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# Organizational Refinements of Nash Equilibrium

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# Organizational Refinements of Nash Equilibrium*


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**Abstract**

Strong Nash equilibrium (see Aumann, 1959) and coalition-proof Nash equilibrium (see Bernheim et al., 1987) rely on the idea that players are allowed to form coalitions and make joint deviations. They both consider a case in which any coalition can be formed. Yet there are many real-life examples where the players cannot form certain types of coalitions/subcoalitions. There may also be instances, when all coalitions are formed, where *conflicts of interest* arise and prevent a player from choosing an action that simultaneously meets the requirements of the two coalitions to which he or she belongs. Here we address these criticisms by studying an *organizational framework* where some coalitions/subcoalitions are not formed and where the coalitional structure is formulated in such a way that no conflicts of interest remain. We define an *organization* as a collection of partitions of a set of players ordered in such a way that any partition is coarser than the partitions that precede it. For a given organization, we introduce the notion of *organizational Nash equilibrium.* We analyze the existence of equilibrium in a subclass of games with strategic complementarities and illustrate how the proposed notion refines the set of Nash equilibria in some examples of normal form games.



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# 1 Introduction

In the final scene of the western movie *The Good, the Bad and the Ugly*, three cowboys resolve a conflict between them via a truel. They all claim rights to a sum of money that the survivor(s) of the truel will collect. The situation can clearly be described as a strategic game. The Good, it turns out, cooperates with the Ugly, while the Bad takes part in no coalition. Now, knowing that only a single coalition is formed, would not the notions of *strong Nash equilibrium* (see Aumann, 1959) or of *coalition-proof Nash equilibrium* (see Bernheim et al., 1987) be misleading? Perhaps more importantly, would not it be possible to find a new equilibrium notion that makes more precise predictions than the notion of *Nash equilibrium* (see Nash, 1951)? More generally, if it is *known* that there are players who cooperate with some of their fellow players but not with others, then how can we make correct and/or precise predictions using a notion that presumes that every player acts entirely on his or her own or using a notion that presumes that the players participate in any combination of coalitions?

Numerous papers on non-cooperative game theory focus on approaches for refining the set of Nash equilibria (see Aumann, 1959; Selten, 1965, 1975; Myerson, 1978; Kohlberg and Mertens, 1986; Bernheim et al., 1987, among others). Some of these equilibrium refinements allow players to form coalitions and make joint deviations. Among these coalitional refinements, in this paper, we are mainly concerned with *strong Nash equilibrium* (SNE) and *coalition-proof Nash equilibrium* (CPNE).[1] Both of these equilibrium notions satisfy a certain type of coalitional stability. For instance, at a SNE, the members of any particular coalition should generally be presumed to prefer not to deviate collectively. As coalitions do not face too many restrictions in choosing their joint deviations, the set of SNE generally turns out to be empty. Expanding from this observation, Bernheim et al. (1987) propose the notion of CPNE according to which coalition members cannot make binding commitments (i.e., agreements must be *self-enforcing*[2]). Accordingly, if no coalition is able to deviate from a strategy profile via self-enforcing contracts, then that strategy profile is said to be coalition-proof.

An important observation would be that SNE and CPNE both consider a case in which any coalition can be formed. Yet in real-life situations, we see many instances where some coalitions are not or cannot be formed. Moreover, even if a particular coalition is formed, this does not necessarily imply that all of its subcoalitions will be formed. Indeed, a game might have players that hate/dislike each other or who

[1]These are known to be the most prominent coalitional refinements of Nash equilibrium. For studies on these refinements, see Bernheim and Whinston (1987); Greenberg (1989); Dutta and Sen (1991); Konishi et al. (1997a,b, 1999) among others. In addition to these equilibrium notions, we can find other refinements of Nash equilibrium also utilizing coalitional structures: strong Berge equilibrium (Berge, 1957), the largest consistent set (Chwe, 1994), negotiation-proof Nash equilibrium (Xue, 2000), etc.

[2]A joint strategy profile of a coalition is *self-enforcing* if the members of the deviating coalition do not desire further deviations.

simply cannot communicate to form a coalition. Consider, for example, two countries with a history of bad relations. These countries might not prefer to create a two-player coalition; or even if they meet at a global association, they might still refuse to form the two-player subcoalition.[3] Following the studies on *conference structure*s in the vein of Myerson (1980), consider also two academic scholars at a conference, neither of whom has met the other or any colleague who could have introduced them or brought them together. who have never met each other or anyone that could have connected them in a conference. Even if they belong to the same society, these scholars cannot or choose not to collaborate. In other examples, we note that some coalitions cannot be formed because of some rules or regulations. The competition laws in many countries, for example, prohibit cooperation between firms that compete in the same market. Firms are free to cooperate, however, with other firms with whom they do not compete. Along a similar line, in sport competitions, a player in a team is forbidden to form a coalition with a player of the opponent team while he or she freely cooperates with the other members of his or her own team.

Another important observation on SNE and CPNE lies within the *actions* of the players. Considering a coalition and its subcoalition, the notion of SNE allows both coalitions to determine joint strategy profiles in such a way that a member of the subcoalition cannot take an action that would simultaneously fulfill the interests of both coalitions (*vertical conflict of interest*). The notion of CPNE actually overcomes this conflict by restricting each coalition to respect the rationality of its subcoalitions/members. Be that as it may, since CPNE allows for coalitions that have a non-empty intersection, a player participating in two coalitions may not be able to take an action that would simultaneously fulfill the interests of both coalitions (*horizontal conflict of interest*).

With regard to the former observation, is it truly reasonable and feasible to control for all coalitions? If a coalition is not or cannot be formed, why would its members' *hypothetical* best actions be effective in the equilibrium behavior? With regard to the latter observation, can there be a specific structure that eliminates both vertical and horizontal conflicts of interests simultaneously? In this paper we seek to address these observations and associated questions by formulating a new equilibrium refinement. Our notion (i) resolves the problems of vertical and horizantal conflicts of interests, and (ii) proves to be more useful than the notions of SNE and CPNE (and even than Nash equilibrium) in cases where some coalitions are formed and the others are not.

Note that the former observation calls for a general coalitional structure that does not necessarily include some coalitions; whereas the latter observation calls for a specific framework that restricts the set of coalitional structures to be studied. More precisely, in order to eliminate vertical conflicts of interest, every coalition

[3]One may argue that the countries would form a two-player coalition if it benefited them, but (i) forming a coalition does not necessarily make them better off (at the equilibrium), given that there is strategic interaction between the players; and (ii) participating in the same coalition could be somehow costly. Note that if participating in the same coalition does turn out to be costly, the cost cannot be implemented into the payoff functions of the game.

should respect to the rationality of its subcoalitions/members (as it does in the case of CPNE). In addition to this, to eliminate horizontal conflicts of interest, the coalitional structure should be formulated in such a way that for any pair of active (or formed) coalitions, it is either the case that the coalitions are disjoint or one coalition contains the other. This is what we refer to as an *organizational structure*.

The intuition behind the *organizational structure* is as follows. In a non-cooperative game, players may prefer to form coalitions if they are allowed to. We assume that if a player is a member of a coalition, then he or she cannot be a member of another coalition. Accordingly, the set of these coalitions turns out to be a partition of the player set. As coalitions may prefer to unite to form greater coalitions, in the next step we have another partition of the player set, one coarser than the former partition. This recursively leads to a collection of partitions that are all coarser than the partitions that preceded them. It leads, in other words, to an *organization*. Consider, for example, a university as a set of faculty members, each of whom belongs to one department. Each department, in turn, belongs to either the school of social sciences or the school of natural sciences. Another example would be a company with divisions, departments, units, and employees.[4]

In this paper, we take the organizational structure as given.[5] Accordingly, for any organization, we define the notion of *organizational Nash equilibrium* (ONE), a solution concept for which we utilize strict Pareto dominance to describe the preferences of coalitions (Section 3). We provide a monotonicity property for the proposed notion in such a way that as we consider *greater* organizations, the equilibrium set is more

[4]More solid real-life examples can be provided. In a doubles tennis match, a total of four players play on two teams, and no player can participate in the same coalition as either of his or her opponent. As an example of a larger organization, we can consider a football (i.e., soccer) game with twenty-two players. Teammates playing in the same position, such as defenders, midfielders, and strikers, form into small coalitions; and these small coalitions join together to form the teams. Clearly, no coalition includes players from both teams. As another example, we can consider the market for mobile phone services. Consumers need mobile phones and lines to receive this service, so that phone producers and telecommunication companies operate in the market to serve them. Two telecommunication companies are forbidden to form a coalition, whereas a telecommunication company is allowed to form a coalition with a phone producer. To establish an organization in such a scenario, we need a regularity condition, namely, an agreement between a telecommunication company and a phone producer that restricts each of them from entering other agreements with third parties without the other. When one thinks of a regularity condition of this type, a coalitional structure emerging from the transfer market (as in sport competitions) seems like a better example: If a team signs a contract with a player, the player cannot sign another contract with another team. And if the team signs a contract with another player, the contract would not preclude the presence of the former player. Finally, we recall the example mentioned at the beginning of the paper, the truel taking place in the final scene of the western movie *The Good, the Bad and the Ugly.* There appears an organization, since the only coalition formed is the coalition between the Good and the Ugly.

[5]Our approach may seem rather *ad-hoc*. It must be understood, however, that we have no intention of imposing a certain type of coalitional structure. We simply argue that (i) there are many real-life examples in which some coalitions/subcoalitions are not included in a coalitional structure and (ii) organizations make up an important subset of such coalitional structures.

refined; we analyze the existence of equilibrium for a subclass of games with strategic complementarities; and we study some examples of normal form games through which we understand how our organizational refinement works and how its predictions are different from those made by SNE and CPNE (Section 4). We conclude in Section 5.

# 2 Preliminaries

Let $\Gamma = (N, (X_i)_{i\in N}, (u_i)_{i\in N})$ be an $|N|$-player normal form game in which $N$ denotes the finite set of players, $X_i$ denotes the strategy set for player $i \in N$, and $u_i : \prod_{i\in N} X_i \to \mathbb{R}$ denotes the utility function for player $i \in N$. For any coalition $S \subset N$, let $X_S \equiv \prod_{i\in S} X_i$ denote the set of strategy profiles for the members of this coalition. For any $S \subset N$, set $X_{-S} = X_{N\setminus S}$. And further, set $X_N = X$.

First, we define a Nash equilibrium.

**Definition 2.1.** Given a normal form game $\Gamma$, a strategy profile $x^* \in X$ is a **Nash equilibrium** if for every $i \in N$ and every $x'_i \in X_i$: $u_i(x^*) \geq u_i(x'_i, x^*_{-i})$.

Being one of the most prominent coalitional refinements in the literature, the notion of strong Nash equilibrium (SNE) is defined as follows.

**Definition 2.2.** Given a normal form game $\Gamma$, a strategy profile $x^* \in X$ is a **strong Nash equilibrium** (SNE) if for no coalition $S \subset N$, there exists some $x'_S \in X_S$ such that for every $i \in S$: $u_i(x'_S, x^*_{-S}) > u_i(x^*)$.

Bernheim et al. (1987) introduce *self-enforceability* in order to weaken the coalitional stability that SNE requires. They accordingly define coalition-proof Nash equilibrium (CPNE) that is weaker than the notion of SNE. Now, before proceeding to the definition of CPNE, we define a reduced game.

**Definition 2.3.** Given a normal form game $\Gamma$, a coalition $S \subset N$, and a strategy profile $x_{-S} \in X_{-S}$, the **reduced game** $\Gamma_S|_{x_{-S}} = (S, (X_i)_{i\in S}, (v_i)_{i\in S})$ is defined in such a way that for every $i \in S$, $v_i : X_S \to \mathbb{R}$ is given by $v_i(x'_S) = u_i(x'_S, x_{-S})$.

The following is the definition of CPNE.

**Definition 2.4.** Given a normal form game $\Gamma$,

(i) If $\Gamma$ is a single-player game, then a strategy profile $x^* \in X$ is a **coalition-proof Nash equilibrium** (CPNE) if and only if $x^*$ maximizes $u_1$.

(ii) Let $|N| > 1$ and assume that the set of CPNE is defined for any game with less than $|N|$ players. Define a strategy profile $x^* \in X$ to be **self-enforcing** if for every $S \subsetneq N$: $x^*_S \in \text{CPNE}(\Gamma_S|_{x^*_{-S}})$. Then a strategy profile $x^* \in X$ is a **CPNE** if and only if it is self-enforcing and there is no other self-enforcing strategy profile $x \in X$ such that for every $i \in N$: $u_i(x) > u_i(x^*)$.

Despite both notions' plausible refinement structures, there are normal form games in which these refinements (i) cannot make any prediction or (ii) make undesirable predictions. The normal form game given in Table 1, for example, has two Nash equilibria: $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$; but it has no SNE. More precisely, the grand coalition $N$ deviates from $(x_1, x_2, x_3)$ to $(y_1, y_2, y_3)$, whereas the coalition $\{1, 2\}$ deviates from $(y_1, y_2)$ to $(x_1, x_2)$ when Player 3 sticks to $y_3$. As a result, none of the Nash equilibria is coalitionally stable in the sense of SNE. Furthermore, this game has a unique CPNE: $(x_1, x_2, x_3)$. In particular, we can see that the deviation by $N$ from $(x_1, x_2, x_3)$ to $(y_1, y_2, y_3)$ is not self-enforcing, because the subcoalition $\{1, 2\}$ further deviates to $(x_1, x_2)$. Since there is no other coalition that would like to deviate, the profile $(x_1, x_2, x_3)$ turns out to be a CPNE. Also note that the unique CPNE is strictly Pareto dominated by the other Nash equilibrium, $(y_1, y_2, y_3)$.

Table 1

$x_3$

| | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 0, 0, 0 | 0, 0, 0 |
| $x_1$ | 1, 1, 1 | 0, 0, 0 |

$y_3$

| | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 0, 0, 0 | 2, 2, 2 |
| $x_1$ | 3, 3, 0 | 0, 0, 0 |

We can also provide an example for which there exists no CPNE, hence no SNE.

Table 2

$x_3$

| | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 0, 0, 0 | 0, 0, 1 |
| $x_1$ | 1, 1, 0 | 0, 0, 1 |

$y_3$

| | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 0, 1, 2 | 2, 0, 1 |
| $x_1$ | 2, 0, 0 | 0, 2, 1 |

For instance, in the normal form game given in Table 2, we find two Nash equilibria: $(x_1, x_2, x_3)$ and $(y_1, y_2, x_3)$. The coalition $\{2, 3\}$ makes a self-enforcing deviation from $(x_2, x_3)$ to $(y_2, y_3)$ when Player 1 sticks to $x_1$. In a similar manner, the same coalition deviates from $(y_2, x_3)$ to $(x_2, y_3)$ when Player 1 sticks to $y_1$.

# 3 Organizational Refinements of Nash Equilibrium

In this section, we first introduce a notation relevant to our definition of organizational refinement. While doing so, we also provide alternative definitions for the notions of SNE and CPNE. These equivalent definitions are consistent with the notation we introduce which makes them apparently comparable to the notion of organizational Nash equilibrium (ONE) we propose later in this paper.

## 3.1 Criticisms of SNE and CPNE

For any strategy profile $x \in X$ and any set of strategy profiles $Y \subset X$, define $B_N(x, Y)$ and $B_N(Y)$ as follows:

$$B_N(x, Y) = B_N(Y) = \{y \in Y \mid \nexists z \in Y, \forall i \in N : u_i(y) < u_i(z)\}.$$

For any coalition $S \subset N$ with $|S| < |N|$, any strategy profile $x \in X$, and any set $Y_S \subset X_S$, define[6]

$$B_S(x, Y_S) = \{y \in X \mid y_S \in Y_S \text{ and } \nexists z_S \in Y_S, \forall i \in S : u_i(y_S, x_{-S}) < u_i(z_S, x_{-S})\}.$$

We refer to $B_S(x, Y_S)$ as the set of *rational* (or weakly Pareto optimal) responses of $S$ to the strategy profile $x \in X$ within the set $Y_S \times X_{-S}$. For any strategy profile $x \in X$, define

$$B(x) = \bigcap_{S \subset N} B_S(x, X_S).$$

We now prove that a SNE is a *fixed point*[7] of this correspondence.

**Proposition 3.1.** *A strategy profile $x \in X$ is a strong Nash equilibrium if and only if $x \in B(x)$.*

*Proof.* Take any $x \in X$ such that $x \in B(x)$. Suppose that $x$ is not a SNE. Then $\exists S \subset N$, $\exists z_S \in X_S$ such that $\forall i \in S : u_i(z_S, x_{-S}) > u_i(x_S, x_{-S})$. We then have $x \notin B_S(x, X_S)$; a contradiction.

Conversely, take any $x \in \text{SNE}(\cdot)$. Suppose that $x \notin B(x)$. Then $\exists S \subset N$ such that $x \notin B_S(x, X_S)$; that is, $\exists z_S \in X_S, \forall i \in S : u_i(z_S, x_{-S}) > u_i(x_S, x_{-S})$; a contradiction. □

Unfortunately, without additional restrictions, the correspondence above is mostly empty-valued. Two types of conflicts of interest, what we call *vertical* and *horizontal* conflicts of interest, are likely to cause this.

Table 3

| | | |
|---|---|---|
| $D$ | 4, 0 | 1, 1 |
| $C$ | 2, 2 | 0, 4 |
| | $C$ | $D$ |

A vertical conflict of interest arises between a coalition and its subcoalitions. Consider, for example, *the Prisoner's Dilemma* represented by the matrix given in

[6] Note that $B_S(x, Y_S)$ does not impose any restriction on the joint strategy profile for the non-members of the coalition $S$. More precisely, for any $y = (y_S, y_{-S}) \in B_S(x, Y_S)$ and any $y'_{-S} \in X_{-S}$, we have $(y_S, y'_{-S}) \in B_S(x, Y_S)$ as well.

[7] A fixed point $x$ of a correspondence of $F : X \to X$ is defined to satisfy $x \in F(x)$.

Table 3. Take the grand coalition $N$ and observe that the unique Nash equilibrium $(D, D)$ is the only strategy profile to be strictly Pareto dominated by another. Thus $B_N(X)$ includes all strategy profiles except $(D, D)$; i.e., at each of the three strategy profiles in $B_N(X)$, at least one player is required to cooperate against his or her individual rationality. Hence, there is always a player that faces a conflict of interest between coalitional rationality and his or her own individual rationality. More precisely, for any $x \in B_N(X)$, we either have $x \notin B_1(x, X_1)$ or $x \notin B_2(x, X_1)$, or both. This surely implies that for any $x \in X$: $B(x) = \varnothing$. Note also that for games with more players, the same type of conflict may arise between a coalition of at least three players and its subcoalitions of multiple players.

Vertical conflicts of interest can be eliminated. In a two-player game, for example, no vertical conflict of interest arises if the grand coalition respects the rationality of each player by restricting itself to the set of individually rational strategy profiles (i.e., the set of Nash equilibria). More generally, no vertical conflict of interest arises if each coalition respects the rationality of its proper subcoalitions, i.e., restricts itself to the strategy profiles from which none of its subcoalitions have an incentive to deviate.

As discussed below, this idea is closely related to self-enforceability in CPNE. We can formalize the idea in the following way. For a set of strategy profiles $Y \subset X$ and a coalition $S \subset N$, define

$$[Y]_S = \{y_S \in X_S \mid \exists y_{-S} \in X_{-S} : (y_S, y_{-S}) \in Y\}.$$

Take any $x \in X$. For any $i \in N$, define $R_i(x) = B_i(x, X_i)$. For any coalition $S \subset N$ with $|S| = 2$, define

$$E_S(x) = \left\{ y \in X \;\middle|\; (y_S, x_{-S}) \in \bigcap_{i \in S} R_i((y_S, x_{-S})) \right\}.$$

The set $[E_S(x)]_S$ can be interpreted as the set of Nash equilibria of the reduced game played by the coalition $S$, given that the actions of the other players are fixed to $x_{-S}$. Then define

$$R_S(x) = B_S\left(x, [E_S(x)]_S\right).$$

This is the set of *rational* responses of coalition $S$ among the strategy profiles which its members can jointly reach and which are *rational* for all of its members. In other words, coalition $S$ restricts itself to the strategy profiles *acceptable* to all of its members. For any coalition $S \subset N$ with $|S| = 3$, define

$$E_S(x) = \left\{ y \in X \;\middle|\; (y_S, x_{-S}) \in \bigcap_{C \subsetneq S} R_C((y_S, x_{-S})) \right\}. \tag{3.1}$$

Similarly, the set $[E_S(x)]_S$ can be interpreted as the *equilibrium* set of the corresponding reduced game. Likewise,

$$R_S(x) = B_S\left(x, [E_S(x)]_S\right). \tag{3.2}$$

This is the set of *rational* responses of coalition $S$ among the strategy profiles that the members of $S$ can jointly reach and that are *rational* for all of the subcoalitions of $S$. In other words, coalition $S$ restricts itself to the strategy profiles *acceptable* to all of its subcoalitions. Using (3.1) and (3.2) inductively, define $E_S(\cdot)$ and $R_S(\cdot)$ for $S \subset N$ with $|S| = 4, 5, \ldots, |N|$. Finally, for any strategy profile $x \in X$, define

$$R(x) = \bigcap_{S \subset N} R_S(x).$$

We first prove the following lemma.

**Lemma 3.1.** *For any coalition $S \subset N$ with $|S| \geq 2$ and any strategy profile $x \in X$, we have*

$$\bigcap_{C \subsetneq S} R_C(x) = \bigcap_{\substack{C \subset S \\ |C| = |S| - 1}} R_C(x).$$

*Proof.* It is trivial that the left-hand side is included in the right-hand side.

Conversely, take any

$$x \in \bigcap_{\substack{C \subset S \\ |C| = |S| - 1}} R_C(x),$$

and suppose that

$$x \notin \bigcap_{C \subsetneq S} R_C(x).$$

Then $\exists C \subsetneq S$ such that $x \notin R_C(x)$. Note that there exists $C' \subset S$ such that $C \subset C'$ and $|C'| = |S| - 1$. Since the definition is recursive, it follows that $x \notin R_{C'}(x)$; a contradiction. □

As we now show, a *fixed point* of the correspondence above turns out to be a CPNE, as formally defined in the previous section.

**Proposition 3.2.** *A strategy profile $x \in X$ is a coalition-proof Nash equilibrium if and only if $x \in R(x)$.*

*Proof.* We prove this result by induction. Clearly, the statement holds when $|N| = 1$. For some $k \in \mathbb{N}$, assume that it holds when $|N| \leq k - 1$ and consider the case where $|N| = k$.

Take any $x \in X$ such that $x \in R(x)$. And suppose that $x$ is not a CPNE. Then either $(i)$ $x$ is not self-enforcing; or $(ii)$ $x$ is self-enforcing but there exists another self-enforcing strategy profile $z \in R(x)$ such that $\forall i \in N : u_i(z) > u_i(x)$. If $(i)$ is the case, then $\exists S \subsetneq N$ such that $x_S \notin \mathrm{CPNE}(\Gamma_S|_{x_{-S}})$. But then $x \notin R_S(x)$ by the induction hypothesis; a contradiction. If, on the other hand, case $(ii)$ applies, then it must be that $x \notin R_N(X)$; a contradiction.

Conversely, take any $x \in \text{CPNE}(\cdot)$. And suppose that $x \notin R(x)$. Then either $(i)$ $x \notin R_N(x)$; or $(ii)$ $x \notin \bigcap_{S \subsetneq N} R_S(x)$. If case $(i)$ applies, then there exists another strategy profile $z \in R(x)$ such that $\forall i \in N : u_i(z) > u_i(x)$. By the induction hypothesis, $z$ must be self-enforcing; a contradiction. On the other hand, if $(ii)$ is the case, then

$$x \notin \bigcap_{\substack{S \subset N \\ |S| = |N| - 1}} R_S(x)$$

by Lemma 3.1. By a similar reasoning, it follows for some $S \subset N$ with $|S| = |N| - 1$ that $x_S$ is not a CPNE of the corresponding reduced game. This implies that $x$ is not self-enforcing; a contradiction. □

Although the definition of CPNE eliminates vertical conflicts of interest, there may still be *horizontal* conflicts of interest. By definition, there can be no horizontal conflict of interest between a coalition and its subcoalitions. A horizontal conflict of interest arises between two coalitions with a non-empty intersection. And it can arise as soon as the game has three players. Consider, as an example, the following normal form game given in Table 4. There exist two coalitions $A = \{1, 2\}$ and $B = \{2, 3\}$ with the corresponding sets of rational responses: $R_A((y_1, y_2, x_3)) = \{(x_1, z_2, \cdot)\}$ and $R_B((y_1, y_2, x_3)) = \{(\cdot, x_2, y_3)\}$. Surely, these sets have an empty intersection. Player 2 in this example belongs to both coalitions $A$ and $B$, and each coalition requires him or her to behave differently than the other coalition requires. In other words, Player 2 faces a horizontal conflict of interest because he or she has no way of choosing an action that simultaneously meets the requirements of both coalitions.

Table 4

$x_3$

| | $x_2$ | $y_2$ | $z_2$ |
|---|---|---|---|
| $z_1$ | 1, 5, 8 | 0, 2, 0 | 7, 2, 3 |
| $y_1$ | 0, 0, 3 | 5, 1, 8 | 4, 0, 0 |
| $x_1$ | 0, 4, 3 | 0, 7, 3 | 7, 7, 0 |

$y_3$

| | $x_2$ | $y_2$ | $z_2$ |
|---|---|---|---|
| $z_1$ | 2, 0, 4 | 0, 1, 0 | 0, 0, 0 |
| $y_1$ | 0, 2, 9 | 1, 0, 4 | 0, 0, 0 |
| $x_1$ | 2, 0, 9 | 0, 0, 0 | 6, 6, 2 |

Since a horizontal conflict of interest arises between coalitions with a non-empty intersection, an easy way to eliminate this type of conflicts of interest would be to restrict coalition formation in such a way that all active coalitions are mutually disjoint. We have no need, however, to impose such a restrictive requirement: if an active coalition is a proper subset of another active coalition, these coalitions would still be unencumbered by horizontal conflicts of interest between them.

To sum up, if (i) one uses a similar formulation to $R(\cdot)$ above, and if (ii) any pair of active coalitions is either disjoint or includes one coalition as a subset of the other, then neither vertical nor horizontal conflicts of interest arise. Accordingly, we say that a collection $\mathscr{S}$ of coalitions in $N$ is *conflict-free* if for any $A, B \in \mathscr{S}$, we have one of the following three properties: $(i)$ $A \cap B = \varnothing$; $(ii)$ $A \subset B$; or $(iii)$ $B \subset A$.

**Proposition 3.3.** *Let $\mathscr{S}$ be a collection of coalitions in $N$ and assume that it includes all singleton coalitions. Then $\mathscr{S}$ is conflict-free if and only if there exists a finite sequence $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$ of partitions of $N$ with the following properties:*

(*a*) *For any $S \in \mathscr{S}$, there exists $P \in \mathcal{O}$ with $S \in P$.*

(*b*) *For any $i \in \{0, \ldots, k-1\}$, $P_i$ is finer than $P_{i+1}$.*

*Proof.* Assume that $\mathscr{S}$ is conflict-free. We now construct a finite sequence $\mathcal{O}$ satisfying the properties above. First, let $P_0$ consist of all singleton coalitions. Let $P_1$ include all two-player coalitions in $\mathscr{S}$. For some $i \in N$, if $i \notin S$ for some $S \in P_1$, then let $\{i\}$ be included in $P_1$ as well. Then $P_1$ turns out to be a partition of $N$. Let $P_2$ include all three-player coalitions in $\mathscr{S}$. For some $S \in P_1$, if $S \not\subset S'$ for some $S' \in P_2$, then let $S$ be included in $P_2$ as well. Then $P_2$ turns out to be a partition of $N$. This process continues until there is no coalition remaining in $\mathscr{S}$. Accordingly, both (a) and (b) are satisfied by construction.

Conversely, take any two coalitions $S$ and $S'$ such that $S \in P$ and $S' \in P'$ for some $P, P' \in \mathcal{O}$. If $P = P'$, then $S \cap S' = \varnothing$. If $P \neq P'$, then one of them is finer than the other. Without loss of generality, assume that $P$ is finer than $P'$. Then there exists $S'' \in P'$ such that $S \subset S''$. If $S' = S''$, then $S \subset S'$. If not, noting that $S' \cap S'' = \varnothing$, we have $S \cap S' = \varnothing$. Hence $\mathscr{S}$ is conflict-free. $\square$

We define an *organization* $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$ of $N$ as an ordered collection of partitions of $N$ with properties (*a*) and (*b*) above, where $P_0 = \{\{1\}, \ldots, \{|N|\}\}$. Now, for a given organization $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$, let

$$\mathscr{S}^{\mathcal{O}} = \{S \subset N \mid \exists P \in \mathcal{O} \text{ such that } S \in P\}. \tag{3.3}$$

Given a partition $P \in \mathcal{O}$, let $P_-$ be the coarsest partition in $\mathcal{O}$ that is finer than $P$ and let $P_+$ be the finest partition in $\mathcal{O}$ that is coarser than $P$. To put it differently, $P_-$ is the layer just below $P$ and $P_+$ is the layer just above $P$.

Moreover, given a partition $P \in \mathcal{O}$ and a coalition $S \in P$, we define *suborganization* $\mathcal{O}_{S\in P}$ as an ordered collection $\{P'_0, P'_1, \ldots, P'_-\}$ of partitions of $S$ such that for each partition $P'_t$ therein, we have $P'_t \subset P_t$. Finally, we define

$$\rho^{\mathcal{O}}(S, P) = \{C \in \mathscr{S}^{\mathcal{O}} \mid C \in P_- \text{ and } C \subset S\}, \tag{3.4}$$

to be the coarsest partition in $\mathcal{O}_{S\in P}$; which is indeed a partition of $S$.

For a concrete example, consider the organization illustrated in Figure 1. Let $\mathcal{O} = \{P_0, P_1, P_2, P_3\}$ where

$$\begin{aligned}
P_0 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, \{10\}, \{11\}\},\\
P_1 &= \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}, \{9, 10\}, \{11\}\},\\
P_2 &= \{\{1, 2, 3, 4, 5\}, \{6, 7, 8\}, \{9, 10\}, \{11\}\}, \text{ and}
\end{aligned}$$

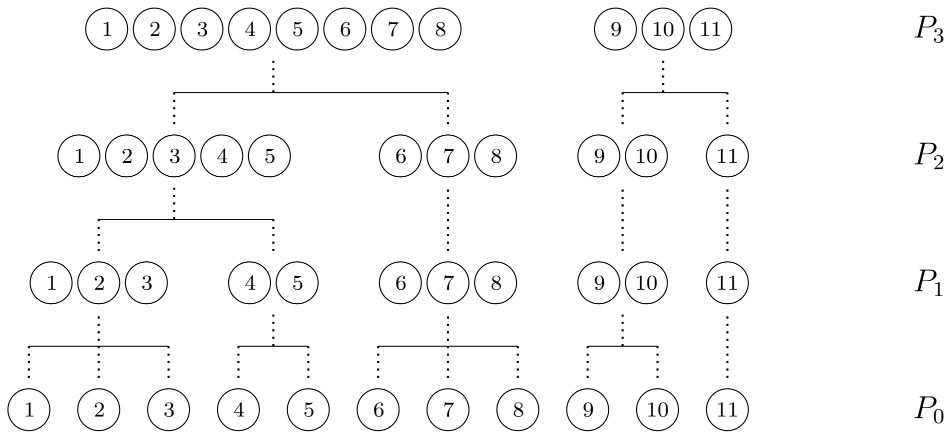

Figure 1: Example of an Organization

$$P_3 \quad = \quad \{\{1, 2, 3, 4, 5, 6, 7, 8\}, \{9, 10, 11\}\}.$$

Moreover, if we consider $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $P = P_3$, then the corresponding suborganization is $\mathcal{O}_{S \in P} = \{P'_0, P'_1, P'_-\}$ such that

$$\begin{aligned} P'_0 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}\}, \\ P'_1 &= \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}, \text{ and} \\ P'_- &= \{\{1, 2, 3, 4, 5\}, \{6, 7, 8\}\}. \end{aligned}$$

## 3.2 Organizational Nash Equilibrium

In this subsection, we present a new refinement of Nash equilibrium with no vertical or horizontal conflict of interest. Consider a normal form game $\Gamma$ and an organization $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$. Take any strategy profile $x \in X$. For any player $i \in N$, define $R_i^{\mathcal{O}}(x) = B_i(x, X_i)$. Then for any coalition $S \in P_1$, the sets $E_S^{\mathcal{O}}(x)$ and $R_S^{\mathcal{O}}(x)$ are defined as follows:

$$\begin{aligned} E_S^{\mathcal{O}}(x) &= \left\{ y \in X \;\middle|\; (y_S, x_{-S}) \in \bigcap_{\{i\} \in \rho^{\mathcal{O}}(S,P)} R_i^{\mathcal{O}}((y_S, x_{-S})) \right\} \\ R_S^{\mathcal{O}}(x) &= B_S\left(x, \left[E_S^{\mathcal{O}}(x)\right]_S\right) \end{aligned}$$

Moreover, for any coalition $S \in \mathscr{S}^{\mathcal{O}} \setminus P_1$, the sets $E_S^{\mathcal{O}}(x)$ and $R_S^{\mathcal{O}}(x)$ are inductively defined as follows:

$$\begin{aligned} E_S^{\mathcal{O}}(x) &= \left\{ y \in X \;\middle|\; (y_S, x_{-S}) \in \bigcap_{C \in \rho^{\mathcal{O}}(S,P)} R_C^{\mathcal{O}}((y_S, x_{-S})) \right\} \\ R_S^{\mathcal{O}}(x) &= B_S\left(x, \left[E_S^{\mathcal{O}}(x)\right]_S\right) \end{aligned}$$

Accordingly, for any $x \in X$, define

$$R^{\mathcal{O}}(x) = \bigcap_{S \in \mathscr{S}^{\mathcal{O}}} R_S^{\mathcal{O}}(x).$$

We call a strategy profile $x \in X$ satisfying $x \in R^{\mathcal{O}}(x)$ an *$\mathcal{O}$-organizational Nash equilibrium*, or simply an *organizational Nash equilibrium*. Let $\text{ONE}^{\mathcal{O}}(\Gamma)$ denote the set of $\mathcal{O}$-organizational Nash equilibria of $\Gamma$.

We first show that ONE is indeed a refinement of Nash equilibrium.

**Proposition 3.4.** *For any normal form game $\Gamma$ and any organization $\mathcal{O}$,*

$$\mathit{ONE}^{\mathcal{O}}(\Gamma) \subset \mathit{NE}(\Gamma)$$

*Proof.* Omitted. □

Note that given two organizations, the respective equilibrium sets may turn out to be very different. Yet as we show in the following, the sets of ONE coincide for *equivalent* organizations.

**Definition 3.1.** Two organizations $\mathcal{O}$ and $\mathcal{O}'$ are **equivalent** if $\mathscr{S}^{\mathcal{O}} = \mathscr{S}^{\mathcal{O}'}$.

**Remark 3.1.** Given a normal form game $\Gamma$ and two equivalent organizations $\mathcal{O}, \mathcal{O}'$:

$$\text{ONE}^{\mathcal{O}}(\Gamma) = \text{ONE}^{\mathcal{O}'}(\Gamma).$$

*Proof.* Consider any coalition $S \in \mathscr{S}^{\mathcal{O}}$. Since $\mathcal{O}$ and $\mathcal{O}'$ are equivalent, $S \in \mathscr{S}^{\mathcal{O}'}$. Moreover for each subcoalition $C \subset S$, if $C \in \mathscr{S}^{\mathcal{O}}$, then $C \in \mathscr{S}^{\mathcal{O}'}$. This implies that in organizations $\mathcal{O}$ and $\mathcal{O}'$, the coalition $S$ considers the rational responses of *the same* subcoalitions when making a joint decision. Then $R_S^{\mathcal{O}} \equiv R_S^{\mathcal{O}'}$. Since S is arbitrarily chosen, it follows that $R^{\mathcal{O}} \equiv R^{\mathcal{O}'}$. Hence the sets of ONE coincide. □

We now focus on the elimination of vertical and horizontal conflicts of interest. Although the former result in Proposition 3.5 is also valid for CPNE, the latter is only valid for ONE. The reason is that CPNE eliminates only vertical conflicts of interest, whereas ONE can eliminate both types of conflict of interest.

**Proposition 3.5.** *For any normal form game $\Gamma$ and any organization $\mathcal{O}$, there exists neither (i) vertical nor (ii) horizontal conflicts of interest within the analysis of ONE. Formally, $(i)$ given two coalitions $S, S' \in \mathscr{S}^{\mathcal{O}}$ such that $S' \subsetneq S$, if $(y_S, \cdot) \in R_S^{\mathcal{O}}(x)$ for some $x \in X$, then $(y_{S'}, \cdot) \in R_{S'}^{\mathcal{O}}(y_S, x_{-S})$; and $(ii)$ given two coalitions $S, S' \in \mathscr{S}^{\mathcal{O}}$ such that $S' \not\subset S$ and $S \not\subset S'$, for any $x \in X$: $R_S^{\mathcal{O}}(x) \cap R_{S'}^{\mathcal{O}}(x) \neq \varnothing$.*

*Proof.* For $(i)$, consider any coalition $S \in P_1$ and set $S' = \{i\}$ for some member $i \in S$. Take any $x \in X$ and any $(y_S, \cdot) \in R_S^{\mathcal{O}}(x)$. Noting that $R_S^{\mathcal{O}}(x) = B_S\left(x, \left[E_S^{\mathcal{O}}(x)\right]_S\right)$ by definition, we find that

$$(y_S, \cdot) \in \bigcap_{i \in S} R_i^{\mathcal{O}}((y_S, x_{-S})).$$

This implies that $(y_{S'}, \cdot) \in R^{\mathcal{O}}_{S'}(y_S, x_{-S})$.

Now, consider any coalition $S \in P_2$. If $S' \subset S$ is a singleton, then the result similarly follows. If not, then $S' \in P_1$. Take any $x \in X$ and any $(y_S, \cdot) \in R^{\mathcal{O}}_S(x)$. Noting that $R^{\mathcal{O}}_S(x) = B_S\left(x, \left[E^{\mathcal{O}}_S(x)\right]_S\right)$ by definition, we find that

$$(y_S, \cdot) \in \bigcap_{\substack{C \subsetneq S \\ C \in P_0 \cup P_1}} R^{\mathcal{O}}_C((y_S, x_{-S})).$$

This implies that $(y_{S'}, \cdot) \in R^{\mathcal{O}}_{S'}(y_S, x_{-S})$.

The rest follows recursively.

As for $(ii)$, consider two coalitions $S, S' \in \mathscr{S}^{\mathcal{O}}$ such that $S' \not\subset S$ and $S \not\subset S'$. By the definition of organizations, $S$ and $S'$ must be disjoint. The proof concludes with the observation that each $R^{\mathcal{O}}_S$ concerns only the relevant part of the strategy profiles for $S$; i.e., if $(y_S, y_{-S}) \in R^{\mathcal{O}}_S(x)$, then for every $y'_{-S} \in X_{-S}$: $(y_S, y'_{-S}) \in R^{\mathcal{O}}_S(x)$. □

According to the definition of SNE, any coalition of players can jointly deviate to any of their joint strategy profiles. On the other hand, our organizational refinement restricts the set of coalitions that can deviate and the set of strategy profiles to which a particular coalition can deviate. Accordingly, our notion of ONE turns out to be weaker than the notion of SNE.[8]

**Proposition 3.6.** *For any normal form game $\Gamma$ and organization $\mathcal{O} = \{P_0, P_1, \ldots, P_k\}$,*

$$SNE(\Gamma) \subset ONE^{\mathcal{O}}(\Gamma).$$

*Proof.* Take any $x^* \in \mathrm{SNE}(\Gamma)$. Suppose that $x^*$ is not an $\mathcal{O}$-organizational Nash equilibrium of $\Gamma$. We then see that there should exist some partition(s) $P_t \in \mathcal{O}$ such that there exists some $S \in P_t$ satisfying $x^*_S \notin R^{\mathcal{O}}_S(x^*)$. We take the one with the smallest $t$ and denote it by $\bar{P}$. The corresponding coalition is denoted by $\bar{S}$.

Then for every $S' \in \bar{P}_-$ with $S' \subset \bar{S}$:

$$x^*_{S'} \in \mathrm{ONE}^{\mathcal{O}_{S' \in \bar{P}_-}}(\Gamma_{S'}|_{x^*_{-S'}}).$$

We thus obtain $y_{\bar{S}} \in X_{\bar{S}}$ such that

$$\begin{aligned} &(i)\ \forall i \in \bar{S} : u_i(y_{\bar{S}}, x^*_{-\bar{S}}) > u_i(x^*) \text{ and} \\ &(ii)\ \forall S' \in \bar{P}_- \text{ with } S' \subset \bar{S} : y_{S'} \in \mathrm{ONE}^{\mathcal{O}_{S' \in \bar{P}_-}}(\Gamma_{S'}|_{x^*_{-S'}}). \end{aligned}$$

We then conclude that $x^*_{\bar{S}}$ is not Pareto optimal for $\bar{S}$, given that the complementary coalition chooses $x^*_{-\bar{S}}$. This is a contradiction; which completes the proof that $x^*$ is an $\mathcal{O}$-organizational Nash equilibrium. □

[8] This also implies that an ONE exists for every normal form game that possesses a SNE.

## 3.3 Illustrative Examples

From Proposition 3.6, we understand the relation between the predictions of ONE and SNE. In this section we consider two examples of normal form games to better understand how ONE refines the set of Nash equilibria and how the predictions of ONE differ from the predictions of CPNE. We will further discuss the insights gained from these equilibrium analyses later, in our Concluding Remarks.

We first recall the three-player normal form game given in Table 4, where we observed horizontal conflicts of interest. This game has three Nash equilibria: $(z_1, x_2, x_3)$, $(y_1, y_2, x_3)$, and $(x_1, z_2, y_3)$. The coalition $\{1, 2\}$ deviates from the first and the second, the coalition $\{2, 3\}$ deviates from the second and the third, and the coalition $\{1, 3\}$ deviates from the first and the third. Further note that all of these deviations are self-enforcing. Therefore, this game possesses no SNE or CPNE.

Table 4 [Revisited]

| $x_3$ | $x_2$ | $y_2$ | $z_2$ |
|---|---|---|---|
| $z_1$ | 1, 5, 8 | 0, 2, 0 | 7, 2, 3 |
| $y_1$ | 0, 0, 3 | 5, 1, 8 | 4, 0, 0 |
| $x_1$ | 0, 4, 3 | 0, 7, 3 | 7, 7, 0 |

| $y_3$ | $x_2$ | $y_2$ | $z_2$ |
|---|---|---|---|
| $z_1$ | 2, 0, 4 | 0, 1, 0 | 0, 0, 0 |
| $y_1$ | 0, 2, 9 | 1, 0, 4 | 0, 0, 0 |
| $x_1$ | 2, 0, 9 | 0, 0, 0 | 6, 6, 2 |

In this example, we consider

$$\begin{aligned}\mathcal{O}_1 &= \{P_0, \{\{1,2\},\{3\}\}, \{N\}\},\\ \mathcal{O}_2 &= \{P_0, \{\{1\},\{2,3\}\}, \{N\}\},\\ \mathcal{O}_3 &= \{P_0, \{\{1,3\},\{2\}\}, \{N\}\};\end{aligned}$$

and we find the following sets of ONE:

$$\begin{aligned}\text{ONE}^{\mathcal{O}_1}(\cdot) &= \{(x_1, z_2, y_3)\},\\ \text{ONE}^{\mathcal{O}_2}(\cdot) &= \{(z_1, x_2, x_3)\},\\ \text{ONE}^{\mathcal{O}_3}(\cdot) &= \{(y_1, y_2, x_3)\}.\end{aligned}$$

The arguments are as follows: In $O_1$, the only active two-player coalition is $\{1, 2\}$. This coalition deviates from $(z_1, x_2, x_3)$ and $(y_1, y_2, x_3)$, but not from $(x_1, z_2, y_3)$. Neither of the other two-player coalitions can deviate from these strategy profiles, since they are not formed. Noting that every Nash equilibrium is Pareto optimal for the grand coalition, we conclude that $(x_1, z_2, y_3)$ is the unique coalitionally stable outcome for this particular organization. As for $O_2$ and $O_3$, similar reasoning applies.

Note here that, as the example above illustrates, it may be the case that each Nash equilibrium is supported by some organizational structure as the unique coalitionally stable outcome. This does not necessarily hold, however, for all normal form games, given that the notions of ONE and CPNE coincide in two-player games:

**Remark 3.2.** ONE and CPNE coincide in a two-player normal form game because the only non-trivial organization includes all possible coalitions.

We now provide an example in which the non-empty sets of CPNE and ONE are disjoint. This observation implies that one equilibrium set does not necessarily include the other. To illustrate, we revisit the normal form game given in Table 1:

Table 1 [Revisited]

| $x_3$ | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 0, 0, 0 | 0, 0, 0 |
| $x_1$ | 1, 1, 1 | 0, 0, 0 |

| $y_3$ | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 0, 0, 0 | 2, 2, 2 |
| $x_1$ | 3, 3, 0 | 0, 0, 0 |

Recall that there are two Nash equilibria: $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$. And there exists a unique CPNE: $(x_1, x_2, x_3)$, since the coalition $\{1, 2\}$ makes a self-enforcing deviation from $(y_1, y_2, y_3)$; whereas the subcoalition $\{1, 2\}$ blocks the deviation of the grand coalition from $(x_1, x_2, x_3)$ to $(y_1, y_2, y_3)$, as the subcoalition would further deviate from $(y_1, y_2, y_3)$. On the other hand, if we analyze ONE of this game for the organization $\{P_0, \{\{1\}, \{2, 3\}\}, \{N\}\}$, the unique ONE turns out to be $(y_1, y_2, y_3)$. This result follows because now that the coalition $\{1, 2\}$ is inactive, they would not deviate from $(y_1, y_2, y_3)$ and could not block the deviation of the grand coalition from $(x_1, x_2, x_3)$ to $(y_1, y_2, y_3)$. By a similar reasoning, the unique ONE of this game would be $(y_1, y_2, y_3)$ also for the organizations $\{P_0, \{\{1, 3\}, \{2\}\}, \{N\}\}$ and $\{P_0, \{N\}\}$. Note also how this example highlights the importance and usefulness of our organizational refinement. Apparently, the unique ONE strictly Pareto dominates the unique CPNE.

# 4 The Results

## 4.1 Existence of Equilibrium

Although the formulation of our organizational refinement eliminates both types of conflicts of interest, a normal form game might not have an ONE for some organizations. For an example, consider the normal form game given in Table 5 which has a unique Nash equilibrium: $x \equiv (x_1, x_2, x_3)$.

Table 5

| $x_3$ | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 0, 0, 0 | 2, 2, 0 |
| $x_1$ | 1, 1, 1 | 0, 0, 0 |

| $y_3$ | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 0, 1, 1 | 1, 0, 1 |
| $x_1$ | 1, 0, 0 | 0, 1, 1 |

Considering the organization $\mathcal{O} = \{P_0, \{\{1,2\},\{3\}\}\}$, we have $R_1^{\mathcal{O}}(x) = (x_1, \cdot, \cdot)$, $R_2^{\mathcal{O}}(x) = (\cdot, x_2, \cdot)$, and $R_{\{1,2\}}^{\mathcal{O}}(x) = (y_1, y_2, \cdot)$. Then $R^{\mathcal{O}}(x) = \varnothing$. Since $x$ is the unique Nash equilibrium, the non-existence of ONE follows.

The foregoing observation naturally compels us to find classes of normal form games where an ONE exists. In this section of the paper, we prove the existence of our organizational refinement in a subclass of games with strategic complementarities (see Topkis, 1998; Amir, 2005; Vives, 2005, among others).[9] The following are the definitions that will be utilized throughout this subsection.

A set is a *lattice* if it contains the supremum and the infimum of every pair of its elements. A lattice is *complete* if each non-empty subset has a supremum and an infimum.[10] Moreover, a subset $Y$ of a lattice $X$ is a *subcomplete sublattice* of $X$ if for every non-empty subset $Y'$ of $Y$, the supremum of $Y'$ and the infimum of $Y'$ exist and are contained in $Y$. Let $X$ be a lattice and $T$ be a partial order. A function $f : X \to \mathbb{R}$ is called *quasi-supermodular* if for every $x, y \in X$: $f(x) \geq f(x \wedge y)$ implies that $f(x \vee y) \geq f(y)$ and $f(x) > f(x \wedge y)$ implies that $f(x \vee y) \geq f(y)$. We say that a function $f : X \times T \to \mathbb{R}$ satisfies the *single crossing property* in $(x, t)$ if for every $x, x' \in X$ and $t, t \in T$ with $x > x'$ and $t > t'$: $f(x, t') \geq f(x', t')$ implies that $f(x, t) \geq f(x', t)$ and $f(x, t') > f(x', t')$ implies that $f(x, t) > f(x', t)$.

The following definition of games with strategic complementarities is provided by Milgrom and Shannon (1994) and Milgrom and Roberts (1996).

**Definition 4.1.** A normal form game $\Gamma$ is a **game with strategic complementarities** if for every $i \in N$: *(i)* $X_i$ is a non-empty complete lattice; *(ii)* $u_i$ is upper semi-continuous in $x_i$ and continuous in $x_{-i}$; and *(iii)* $u_i$ is quasi-supermodular in $x_i$ and has the single crossing property in $(x_i, x_{-i})$.

Milgrom and Shannon (1994) show that the smallest and the largest serially undominated strategy profiles[11] exist in a game with strategic complementarities and are respectively the smallest and the largest Nash equilibria of the game (see their Theorem 12). Furthermore, as shown by Milgrom and Roberts (1996), an additional monotonicity assumption would suffice for the existence of CPNE in a subclass of games with strategic complementarities. In particular, these authors assume that each utility function $u_i$ is non-decreasing/non-increasing in $x_{-i}$ (see their Theorem 2). In the following, we prove the existence of our organizational refinement by weakening this monotonicity assumption.

[9] Games with strategic complemetarities are commonly utilized in the literature both for the existence of Nash equilibrium (see Zhou, 1994; Echenique, 2005; Calciano, 2007; Keskin et al., 2014, among others) and for the existence of some of the refinements of Nash equilibrium; such as minimally altruistic Nash equilibrium (see Karagozoglu et al., 2013), perfect equilibrium (see Carbonell-Nicolau and McLean, 2014), and strong Berge equilibrium (see Keskin and Saglam, 2014).

[10] Note that a complete lattice $X$ is compact in its interval topology which is the topology generated by taking the closed intervals, $[y, z] = \{x \in X : y \leq x \leq z\}$ with $y, z \in X$ as a subbasis of closed sets (see Birkhoff (1967)).

[11] A strategy profile is said to be *serially undominated* if it survives the iterated elimination of strictly dominated strategies.

**Proposition 4.1.** *Consider a game with strategic complementarities* $\Gamma$ *and an organization* $\mathcal{O}$. *Assume that for every* $i \in N$ *and every* $S \in \mathscr{S}^{\mathcal{O}}$ *that includes* $i$*: either* $(i)$ $u_i$ *is non-decreasing in* $x_{-S}$*, or* $(ii)$ $u_i$ *is non-increasing in* $x_{-S}$*. Then there exists an* $\mathcal{O}$*-organizational Nash equilibrium for this game.*

*Proof.* Assume case $(i)$ above and consider the largest Nash equilibrium of the game, denoted by $x^*$. As we know from Milgrom and Shannon (1994), $x^*$ is also the largest serially undominated strategy profile in this game. Consider any coalition $S \in P_1$ and the corresponding reduced game $\Gamma_S|_{x^*_{-S}} = (S, (X_i)_{i \in S}, (v_i)_{i \in S})$. By definition, this is a reduced game with strategic complementarities in which each $v_i$ is non-decreasing in $x_{-i}$. We also know that $x^*_S$ is a Nash equilibrium for $\Gamma_S|_{x^*_{-S}}$; which means that $x^*_S$ would survive the iterated elimination of strictly dominated strategies in the reduced game. As a matter of fact, $x^*_S$ turns out to be the largest serially undominated strategy profile in this game.[12] It then follows from Milgrom and Shannon (1994) that $x^*_S$ is the largest Nash equilibrium for the reduced game. Applying Theorem A2 from Milgrom and Roberts (1996), each subcoalition $C \subset S$ prefers playing $x^*_C$ to any other strategy profile in the reduced game. Their theorem surely applies to the coalition $S$ itself. Accordingly, for any Nash equilibrium $y_S$ of the reduced game $\Gamma_S|_{x^*_{-S}}$, we have for every $i \in S$: $u_i(y_S, x^*_{-S}) \leq u_i(x^*_S)$. Therefore, $x^*_S$ is a coalitional best response for $S$. Since $S$ is arbitrarily chosen, for every coalition $S \in P_1$: $(x^*_S, \cdot) \in R^{\mathcal{O}}_S(x^*)$.

Now, consider any coalition $S' \in P_2$ and the corresponding reduced game $\Gamma_{S'}|_{x^*_{-S'}}$. Noting that $\mathcal{O}_{S' \in P_2}$ is the suborganization for this coalition and considering the arguments above, we know that $x^*_{S'}$ is an $\mathcal{O}_{S' \in P_2}$-organizational Nash equilibrium of this game. This follows because for every member $i \in S'$: $(x^*_i, \cdot) \in R^{\mathcal{O}}_i(x^*)$ as well as for every active subcoalition $C' \subset S'$ in this suborganization: $(x^*_{C'}, \cdot) \in R^{\mathcal{O}}_{C'}(x^*)$. Applying Theorem A2 from Milgrom and Roberts (1996) once again, we conclude that the coalition $S'$ prefers playing $x^*_{S'}$ to any other ONE in the reduced game $\Gamma_{S'}|_{x^*_{-S'}}$. It similarly follows that for every coalition $S' \in P_2$: $(x^*_{S'}, \cdot) \in R^{\mathcal{O}}_{S'}(x^*)$.

Finally, it recursively follows that for every coalition $S'' \in \mathscr{S}^{\mathcal{O}}$: $(x^*_{S''}, \cdot) \in R^{\mathcal{O}}_{S''}(x^*)$. Hence, $x^* \in R^{\mathcal{O}}(x^*)$, which in turn implies that $x^* \in \text{ONE}(\Gamma)$.

Arguments for case $(ii)$ similarly follow. □

In the following normal form game given in Table 6, we can demonstrate how the existence result works:

Table 6

$x_3$

| | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 3, 4, 0 | 5, 5, 0 |
| $x_1$ | 4, 4, 1 | 5, 3, 3 |

$y_3$

| | $x_2$ | $y_2$ |
|---|---|---|
| $y_1$ | 0, 1, 1 | 2, 2, 2 |
| $x_1$ | 1, 1, 0 | 1, 0, 2 |

[12] The reader is referred to the Appendix for the proof of this claim.

This example is a game with strategic complementarities since each utility function $u_i$ is quasi-supermodular in $x_i$ and has the single crossing property in $(x_i, x_{-i})$. Furthermore, $u_1$ is non-decreasing in $x_2$ and $u_2$ is non-decreasing in $x_1$. Also note that $u_3$ is not monotone in $x_1$ or $x_2$, since $(x_1, y_2, x_3)$ yields the highest utility for Player 3. Accordingly, this normal form game satisfies the conditions of our existence result for the organization $\mathcal{O}^* = \{P_0, \{\{1,2\},\{3\}\}\}$, but not for the other possible organizations. Neither does it satisfy the conditions for the existence of CPNE provided by Milgrom and Roberts (1996). There are two Nash equilibria: $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$. Now, analyzing the set of $\mathcal{O}^*$-organizational Nash equilibria, we see that the coalition $\{1,2\}$ deviates from $(x_1, x_2, x_3)$ to $(y_1, y_2)$ when Player 3 sticks to $x_3$. Noting that they would not deviate from the other Nash equilibrium, we find that $(y_1, y_2, y_3)$ is the unique ONE for this game. We also know, meanwhile, that no CPNE exists, because $(i)$ the coalition $\{1,2\}$ still makes the aforementioned deviation and $(ii)$ the coalition $\{1,3\}$ makes a self-enforcing deviation from $(y_1, y_2, y_3)$ to $(x_1, x_3)$ when Player 2 sticks to $y_2$.

Table 7

| | | $x_3$ | | $y_3$ | |
|---|---|---|---|---|---|
| | | $x_2$ | $y_2$ | $x_2$ | $y_2$ |
| $y_4$ | $y_1$ | 3, 3, 2, 0 | 3, 3, 0, 2 | 1, 0, 1, 1 | 2, 2, 1, 2 |
| | $x_1$ | 3, 3, 2, 0 | 3, 3, 2, 0 | 0, 0, 1, 1 | 0, 1, 1, 1 |
| $x_4$ | $y_1$ | 2, 5, 1, 1 | 5, 5, 1, 0 | 4, 3, 0, 0 | 5, 5, 0, 0 |
| | $x_1$ | 4, 4, 1, 1 | 5, 2, 1, 1 | 3, 3, 0, 0 | 3, 4, 0, 0 |

As another example satisfying the conditions of Proposition 4.1, consider the four-player normal form game given in Table 7. Once again, $u_1$ is non-decreasing in $x_2$ and $u_2$ is non-decreasing in $x_1$. We can also see that neither $u_3$ nor $u_4$ is monotone in either of the other players' strategies. This game has two Nash equilibria: $(x_1, x_2, x_3, x_4)$ and $(y_1, y_2, y_3, y_4)$. Considering the organization $\mathcal{O}^*$ above, we note that the coalition {1,2} deviates from $(x_1, x_2, x_3, x_4)$ to $(y_1, y_2)$ when Players 3 and 4 stick to $(x_3, x_4)$, and we find that $(y_1, y_2, y_3, y_4)$ is the unique ONE for this game. Now, if we consider a greater organization by adding $P_2 = \{\{1,2,3\},\{4\}\}$ into the existing organization $\mathcal{O}^*$, we see that the unique ONE ceases to exist. This is because the coalition $\{1,2,3\}$ deviates from $(y_1, y_2, y_3, y_4)$ to $(x_1, x_2, x_3)$ when Player 4 sticks to $y_4$. As we see in this particular example, adding a new coalition for which the utility functions of its members do not have a monotonicity relation as described in Proposition 4.1 *might* lead to the non-existence of equilibrium.

Another interesting note regarding this example is that if one considers the organization $\{P_0, \{\{1,2,3\},\{4\}\}\}$, then the weakly Pareto optimal Nash equilibrium

$(x_1, x_2, x_3, x_4)$ is realized as the unique ONE. Accordingly, we can claim that removing small coalitions from an organization or replacing small coalitions in an organization with larger coalitions *might* turn out to be socially beneficial.

## 4.2 A Monotonicity Property

We start with a nice result indicating that the introduced refinement structures follow in a monotonic fashion. In a normal form game, given two organizations $\mathcal{O}$ and $\mathcal{O}'$, we say that $\mathcal{O}$ is *greater* than $\mathcal{O}'$ if $(i)$ $P \in \mathcal{O}'$ implies that $P \in \mathcal{O}$ and $(ii)$ $\big[P' \in \mathcal{O}$ and $P' \notin \mathcal{O}'\big]$ implies that $P'$ is coarser than the coarsest partition in $\mathcal{O}'$. We now show that the equilibrium set is more refined for *greater* organizations.[13]

**Proposition 4.2.** *For any normal form game $\Gamma$, if an organization $\mathcal{O}$ is greater than another organization $\mathcal{O}'$, then*

$$ONE^{\mathcal{O}}(\Gamma) \subset ONE^{\mathcal{O}'}(\Gamma).$$

*Proof.* Take any $x^* \in \mathrm{ONE}^{\mathcal{O}}(\Gamma)$. By definition, $x^* \in R^{\mathcal{O}}(x^*)$. This implies that for every $S \in \mathscr{S}^{\mathcal{O}}$: $(x^*_S, \cdot) \in R^{\mathcal{O}}_S(x^*)$. Since $\mathcal{O}$ is greater than $\mathcal{O}'$, we know that $\mathscr{S}^{\mathcal{O}'} \subset \mathscr{S}^{\mathcal{O}}$ and that for every $S \in \mathscr{S}^{\mathcal{O}} \setminus \mathscr{S}^{\mathcal{O}'}$: $\nexists S' \in \mathscr{S}^{\mathcal{O}'}$ such that $S \subset S'$. Accordingly, for every $S' \in \mathscr{S}^{\mathcal{O}'}$: $(x^*_{S'}, \cdot) \in R^{\mathcal{O}'}_{S'}(x^*)$. Therefore, $x^* \in R^{\mathcal{O}'}(x^*)$; i.e., $x^* \in \mathrm{ONE}^{\mathcal{O}'}(\Gamma)$. It thus follows that $\mathrm{ONE}^{\mathcal{O}}(\Gamma) \subset \mathrm{ONE}^{\mathcal{O}'}(\Gamma)$. $\square$

This monotonicity property leads to the following observation.

**Corollary 4.1.** *Consider a normal form game $\Gamma$ that possesses a Nash equilibrium. Take any increasing sequence of organizations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_t$ such that $\mathcal{O}_1 = \{P_0\}$ and for any $i \in \{1, \ldots, t-1\}$: $\mathcal{O}_{i+1}$ is greater than $\mathcal{O}_i$. We have*

$$ONE^{\mathcal{O}_{i+1}}(\Gamma) \subset ONE^{\mathcal{O}_i}(\Gamma)$$

*for any $i \in \{1, \ldots, t-1\}$. In addition, the sequence has a maximum organization for which the set of ONE is non-empty.*

In normal form games with multiple equilibria, one can form the second layer of an organization in order to decrease the number of equilibria. If the multiplicity is still preserved, one can continue with the upper layers of the organization until the organization is about to become "too big" such that it will fail to take an action (or, reach an equilibrium). The corollary above indicates that the formation process stops at a unique point before an ONE ceases to exist. The following example demonstrates.

[13] It is important here that we do not compare any particular partitions when comparing two organizations. Our definition of "being greater" indicates that an organization is greater than another if the former completely preserves the structure of the latter and additionally includes coarser partitions. Alternative definitions *may* yield different results.

**Example 4.1.** Consider the five-player normal form game given in Table 8 for which there are three Nash equilibria: $(x_1, x_2, y_3, y_4, x_5)$, $(y_1, y_2, x_3, y_4, x_5)$, $(y_1, y_2, y_3, x_4, x_5)$. We now set

$$\begin{aligned} P_0 &= \{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}, \\ P_1 &= \{\{1, 2\}, \{3\}, \{4\}, \{5\}\}, \\ P_2 &= \{\{1, 2, 3\}, \{4\}, \{5\}\}, \text{ and} \\ P_3 &= \{\{1, 2, 3, 4\}, \{5\}\}. \end{aligned}$$

Table 8

| | | | $x_3$ | | $y_3$ | |
|---|---|---|---|---|---|---|
| | | | $x_2$ | $y_2$ | $x_2$ | $y_2$ |
| $y_5$ | $y_4$ | $y_1$ | 1, 0, 1, 0, 0 | 0, 1, 1, 0, 0 | 1, 0, 0, 1, 0 | 0, 1, 0, 1, 0 |
| | | $x_1$ | 0, 1, 1, 0, 0 | 1, 0, 1, 0, 0 | 0, 1, 0, 1, 0 | 1, 0, 0, 1, 0 |
| | $x_4$ | $y_1$ | 1, 0, 0, 1, 0 | 0, 1, 0, 1, 0 | 1, 0, 1, 0, 0 | 0, 1, 1, 0, 0 |
| | | $x_1$ | 0, 1, 0, 1, 2 | 1, 0, 0, 1, 0 | 0, 1, 1, 0, 0 | 1, 0, 1, 0, 0 |
| $x_5$ | $y_4$ | $y_1$ | 0, 0, 0, 1, 1 | 1, 1, 1, 1, 1 | 0, 0, 1, 0, 1 | 1, 1, 0, 1, 1 |
| | | $x_1$ | 2, 2, 0, 0, 1 | 0, 0, 0, 1, 1 | 3, 3, 3, 3, 1 | 0, 0, 1, 0, 1 |
| | $x_4$ | $y_1$ | 0, 0, 1, 0, 1 | 2, 2, 0, 0, 1 | 0, 0, 0, 1, 1 | 2, 2, 2, 2, 1 |
| | | $x_1$ | 4, 4, 4, 4, 1 | 0, 0, 1, 0, 1 | 1, 1, 0, 0, 1 | 0, 0, 0, 1, 1 |

First consider $\mathcal{O}_1 = \{P_0\}$. Here, the set of ONE surely coincides with the set of Nash equilibria. Next, consider $\mathcal{O}_2 = \{P_0, P_1\}$. We see that the coalition $\{1, 2\}$ deviates to $(x_1, x_2)$ when Players 3, 4, and 5 stick to $(x_3, y_4, x_5)$. Since there is no further deviation, $(y_1, y_2, x_3, y_4, x_5)$ is not coalitionally stable in the sense of ONE. And since there is no other deviation, the remaining two Nash equilibria turn out to be ONE for this game:

$$(x_1, x_2, y_3, y_4, x_5), (y_1, y_2, y_3, x_4, x_5).$$

If we consider $\mathcal{O}_3 = \{P_0, P_1, P_2\}$, we see that the coalition $\{1, 2, 3\}$ deviates to $(x_1, x_2, x_3)$ when Players 4 and 5 stick to $(x_4, x_5)$. Since there is no further deviation, $(y_1, y_2, y_3, x_4, x_5)$ is not coalitionally stable in the sense of ONE. And since there is no other deviation, there exists a unique ONE:

$$(x_1, x_2, y_3, y_4, x_5).$$

Finally, when we consider $\mathcal{O}_4 = \{P_0, P_1, P_2, P_3\}$, the coalition $\{1, 2, 3, 4\}$ deviates to $(x_1, x_2, x_3, x_4)$ when Player 5 sticks to $x_5$. Since there is no further deviation, the set of ONE turns out to be empty.

It is also worth noting that out of all of the deviations above, the five-player normal form game has neither a SNE nor a CPNE.

The following observation utilizes a different perspective.

**Corollary 4.2.** *Consider a normal form game $\Gamma$ that possesses a Nash equilibrium. Take any organization $\bar{\mathcal{O}}$ and any decreasing sequence of organizations $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_t$ such that $\mathcal{O}_1 = \bar{\mathcal{O}}$, $\mathcal{O}_t = \{P_0\}$, and for any $i \in \{2, \ldots, t\}$: $\mathcal{O}_i$ is greater than $\mathcal{O}_{i-1}$. We have*

$$ONE^{\mathcal{O}_{i-1}}(\Gamma) \subset ONE^{\mathcal{O}_i}(\Gamma)$$

*for any $i \in \{2, \ldots, t\}$. In addition, the sequence has a maximum organization for which the set of ONE is non-empty.*

In some normal form games, there may exist a "too big" organization that will fail to take an action (or, reach an equilibrium). In such cases, it might help to dissolve all of the coalitions in the final layer of the organization and play the game as a *smaller* organization. The corollary above indicates that the dissolving process stops at a unique point for which the normal form game possesses an ONE.

# 5 Concluding Remarks

We have studied cases in which some coalitions are not or cannot be formed. Taking the organizational structures as given, we have introduced a refinement of Nash equilibrium. We have showed the existence of equilibria in certain classes of games. Moreover, through remarks and examples, we have further analyzed how our notion refines the set of Nash equilibria.

Organizational refinements can lead to many interesting and fruitful questions. First, one can study the robustness of equilibrium. More precisely, some Nash equilibria may remain to be an equilibrium for any given organization, whereas some others may fail to be an equilibrium as soon as any organization is formed. One can therefore consider the former to be the most robust Nash equilibrium, and the latter to be the least robust. In that sense, any two Nash equilibria can be compared in terms of robustness to organizational deviations. Such an analysis may also provide general insights for certain classes of games.

Second, one can study the endogenous formation of organizations. There can be several methods for this exercise. Either (i) players may have pre-defined preferences over the set of coalitions/organizations that *somehow* induce organizational structures; or (ii) as the set of equilibria is now known for any given organization, players may form coalitions/organizations strategically by opting for organizational structures that yield *the best* set of equilibria. As an example, recall the game given in Table 4. Either of the three Nash equilibria can be captured by a certain organization. Among the three Nash equilibria, $(x_1, z_2, y_3)$ is Pareto optimal for the coalition $\{1, 2\}$. And Players 1 and 2 are able to reach there by forming the two-player coalition, thereby blocking the formation of $\{1, 3\}$ and $\{2, 3\}$.

Third, one can analyze policy implications. Notice that the formation of coalitional/organizational structures does not have to be strategic (as described above). For instance, a *social planner* may be interested in forming a socially optimal organization. As an example, recall the game given in Table 1. Given the existence of an organization for which the unique ONE strictly Pareto dominates the unique CPNE, a social planner would prefer to forbid the formation of $\{1, 2\}$.

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# Appendix

We first note that for a player $i \in N$, a strategy $x_i \in X_i$ is *strictly dominated* if there exists another strategy $x_i' \in X_i$ such that for every $x_{-i} \in X_{-i}$:

$$u_i(x_i', x_{-i}) > u_i(x_i, x_{-i}).$$

The following lemma is used in the proof of Proposition 4.1.

**Lemma 5.1.** *In a normal form game $\Gamma$, let $x^*$ be a Nash equilibrium which is also the largest serially undominated strategy profile. Then $x_S^*$ turns out to be the largest serially undominated strategy profile of the reduced game $\Gamma_S|_{x_{-S}^*}$ for any coalition $S \subset N$.*

*Proof.* Take any coalition $S \subset N$. Note that $x_S^*$ is a Nash equilibrium of the reduced game $\Gamma_S|_{x_{-S}^*}$. Thus, it is a serially undominated strategy profile.

We now describe a particular procedure of iterated elimination of strictly dominated strategies: We start with $\Gamma^0 \equiv \Gamma$. At stage 1, only Player 1's dominated strategies are eliminated. The resulting game is labeled as $\Gamma^1$. At stage 2, only Player 2's dominated strategies are eliminated. The resulting game is labeled as $\Gamma^2$. After each player has one elimination stage, we reach $\Gamma^n$. From stage $n+1$ onwards, the same procedure follows. More generally, for any $k \in \mathbb{N}_0$, only Player $i$'s dominated strategies are eliminated in stage $i + kn$. The procedure continues until $\Gamma^\infty$.

Without loss of generality, assume that Player 1 is a member of $S$ and that $x_1 \in X_1$ is strictly dominated by some $x_1' \in X_1$. In the reduced game $\Gamma_S|_{x_{-S}^*}$, we have

$$u_1(x_1', x_{S\setminus\{1\}}, x_{-S}^*) > u_1(x_1, x_{S\setminus\{1\}}, x_{-S}^*)$$

for every $x_{S\setminus\{1\}} \in X_{S\setminus\{1\}}$. This shows that $x_1$ remains to be strictly dominated in the reduced game.

Now, we start with $\Gamma_S^0 \equiv \Gamma_S|_{x_{-S}^*}$. At stage 1, only Player 1's dominated strategies in $\Gamma^0$ are eliminated. The resulting game is labeled as $\Gamma_S^1$. Notice that Player 1 *may* have additional dominated strategies in the reduced game, but even so, we do not eliminate those at this stage. Notice further that $\Gamma_S^1$ is a reduced game of $\Gamma^1$. For the next stage, if Player 2 is a member of $S$, then only Player 2's dominated strategies in $\Gamma^1$ are eliminated and the resulting game is labeled as $\Gamma_S^2$; but if otherwise, then we simply set $\Gamma_S^2 = \Gamma_S^1$. Notice that, in either case, $\Gamma_S^2$ becomes a reduced game of $\Gamma^2$. This procedure eventually yields a reduced game $\Gamma_S^\infty$ of $\Gamma^\infty$.

Given a player $i \in S$, we know that whichever strategy $y_i \not\leq x_i^*$ is eliminated in some stage of the iterated elimination of strictly dominated strategies for $\Gamma$, the same strategy should be eliminated in the corresponding stage of the iterated elimination of strictly dominated strategies for $\Gamma_S|_{x_{-S}^*}$. Also knowing that $x_S^*$ is serially undominated, $x_S^*$ turns out to be the largest serially undominated strategy profile. $\square$