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On the Real Determinacy and Indeterminacy of Stationary Equilibria in Monetary Models

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Abstract

It is known that stationary equilibria are indeterminate in some monetary models, especially in money search models with divisible money. However, most of the indeterminacy results are limited to the case that money holdings distributions have finite supports. In the case of infinite supports, both determinacy and indeterminacy results are known. In this paper, using the Borsuk-Ulam theorem in Banach Space, I investigate what determines the differences.

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1 Introduction

It is known that real indeterminacy of stationary equilibria arises in some monetary models, especially in money search models with divisible money. (See, for example, Green and Zhou (1998, 2002), Kamiya and Shimizu (2006), Matsui and Shimizu (2005), and Zhou (1999).) However, most of the indeterminacy results are limited to the case that the supports of equilibrium money holdings distributions are finite sets. The exceptions are Green and Zhou (1998) and Kamiya and Shimizu (2011). In the former model, the money holdings distribution has a support $\{0, p, 2p, \ldots\}$ for some $p > 0$, and in the latter model, it is a non-discrete money holdings distribution. On the other hand, Kamiya and Shimizu (2013) suggest that a dynamic all-pay auction model with fiat money could have a determinate equilibrium, where the support is an interval. The purpose of this paper is to investigate what determines the differences.

In the case that a money holdings distribution has a finite support $\{0, p, \ldots, Np\}$ for some $p > 0$ and some integer $N > 0$, Kamiya and Shimizu (2006) show that the condition for stationarity of money holdings distribution always has a hidden identity (conservation law) in money search models. Therefore, the number of equations in stationary condition is one less than the number of variables due to the identity, and, applying the implicit function theorem, there is a continuum of stationary equilibria. [1] More precisely, the variables are $h_0, h_1, \ldots,$ and h_N, and strategies, where h_n is a measure of agents with np amount of money. On the other hand, the conditions for stationary equilibria are $f_0^O - f_0^I = 0, f_1^O - f_1^I = 0, \ldots,$ and $f_N^O - f_N^I = 0$, and the Nash equilibrium condition, where f_n^O and f_n^I are the outflow of agents from np and the inflow of agents to np, respectively. The number of equations in Nash equilibrium condition is typically equal to the number of variables in strategies. Thus, from the identity, the number of equations is one less than the number of variables. That is, the dimension of the range is one less than that of the domain. Thus under some regularity condition the implicit function theorem can be applied, and the set of stationary equilibria is one-dimensional.

In the case that a support $B \subset R_+$ has an infinite number of elements, the variable is a distribution on B and is an element in an infinite dimensional space. On the other hand, the condition for stationary distribution cannot be expressed by a finite number of equations. Thus both the numbers of variables and equations cannot be counted. In other words, the dimensions of the domain and range cannot

[1] Lagos and Wright (2005) present a model with both Walrasian markets and search markets, and there is a unique stationary equilibrium. In their model, the identity does not hold due to the Walrasian Market.

be directly comparable. In this paper, using the Borsuk-Ulam theorem in Banach space, I present a method to compare two Banach spaces, and to apply the implicit function theorem in Banach spaces to monetary models.

In order to compare the topological properties of two Euclidean spaces, the Borsuk-Ulam theorem in a finite dimensional space can be applied. Let R^n and S^n be an n-dimensional Euclidean space and an n-dimensional unit sphere. The theorem says that, for a continuous function $f : S^n \subset R^{n+1} \to R^n$, there exists an $x \in S^n$ such that $f(x) = f(-x)$. This immediately implies that R^{n+1} is not homeomorphic to R^n.[2] Similarly, if an infinite dimensional version of the Borsuk-Ulam theorem can be applied to infinite dimensional spaces X and Y and a function $f : X \to Y$, then X and Y are topologically different spaces, and the dimension of X is in some sense larger than that of Y. Therefore, the implicit function theorem in infinite dimensional spaces could be applied to X, Y and f. Conversely, if the condition for the Borsuk-Ulam theorem in infinite dimensional space is not satisfied, then the stationary equilibria could be determinate. Indeed, in the all-pay auction in Kamiya and Shimizu (2013), the condition for the Borsuk-Ulam theorem is not satisfied, and the stationary equilibrium is determinate.

The plan of this paper is as follows. In Section 2, some mathematical tools are presented. Section 3 is the main part of this paper and is devoted to a general monetary model and a condition for the implicit function theorem in Banach space. In Section 4, a model with a determinate stationary equilibrium is presented, where the condition for the implicit function theorem is not satisfied. Section 5 concludes the paper.

2 Implicit Function Theorem and Borsuk-Ulam Theorem in Banach Space

Let X and Y be Banach spaces, and $L(X, Y)$ be the set of continuous linear function from X to Y. For $C \in L(X, Y)$, if there exists a $D \in L(Y, X)$ such that $CD = I_Y$, where I_Y is the identity function on Y, then D is called a right inverse of C.

Theorem 1. *: The image of the right inverse of Y, denoted by $D(Y)$, is a closed linear subspace of X and $X = Ker\ C + D(Y)$, where $+$ denotes the direct sum of two linear spaces and $Ker\ C$ is the kernel*

[2] If there exists a homeomorphism $f : R^{n+1} \to R^n$, then for all $x, y \in S^n$ such that $x \neq$y, $f(x) \neq f(y)$ holds. However, there exists an $x \in S^n$ such that $f(x) = f(-x)$. This is a contradiction.

of C.

Proof. [3] For $x \in X$, let $x = (x - DC(x)) + DC(x)$. Then $x - DC(x) \in \text{Ker } C$. Let $K = I_X - DC$. Then $K = L(X, X)$ and $\text{Ker } K = D(Y)$. Clearly, $DC(X) \subset D(Y)$. Moreover, for $y \in Y$, $D(y) = DCD(y)$ and thus $D(Y) \subset DC(X)$. Thus $D(Y)$ is a closed linear space and $X = K(X) + D(Y) = \text{Ker } C + D(Y)$. □

Next, I present the implicit function theorem in Banach space.

Theorem 2. *(Theorem 5.9 in Chapter 1 in Lang (1999)): Let X, Y, and Z be Banach spaces and $f : X \times Y \to Z$ be a Fréchet differentiable function. Suppose $f(x_0, y_0) = 0$ at $(x_0, y_0) \in X \times Y$ and $y \to df_{(x_0,y_0)}(0, y)$ is isomorphism between Banach spaces Y and Z, where $df_{(x_0,y_0)}$ is the Fréchet differential at (x_0, y_0). Then there exist neighborhoods U of x_0 and V of y_0, and a Fréchet differentiable function $g : U \to V$ such that*

$$g(x_0) = y_0 \text{ and } \forall x \in U, f(x, g(x)) = 0.$$

Next, I present an infinite dimensional version of the Borsuk-Ulam Theorem.

Definition 1. *: $f : X \to Y$ is said to be an odd function if $f(-x) = -f(x)$.*

Definition 2. *: $f : X \to Y$ is said to be a completely continuous function if for a weak convergent sequence $\{x^q\}$, $\{f(x^q)\}$ is norm convergent.*

Theorem 3. *(Gel′man (2002)): Let $\alpha : X \to Y$ be a surjective continuous linear function, and $f : S_r(0) \to Y$ be an odd completely continuous function, where $S_r(0)$ is a sphere with radius $r > 0$ in X. If dim(Ker α) ≥ 1, then the equation*

$$\alpha(x) = f(x)$$

has a nonempty solution set $N(\alpha, f)$ and the dimension of the set is larger than or equal to dim(Ker α)$-$ 1.

This theorem is a generalized version of the Borsuk-Ulam theorem. Indeed, for a function $g : S_r(0) \to Y$, let $f(x) = g(x) - g(-x)$. Then f is an odd function, and applying the above theorem there exists an $x_\lambda \in S_r(0)$ such that $\lambda\alpha(x_\lambda) = f(x_\lambda)$, where $\lambda > 0$. Then, if $S_r(0)$ is sequentially compact,

[3] This proof is based on Omori (1978).

then there exists a converging subsequence of $\{x_\lambda\}$, and from the complete continuity the limit point $x^* \in S_r(0)$ satisfies $0 = f(x^*) = g(x^*) - g(-x^*)$.

A linear function is clearly an odd function. Then the following theorem can be proved.

Theorem 4. *(Gel′man (2002)): Let $\alpha : X \to Y$ be a surjective linear continuous function, and $k : X \to Y$ be a completely continuous linear function. If $dim(Ker\ \alpha) \geq 1$, then the function $\alpha + k$ has a non-trivial kernel and $dim(Ker\ \alpha + k) \geq dim(Ker\ \alpha)$.*

The following Theorem will be used in the following section.

Theorem 5. *(Theorem 8.6.2. in Bogachev (2007)): Let (G, ρ) be a separable and complete metric space, where ρ is a metric on a set G. Let Ω be the set of signed Borel measures on (G, ρ) with weak convergence topology. Then the following two statements are equivalent.*

1. *Ω is sequentially compact.*

2. *Ω is uniformly tight and uniformly bounded.*

3 A General Monetary Model and a Condition for Indeterminacy

3.1 A General Model

The set of agents is represented by the interval $[0, 1]$. The support of real money holdings distributions is a closed interval $[0, \bar{m}]$, where $\bar{m} > 0$. A real money holdings distribution is expressed as a Borel probability measure φ on $[0, \bar{m}]$ satisfying $\int d\varphi = 1$. For a given money supply $M > 0$, nominal money holding distribution is obtained by adjusting the support using $p = \frac{M}{\int \eta d\varphi}$. For example, for a real money holdings distribution φ expressed by $(h_0, h_1, \dots, h_N)$, where h_i is a measure of $\{\eta_i\} \subset [0, \bar{m}]$, the corresponding nominal money holdings distribution is $(h_0, h_1, \dots, h_N)$, where h_i is a measure of $\{p\eta_i\}$ with $p = \frac{M}{\int \eta d\varphi} = \frac{M}{\sum_{i=0}^{N} \eta_i h_i}$.
Time is discrete and infinite horizon, denoted by $t = 1, 2, \dots$. In each period, agent $i \in [0, 1]$ chooses $a_i \in A(\eta_i)$, where $A(\eta_i)$ is the set of feasible strategies and $\eta_i \in [0, \bar{m}]$ is her money holding. The temporal utility of agent i is $g(a_i, a_{-i}, \varphi)$, where $a = \{a_i\}_{i \in [0,1]}$ and $a_i \in A(\eta_i)$. Let $f^O(a, \varphi)$ and $f^I(a, \varphi)$ are an outflow and an inflow of agents, respectively. For example, $f^O(a, \varphi)(C)$ $(f^I(a, \varphi)(C))$

is the outflow (inflow) from (to) a Borel set $C \subset [0, \bar{m}]$. Each agent maximizes the discounted sum of temporal utilities with a discount factor $\beta \in (0, 1)$.

Since an agent in an outflow at $\eta \in [0, \bar{m}]$ must be in an inflow at some $\eta' \in [0, \bar{m}]$, $\int d(f^O(a, \varphi) - f^I(a, \varphi)) = 0$ holds. Moreover, as shown in Kamiya and Shimizu (2006), if the total money holding before trades is equal to that after trades even out of equilibria, then $\int \eta d(f^O(a, \varphi) - f^I(a, \varphi)) = 0$ always holds, i.e., an identity. This identity typically holds in money search models and some dynamic auction models with fiat money. Therefore, I make the following assumption.

Assumption 1. *For any strategy a and any Borel probability measure φ, $\int d(f^O(a, \varphi) - f^I(a, \varphi)) = 0$ and $\int \eta d(f^O(a, \varphi) - f^I(a, \varphi)) = 0$ hold.*

Let E be the set of signed Borel measures on $[0, \bar{m}]$. The domain of f^O and f^I can be extended to E as follows. For a signed Borel measure φ, let $|\varphi| = \varphi_+ + \varphi_-$ be the total variation of φ, where φ_+ and φ_- are the positive and negative parts of φ. Let $\|\varphi\|$ be the total variation norm of φ. First, f^O and f^I can be extended to $\tilde{f}^O$ and $\tilde{f}^I$ on the space of signed Borel measures φ with total variation norm one, i.e., $\|\varphi\| = 1$, as follows:

$$\tilde{f}^O(a, \varphi) = f^O(a, |\varphi|) \text{ and } \tilde{f}^I(a, \varphi) = f^I(a, |\varphi|).$$

$\tilde{f}^O$ and $\tilde{f}^I$ can be extended to $\hat{f}^O$ and $\hat{f}^I$ on the space of signed Borel measure E as follows:

$$\hat{f}^O(a, \varphi) = \|\varphi\| \tilde{f}^O\left(a, \frac{\varphi}{\|\varphi\|}\right) \text{ and } \hat{f}^I(a.\varphi) = \|\varphi\| \tilde{f}^I\left(a, \frac{\varphi}{\|\varphi\|}\right).$$

Define $\hat{f}^O(a, 0) = \hat{f}^I(a, 0) = 0$. Below, I use f^O and f^I instead of $\hat{f}^O$ and $\hat{f}^I$. Note that, from the construction, Assumption 1 holds for all signed Borel measures.

Let F be the product space of two sets, the set of signed Borel measure φ satisfying $\int d\varphi = 0$ and $\int \eta d\varphi = 0$, and the set of real numbers R. That is, F is the set of values of $f^O(a, \varphi) - f^I(a, \varphi)$ and $1 - \int d\varphi$. Note that E is a Banach space with total variation norm, and so is F with the product topology of total variation norm and Euclidean norm on R. Let $f(a, \varphi) = (f^O(a, \varphi) - f^I(a, \varphi), 1 - \int d\varphi)$. For a given a, a stationary distribution is a solution to $f(a, \varphi) = (0, 0)$, where the first 0 is the zero measure and the second zero is the real number 0.

The stationary equilibrium is defined as follows.

Definition 3. *Let $a_i : [0, \bar{m}] \to A, i \in [0, 1]$, and $a = \{a_i\}_{i \in [0,1]}$. A pair (a, φ) is said to be a stationary equilibrium, where φ is a Borel probability measure on $[0, \bar{m}]$, if*

1. $\forall i \in [0,1]$, a_i *maximizes discounted sum of expected utilities for given* a_{-i}, φ, *and her money holding,*

2. $f^O(a,\varphi) - f^I(a,\varphi)$ *is a zero measure, i.e.,* $(f^O(a,\varphi) - f^I(a,\varphi))(C) = 0$ *for all Borel set* C.

In a stationary equilibrium, if the discounted sum is not constant in money holdings, then it is called a monetary equilibrium. Below, I suppose that an equilibrium strategy a^* is fixed, and thus f^O and f^I are functions of only a measure. I redefine $f(\varphi) = f(a^*, \varphi)$.

3.2 A Condition for Implicit Function Theorem

In order to apply the Borsuk-Ulam theorem in Banach space, it is shown that there exists a surjective linear continuous function $\alpha : E \to F$.

Lemma 1. *There exists a surjective linear continuous function* $\alpha : E \to F$. *Moreover, the dimension of the kernel of* α *is larger than or equal to one.*

Proof. From the construction, $F = E_1 \times R$, where E_1 is the linear subspace in E which consists of $\nu \in E$ satisfying $\int d\nu = 0$ and $\int \eta d\nu = 0$.

Let $\bar{\varphi}$ be a measure with a support $\{0\}$ and a mass $h_0 > 0$ on $\{0\}$. Consider the product space of E_1 and the space spanned by the measure $\bar{\varphi}$. Denote the product space by E_2. $\varphi \in E$ can be expressed as $\varphi = \varphi_1 + c\bar{\varphi} + \varphi_{-2}$, where $\varphi_1 \in E_1$, c is a scalar, and $\varphi_{-2} \in E \setminus E_2$. Let $\alpha(\varphi) = (\varphi_1, c) \in F$. Then α is a surjective and continuous linear function.

Below, I show that $E \setminus E_2$ is at least one dimensional. Let φ' be a measure with a support $\{q\}$ and a mass $h_q > 0$ on $\{q\}$, where $q > 0$ is in $[0, \bar{m}]$. Suppose φ' is in E_2. Then $\varphi' = \varphi_1' + c'\bar{\varphi}$ for some $\varphi_1' \in E_1$ and c'. Then φ_1' has masses $-c'h_0$ and h_q on $\{0\}$ and $\{q\}$, respectively, and the support is $\{0, q\}$. However, $\int \eta d\varphi' = -0c'h_0 + qh_q = qh_q > 0$ contradicts the definition of E_1 and therefore $\varphi' \in E \setminus E_2$. Thus $E \setminus E_2$ is at least one dimensional and the dimension of kernel of α is at least one. $\square$

Below, I show that the kernel of df_{φ^*}, denoted by Ker df_{φ^*}, is at least one-dimensional.

Lemma 2. *Suppose* df_{φ^*} *is a completely continuous function. Then Ker* df_{φ^*} *is at least one-dimensional.*

Proof. Let S be the set of signed Borel measures of which total variation norms are one. Since df_{φ^*} is a completely continuous function, then, from Lemma **??**, Theorem 4 can be applied and there exists a nontrivial solution to

$$(\alpha + df_{\varphi^*})(\varphi) = 0.$$

Since $\lambda\alpha, \lambda > 0$, is a surjective linear continuous function, then there exists a nontrivial solution to

$$(\lambda\alpha + df_{\varphi^*})(\varphi) = 0.$$

Since the solution is nontrivial, then from the linearity of $\lambda\alpha + df_{\varphi^*}$ there exists a solution $\varphi_\lambda \in S$. In our environment, the support of measure is a compact set $[0, \bar{m}]$. Thus, from Theorem 5, S is sequentially compact. Indeed, the set of signed Borel measures on $[0, \bar{m}]$ is uniformly tight and uniformly bounded.[4] Let $\lambda \to 0$. Then, from sequential compactness there exists a subsequence $\{\varphi_{\lambda^q}\}$ weakly converging to some $\hat{\varphi} \in S$. Then, from complete continuity and $\lambda^q \to 0$, $df_{\varphi^*}(\hat{\varphi}) = 0$. Since $\hat{\varphi} \neq 0$, the linear space spanned by $\hat{\varphi}$ is one dimensional and Ker df_{φ^*} is at least one-dimensional. □

If df_{φ^*} is completely continuous and has a right inverse, then from Theorems 1 and 2 the set of stationary equilibria is at least one-dimensional.

Theorem 6. *Let $U \subset E$ be an open set. Suppose f is a Fréchet differentiable function from U to F and $f(\varphi^*) = 0$. Suppose $df_{\varphi^*} \in L(E, F)$ is a completely continuous function and has a right inverse D. Then $E = Ker\ df_{\varphi^*} + D(F)$ and $\varphi^* = (x_0, y_0)$, where $x_0 \in Ker\ df_{\varphi^*}$ and $y_0 \in D(F)$.[5] Moreover, there exist neighborhoods U_1 of $Ker\ df_{\varphi^*}$, U_2 of $D(F)$, and V of F such that $U_1 + U_2 \subset U$ and there exists a unique Fréchet differentiable function $g : Ker\ df_{\varphi^*} \to F$ such that*
(*) $g(x_0) = y_0$, $(x_0, g(x_0)) \in U$, and $f(x, g(x)) = 0$ for all $x \in U_1$.

Proof. From Theorem 1, $E = \text{Ker } df_{\varphi^*} + D(F)$ and $\varphi^* = (x_0, y_0)$, where $x_0 \in \text{Ker } df_{\varphi^*}$ and $y_0 \in D(F)$. First, I show that both Ker df_{φ^*} and $D(F)$ are Banach spaces and the implicit function theorem can be applied. Beside the completeness of the space, all the properties of Banach space are satisfied.

[4]The set of signed Borel measures on $[0, \bar{m}]$ is called uniformly tight if for all $\varepsilon > 0$ there exists a compact set K_ε such that $|\varphi|([0, \bar{m}] \setminus K_\varepsilon) < \varepsilon$ for all signed Borel measure φ in the set, where $|\varphi|$ is the total variation of φ. Clearly, $K_\varepsilon = [0, \bar{m}]$ satisfies the condition.

[5]The product and direct sum of finite number of linear spaces are the same. Therefore, I will use $x + y$ and (x, y) for the same object.

Consider a Cauchy sequence $a^q \in \text{Ker } df_{\varphi^*}, q = 1, 2, \ldots,$ where a^q norm converges to some $a \in E$. Since norm convergence implies weak convergence, then from the complete continuity of df_{φ^*}, $df_{\varphi^*}(a^q)$ norm converges to $df_{\varphi^*}(a)$. That is,

$$0 = \lim_{q\to\infty} \|df_{\varphi^*}(a^q) - df_{\varphi^*}(a)\| = \| - df_{\varphi^*}(a)\|$$

Therefore, $a \in \text{Ker } df_{\varphi^*}$ and thus $\text{Ker } df_{\varphi^*}$ is complete. Since $E = \text{Ker } df_{\varphi^*} + D(F)$ is complete and $D(F)$ is closed from Theorem 1, then $D(F)$ is complete.

Next, D is clearly an isomorphism between F and $D(F)$ so that $y \to df_{\varphi^*}(0, y) = df_{(x_0,y_0)}(0, y)$ is also an isomorphism between F and $D(F)$. Thus, applying Theorem 2, (*) holds. $\square$

From Lemma **??**, $\text{Ker } df_{\varphi^*}$ is at least one-dimensional. Therefore, from Theorem **??**, the set of stationary equilibria is at least one-dimensional.

Theorem 7. *Let $U \subset E$ be an open set. Suppose f is a Fréchet differentiable function from U to F and $f(\varphi^*) = 0$. Suppose $df_{\varphi^*} \in L(E, F)$ is a completely continuous function and has a right inverse. Then the set of stationary equilibria is at least one-dimensional.*

4 A Model with a Determinate Equilibrium

Kamiya and Shimizu (2013) suggest that a dynamic all-pay auction model with fiat money has a determinate stationary equilibrium. In the model, all bidders must pay regardless of whether they win the prize. Although the identity exists, the stationary equilibrium is determinate. This is due to the fact that the support of money holdings distribution in the all-pay auction markets cannot be confined to a finite set and the outflow minus outflow is NOT a completely continuous function.

Below, I precisely explain a dynamic all-pay auction model with fiat money. In the model, there is only one type of indivisible good. At the beginning of each period, an agent chooses either to be a buyer or a seller. If she becomes a seller, then she can produce one unit of good with cost $c \geq 0$. When she becomes a buyer, without knowing the other agents' choices, she chooses a nonnegative bid price which cannot exceeds her money holding. If she buys a good, then she obtains utility $u > 0$, where $\beta u > c$. Let the measure of sellers be m_s and the distribution of bid price be φ_b, where $m_s + \int d\varphi_b = 1$. Suppose $m_s > \int d\varphi_b$. Then all the buyers obtain the goods, all the sellers equally share the total bids and randomly chosen sellers with measure $1 - m_s$ produces goods, and the transaction price is the lowest bid. Note that if the lowest bid does not exist, then the infimum of the bids is the transaction price. Suppose $m_s \leq \int d\varphi_b$. Then all the sellers produce goods and the transaction price is determined as the highest p satisfying $m_s \leq \int_p^\infty d\varphi_b$. Note that if there is no mass of φ_b at the highest p, then $m_s = \int_p^\infty d\varphi$. Then the buyers with a bid price larger than or equal to p obtain goods. If there is a mass at the highest p, then the buyers who bid larger than p obtain goods and the buyers who bid p are randomly chosen and obtain goods. Note that, in the all-pay auction, the total money holdings before trades is always equal to that after trades, i.e., the identity in the previous section holds.

As in the general model in the previous section, a stationary equilibrium is a pair of agents' strategies and a money holdings distribution satisfying that (i) each agent maximizes discounted sum of expected utilities given the other agents' strategies, her money holding, and a money holdings distribution, and (ii) the distribution is stationary. Below, I focus on monetary equilibria.

Suppose the strategy is symmetric across agents. Then, from the stationarity of distribution, the transaction price p is uniquely determined, since, for a given strategy, it only depend on money holdings distribution. Thus, in a stationary equilibrium, the support of money holdings distribution is $\{0, p, 2p, \ldots\}$ or some distribution which has essentially the same support, e.g., $\{\varepsilon, p+\varepsilon, 2p+\varepsilon, \ldots\}$, where ε has no value. For simplicity, I only consider $\{0, p, 2p, \ldots\}$. Since each agent can choose either to be a buyer or a seller, then an agent with $\eta \geq p$ always chooses to be a buyer. This is because if she becomes a seller, she postpones the opportunity of consumption. Thus the support of stationary equilibrium distribution is $\{0, p\}$. Let $(h_0, 1-h_0)$ be the money holdings distribution, where h_0 is a measure of agents without money and $1-h_0$ is a measure of agents with p amount of money. Suppose all agents without money choose to be sellers and all agents with p choose to be buyers. Below, it will be checked that the above choices are optimal strategies. From the definition of all-pay auction, the sellers' money holding become $\frac{p(1-h_0)}{h_0}$ after trades. From stationarity, $\frac{p(1-h_0)}{h_0} = p$ holds and thus $h_0 = \frac{1}{2}$. Note that, from $p(1-h_0) = M$, $p = 2M$ holds.

Below, the optimality of the above equilibrium strategy is rigorously checked. Let $v(np), n = 0, 1, 2, \ldots$ be the value of np. Then the Bellman equation is as follows.

$$\begin{aligned} v(0) &= \max\{-c + \beta v(p), \beta v(0)\} \\ v(np) &= \max\{u + \beta v((n-1)p), -c + \beta v((n+1)p), \beta v(np)\}, n = 1, 2 \ldots. \end{aligned} \tag{1}$$

The strategy is (i) an agent without money chooses to be a seller and (ii) an agent with $np, n \geq 1$, amount of money chooses to be a buyer and bids p. From the strategy, $v(0) = \frac{\beta u - c}{1-\beta^2}$, $v(p) = \frac{u - \beta c}{1-\beta^2}, \ldots, v(np) =$

$\sum_{k=1}^{n-1} \beta^{k-1} u + \beta^{n-1} \frac{u-\beta c}{1-\beta^2}, \dots$ hold. It can be easily checked that the strategy is optimal.

Below, it is shown that the outflow minus inflow is NOT completely continuous. To see this, consider a sequence of probability measure $\{\varphi^q\}$, where φ^q has masses $\frac{1}{2} - \frac{1}{2^q}$ at 0 and $\frac{1}{2} + \frac{1}{2^q}$ at $\frac{M}{\frac{1}{2}+\frac{1}{2^q}}$, respectively. Clearly, $\{\varphi^q\}$ weakly converges to the measure of which masses are $\frac{1}{2}$ at 0 and $\frac{1}{2}$ at $2M$. On the other hand, the sequence of outflow minus inflow has masses $\frac{1}{2} - \frac{1}{2^q} - (\frac{1}{2} + \frac{1}{2^q}) = -\frac{2}{2^q}$ at 0, $\frac{1}{2} + \frac{1}{2^q}$ at $\frac{M}{\frac{1}{2}+\frac{1}{2^q}}$, and $-(\frac{1}{2} - \frac{1}{2^q})$ at $\frac{M}{\frac{1}{2}-\frac{1}{2^q}}$, respectively. As $q \to \infty$, the sequence of the outflow minus inflow clearly converges weakly to zero measure, but it is not norm convergent. Indeed, for all q, one can take an open neighborhood of $p = 2M$ such that the total variation measure is one. Thus the outflow minus inflow is not completely continuous and the argument in the previous section cannot be applied.

5 Conclusion

In this paper, I investigate monetary models with fiat money and present a condition for indeterminacy of stationary equilibria. More precisely, the conditions for implicit function theorem in Banach space are the complete continuity of Fréchet differential and the existence of the right inverse. Moreover, if the conditions are not satisfied, then the stationary equilibrium could be determinate. Indeed, in Section 4, I present a dynamic all-pay auction model with a determinate equilibrium.

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