where a, b, F, and m^* are parameters.

The candidate for a equilibrium money holding distribution satisfies Equation (14):

the support of the distribution H is $\{0\} \cup [\underline{z}, \overline{Z}]$.

In the following steps, it will be shown that H, $\underline{z}$, and $\overline{Z}$ can be freely chosen, i.e., indeterminate, if they satisfy (1), (2), and Equation (15)

$m^* \leq \underline{z} < \overline{Z} \leq 1 - m^*$.

Next, we derive the relationship between the slopes a and b, The slope a is the marginal life-time utility of money for sellers, and b is that for buyers. We show that $a < b$. A seller needs to wait at least one period for being a buyer and purchasing goods; hence, the gain from holding money is discounted by the time preference.

Lemma A.2. *The coefficients a and b satisfy*

$$a = \chi b, \tag{A.9}$$

where $\chi \equiv \frac{\alpha\beta(1-\theta)[1-H(0)]}{\{\alpha\beta(1-\theta)[1-H(0)]+1-\beta\}} < 1$.

Proof. Consider a meeting in which a seller holds $m_s < m^*$ and a buyer holds $m_b \in [\underline{z}, \overline{Z}]$. Note that from (15), the money holding of the seller after the trade does not exceeds the maximum money holding 1. By (A.6),

$$v(m_s) = \alpha \int_{m_b \in [\underline{z}, \overline{Z}]} [-d - cx(m_s, m_b, H) + \beta(v(m_s + m_b) - v(m_s))]h(m_b)dm_b + \beta v(m_s)$$

Thus

$$
\begin{aligned}
am_s &= \alpha(1-\theta)\int_{m_b\in[\underline{z},\overline{Z}]}\beta[v(m_s+m_b)-cv(m_b)]h(m_b)dm_b+\alpha(1-\theta)[h(\overline{Z})-h(\underline{z})][-\beta v(m_s)+c\beta v(0)\\
&= \alpha(1-\theta)\int_{m_b\in[\underline{z},\overline{Z}]}\beta v(0)+F+b(m_s+m_b)-c[v(0)+F+bm_b]\}h(m_b)dm_b\\
&\quad -\alpha(1-\theta)[1-H(0)][a\beta m_s+(\beta c-1)v(0)+ck-d]+\beta am_s+\beta v(0)\\
&= \alpha(1-\theta)[1-H(0)](1-c)\beta[v(0)+F]+\alpha(1-\theta)[1-H(0)]\beta bm_s+\alpha(1-\theta)(1-c)\beta b(M/\overline{m})\\
&\quad -\alpha(1-\theta)[1-H(0)](a\beta m_s+(\beta c-1)v(0)+ck-d)+\beta am_s+\beta v(0)
\end{aligned}
$$

Comparing the coefficients of m_s in the LHS and the RHS yields

$$a=\alpha(1-\theta)[1-H(0)]\beta(b-a)+\beta a.$$

□

Step 3: The conditions on the parameters which guarantee the pay-all property We first present a sufficient condition that there is no trade if $m_s+q(m_s,m_b,H)<m^*$ or $m_s\geq m^*$.

Lemma A.3. *Suppose*

$$b(1-m^*)<d. \tag{A.10}$$

Then the bargaining can reach the agreement if only if $m_s<m^*<m_s+q(m_s,m_b,H)$.

Proof. Suppose $m_s<m^*<m_s+q(m_s,m_b,H)$ does not hold and the bargaining reaches the agreement with $x(m_b,m_s)>0$ and $q(m_s,m_b,H)$. The seller's surplus does not contain F. Then the maximum increase in the discounted utility by earning money is am^*, $b(\overline{Z})$ or $b(1-m^*)$. By $a<b$ and (15), $b(1-m^*)$ is the largest among them. Then, the maximum amount of the surplus does not exceed

$$-d+b(1-m^*),$$

which is negative under (A.10). Note that $-d$ is the cost of production in the case of $x=0$. □

We next present a condition that the all-pay property holds in a bargaining between a seller without money and a buyer with $m_b \geq m^*$.

Lemma A.4. *If*

$$[(1-\theta)c+\theta]\beta F > (1-\theta)ck+\theta d, \tag{A.11}$$

then $x(m_s, m_b, H) > 0$ *and* $q(m_s, m_b, H) = m_b$ *hold in the bargaining between a seller with* $m_s = 0$ *and a buyer with* $m_b \geq m^*$.

Proof. If $x(m_s, m_b, H) = 0$ in the Nash bargaining, then $q(m_s, m_b, H) = 0$, i.e., no trade, and thus the surpluses of the agents are zero. As shown below, they can be positive and the buyer and the seller trade.

If $x(m_s, m_b, H) > 0$, then by Lemma (A.1), $q(m_s, m_b, H) = m_b$. Indeed, $x(m_s, m_b, H) > 0$ will be shown below, and the surpluses are positive.

From Lemma (A.1),

$$\begin{aligned} cx &= (1-\theta)c\beta(F+bm_b)+\theta\beta(F+bm_b)-[(1-\theta)ck+\theta d] \\ &= [(1-\theta)c+\theta]\beta(F+bm_b)-[(1-\theta)ck+\theta d]. \end{aligned}$$

From (A.11), this is positive and thus $x(m_s, m_b, H) > 0$.

$$[(1-\theta)c+\theta]\beta(F+bm^*) > [(1-\theta)ck+\theta d]$$

The seller's surplus is positive, because

$$-d-cx+\beta(v(m_b)-v(0))$$

$$= (1-\theta)\left\{(1-c)\beta[F+b(m_b-m^*)]+(ck-d)\right\} > 0$$

holds by $m_b \geq m^*$. Since the seller's surplus is positive, so is the buyer's one. Thus $x(m_s, m_b, H) > 0$ and $q(m_s, m_b, H) = m_b$ hold. □

Next, we show that the pay-all property holds in the bargaining between a seller with $m_s < m^*$ and a buyer with $m_b \in [\underline{z}, \overline{Z}]$.

Lemma A.5. *If* (A.11) *and*

$$[(1-\theta)c+\theta]\beta F < (1-\theta)ck+\theta d+(1-\theta)(2-c)a\beta m^* - [(1-\theta)c+\theta]b\beta(\overline{Z}-m^*) \quad (A.12)$$

Hold, then, in the bargaining between a seller with $m_s < m^*$ and a buyer with $m_b \in [\underline{z}, \overline{Z}]$, $x(m_s, m_b, H) > 0$ and $q(m_s, m_b, H) = m_b$ hold.

Proof. If $x(m_s, m_b, H) = 0$ in the Nash bargaining, then $q(m_s, m_b, H) = 0$, i.e., no trade, and thus the surpluses of the agents are zero. As shown below, they can be positive and the buyer and the seller trade.

If q satisfies $m_s+q < m^*$, then from Lemma (A.3), it can not be $q(m_s, m_b, H)$. Therefore, below we consider the case $m_s+q \geq m^*$. As for a buyer, we consider the cases (i) $m_b-q < m^*$ and (ii) $m_b - q \geq m^*$.

Case (i): If $x(m_s, m_b, H) > 0$, then from Lemma (A.1), $q(m_s, m_b, H) = m_b$ holds. Indeed, $x(m_s, m_b, H) > 0$ will be shown below, and the surpluses are positive. From Lemma (A.1),

$$\begin{aligned} cx &= (1-\theta)c\beta(F+bm_b) + \theta\beta\left[F + b(m_b+m_s) - am_s\right] - [(1-\theta)ck+\theta d] \\ &= [(1-\theta)c+\theta]\beta(F+bm_b) + \theta(b-a)\beta m_s - [(1-\theta)ck+\theta d] \end{aligned}$$

hold. From (A.11), this is positive, and thus $x(m_s, m_b, H) > 0$.

The seller's surplus is

$$-d-cx+\beta v(m_b+m_s)-\beta v(m_s)) = (1-\theta)\left[(1-c)\beta(F+bm_b) + (b-a)\beta m_s + (ck-d)\right] > 0. \quad (A.13)$$

Similarly, the buyer's surplus is positive.

Case (ii): In this case, $q \leq m_b - m^*$ and $m_s + q > m^*$. Note that the buyer's surplus does not contain $-F$. The seller's surplus is

$$-d - cx + \beta v(m_s+q) - \beta v(m_s) = -d - cx + a\beta m_s + \beta F + \beta b(m_s+q)$$

$$< -d + \beta F + (b-a)\beta m_s + b\beta(m_b - m^*) \tag{A.14}$$

Below. we show that the surplus in (A.14) is smaller than that in (A.13), i.e., Case (ii) does not occur in the bargaining. The inequality is expressed as:

$$-d+\beta F+(b-a)\beta m_s+b\beta(m_b-m^*) < (1-\theta)\left[(1-c)\beta(F+bm_b)+(b-a)\beta m_s+(ck-d)\right].$$

Since the RHS is equal to

$$(1-\theta)(1-c)\beta F + (1-\theta)(1-c)b\beta m_b + (1-\theta)(b-a)\beta m_s + (1-\theta)(ck-d),$$

The above inequality is equivalent to

$$[(1-\theta)c+\theta]\beta F < (1-\theta)ck + \theta d + bm^* - \theta(b-a)\beta m_s - [(1-\theta)c+\theta]b\beta m_b. \tag{A.15}$$

Since $m_s < m^*$,

$$[(1-\theta)c+\theta]\beta F < (1-\theta)ck + \theta d + b\beta m^* - \theta(b-a)\beta m^* - [(1-\theta)c+\theta]b\beta m_b \tag{A.16}$$

is a sufficient condition for (A.15). Since $m_b < \overline{Z}$,

$$[(1-\theta)c+\theta]\beta F < (1-\theta)ck + \theta d + [a+(1-\theta)b]\beta m^* - [(1-\theta)c+\theta]b\beta\overline{Z}$$

is a sufficient condition for (A.16), and it is equivalent to

$$[(1-\theta)c+\theta]\beta F < (1-\theta)ck+\theta d+a\beta m^*+[(1-\theta)(1-c)-\theta]b\beta m^*-[(1-\theta)c+\theta]b\beta(\overline{Z}-m^*). \tag{A.17}$$

Since $b > a$,

$$[(1-\theta)c+\theta]\beta F < (1-\theta)ck+\theta d+a\beta m^*+[(1-\theta)(1-c)-\theta]a\beta m^*-[(1-\theta)c+\theta]b\beta(\overline{Z}-m^*)$$

is a sufficient condition for (A.17), and it is equivalent to (A.12). □

Step 4: The conditions in the Lemmas

Lemma A.6. *Under the assumptions in Proposition 1, there exists a continuum of* $(\underline{z}, \overline{Z}, H, m^*, M, a, b, F)$ *satisfying the following conditions in the previous Lemmas:*

- $b > 0$
- (15)*:* $m^* \leq \underline{z} < \overline{Z} \leq 1 - m^*$
- *H is a Borel measure on R satisfying that the support is* $\{0\} \cup [\underline{z}, \overline{Z}]$*,* $\int dH = 1 - H(0)$*, and* $\int m dH = M$
- (A.10)*:* $b(1 - m^*) < d$
- (A.11)*:* $[(1-\theta)c + \theta]\beta F > (1-\theta)ck + \theta d$
- (A.12)*:* $[(1-\theta)c+\theta]\beta F < (1-\theta)ck + \theta d + (1-\theta)(2-c)a\beta m^* - [(1-\theta)c+\theta]b\beta(\overline{Z} - m^*)$

Proof. For simplicity, we denote $x(m) = x(0, m, H)$. Assuming $x(m) > 0$, we first calculate F. From the first order condition,

$$-d - cx(m) + \beta(v(m) - v(0)) = \left(\frac{(1-\theta)c}{\theta}\right)[k + x + \beta(v(0) - v(m))].$$

On the other hand,

$$\begin{aligned} v(0) &= \alpha \textstyle\int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta v(m)]dH + (1 - \alpha(1 - H(0)))\beta v(0) \\ &= \alpha \textstyle\int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta(v(m) - v(0))]dH + \beta v(0). \end{aligned}$$

Thus, from (5),

$$v(0) = \frac{1}{1-\beta}\alpha \int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta(v(m) - v(0))dH$$

$$= \left(\frac{(1-\theta)c}{\theta H(0)}\right) \int_{\underline{z}}^{\overline{Z}} [v(0) + F + bm]dH$$

$$= \left(\frac{(1-\theta)c}{\theta H(0)}\right) \{Mb + [1 - H(0)][v(0) + F]\}.$$

Thus

$$\Leftrightarrow \ [1-H(0)]F = \left(\frac{\theta H(0) - (1-\theta)c[1-H(0)]}{(1-\theta)c}\right) v(0) - Mb$$

holds. Define $\phi \equiv \frac{\theta H(0)}{(1-\theta)[1-H(0)]c}$. Then, from the above,

$$F = (\phi - 1)\, v(0) - \left(\frac{M}{1-H(0)}\right) b,$$

and thus

$$b = \frac{1-H(0)}{M}\left[(\phi-1)\, v(0) - F\right]. \tag{A.18}$$

For some $\varepsilon > 0$, (A.11) can be rewritten as

$$[(1-\theta)c + \theta]\beta F = (1-\theta)ck + \theta d + \varepsilon. \tag{A.19}$$

Substituting Equation (A.19) into (A.18) derives

$$b = \frac{1-H(0)}{M}\left[(\phi-1)\, v(0) - \frac{(1-\theta)ck + \theta d + \varepsilon}{[(1-\theta)c+\theta]\beta}\right]. \tag{A.20}$$

For a sufficiently small $\delta > 0$, let $m^* = \frac{1}{2} - \delta, \bar{Z} = \frac{1}{2} - \frac{\delta}{3}, \underline{z} = \frac{1}{2} - \frac{2\delta}{3}$. Then $\bar{Z} - m^* = \frac{2}{3}\delta$.

We can choose M satisfying

$$[1-H(0)]\underline{z} < M < [1-H(0)]\bar{Z}. \tag{A.21}$$

Then there exists a measure H on $[\underline{z}, \bar{Z}]$ satisfying $\int_{\underline{z}}^{\overline{Z}} dH = M$ and $\int_{\underline{z}}^{\overline{Z}} dH = 1 - H(0)$.

Next, we show that there exists $b > 0$ satisfying (A.12). Let

$$\chi \equiv \frac{\alpha\beta(1-\theta)[1-H(0)]}{\alpha\beta(1-\theta)[1-H(0)] + 1 - \beta}$$

Then the sum of the last two terms in the right hand side of (A.12) is denoted by LTT:

$$\begin{aligned} LTT &= (1-\theta)(2-c)\chi\beta m^* b - [(1-\theta)c+\theta]\beta(Z-m^*)b \\ &= (1-\theta)(2-c)\chi\beta\left(\tfrac{1}{2}-\delta\right)b - [(1-\theta)c+\theta]\beta\left(\tfrac{2}{3}\delta\right)b \\ &= (1-\theta)(2-c)\chi\beta\tfrac{b}{2} - \left[(1-\theta)(2-c)\chi\beta + [(1-\theta)c+\theta]\beta\left(\tfrac{2}{3}\right)\right]\delta b. \end{aligned}$$

Note that the conditions(A.11) and (A.12) are.

$$[(1-\theta)c+\theta]\beta F = (1-\theta)ck+\theta d+\varepsilon,$$

$$[(1-\theta)c+\theta]\beta F < (1-\theta)ck+\theta d+LTT$$

If, for some $\varepsilon > 0$ and $\gamma > 1$,

$$(1-\theta)(2-c)\chi\beta\frac{b}{2} = \gamma\varepsilon, \tag{A.22}$$

then both conditions hold for sufficiently small $\delta > 0$. Solving (A.22) for ε and substituting it to (A.20) yields

$$b = \frac{1-H(0)}{M}\left[(\phi-1)\,v(0) - \frac{(1-\theta)ck+\theta d+(1-\theta)(2-c)\chi\beta\left(\frac{b}{2\gamma}\right)}{[(1-\theta)c+\theta]\beta}\right].$$

Thus

$$\left[1+\left(\frac{1-H(0)}{M}\right)\left(\frac{(1-\theta)(2-c)\chi}{[(1-\theta)c+\theta]}\right)\left(\frac{1}{2\gamma}\right)\right]b$$

$$= \frac{1-H(0)}{M}\left[(\phi-1)\,v(0) - \frac{(1-\theta)ck+\theta d}{[(1-\theta)c+\theta]\beta}\right]$$

For $b > 0$, ϕ must be larger than one and

$$(\phi-1)v(0) > \frac{(1-\theta)ck+\theta d}{[(1-\theta)c+\theta]\beta}.$$

Finally, $b(1-m^*) < d$ holds for a γ close to one and a sufficiently small $\delta > 0$ if

$$\frac{\frac{1-H(0)}{M}\left[(\phi-1)\,v(0)-\frac{(1-\theta)ck+\theta d}{[(1-\theta)c+\theta]\beta}\right]}{\left[1+\left(\frac{1-H(0)}{M}\right)\left(\frac{(1-\theta)(2-c)\chi}{2[(1-\theta)c+\theta]}\right)\right]} < 2d.$$

□

Step 5: The existence of equilibria

Below, we show that the linear value function considered in Proposition 1 satisfies the Bellman equation (5) and (6), (Lemmas A.8 and A.9) and the money holdings distribution presented in the previous step is stationary. (Lemma A.7) That is, there exist equilibria. Note that there exist a continuum of pay-all equilibria, since $(\underline{z}, \overline{Z}, H, m^*, M, a, b, F)$ satisfying the conditions in Lemma A.6 is also a continuum.

In the following three Lemmas, we assume the assumptions in Proposition 1, and thus all the Lemmas in the previous Steps can be used.

Lemma A.7. *Under the assumptions in Proposition 1,* $(m^*, \underline{z}, \overline{Z}, H)$ *given in the previous step is stationary.*

Proof. In the distribution H, some agents do not have money, and the other agents have money in $[\underline{z}, \overline{Z}]$. From Lemma A.3, a buyer without money does not trade. Similarly, a seller with money in $[\underline{z}, \overline{Z}]$ does not trade. Thus, on an equilibrium path, a trade occurs only in the case that a seller does not have money and a buyer has money $m_b \in [\underline{z}, \overline{Z}]$. From Lemma A.4, the pay-all property holds and, after the trade, the seller has money $m_b \in [\underline{z}, \overline{Z}]$ and the buyer does not have money. Therefore, the money holdings distribution remains the same. □

Lemma A.8. *Under the assumptions in Proposition 1, the linear value function defined as Equation (13) is consistent with Equation (5).*

Proof. We substitute Equation (13) to the right-hand side of Equation (5) and then check it is actually $v(m)$. For convinience, Equation (5) is

$$v(m) = \alpha H(0)[k + x(m) + \beta v(0)] + [1 - \alpha H(0)]\beta v(m)$$

$$= \alpha H(0)\{k + x(m) + \beta[v(0) - v(m)]\} + \beta v(m)$$

Consider the buyer's surplus from the bargaining, $k + x(m) + \beta[v(0) - v(m)]$. Under pay-all equilibrium, the buyer pays m. By Equation (A.7), this buyer's surplus is

$$k + x(m) + \beta[v(0) - v(m)] = \frac{\theta}{c}\big\{\beta[v(m) - v(0)] - c\beta[v(m) - v(0)] + ck - d\big\}$$

$$= \frac{\theta}{c}\big[\beta(F + bm) + c\beta(F + bm) + ck - d\big]$$
$$= \frac{\theta}{c}\left[(1 - c)\beta(F + bm) + (ck - d)\right].$$

Then, the right-hand-side of Equation (5) is

$$\alpha H(0)\{k + x(m) + \beta[v(0) - v(m)]\} + \beta v(m)$$

$$= \alpha H(0)\left(\frac{\theta}{c}\right)\left[(1 - c)\beta(F + bm) + (ck - d)\right] + \beta v(m),$$

by Lemma 3,

$$= \alpha\left(\frac{(1 - \beta)c}{\alpha\beta\theta(1 - c)}\right)\frac{\theta}{c}\left[(1 - c)\beta(F + bm) + (ck - d)\right] + \beta v(m)$$

$$= (1 - \beta)\left(F + bm + \frac{ck - d}{\beta(1 - c)}\right) + \beta v(m),$$

by Lemma 5,

$$= (1 - \beta)\left[F + bm + v(0)\right] + \beta v(m),$$

$$= (1 - \beta)v(m) + \beta v(m),$$

$$= v(m),$$

which is Equation (5). □

Lemma A.9. *Under the assumptions in Proposition 1, the linear value function defined as Equation (13) is consistent with Equation (6).*

Proof. Before the derivation of Equation (13), we first consider the seller's surplus. By

Equation (A.6),

$$= -d - cx(m) + \beta[v(m) - v(0)]$$

$$= (1-\theta)\big\{\beta[v(m) - v(0)] - c\beta[v(m) - v(0)] + (ck - d)]\big\},$$

given Equation (13),

$$= (1-\theta)[(1-c)\beta(F + bm) + ck - d],$$

$$= (1-\theta)[(1-c)\beta(v(0) + F + bm) - (1-c)\beta v(0) + ck - d],$$

by Lemma 5,

$$= (1-\theta)[(1-c)\beta v(m) - (ck - d) + ck - d],$$

$$= (1-\theta)(1-c)\beta v(m). \tag{A.23}$$

Now we show the consistency. Note that Equation (6) is

$$v(0) = \alpha \int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta v(m)]dH + \{1 - \alpha[1 - H(0)]\}\beta v(0).$$

Here, we substitute Equation (A.23) to the right-hand side of Equation (6) and derive $v(0)$. The right-hand side is equivalent to

$$\alpha \int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta v(m)]dH + \beta v(0) - \int_{\underline{z}}^{\overline{Z}} \alpha\beta v(0)dH$$

$$= \alpha \int_{\underline{z}}^{\overline{Z}} \{-d - cx(m) + \beta[v(m) - v(0)]\}dH + \beta v(0).$$

By Equation (A.23), it is equivalent to

$$= \alpha \int_{\underline{z}}^{\overline{Z}} (1-\theta)(1-c)\beta v(m)dH + \beta v(0).$$

$$= \alpha(1-\theta)(1-c)\beta \int_{\underline{z}}^{\overline{Z}} v(m)dH + \beta v(0),$$

by Lemma 4,

$$= \alpha(1-\theta)(1-c)\beta \left[\frac{\theta H(0)v(0)}{(1-\theta)c}\right] + \beta v(0)$$

$$= \left[\alpha\left(\frac{1-c}{c}\right)\theta H(0) + 1\right]\beta v(0),$$

by Lemma 5,

$$= \left[\alpha\left(\frac{1-c}{c}\right)\theta\left(\frac{(1-\beta)c}{\alpha\beta\theta(1-c)}\right) + 1\right]\beta v(0)$$

$$= \left[\frac{1-\beta}{\beta} + 1\right]\beta v(0)$$

$$= v(0).$$

This is the left-hand side of Equation (6). □

Finally, $c = 0.3, d = 0.7, k = 40, \alpha = 0.8, \beta = 0.9.\theta = 0.9$, and $M = 0.5$ satisfy the conditions in the proposition.

The proof of Lemma 7

By the government budget constraint (23), (25) is rewritten as

$$v(0) = \alpha\int_{\underline{z}}^{\overline{Z}} \big(-d - cx(m) + \beta\{v(0) + F + bg[m - tx(m)]\}\big)dH + \{1-\alpha[1-H(0)]\}\beta v(0)$$

$$= \alpha\int_{\underline{z}}^{\overline{Z}} \big\{-d - cx(m) + \beta[v(0) + F + bgm]\big\}dH - \alpha\int_{\underline{z}}^{\overline{Z}} \beta bgtx(m)dH + \{1-\alpha[1-H(0)]\}\beta v(0)$$

$$= \alpha\int_{\underline{z}}^{\overline{Z}} \big\{-d - cx(m) + \beta[v(0) + F + bgm]\big\}dH - \alpha\beta b(g-1)M + \{1-\alpha[1-H(0)]\}\beta v(0)$$

$$= \alpha\int_{\underline{z}}^{\overline{Z}} \big\{-d - cx(m) + \beta[v(0) + F + bgm]\big\}dH - \alpha\beta b(g-1)\int_{\underline{z}}^{\overline{Z}} mdH + \{1-\alpha[1-H(0)]\}\beta v(0)$$

$$= \alpha\int_{\underline{z}}^{\overline{Z}} \big\{-d - cx(m) + \beta[v(0) + F + bm]\big\}dH + \{1-\alpha[1-H(0)]\}\beta v(0)$$

$$= \alpha \int_{\underline{z}}^{\overline{Z}} \left[-d - cx(m) + \beta v(m)\right] dH + \{1 - \alpha[1 - H(0)]\}\beta v(0)$$

The proof of Proposition 3

First, We will derive key equations.

Lemma A.10.

$$H(0) = \frac{(1-\beta)(c + (1-\theta)\beta g t v')}{\alpha\beta\theta(1-c)} \tag{A.24}$$

Proof. Shown below. □

The seller's population $H(0)$ depends on g and t, that is, the policy alters the distribution of money holdings. By changing the seller's marginal welfare, the policy affects the relative surplus in (24). Then, the associated distribution is also reshaped so as to keep the stationarity.

Lemma A.11.

$$\frac{\int_{\underline{z}}^{\overline{Z}} v(m) dH}{H(0)v(0)} = \frac{\theta}{(1-\theta)(c + \beta g t v')} \tag{A.25}$$

Proof. Shown below. □

This Lemma shows that the shift in bargaining surplus also affects the ratio of discounted values between the two groups. The numerator, $\int_{\underline{z}}^{\overline{Z}} v(m)dH$, and the denominator, $H(0)v(0)$, represents the aggregate discounted sum of utilities of buyers and sellers, repsectively. The long-run policy with $g > 1$ and $t > 0$ shifts welfare from buyers to sellers.

Lemma A.12.

$$v(0) = \frac{ck - d}{\beta(1-c)}. \tag{A.26}$$

Proof. Shown below. □

The proof of Lemma A.10

First, we derive the consumption elasticity of discounted value $\xi(m) = \dfrac{x'(m)}{v'(m)}$. By (24),

$$(c + (1-\theta)\beta g t v')x' = (\theta\beta + (1-\theta)c\beta + (1-\theta)\beta^2 g t v')v',$$

$$\Leftrightarrow\ \xi(m) = \frac{\beta\theta(1-c)}{c + (1-\theta)\beta g t v'} + \beta$$

Since the buyer's value function is unchanged, we can also derive $\xi(m)$ as in (7). Therefore,

$$\frac{\beta\theta(1-c)}{c + (1-\theta)\beta g t v'} + \beta = \xi(m) = \frac{1-\beta}{\alpha H(0)} + \beta,$$

which leads to the expression of $H(0)$.

The proof of Lemma A.11

From (5)

$$\left[\frac{(1-\beta)(1-\theta)c)}{\alpha H(0)}\right] v(m) = (1-\theta)c\{k + x(m) + \beta[v(0) - v(m)]\}$$

$$\Rightarrow\ \left[\frac{(1-\beta)(1-\theta c)}{\alpha H(0)}\right] \int_{\underline{z}}^{\overline{Z}} v(m) = \int_{\underline{z}}^{\overline{Z}} (1-\theta)c\{k + x(m) + \beta[v(0) - v(m)]\} \qquad \text{(A.27)}$$

By (6),

$$v(0) = \alpha \int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta v(m)]dH + \beta v(0) - \alpha \int_{\underline{z}}^{\overline{Z}} \beta v(0) dH$$

$$= \alpha \int_{\underline{z}}^{\overline{Z}} \{-d - cx(m) + \beta[v(m) - v(0])\} dH + \beta v(0)$$

$$\Leftrightarrow\ \left[\frac{(1-\beta)\theta}{\alpha}\right] v(0) = \theta \int_{\underline{z}}^{\overline{Z}} \{-d - cx(m) + \beta[v(m) - v(0])\} dH \qquad \text{(A.28)}$$

Integrating the FOC of Nash bargaining solution, (24), derives

$$\int_{\underline{z}}^{\overline{Z}} (1-\theta)(c+\beta g t v')[k+x(m)+\beta[v(0)-v(m))]dH$$
$$= \int_{\underline{z}}^{\overline{Z}} \theta\{-d-cx(m)+\beta[v(m)-v(0)]\}dH \tag{A.29}$$

Substitute Equation (A.27) and (A.28) into (A.29), then

$$\left[\frac{(1-\beta)(1-\theta)c}{\alpha H(0)} + \frac{\beta g t v'(1-\beta)(1-\theta)c}{c\alpha H(0)}\right]\int_{\underline{z}}^{\overline{Z}} v(m)dH = \left[\frac{(1-\beta)\theta}{\alpha}\right]v(0),$$

which leads to the Lemma .

The proof of Lemma A.12

By (5),

$$\left(\frac{1-\beta}{H(0)}\right)v(m) = \alpha\{k+x(m)+\beta[v(0)-v(m))]\}$$

$$\Leftrightarrow \left(\frac{(1-\beta)c}{H(0)}\right)v(m) = \alpha\{ck+cx(m)+\beta cv(0)-\beta cv(m)\}$$

$$\Leftrightarrow \alpha[ck+\beta cv(0)-\beta cv(m)] - \left(\frac{(1-\beta)c}{H(0)}\right)v(m) = \alpha[-cx(m)]$$

$$= \alpha[d-d-cx(m)+\beta v(m)-\beta v(m)]$$

$$= \alpha[-d-cx(m)+\beta v(m)] + \alpha[d-\beta v(m)]$$

Then,

$$\alpha[-d-cx(m)+\beta v(m)]$$
$$= \alpha ck + \alpha\beta cv(0) - \left(\frac{(1-\beta)c}{H(0)}\right)v(m) - \alpha d + \alpha\beta(1-c)v(m)]$$
$$= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right]v(m) + \alpha[ck-d+\beta cv(0)]. \tag{A.30}$$

By integrating both sides of (A.30), we get

$$\alpha \int_{\underline{z}}^{\overline{Z}} [-d - cx(m) + \beta v(m)]dH$$

$$= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right] \int_{\underline{z}}^{\overline{Z}} v(m)dH + \alpha[1-H(0)][ck - d + \beta cv(0)].$$

By (6),

$$\{1 - \beta + \alpha[1-H(0)]\beta\}v(0)$$

$$= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right] \int_{\underline{z}}^{\overline{Z}} v(m)dH + \alpha[1-H(0)][ck - d + \beta cv(0)].$$

$$\Leftrightarrow \{1 - \beta + \alpha[1-H(0)](1-c)\beta\}v(0)$$

$$= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right] \int_{\underline{z}}^{\overline{Z}} v(m)dH + \alpha[1-H(0)](ck - d).$$

Let

$$X = \left(1 + \frac{\beta G v'}{c}\right).$$

Then by Lemma A.11,

$$\{1 - \beta + \alpha[1-H(0)](1-c)\beta\}v(0)$$

$$= \left[\alpha\beta(1-c) - \frac{(1-\beta)c}{H(0)}\right] \left[\frac{\theta H(0)v(0)}{(1-\theta)cX}\right] + \alpha[1-H(0)](ck - d)$$

$$= [\alpha\beta(1-c)H(0) - (1-\beta)c] \left[\frac{\theta v(0)}{(1-\theta)cX}\right] + \alpha[1-H(0)](ck - d).$$

Then,

$$\Leftrightarrow \{1 - \beta + \alpha[1-H(0)](1-c)\beta\}(1-\theta)cXv(0)$$

$$= [\alpha\beta(1-c)H(0) - (1-\beta)c]\,\theta v(0) + \alpha[1-H(0)](1-\theta)cX(ck - d).$$

Solving the above with respect to $v(0)$, we obtain

$v(0) = Xc\alpha\left(\theta - 1\right)\frac{d-ck}{\theta(c(\beta-1)-H(0)\alpha\beta(c-1))+Xc(\theta-1)(-\beta+\alpha\beta(H(0)-1)(c-1)+1)}\left(H(0) - 1\right).$

Substituting $H(0) = \frac{(1-\beta)(c+(1-\theta)\beta Gv')}{\alpha\beta\theta(1-c)}$, the RHS of the above is equal to
$\frac{Xc(d-ck)(c-c\beta+Gv'\beta-\theta\alpha\beta-Gv'\beta^2-Gv'\theta\beta+c\theta\alpha\beta+Gv'\theta\beta^2)}{\beta(c-1)(c\theta+Xc^2-Xc\theta-c\theta\beta-Xc^2\beta+Gv'\theta\beta+Xc\theta\beta-Gv'\theta\beta^2+Xc^2\theta\alpha\beta+Gv'Xc\beta-Xc\theta\alpha\beta-Gv'Xc\beta^2-Gv'Xc\theta\beta+Gv'Xc\theta\beta^2)}$

Substituting $X = 1 + \frac{\beta Gv'}{c}$, into
$\frac{(c-c\beta+Gv'\beta-\theta\alpha\beta-Gv'\beta^2-Gv'\theta\beta+c\theta\alpha\beta+Gv'\theta\beta^2)}{(c\theta+Xc^2-Xc\theta-c\theta\beta-Xc^2\beta+Gv'\theta\beta+Xc\theta\beta-Gv'\theta\beta^2+Xc^2\theta\alpha\beta+Gv'Xc\beta-Xc\theta\alpha\beta-Gv'Xc\beta^2-Gv'Xc\theta\beta+Gv'Xc\theta\beta^2)}$
we obtain
$\frac{c-c\beta+Gv'\beta-\theta\alpha\beta-Gv'\beta^2-Gv'\theta\beta+c\theta\alpha\beta+Gv'\theta\beta^2}{(c+Gv'\beta)(c-c\beta+Gv'\beta-\theta\alpha\beta-Gv'\beta^2-Gv'\theta\beta+c\theta\alpha\beta+Gv'\theta\beta^2)} = \frac{1}{c+Gv'\beta}$.
Therefore,

$v(0) = \frac{Xc(ck-d)}{\beta(1-c)(c+Gv'\beta)}$.
Since $X = \left(1 + \frac{\beta Gv'}{c}\right)$,

$v(0) = \frac{(ck-d)}{\beta(1-c)}$.

The proof of Proposition 3

Define the total welfare as $W = H(0)v(0) + \int_{\underline{z}}^{\overline{Z}} v(m)dH$. Then, from Lemma A.11 and Lemma A.12, it can be rewritten as

$$W = QH(0)v(0), \tag{A.31}$$

where $Q = \left(1 + \frac{\theta}{(1-\theta)(c+\beta Gv')}\right)$.

Then,

$$\frac{\partial W}{\partial t} = \frac{\partial Q}{\partial t}H(0)v(0) + \frac{\partial H(0)}{\partial t}Qv(0) + \frac{\partial v(0)}{\partial t}QH(0). \tag{A.32}$$

From the definition of Q,

$$\frac{\partial Q}{\partial t} = \frac{-\theta\beta G'v'}{(1-\theta)(c+\beta Gv')^2}. \tag{A.33}$$

From Lemma A.10,

$$\frac{\partial H(0)}{\partial t} = \frac{(1-\beta)(1-\theta)\beta G'v'}{\alpha\beta\theta(1-c)}. \tag{A.34}$$

From (A.26),

$$\frac{\partial v(0)}{\partial t} = 0. \tag{A.35}$$

Below, we obtain $\frac{\partial W}{\partial t}$. From (A.33), (A.34), and (A.35),

$$\frac{\partial Q}{\partial t}H(0)v(0) + \frac{\partial H(0)}{\partial t}Qv(0) + \frac{\partial v(0)}{\partial t}QH(0) = \frac{\beta G'v'(1-\beta)(Gv'\beta(1-\theta)^2(2c+Gv'\beta)+c((c-1)\theta^2-2c\theta+c))}{\alpha\theta(1-c)(1-\theta)(c+GV\beta)^2} \tag{A.36}$$

This is positive, if and only if

$$Gv'\beta\,(1-\theta)^2\,(2c+Gv'\beta) + c((c-1)\,\theta^2 - 2c\theta + c) > 0. \tag{A.37}$$

Considering (A.37)$=0$ as an equation with respect to θ, it has two solutions $\frac{c+\beta Gv'}{c-\sqrt{c}+\beta Gv'}$ and $\frac{c+\beta Gv'}{c+\sqrt{c}+\beta Gv'}$. Suppose $G' > 0$. Then $\frac{\partial W}{\partial t} > 0$, if and only if $\frac{c+\beta Gv'}{c-\sqrt{c}+\beta Gv'} < \theta < \frac{c+\beta Gv'}{c+\sqrt{c}+\beta Gv'}$, since the coefficient of θ^2 is negative.