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Abstract: We first consider a closed model, where households' time discount depends on externality in consumption. We can prove that there is a unique steady state, which is a saddle point. Then, we extend the model to a two country world, and derive the condition about the effects of consumption externality under which there is a unique free trade steady state with saddle-point stability.

Key-words: time preference, consumption externality, two-country model, Heckscher–Ohlin

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1 Introduction

This paper presents a closed two-sector model, where households' time discount depends on externality in consumption. Then, we extend it to the dynamic Heckscher–Ohlin (H–O) model of international trade.

With a constant time discount rate, the dynamic H–O model yields a continuum of steady states under free trade, and initial capital stocks in each country affect the steady state values of capital stocks and the levels of welfare: Initially capital abundant country will be capital abundant in the steady state, and vice versa. As Baxter (1992) pointed out, one problematic property of the dynamic H–O model is that the long-run production/trade structure drastically changes, if there is a small difference in the interest rates across countries, which easily happens when the depreciation rate on capital or the capital tax rate in each country differs. Chen et al. (2008) resolve the problem by introducing endogenous time preference originated by Uzawa (1968). In their model, there is a unique steady state with saddle-point stability, and the steady state remains to exist as long as such differences are not large.

Quite a number of empirical studies find strong evidence that households are heterogeneous in terms of impatience, see for instance, Hausman (1979), Becker and Mulligan (1997), and Barsky et al. (1997). To ensure stability, many theoretical studies, including Chen et al. (2008), assumed increasing marginal impatience (IMI), which implies that the rate of time preference is increasing in wealth. However, empirical studies support the validity of decreasing marginal impatience (DMI): For example, Lawrance (1991) and Samwick (1998) find that households become more patient as their income goes up.

Das (2003), Hirose and Ikeda (2008), and Chang (2009) examined closed models with DMI and verified that the steady state can be a saddle point when the degree of DMI is sufficiently weak. However, DMI intrinsically yields unstable outcome when two or more heterogeneous households exist as in two-country models of international trade.

In this study, we assume households' time discount depends on the average level of consumption in their economy, which captures the idea that households' saving behavior is strongly affected by their social environment. Moreover, under free trade, we assume it also depends on the level in the other country, which makes our two-country model stable, even when households become patient as their average income rises like DMI. The latter assumption on consumption externality is crucial for the result on stability in our two-country model of trade. Indeed, we obtain substantially the same result under free trade, if households' time preference is endogenized respect to their own consumption but it is affected exogenously by the average level of consumption in the other country. However, the model becomes less tractable with endogenous time preference, because it needs some regularity conditions, see Uzawa (1968) and Epstein (1987).

There are some studies where the subjective discount rate is not constant and depends on social variables, see, for instance, Shi (1999), Schmitt-Grohe and Uribe (2003), and Meng (2006). To the best of our knowledge, this paper is the first attempt to introduce consumption externality in time preference rate into a two-country model of international trade. And our dynamic H–O model has a unique steady state with saddle-point stability under free trade, where both countries are incompletely specialized in production. Moreover, the condition for stability can be met, even when households become more patient as the average level of consumption in their economy rises.

Section 2 sets up the closed two-sector model with consumption externalities. In section 3, we extend the model to the two-country model of international trade, and derive the condition under which there exists a unique steady state with saddle-point stability. Section 4 concludes the paper.

2 The Two Sector Model with Consumption Externality

In this section, we formulate a two-sector model with consumption externality. Consumption externality in our model is analogous to the model in Meng (2006) in the sense that it does not affect households' instantaneous utility, but it does their discount rate. More specifically, we assume the discount rate will depend on the average level of consumption in their economy. This captures the idea that households' saving behavior is strongly affected by their social environment. There are two sectors, one of which produces a consumption good, say good 1, and the other produces a capital good, say good 2. Good 2 is numeraire. Both sectors use a fixed factor (labor, l) and a reproducible factor (capital, k).

2.1 The closed model

We assume that households' preferences are characterized by a concave felicity function u and a time discount function ρ. Moreover, we assume

Assumption 1: The felicity function is strictly increasing and concave: $u'(c) > 0 > u''(c)$ for any $c > 0$.

Assumption 2: The discount function is monotone: $\rho'(c) \geq 0$ for any $c > 0$ or $\rho'(c) \leq 0$ for any $c > 0$, and $\rho(0) < \infty$.

The representative household is assumed to maximize the discounted sum of its utilities

$$\max \int_0^\infty u(c)X dt,$$

subject to

$$\dot{k} = Rk + wl - pc - \delta k, \tag{1}$$

$$\dot{X} = -\rho(\bar{c})X, \tag{2}$$

where R, w, p, δ, and $\bar{c}$ denote the rental rate, the wage rate, the price of pure consumption good 1, the depreciation rate, and the average level of consumption in the economy, respectively. Thus, the discount rate in the model is not constant, but it does not depend on her own level of consumption.

The Hamiltonian associated with our optimization problem is[1]

$$\mathcal{H} = u(c)X + \lambda(Rk + wl - pc - \delta k) - \mu\rho(\bar{c})X,$$

where λ and μ are the co-state variables. The necessary conditions for optimality are

$$\frac{\partial \mathcal{H}}{\partial c} = u'(c)X - \lambda p = 0, \tag{3}$$

$$\frac{\partial \mathcal{H}}{\partial k} = \lambda(R - \delta) = -\dot{\lambda}, \tag{4}$$

$$\frac{\partial \mathcal{H}}{\partial X} = u(c) - \mu\rho(\bar{c}) = -\dot{\mu}. \tag{5}$$

Let

$$\nu \equiv \lambda / X.$$

Then (3) and (4) can be rewritten as

$$0 = u'(c) - \nu p, \tag{6}$$

$$\dot{\nu} = \nu[\rho(\bar{c}) + \delta - R]. \tag{7}$$

To simplify the following analysis, we assume

Assumption 3: Production technologies take the Cobb-Douglas form, and pure consumption good 1 is labor intensive.

Then, as long as both goods are produced, the rental on capital and the wage rate are given by the functions of p :

$$R = R(p) \text{ with } R'(p) < 0 \text{ for any } p > 0,$$
$$w = w(p) \text{ with } pw'(p)/w(p) > 1 \text{ for any } p > 0.$$

Also, the outputs of good 1 and good 2, y_1 and y_2, are given by

$$y_1 = y_1(p,k) \equiv R'(p)k + w'(p)l \text{ with } y_{1p}(p,k) \equiv \frac{\partial y_1(p,k)}{\partial p} > 0,$$
$$y_2 = y_2(p,k) \equiv R(p)k + w(p)l - p\left[R'(p)k + w'(p)l\right],$$

respectively.

[1] Since we assume ρ depends only on externality in consumption, we need no further assumption on the shapes of u and ρ to satisfy the concavity of the Hamiltonian in c.

Let $\kappa_i(p)$, $i = 1, 2$, denote the capital labor ratio in sector i when the rental rate and the wage rate are given by $r(p)$ and $w(p)$, respectively. Then, we have

$$y_1(p, \kappa_2(p)l) = 0 \text{ and } y_2(p, \kappa_1(p)l) = 0,$$

which yields, under Assumption 3,

$$0 < \kappa_1(p) = \frac{pw'(p) - w(p)}{R(p) - pR'(p)} < \kappa_2(p) = -\frac{w'(p)}{R'(p)}.$$

So, the economy produces both good 1 and good 2 when the price p and capital stock k satisfy $k/l \in (\kappa_1(p), \kappa_2(p))$.

Using the above, the dynamic general equilibrium system can be described as

$$\dot{k} = R(p)k + w(p)l - pc - \delta k, \tag{8}$$

$$\dot{\nu} = \nu\left[\rho(c) + \delta - R(p)\right], \tag{9}$$

$$\dot{\mu} = \mu\rho(c) - u(c), \tag{10}$$

$$0 = u'(c) - \nu p, \tag{11}$$

$$0 = R'(p)k + w'(p)l - c, \tag{12}$$

where (12) is the market clearing condition for good 1. The system determines one state variable, k, and four jump variables, ν, μ, c, p.

2.2 The steady state

We define the steady state of the closed model as when all variables are constant. Then the steady state is a solution to the following system of equations

$$0 = R(p)k + w(p)l - pc - \delta k, \tag{13}$$

$$0 = \nu\left[\rho(c) + \delta - R(p)\right], \tag{14}$$

$$0 = \mu\rho(c) - u(c), \tag{15}$$

$$0 = u'(c) - \nu p,$$

$$0 = R'(p)k + w'(p)l - c.$$

From (12), we have

$$k = \frac{c - w'(p)l}{R'(p)}. \tag{16}$$

Substituting (16) into (13) and rearranging it yields

$$c = \phi(p) \equiv \frac{w(p)}{p}\left[1 - \frac{R(p) - \delta}{pR'(p)} \cdot \frac{pw'(p)}{w(p)}\right]\left[1 - \frac{R(p) - \delta}{pR'(p)}\right]^{-1} l. \tag{17}$$

Also, from (14) we see that

$$p = \psi(c) \equiv R^{-1}\left(\rho(c) + \delta\right), \tag{18}$$

where R^{-1} denotes the inverse function of R, since (11) implies that $\nu \neq 0$ at any steady state.

Then, we can conclude that if $p = \psi(\phi(p))$ has a solution, which is a steady state price $\tilde{p}$, and the steady state values of k, c, ν, and μ are given by[2]

$$\begin{aligned}
\tilde{k} &= \frac{\phi(\tilde{p}) - w'(\tilde{p})l}{R'(\tilde{p})}, \\
\tilde{c} &= \phi(\tilde{p}), \\
\tilde{\nu} &= \frac{u'(\phi(\tilde{p}))}{\tilde{p}}, \\
\tilde{\mu} &= \frac{u(\phi(\tilde{p}))}{\rho(\phi(\tilde{p}))}.
\end{aligned}$$

Concerning the existence and uniqueness of the steady state, we can obtain the lemma below.[3]

Lemma 1 *(i) There exists a steady state; (ii) It is unique, if for every $c \geq 0$, $\rho'(c) > -\varepsilon$ holds with a sufficiently small $\varepsilon > 0$; (iii) At the steady state,*

$$\rho\rho' y_{1p} + R'(\tilde{p}R' - \rho) > 0 \tag{19}$$

holds.

Proof. *See the Appendix.* ■

We evaluate the elements of a Jacobian for the dynamic system, (8)–(12), to study the local dynamics around the steady state.

Differentiating the system gives the Jacobian

$$\tilde{j} = \begin{bmatrix} \rho & 0 & 0 & -\tilde{p} & 0 \\ 0 & 0 & 0 & \tilde{\nu}\rho' & -\tilde{\nu}R' \\ 0 & 0 & \rho & \tilde{\mu}\rho' - u' & 0 \\ 0 & -\tilde{p} & 0 & u'' & -\tilde{\nu} \\ R' & 0 & 0 & -1 & y_{1p} \end{bmatrix}$$

and the characteristic equation $j(x) \equiv \det\left[xI - \tilde{j}\right]$.

[2] It can be easily verified that $\tilde{k}/l \in (\kappa_1(\tilde{p}), \kappa_2(\tilde{p}))$.

[3] In the case of $\rho' \geq 0$, the uniqueness of the steady state always holds as in two-sector models with constant time discount or endogenous time preferences.

Then, we can easily find that the steady state is a saddle point if (19) holds, because

$$\begin{aligned} j(x) &= \begin{vmatrix} \rho - x & 0 & 0 & -\tilde{p} & 0 \\ 0 & -x & 0 & \tilde{\nu}\rho' & -\tilde{\nu}R' \\ 0 & 0 & \rho - x & \tilde{\mu}\rho' - u' & 0 \\ 0 & -\tilde{p} & 0 & u'' & -\tilde{\nu} \\ R' & 0 & 0 & -1 & y_{1p} \end{vmatrix} \\ &= (x - \rho)m(x), \end{aligned}$$

where

$$m(x) \equiv (\tilde{\nu} - u'' y_{1p})x^2 - [\rho(\tilde{\nu} - u'' y_{1p}) - \tilde{p} y_{1p} \tilde{\nu}\rho']\, x - \tilde{\nu}\tilde{p}\,[\rho\rho' y_{1p} + R'(\tilde{p}R' - \rho)]\,,$$

and $m(x) = 0$ has one negative root if and only if (19) holds.[4]

Therefore, we obtain

Proposition 1 *There exists an $\varepsilon > 0$ such that with $\rho'(c) > -\varepsilon$ for every $c \geq 0$, the steady state is unique and a saddle point.*

3 The Dynamic Two Country Heckscher–Ohlin Model with Consumption Externality

In this section, we formulate a dynamic H–O model with consumption externality. By dynamic H–O model, we mean that each country has access to the same technology for producing two goods. Factors of production are assumed to be mobile between sectors within a country, but immobile internationally, and there are no markets for international borrowing and lending. We refer to the representative country as the home country, and the corresponding behavioral relations for the other (foreign) country will be denoted by an asterisk (*). We will show that as long as the differences between the home and foreign countries are not large, there exists a free trade steady state where both countries are incompletely specialized in production. And we will derive the conditions under which the steady state is unique and a saddle point. In the following, we assume the home and foreign countries are identical.[5]

3.1 The free trade model

We assume that households in the home and foreign countries have identical preferences, and that the discount function, $\hat{\rho}$, depends not only on the average level of consumption in their own country,

[4] Notice that the quadratic coefficient of $m(x)$ is positive due to $u'' < 0 < y_{1p}$.

[5] Notice that the results will hold when the differences between two countries are not large, because the Jacobian determinant for the dynamic system of the model under free trade is not zero at the steady state, which will be proved in the Appendix on the stability of the steady state.

but also on the level in the other country, that is, $\hat{\rho} : (\bar{c}, \bar{c}^*) \in \mathbb{R}^2_+ \to \mathbb{R}_{++}$. Moreover, we assume

Assumption 4: The discount function is monotone in the sense that $d\hat{\rho}(c, c)/dc \geq 0$ for any $c > 0$ or $d\hat{\rho}(c, c)/dc \leq 0$ for any $c > 0$, and $\hat{\rho}(0, 0) < \infty$.

Assumption 5: The discount function satisfies $\hat{\rho}_1(c, c^*) \neq \hat{\rho}_2(c, c^*)$ if $c = c^*$, where $\hat{\rho}_1(c, c^*) \equiv \partial\hat{\rho}(c, c^*)/\partial c$ and $\hat{\rho}_2(c, c^*) \equiv \partial\hat{\rho}(c, c^*)/\partial c^*$ for any $(c, c^*) \in \mathbb{R}^2_+$.

Under free trade environment, the representative household in the home country is assumed to maximize the discounted sum of its utilities

$$\max \int_0^\infty u(c) X dt,$$

subject to

$$\dot{k} = Rk + wl - pc - \delta k, \tag{20}$$

$$\dot{X} = -\hat{\rho}(\bar{c}, \bar{c}^*) X. \tag{21}$$

Then, the necessary conditions for optimality are similar to those in the closed model as follows.

$$0 = u'(c) - \nu p, \tag{22}$$

$$\dot{\nu} = \nu[\hat{\rho}(\bar{c}, \bar{c}^*) + \delta - R], \tag{23}$$

$$\dot{\mu} = \mu\hat{\rho}(\bar{c}, \bar{c}^*) - u(c). \tag{24}$$

Also, we obtain the necessary conditions for optimality of foreign households with the budget constraint,

$$\dot{k}^* = R^* k^* + w^* l - pc^* - \delta k^*, \tag{25}$$

as follows.

$$0 = u'(c^*) - \nu^* p, \tag{26}$$

$$\dot{\nu}^* = \nu^*[\hat{\rho}(\bar{c}^*, \bar{c}) + \delta - R^*], \tag{27}$$

$$\dot{\mu}^* = \mu^* \hat{\rho}(\bar{c}^*, \bar{c}) - u(c^*). \tag{28}$$

As long as both countries are incompletely specialized in production, the outputs of good 1 in the home and foreign country are given by

$$y_1 = y_1(p, k) \text{ and } y_1^* = y_1(p, k^*).$$

3.2 The free trade steady state

Using the above, our dynamic general equilibrium system can be described as

$$\dot{k} = R(p)k + w(p)l - pc - \delta k, \tag{29}$$

$$\dot{k}^* = R(p)k^* + w(p)l - pc^* - \delta k^*, \tag{30}$$

$$\dot{\nu} = \nu \left[\hat{\rho}(c, c^*) + \delta - R(p)\right], \tag{31}$$

$$\dot{\nu}^* = \nu^* \left[\hat{\rho}(c^*, c) + \delta - R(p)\right], \tag{32}$$

$$\dot{\mu} = \mu\hat{\rho}(c, c^*) - u(c), \tag{33}$$

$$\dot{\mu}^* = \mu^*\hat{\rho}(c^*, c) - u(c^*), \tag{34}$$

$$0 = u'(c) - \nu p, \tag{35}$$

$$0 = u'(c^*) - \nu^* p, \tag{36}$$

$$0 = R'(p)(k + k^*) + 2w'(p)l - (c + c^*), \tag{37}$$

where (37) is the world market clearing condition for good 1. The system determines two state variables, k and k^*, and seven jump variables, $\nu, \nu^*, \mu, \mu^*, c, c^*, p$.[6]

We define the steady state of the model as when all variables are constant. Then the free trade steady state is a solution to the following system of equations

$$0 = R(p)k + w(p)l - pc - \delta k, \tag{38}$$

$$0 = R(p)k^* + w(p)l - pc^* - \delta k^*, \tag{39}$$

$$0 = \nu \left[\hat{\rho}(c, c^*) + \delta - R(p)\right], \tag{40}$$

$$0 = \nu^* \left[\hat{\rho}(c^*, c) + \delta - R(p)\right], \tag{41}$$

$$0 = \mu\hat{\rho}(c, c^*) - u(c), \tag{42}$$

$$0 = \mu^*\hat{\rho}(c^*, c) - u(c^*), \tag{43}$$

$$0 = u'(c) - \nu p,$$

$$0 = u'(c^*) - \nu^* p,$$

$$0 = R'(p)(k + k^*) + 2w'(p)l - (c + c^*).$$

From (37)–(39), we have

$$\frac{c + c^*}{2} = \phi(p)$$

So, from (40) and (41), we see that there may exist an asymmetric steady state, if the following conditions are met: For some $p > 0$, there is a value of c that satisfies

$$\hat{\rho}(c, 2\phi(p) - c) = \hat{\rho}(2\phi(p) - c, c) = R(p) - \delta \quad \text{with } c \neq \phi(p).$$

[6] We focus here the steady state where both countries are incompletely specialized in production. Without further assumption on the discount function, there may exist a steady state where at least one country is completely specialized. One sufficient condition to exclude such a steady state is $c \gtreqless c^* \Leftrightarrow \hat{\rho}(c, c^*) \gtreqless \hat{\rho}(c^*, c)$.

This is an extremely rare case, so that we focus on the symmetric steady state.

So, we can set $c = c^*$ at the free trade steady state. Then, $\dot{\nu} = \dot{\nu}^* = 0$ holds, if

$$p = \hat{\psi}(c) \equiv R^{-1}\left(\hat{\rho}(c,c) + \delta\right).$$

Therefore, we can conclude that if $p = \hat{\psi}(\phi(p))$ has a solution, which is a steady state price $\tilde{p}^T$ and the other steady state values are given by

$$\begin{aligned}
\tilde{k}^T &= \tilde{k}^{*T} = \frac{\phi(\tilde{p}^T) - w'(\tilde{p}^T)l}{R'(\tilde{p}^T)}, \\
\tilde{c}^T &= \tilde{c}^{*T} = \phi(\tilde{p}^T), \\
\tilde{\nu}^T &= \tilde{\nu}^{*T} = \frac{u'(\phi(\tilde{p}^T))}{\tilde{p}^T}, \\
\tilde{\mu}^T &= \tilde{\mu}^{*T} = \frac{u(\phi(\tilde{p}^T))}{\rho(\phi(\tilde{p}^T))}.
\end{aligned}$$

To simplify the notation below, we assume $\rho(c) = \hat{\rho}(c,c)$ for any $c > 0$, which implies that $\tilde{p}^T = \tilde{p}$, and hence all of the steady-state values above are the same as in the closed model.

Then, we can obtain the lemma below.[7]

Lemma 2 *(i) There exists a symmetric steady state under free trade; (ii) It is unique, if for every* $c \geq 0$, $\hat{\rho}_1(c,c) + \hat{\rho}_2(c,c) > -\varepsilon$ *holds with a sufficiently small* $\varepsilon > 0$; *(iii) At the steady state,*

$$\hat{\rho}(\hat{\rho}_1 + \hat{\rho}_2)y_{1p} + R'(\tilde{p}R' - \hat{\rho}) > 0 \tag{44}$$

holds.

Proof. *See the Appendix.* ■

We evaluate the elements of a Jacobian for the dynamic system, (29)–(37), to study the local dynamics around the free trade steady state. Differentiating the system gives the Jacobian $\tilde{J}$ for the dynamic system and the characteristic equation $J(x) \equiv \det\left[xI - \tilde{J}\right]$. Then, we have

Lemma 3 *At the symmetric steady state, we have*

$$J(x) = -2(x - \hat{\rho})^3\left[u''x - \tilde{\nu}\tilde{p}(\hat{\rho}_1 - \hat{\rho}_2)\right]M(x),$$

where

$$M(x) \equiv (\tilde{\nu} - u''y_{1p})x^2 - \left[\hat{\rho}(\tilde{\nu} - u''y_{1p}) - \tilde{\nu}\tilde{p}(\hat{\rho}_1 + \hat{\rho}_2)y_{1p}\right]x - \tilde{\nu}\tilde{p}\left[\hat{\rho}(\hat{\rho}_1 + \hat{\rho}_2)y_{1p} + R'(\tilde{p}R' - \hat{\rho})\right]$$

Proof. *See the Appendix.* ■

[7] In the case of $d\hat{\rho}(c,c)/dc \geq 0$ for $^\forall c > 0$, the uniqueness of the steady state always holds as in the closed model in section 2.

Based on the lemmas above, we obtain the main result of the paper as follows.

Proposition 2 *There exists an $\varepsilon > 0$ such that with $\hat{\rho}_1(c, c) + \hat{\rho}_2(c, c) > -\varepsilon$ for every $c \geq 0$, the symmetric steady state under free trade is unique. And it will be a saddle point if $\hat{\rho}_1 > \hat{\rho}_2$ holds at the steady state, but it will be unstable otherwise.*

Proof. With (44), two roots of $M(x) = 0$, say x_1 and x_2, satisfies $x_1 < 0 < x_2$. So, $J(x) = 0$ has two negative roots and four positive roots when $\hat{\rho}_1 > \hat{\rho}_2$ holds, but it has only one negative root otherwise. ■

Thus, for the stability of the steady state, the sign of $\hat{\rho}_1$ and/or $\hat{\rho}_2$, which denote the effects of the average levels of consumption in households' own country and the other country on their discount rate, does not matter, but the relative magnitude of $\hat{\rho}_1$ and $\hat{\rho}_2$ matters.

Since all of the roots of $J(x) = 0$ are not zero implies that the Jacobian determinant is not zero at the symmetric steady state, the result will hold as long as the differences between two countries are not large. So, with such differences, we can examine an asymmetric steady state, where both countries are incompletely specialized and trade of two goods occurs, although there is no trade at the symmetric steady state above.

4 Concluding remarks

We have examined a closed model, where households' time discount depends on externality in consumption. We have proven that there is a unique steady state, which is a saddle point. Then, we have extended the model to a two country world, and have derived the condition about the effects of consumption externality under which there is a unique free trade steady state with saddle-point stability. We have shown that for the stability of the steady state under free trade, the effects of the average levels of consumption in households' own country and the other country on their discount rate, does not matter, but their relative magnitude matters.

5 Appendix

5.1 Proofs of Lemma 1 and Lemma 2

Let us define p_0 and p_∞ as follows.

$$p_0 \equiv R^{-1}(\rho(0) + \delta),$$
$$p_\infty \equiv R^{-1}(\lim_{c\to\infty} \rho(c) + \delta),$$

each of which uniquely exists under the Cobb-Douglas technologies. By definition,

$$p_0 = \psi(0) \text{ and } p_\infty = \lim_{c\to\infty} \psi(c). \tag{45}$$

Since

$$\psi' = \frac{\rho'}{R'},$$

ψ changes monotonically from p_0 to p_∞ as c varies from zero to infinity.[8]

Substituting (16) into (13) and rearranging it yields

$$0 = (R - \delta - pR')c + [R'w - (R - \delta)w']\, l.$$

Totally differentiating it, we have

$$\begin{aligned} 0 &= (R - \delta - pR')dc + [R''(wl - pc) - (R - \delta)w''l]\, dp \\ &= (R - \delta - pR')dc + [-R''(R - \delta)k - (R - \delta)w''l]\, dp \\ &= (R - \delta - pR')dc - (R - \delta)y_{1p}dp, \end{aligned}$$

which yields

$$\phi' = \frac{(R - \delta)y_{1p}}{(R - \delta - pR')} > 0 \quad \text{for } p \in (p_{\min}, p_{\max}),$$

where

$$p_{\min} \equiv \min\{p_0, p_\infty\} \quad \text{and} \quad p_{\max} \equiv \max\{p_0, p_\infty\}.$$

In the case of $\rho' \geq 0$ except the case where $\rho'(c) = 0$ for any $c > 0$, we obtain

$$\psi' \leq 0$$

and

$$0 \leq p_\infty < p_0 < \infty.$$

Since $\phi' > 0$, we have

$$0 \leq \phi(p_\infty) < \phi(p_0) < \infty.$$

Therefore, the intersection between $c = \phi(p)$ and $p = \psi(c)$ in (p, c) space, which turns into $(\tilde{p}, \tilde{c})$, uniquely exists, and at the intersection, we see

$$\frac{1}{\phi'(\tilde{p})} > \psi'(\tilde{c}) \Leftrightarrow \rho\rho' y_{1p} + R'(\tilde{p}R' - \rho) > 0.$$

In the case of $\rho' \leq 0$ except the case where $\rho'(c) = 0$ for any $c > 0$, we have

$$\psi' \geq 0$$

and

$$0 < p_0 < p_\infty < \infty,$$

[8] In the case where $\rho'(c) = 0$ for any $c > 0$, $p_0 = p_\infty$ holds, and hence the graph of $p = \psi(c)$ becomes a vertical line in (p, c) space.

where $p_0 > 0$ comes from $\rho(0) < \infty$ in Assumption 2. Also, we have

$$0 < \phi(p_0) < \phi(p_\infty) < \infty. \tag{46}$$

Conditions (45) and (46) together imply that at least one intersection between $c = \phi(p)$ and $p = \psi(c)$ in (p, c) space exists, and

$$\frac{1}{\phi'(\tilde{p})} > \psi'(\tilde{c}) \Leftrightarrow \rho\rho' y_{1p} + R'(pR' - \rho) > 0$$

holds at the intersection. So, if $|\rho'|$ is not so large over $p \in (p_0, p_\infty)$, we can conclude that there is no other intersection.

Finally, in the case where $\rho'(c) = 0$ for $^\forall c > 0$, the graph of $p = \psi(c)$ becomes a vertical line in (p, c) space, and hence it intersects with $c = \phi(p)$ only at once, and (19) necessarily holds.

Based on the above, we see that the steady state is unique and (19) holds in all cases above, which proves Lemma 1.

Lemma 2 can be proved similarly, because the slope of $\hat{\psi}$ is given by

$$\psi' = \frac{\hat{\rho}_1 + \hat{\rho}_2}{R'}.$$

5.2 Proof of Lemma 3

We have the Jacobian,

$$\tilde{J} = \begin{bmatrix} \hat{\rho} & 0 & 0 & 0 & 0 & 0 & -\tilde{p} & 0 & 0 \\ 0 & \hat{\rho} & 0 & 0 & 0 & 0 & 0 & -\tilde{p} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\nu}\hat{\rho}_1 & \tilde{\nu}\hat{\rho}_2 & -\tilde{\nu}R' \\ 0 & 0 & 0 & 0 & 0 & 0 & \tilde{\nu}\hat{\rho}_2 & \tilde{\nu}\hat{\rho}_1 & -\tilde{\nu}R' \\ 0 & 0 & 0 & 0 & \hat{\rho} & 0 & \tilde{\mu}\hat{\rho}_1 - u' & \tilde{\mu}\hat{\rho}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{\rho} & \tilde{\mu}\hat{\rho}_2 & \hat{\mu}\hat{\rho}_1 - u' & 0 \\ 0 & 0 & -\tilde{p} & 0 & 0 & 0 & u'' & 0 & -\tilde{\nu} \\ 0 & 0 & 0 & -\tilde{p} & 0 & 0 & 0 & u'' & -\tilde{\nu} \\ R' & R' & 0 & 0 & 0 & 0 & -1 & -1 & 2y_{1p} \end{bmatrix},$$

and the characteristic equation,

$$J(x) = \begin{vmatrix} \hat{\rho}-x & 0 & 0 & 0 & 0 & 0 & -\tilde{p} & 0 & 0 \\ 0 & \hat{\rho}-x & 0 & 0 & 0 & 0 & 0 & -\tilde{p} & 0 \\ 0 & 0 & -x & 0 & 0 & 0 & \tilde{\nu}\hat{\rho}_1 & \tilde{\nu}\hat{\rho}_2 & -\tilde{\nu}R' \\ 0 & 0 & 0 & -x & 0 & 0 & \tilde{\nu}\hat{\rho}_2 & \tilde{\nu}\hat{\rho}_1 & -\tilde{\nu}R' \\ 0 & 0 & 0 & 0 & \hat{\rho}-x & 0 & \tilde{\mu}\hat{\rho}_1-u' & \tilde{\mu}\hat{\rho}_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \hat{\rho}-x & \tilde{\mu}\hat{\rho}_2 & \hat{\mu}\hat{\rho}_1-u' & 0 \\ 0 & 0 & -\tilde{p} & 0 & 0 & 0 & u'' & 0 & -\tilde{\nu} \\ 0 & 0 & 0 & -\tilde{p} & 0 & 0 & 0 & u'' & -\tilde{\nu} \\ R' & R' & 0 & 0 & 0 & 0 & -1 & -1 & 2y_{1p} \end{vmatrix}.$$

Therefore,

$$J(x) = (x-\hat{\rho})^2 \begin{vmatrix} \hat{\rho}-x & 0 & 0 & 0 & -\tilde{p} & 0 & 0 \\ 0 & \hat{\rho}-x & 0 & 0 & 0 & -\tilde{p} & 0 \\ 0 & 0 & -x & 0 & \tilde{\nu}\hat{\rho}_1 & \tilde{\nu}\hat{\rho}_2 & -\tilde{\nu}R' \\ 0 & 0 & 0 & -x & \tilde{\nu}\hat{\rho}_2 & \tilde{\nu}\hat{\rho}_1 & -\tilde{\nu}R' \\ 0 & 0 & -\tilde{p} & 0 & u'' & 0 & -\tilde{\nu} \\ 0 & 0 & 0 & -\tilde{p} & 0 & u'' & -\tilde{\nu} \\ R' & R' & 0 & 0 & -1 & -1 & 2y_{1p} \end{vmatrix}$$

$$= -(x-\hat{\rho})^3 \begin{vmatrix} \hat{\rho}-x & 0 & 0 & -\tilde{p} & -\tilde{p} & 0 \\ 0 & -x & 0 & \tilde{\nu}\hat{\rho}_1 & \tilde{\nu}\hat{\rho}_2 & -\tilde{\nu}R' \\ 0 & 0 & -x & \tilde{\nu}\hat{\rho}_2 & \tilde{\nu}\hat{\rho}_1 & -\tilde{\nu}R' \\ 0 & -\tilde{p} & 0 & u'' & 0 & -\tilde{\nu} \\ 0 & 0 & -\tilde{p} & 0 & u'' & -\tilde{\nu} \\ R' & 0 & 0 & -1 & -1 & 2y_{1p} \end{vmatrix}$$

$$= -(x-\hat{\rho})^3 \begin{vmatrix} \hat{\rho}-x & 0 & 0 & -\tilde{p} & -\tilde{p} & 0 \\ 0 & -x & 0 & \tilde{\nu}\hat{\rho}_1 & \tilde{\nu}\hat{\rho}_2 & -\tilde{\nu}R' \\ 0 & 0 & -x & \tilde{\nu}\hat{\rho}_2 & \tilde{\nu}\hat{\rho}_1 & -\tilde{\nu}R' \\ 0 & 0 & 0 & u''-\frac{\tilde{p}}{x}\tilde{\nu}\hat{\rho}_1 & -\frac{\tilde{p}}{x}\tilde{\nu}\hat{\rho}_2 & -\tilde{\nu}+\frac{\tilde{p}}{x}\tilde{\nu}R' \\ 0 & 0 & 0 & -\frac{\tilde{p}}{x}\tilde{\nu}\hat{\rho}_2 & u''-\frac{\tilde{p}}{x}\tilde{\nu}\hat{\rho}_1 & -\tilde{\nu}+\frac{\tilde{p}}{x}\tilde{\nu}R' \\ 0 & 0 & 0 & -1+\frac{\tilde{p}R'}{\hat{\rho}-x} & -1+\frac{\tilde{p}R'}{\hat{\rho}-x} & 2y_{1p} \end{vmatrix}.$$

Then, we see

$$
\begin{aligned}
&\begin{vmatrix}
\hat\rho - x & 0 & 0 & -\tilde p & -\tilde p & 0 \\
0 & -x & 0 & \tilde\nu\hat\rho_1 & \tilde\nu\hat\rho_2 & -\tilde\nu R' \\
0 & 0 & -x & \tilde\nu\hat\rho_2 & \tilde\nu\hat\rho_1 & -\tilde\nu R' \\
0 & 0 & 0 & u'' - \frac{\tilde p}{x}\tilde\nu\hat\rho_1 & -\frac{\tilde p}{x}\tilde\nu\hat\rho_2 & -\tilde\nu + \frac{\tilde p}{x}\tilde\nu R' \\
0 & 0 & 0 & -\frac{\tilde p}{x}\tilde\nu\hat\rho_2 & u'' - \frac{\tilde p}{x}\tilde\nu\hat\rho_1 & -\tilde\nu + \frac{\tilde p}{x}\tilde\nu R' \\
0 & 0 & 0 & -1 + \frac{\tilde p R'}{\hat\rho - x} & -1 + \frac{\tilde p R'}{\hat\rho - x} & 2y_{1p}
\end{vmatrix} \\
&= \begin{vmatrix}
u''x - \tilde\nu\tilde p\hat\rho_1 & -\tilde\nu\tilde p\hat\rho_2 & -\tilde\nu x + \tilde\nu\tilde p R' \\
-\tilde\nu\tilde p\hat\rho_2 & u''x - \tilde\nu\tilde p\hat\rho_1 & -\tilde\nu x + \tilde\nu\tilde p R' \\
(x-\hat\rho) + \tilde p R' & (x-\hat\rho) + \tilde p R' & -2y_{1p}(x-\hat\rho)
\end{vmatrix} \\
&= \begin{vmatrix}
u''x - \tilde\nu\tilde p(\hat\rho_1 - \hat\rho_2) & -\tilde\nu\tilde p\hat\rho_2 & -\tilde\nu x + \tilde\nu\tilde p R' \\
-u''x + \tilde\nu\tilde p(\hat\rho_1 - \hat\rho_2) & u''x - \tilde\nu\tilde p\hat\rho_1 & -\tilde\nu x + \tilde\nu\tilde p R' \\
0 & (x-\hat\rho) + \tilde p R' & -2y_{1p}(x-\hat\rho)
\end{vmatrix} \\
&= \begin{vmatrix}
u''x - \tilde\nu\tilde p(\hat\rho_1 - \hat\rho_2) & -\tilde\nu\tilde p\hat\rho_2 & -\tilde\nu x + \tilde\nu\tilde p R' \\
0 & u''x - \tilde\nu\tilde p(\hat\rho_1 + \hat\rho_2) & -2\tilde\nu x + 2\tilde\nu\tilde p R' \\
0 & (x-\hat\rho) + \tilde p R' & -2y_{1p}(x-\hat\rho)
\end{vmatrix} \\
&= 2\left[u''x - \tilde\nu\tilde p(\hat\rho_1 - \hat\rho_2)\right]\begin{vmatrix}
u''x - \tilde\nu\tilde p(\hat\rho_1 + \hat\rho_2) & -\tilde\nu x + \tilde\nu\tilde p R' \\
(x-\hat\rho) + \tilde p R' & -y_{1p}(x-\hat\rho)
\end{vmatrix} \\
&= 2\left[u''x - \tilde\nu\tilde p(\hat\rho_1 - \hat\rho_2)\right] M(x).
\end{aligned}
$$

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