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A Simple Optimality-Based No-Bubble Theorem for Deterministic Sequential Economies with Strictly Monotone Preferences*

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Abstract

We establish a simple no-bubble theorem that applies to a wide range of deterministic sequential economies with infinitely lived agents. In particular, we show that asset bubbles never arise if at least one agent can reduce his asset holdings permanently from some period onward. Our no-bubble theorem is based on the optimal behavior of a single agent, requiring virtually no assumption beyond the strict monotonicity of preferences. The theorem is a substantial generalization of Kocherlakota's (1992, *Journal of Economic Theory* 57, 245–256) result on asset bubbles and short sales constraints.

Keywords: Asset bubbles; no-bubble theorem; sequential budget constraints; optimality; monotone preferences

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1 Introduction

Since the global financial crisis of 2007-2008, there has been a surge of interest in rational asset pricing bubbles, or simply "asset bubbles." Numerous economic mechanisms that give rise to asset bubbles are still being proposed. In constructing models of asset bubbles, it is important to understand the conditions under which asset bubbles do or do not exist. While the conditions for the existence of bubbles are mostly restricted to specific models, some general conditions for nonexistence are known.

Most of the results on the nonexistence of bubbles, or no-bubble theorems, for general equilibrium models can be grouped into two categories. A no-bubble theorem of the first category typically states that asset bubbles never arise if the present value of the aggregate endowment process is finite. Wilson's (1981, Theorem 2) result on the existence of a competitive equilibrium can be viewed as an earlier example of a no-bubble theorem of the first category. Santos and Woodford (1997, Theorems 3.1, 3.3) established celebrated no-bubble theorems of this category, which were extended by Huang and Werner (2000, Theorem 6.1) and Werner (2014, Remark 1, Theorem 1) to different settings.

Unlike these results, no-bubble theorems of the second category are mostly based on the optimal behavior of a single agent without relying on the present value of the aggregate endowment process. For example, in a deterministic economy with finitely many agents, Kocherlakota (1992, Proposition 3) showed that asset bubbles can be ruled out if at least one agent can reduce his asset holdings permanently from some period onward.[1] Obstfeld and Rogoff (1986) used a similar idea earlier to rule out deflationary equilibria in a money-in-the-utility-function model.[2] These results rely on the necessity of a transversality condition, and a fairly general no-bubble theorem based on the necessity of a transversality condition was shown by Kamihigashi (2001, p. 1007) for deterministic representative-agent models in continuous time.[3]

In this paper we establish a simple no-bubble theorem of the second category that can be used to rule out asset bubbles in an extremely wide range of deterministic models. We consider the problem of a single agent who

[1]Santos (2006) showed a similar result on the value of money in a general cash-in-advance economy.

[2]See Kamihigashi (2008a, 2008b) for results on asset bubbles in related models.

[3]See Kamihigashi (2002, 2003, 2005) for further results on necessity of transversality conditions.

faces sequential budget constraints and has strictly monotone preferences. We show that asset bubbles never arise if the agent can reduce his asset holdings permanently from some period onward. This result is a substantial generalization of Proposition 3 in Kocherlakota (1992). Our contribution is to show that this no-bubble theorem holds under extremely general conditions.

The rest of the paper is organized as follows. In Section 2 we present a single agent's problem along with necessary assumptions, and formally define asset bubbles. In Section 3 we offer several examples satisfying our assumptions. In Section 4 we state our no-bubble theorem and show several consequences. In Section 6 we offer some concluding comments. Longer proofs are relegated to the appendices.

2 The General Framework

2.1 Feasibility and Optimality

Time is discrete and denoted by $t \in \mathbb{Z}_+$. There is one consumption good and one asset that pays a dividend of d_t units of the consumption good in each period $t \in \mathbb{Z}_+$. Let p_t be the price of the asset in period $t \in \mathbb{Z}_+$. Consider an infinitely lived agent who faces the following constraints:

$$c_t + p_t s_t = y_t + (p_t + d_t)s_{t-1},\ c_t \geq 0, \quad \forall t \in \mathbb{Z}_+, \tag{2.1}$$

$$s \in \mathcal{S}(s_{-1}, y, p, d), \tag{2.2}$$

where c_t is consumption in period t, $y_t \in \mathbb{R}$ is (net) income in period t, s_t is asset holdings at the end of period t with s_{-1} historically given, and $\mathcal{S}(s_{-1}, y, p, d)$ is a set of sequences in $\mathbb{R}$ with $s = \{s_t\}_{t=0}^{\infty}$, $y = \{y_t\}_{t=0}^{\infty}$, $p = \{p_t\}_{t=0}^{\infty}$, and $d = \{d_t\}_{t=0}^{\infty}$. We offer several examples of (2.2) in Subsection 3.1.

Let $\mathcal{C}$ be the set of sequences $\{c_t\}_{t=0}^{\infty}$ in $\mathbb{R}_+$. For any $c \in \mathcal{C}$, we let $\{c_t\}_{t=0}^{\infty}$ denote the sequence representation of c, and vice versa. We therefore use c and $\{c_t\}_{t=0}^{\infty}$ interchangeably; likewise we use s and $\{s_t\}_{t=0}^{\infty}$ interchangeably, and so on. The inequalities $<$ and $\leq$ on the set of sequences in $\mathbb{R}$ (which includes $\mathcal{C}$) are defined as follows:

$$c \leq c' \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_t,\ c_t \leq c'_t, \tag{2.3}$$

$$c < c' \quad \Leftrightarrow \quad c \leq c' \text{ and } \exists t \in \mathbb{Z}_+, c_t < c'_t. \tag{2.4}$$

The agent's preferences are represented by a binary relation $\prec$ on $\mathcal{C}$. More concretely, for any $c, c' \in \mathcal{C}$, the agent strictly prefers c' to c if and only if $c \prec c'$. Assumptions 2.1 and 2.2, stated below, are maintained throughout the paper.

Assumption 2.1. $d_t \geq 0$ and $p_t > 0$ for all $t \in \mathbb{Z}_+$.

We say that a pair of sequences $c = \{c_t\}_{t=0}^{\infty}$ and $s = \{s_t\}_{t=0}^{\infty}$ in $\mathbb{R}$ is a *plan*; that a plan (c, s) is *feasible* if it satisfies (2.1) and (2.2); and that a feasible plan (c^*, s^*) is *optimal* if there exists no feasible plan (c, s) such that $c^* \prec c$. Whenever we take an optimal plan (c^*, s^*) as given, we assume the following.

Assumption 2.2. For any $c \in C$ with $c^* < c$, we have $c^* \prec c$.

This assumption holds if $\prec$ is strictly monotone in the sense that for any $c, c' \in \mathcal{C}$ with $c < c'$, we have $c \prec c'$. While this latter requirement may seem reasonable, there is an important case in which it is not satisfied; see Example 3.2.

2.2 Asset Bubbles

In this subsection we define the fundamental value of the asset and the bubble component of the asset price in period $t \in \mathbb{Z}_+$ using the period t prices of the consumption goods in periods $t, t+1, \ldots$. To be more concrete, let $q_{0,t}$ be the period 0 price of the consumption good in period $t \in \mathbb{Z}_+$. It is well known (e.g., Huang and Werner, 2000, (8)) that the absence of arbitrage implies that there exists a price sequence $\{q_{0,t}\}$ such that

$$\forall t \in \mathbb{Z}_+, \quad q_{0,t} p_t = q_{0,t+1}(p_{t+1} + d_{t+1}), \tag{2.5}$$

$$\forall t \in \mathbb{N}, \quad q_{0,t} > 0, \tag{2.6}$$

$$q_0 = 1. \tag{2.7}$$

Under Assumption 2.1, conditions (2.5) and (2.7) uniquely determine the price sequence $\{q_{0,t}\}$. For the rest of the paper, we consider the price sequence $\{q_{0,t}\}$ given by (2.5) and (2.7).

For $t \in \mathbb{N}$ and $i \in \mathbb{Z}_+$, we define

$$q_{t,t+i} = q_{0,t+i}/q_{0,t}, \tag{2.8}$$

which is the period t price of consumption in period $t+i$. Note that

$$\forall i, j, t \in \mathbb{Z}_+, \quad q_{t,t+i}q_{t+i,t+i+j} = \frac{q_{0,t+i}}{q_{0,t}}\frac{q_{0,t+i+j}}{q_{0,t+i}} = q_{t,t+i+j}. \tag{2.9}$$

Let $t \in \mathbb{Z}_+$. Equations (2.5) and (2.8) give us $p_t = q_{t,t+1}(p_{t+1} + d_{t+1})$. By repeatedly applying this equality and (2.9), we obtain

$$\begin{aligned} p_t &= q_{t,t+1}d_{t+1} + q_{t,t+1}p_{t+1} && (2.10)\\ &= q_{t,t+1}d_{t+1} + q_{t,t+1}q_{t+1,t+2}(p_{t+2} + d_{t+2}) && (2.11)\\ &= q_{t,t+1}d_{t+1} + q_{t,t+2}d_{t+2} + q_{t,t+2}p_{t+2} && (2.12)\\ &\vdots && (2.13)\\ &= \sum_{i=1}^{n} q_{t,t+i}d_{t+i} + q_{t,t+n}p_{t+n}, \quad \forall n \in \mathbb{N}. && (2.14) \end{aligned}$$

Since the above finite sum is increasing in $n \in \mathbb{N}$,[4] it follows that

$$p_t = \sum_{i=1}^{\infty} q_{t,t+i}d_{t+i} + \lim_{n\uparrow\infty} q_{t,t+n}p_{t+n}. \tag{2.15}$$

As is commonly done in the literature, we define the *fundamental value* of the asset in period t as the present discounted value of the dividend stream from period $t+1$ onward:

$$f_t = \sum_{i=1}^{\infty} q_{t,t+i}d_{t+i}. \tag{2.16}$$

The *bubble* component of the asset price in period t is the part of p_t that is not accounted for by the fundamental value:

$$b_t = p_t - f_t. \tag{2.17}$$

It follows from (2.15)–(2.17) that

$$b_t = \lim_{n\uparrow\infty} q_{t,t+n}p_{t+n}. \tag{2.18}$$

[4]In this paper, "increasing" means "nondecreasing," and "decreasing" means "nonincreasing."

Using (2.9) we see that

$$q_{0,t} \lim_{n\uparrow\infty} q_{t,t+n} p_{t+n} = \lim_{n\uparrow\infty} q_{0,t+n} p_{t+n} = \lim_{i\uparrow\infty} q_{0,i} p_i. \tag{2.19}$$

Hence, (2.18) and (2.6) give us

$$\lim_{i\uparrow\infty} q_{0,i} p_i = 0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ b_t = 0. \tag{2.20}$$

3 Examples

In this section we present several examples of (2.2) as well as examples of preferences that satisfy Assumption 2.2. Some of the examples are used in Section 4.

3.1 Constraints on Asset Holdings

The simplest example of (2.2) would be the following:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq 0. \tag{3.1}$$

This constraint is often used in representative-agent models; see, e.g., Lucas (1978) and Kamihigashi (1998).

Kocherlakota (1992) uses a more general version of (3.1):

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma, \tag{3.2}$$

where $\sigma \in \mathbb{R}$. If $\sigma < 0$, then the above constraint is called a short sales constraint. The following constraint is even more general:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma_t, \tag{3.3}$$

where $\sigma_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Note that (3.2) is a special case of (3.3) with $\sigma_t = \sigma$ for all $t \in \mathbb{Z}_+$.

Santos and Woodford (1997, p. 24) consider a (state-dependent) borrowing constraint that reduces to the following in our single-asset setting:

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t \geq -\xi_t, \tag{3.4}$$

where $\xi_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. This constraint is a special case of (3.3) with $\sigma_t = -\xi_t / p_t$.

The (state-dependent) debt constraint considered by Werner (2014) and LeRoy and Werner (2014, p. 313) reduces to the following in our setting:

$$\forall t \in \mathbb{Z}_+, \quad (p_{t+1} + d_{t+1}) s_t \geq -\xi_{t+1}. \tag{3.5}$$

This constraint is another special case of (3.3) with

$$\sigma_t = -\xi_{t+1}/(p_{t+1} + d_{t+1}). \tag{3.6}$$

In addition to (3.2), Kocherlakota (1992) considers the following wealth constraint:

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t + \sum_{i=1}^{\infty} q_{t,t+i} y_{t+i} \geq 0, \tag{3.7}$$

which is another example of (2.2). The left-hand side above is the period t value of the agent's current asset holdings and future income. Note that (3.7) is yet another special case of (3.3) with

$$\forall t \in \mathbb{Z}_+, \quad \sigma_t = -\sum_{i=1}^{\infty} q_{t,t+i} y_{t+i}/p_t. \tag{3.8}$$

See Wright (1987) and Huang and Werner (2000) for related discussion.

3.2 Preferences

Example 3.1. A typical objective function in an agent's maximization problem takes the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{3.9}$$

where $\beta \in (0,1)$ and $u : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing bounded function. Define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad \sum_{t=0}^{\infty} \beta^t u(c_t) < \sum_{t=0}^{\infty} \beta^t u(c'_t). \tag{3.10}$$

Then $\prec$ clearly satisfies Assumption 2.2.

Example 3.2. If u is allowed to be unbounded below in Example 3.1, then the binary relation defined by (3.10) may not satisfy Assumption 2.2. To deal with this problem, let $u_t : \mathbb{R}_+ \to [-\infty, \infty)$ be a strictly increasing function for $t \in \mathbb{Z}_+$. Consider the binary relation $\prec$ defined by

$$c \prec c' \quad \Leftrightarrow \quad \varliminf_{n \uparrow \infty} \sum_{t=0}^{n} [u_t(c_t) - u_t(c'_t)] < 0, \tag{3.11}$$

where we follow the convention that $(-\infty) - (-\infty) = 0$; see Dana and Le Van (2006) for related optimality criteria. The binary relation $\prec$ defined above clearly satisfies Assumption 2.2.

Suppose further that (2.2) is given by (3.1), that each u_t is differentiable on $\mathbb{R}_{++}$, and that there exists an optimal plan (c^*, s^*) such that

$$\forall t \in \mathbb{Z}_+, \quad c^*_t > 0, \ s^*_t = 1. \tag{3.12}$$

Then the standard Euler equation holds:

$$u'_t(c^*_t)p_t = u'_{t+1}(c^*_{t+1})(p_{t+1} + d_{t+1}), \quad \forall t \in \mathbb{Z}_+. \tag{3.13}$$

In view of (2.5) and (2.7), the price sequence $\{q_{0,t}\}$ is given by

$$q_{0,t} = \frac{u'_t(c^*_t)}{u'_0(c^*_0)}, \quad \forall t \in \mathbb{Z}_+. \tag{3.14}$$

The fundamental value f_t takes the familiar form:

$$f_t = \sum_{i=1}^{\infty} \frac{u'_{t+i}(c^*_{t+i})}{u'_t(c^*_t)} d_{t+i}, \quad \forall t \in \mathbb{Z}_+. \tag{3.15}$$

Example 3.3. Let $v : \mathcal{C} \to \mathbb{R}$ be a strictly increasing function. Define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad v(c_0, c_1, c_2, \dots) < v(c'_0, c'_1, c'_2, \dots). \tag{3.16}$$

Note that (3.16) satisfies Assumption 2.2 without any additional condition on v. For example, v can be a recursive utility function (see, e.g., Boyd, 1990). As in Example 3.2, suppose that (2.2) is given by (3.1), that $v(c_0, c_1, \dots)$ is differentiable in each $c_i > 0$, and that there exists an optimal plan (c^*, s^*) satisfying (3.12). For $i \in \mathbb{Z}_+$ define

$$v_i(c^*) = \frac{\partial v(c^*_0, c^*_1, \dots,)}{\partial c^*_i}. \tag{3.17}$$

Then it is easy to see that the following version of the Euler equation holds:

$$v_t(c^*)p_t = v_{t+1}(c^*)(p_{t+1} + d_{t+1}), \quad \forall t \in \mathbb{Z}_+. \tag{3.18}$$

As in Example 3.2, for all $t \in \mathbb{Z}_t$ we have

$$q_{0,t} = \frac{v_t(c^*)}{v_0(c^*)}, \tag{3.19}$$

$$f_t = \sum_{i=1}^{\infty} \frac{v_{t+i}(c^*)}{v_t(c^*)} d_{t+i}. \tag{3.20}$$

4 Implications of Feasibility and Optimality

4.1 A No-Bubble Theorem

To state our no-bubble theorem, we need to introduce some notation. Given any sequence $\{s_t^*\}_{t=0}^{\infty}$ in $\mathbb{R}$, $\tau \in \mathbb{Z}_+$, and $\epsilon > 0$, let $\mathcal{S}_{\tau,\epsilon}(s^*)$ be the set of sequences $\{s_t\}_{t=0}^{\infty}$ in $\mathbb{R}$ such that

$$s_t \begin{cases} = s_t^* & \text{if } t < \tau, \\ \in [s_t^* - \epsilon, s_t^*] & \text{if } t \geq \tau. \end{cases} \tag{4.1}$$

In other words, a sequence $\{s_t\}$ in $\mathcal{S}_{\tau,\epsilon}(s^*)$ coincides with $\{s_t^*\}$ up to period $\tau - 1$ and is required to satisfy $s_t^* - \epsilon \leq s_t \leq s_t^*$ from period τ onward. Now we are ready to state the main result of this paper.

Theorem 4.1. *Let (c^*, s^*) be an optimal plan. Suppose that there exist $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ such that*

$$\mathcal{S}_{\tau,\epsilon}(s^*) \subset \mathcal{S}(s_{-1}, y, p, d). \tag{4.2}$$

Then $b_t = 0$ for all $t \in \mathbb{Z}_+$.

Proof. See Appendix A. □

The proof of Theorem 4.1 is based on a simple idea. If the left equality in (2.20) is violated, we can construct the following alternative plan. Let $\delta > 0$, and let $s_\tau = s_\tau^* - \delta$ and $c_\tau = c_\tau^* + p_\tau \delta$, where τ is given by the statement of the theorem. For $t \neq \tau$, let s_t be determined by the budget constraint

(2.1) with $c_t = c_t^*$. This alternative plan provides the same consumption sequence except in period τ, where consumption is increased by $p_\tau \delta > 0$. The plan, therefore, is strictly preferred over the original plan (c^*, s^*). We derive a contradiction by showing that the alternative plan is feasible for a sufficiently small $\delta > 0$.

Huang and Werner (2000, Theorems 5.1, 6.1) use similar constructions as "Ponzi schemes," but their constructions are not directly linked to the nonexistence of asset bubbles. Santos and Woodford (1997, Lemma 3.8) also use a somewhat similar construction, but their argument requires a sufficient degree of impatience on the agent's preferences. By contrast, Theorem 4.1 does not require any form of impatience.

It seems remarkable that asset bubbles can be ruled out by a simple condition such as (4.2). No explicit utility function is assumed, and the only requirement on the binary relation $\prec$ is Assumption 2.2, which merely requires strict monotonicity at the given optimal consumption plan c^*.

Intuitively, condition (4.2) ensures that the agent can sell a small fraction of the asset in any period $t \geq \tau$. By selling, say, one unit of the asset in period $t \geq \tau$, he gains p_t, while he loses the dividend stream from period $t+1$ onward, whose present discounted value is f_t. This suggests that $p_t \leq f_t$ if the agent's current plan is optimal. This intuitive argument can be formalized if the binary relation $\prec$ is sufficiently smooth and well behaved. As discussed above, however, the actual proof of Theorem 4.1 is more involved since it assumes very little on the binary relation.

4.2 Consequences of Theorem 4.1

In this subsection we provide several consequences of Theorem 4.1. Throughout this subsection we take an optimal plan (c^*, s^*) as given. We start with a simple result assuming that the feasibility constraint on asset holdings (2.2) is given by a sequence of constraints of the form (3.3). As discussed in Subsection 3.1, this simple form covers various constraints on borrowing, debt, and wealth.

Corollary 4.1. *Suppose that (2.2) is given by (3.3) with $\sigma_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Suppose further that*

$$\varliminf_{t \uparrow \infty} (s_t^* - \sigma_t) > 0. \tag{4.3}$$

Then the conclusion of Theorem 4.1 holds.

Proof. Assume (4.3). Let $\epsilon \in (0, \underline{\lim}_{t\uparrow\infty}(s_t^* - \sigma_t))$. Then there exists $\tau \in \mathbb{Z}_+$ such that $s_t^* - \sigma_t \geq \epsilon$, or $s_t^* - \epsilon \geq \sigma_t$, for all $t \geq \tau$. This implies (4.2). The conclusion of Theorem 4.1 therefore holds. □

If there is a constant lower bound on asset holdings s_t, the above result reduces to the following.

Corollary 4.2. *Suppose that (2.2) is given by (3.2) for some $\sigma \in \mathbb{R}$. Suppose further that $\underline{\lim}_{t\uparrow\infty} s_t^* > \sigma$. Then the conclusion of Theorem 4.1 holds.*

Kocherlakota (1992, Proposition 3) in effect shows a special case of the above result under the following additional assumptions: (i) the binary relation $\prec$ is represented by (3.10); (ii) u is continuously differentiable on $\mathbb{R}_{++}$, strictly increasing, concave, and bounded above or below; and (iii) the optimal plan (c^*, s^*) satisfies

$$\forall t \in \mathbb{Z}_+, \quad c_t^* > 0, \tag{4.4}$$

$$\left|\sum_{t=0}^{\infty} \beta^t u(c_t^*)\right| < \infty. \tag{4.5}$$

Since Corollary 4.2 requires none of these additional assumptions, it is a substantial generalization of Proposition 3 in Kocherlakota (1992). He uses the extra assumptions mostly to derive a transversality condition, which is crucial to his approach. By contrast, Corollary 4.2 is based on our Theorem 4.1, which is proved by an elementary perturbation argument that fully exploits the structure of sequential budget constraints. Since Corollary 4.1 and Theorem 4.1 are even more general, they can also be viewed as generalizations of Kocherlakota's result.

Next we present an extremely general no-bubble theorem for representative-agent models.

Proposition 4.1. *Suppose that (2.2) is given by (3.1). Suppose further that*

$$\forall t \in \mathbb{Z}_+, \quad s_t^* = 1. \tag{4.6}$$

Then the conclusion of Theorem 4.1 holds.

Proof. Note that (4.6) and (3.1) imply (4.2) with $\tau = 0$ and $\epsilon = 1$. The conclusion of Theorem 4.1 therefore holds. □

To our knowledge, there is no result in the literature that covers the above result in its full generality. As an immediate consequent, we obtain the familiar asset-pricing formula in the setup of Example 3.2.

Corollary 4.3. *In the setup of Example 3.2 (including (3.13)–(3.15)), we have*

$$\forall t \in \mathbb{Z}_+, \quad p_t = \sum_{i=1}^{\infty} \frac{u'_{t+i}(c^*_{t+i})}{u'_t(c^*_t)} d_{t+i}. \tag{4.7}$$

It is known that a stochastic version of Corollary 4.3 requires additional assumptions; see Kamihigashi (1998) and Montrucchio and Privileggi (2001).[5] Kamihigashi (2001, Section 4.2.1) shows a result similar to Corollary 4.3 for a continuous-time model with a nonlinear constraint. Corollary 4.3 seems useful since the exact assumptions required for the asset-pricing formula (4.7) in the setup of Example 3.2 are not documented in the literature.

The following result is another immediate consequence of Proposition 4.1.

Proposition 4.2. *In the setup of Example 3.3, we have*

$$\forall t \in \mathbb{Z}_+, \quad p_t = \sum_{i=1}^{\infty} \frac{v_{t+i}(c^*)}{v_t(c^*)} d_{t+i}. \tag{4.8}$$

This result can be shown using Boyd's (1990) result on a transversality condition if $v(0, 0, \ldots) = 0$ and if v is recursive, concave, and satisfies a certain growth condition. Proposition 4.2 shows that none of these conditions is needed for the asset-pricing formula (4.8).[6]

5 An Application to a Ramsey Model with Heterogeneous Agents

It should be clear that our results so far can be used to rule out asset bubbles in general or partial equilibrium economies with multiple agents and multiple assets. Indeed, as long as at least one agent can reduce his holdings of one

[5] See Kamihigashi (2011) for sample-path properties of stochastic asset bubbles.

[6] See Duffie and Zame (1989) for the corresponding formula for stochastic continuous-time economies with recursive preferences.

asset permanently from some period onward, Theorem 4.1 guarantees that there is no asset bubble on that particular asset. To show such a result, one can reduce the agent's budget constraint to the form of (2.1) by letting y_t include all the other assets.

Our results also apply to models with capital accumulation. For example, consider the model of Becker et al. (2014). There are heterogeneous agents indexed by $j \in J$, where J is a finite set. Each agent j solves the following maximization problem:

$$\max_{\{c_{j,t}, l_{j,t}, k_{j,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta_j^t u_j(c_{j,t}, 1 - l_{j,t}) \tag{5.1}$$

$$\text{s.t. } \forall t \in \mathbb{Z}_+, \quad c_{j,t} + k_{j,t+1} = (1 + r_t - \delta)k_{j,t} + w_t l_{j,t}, \tag{5.2}$$

$$c_{j,t} \geq 0, l_{j,t} \in [0, 1], k_{j,t+1} \geq 0, \tag{5.3}$$

$$k_{j,0} \geq 0 \quad \text{given}, \tag{5.4}$$

where $\beta_j \in (0, 1)$ is agent j's discount factor, $u_j : \mathbb{R}_+ \times [0, 1] \to [-\infty, \infty)$ is agent j's utility function, $c_{j,t}, l_{j,t}$, and $k_{j,t}$ are agent j's consumption, labor supply, and capital stock, respectively, r_t is the rental rate on capital, $\delta \in (0, 1)$ is the depreciation rate of capital, and w_t is the wage rate.

We could simply borrow the production side and market-clearing conditions of the economy from Becker et al. (2014), but we can present a meaningful result without specifying them. For this purpose we take r_t and w_t as given for all $t \in \mathbb{Z}_+$.

We assume that agent j's utility function $u_j(c, 1-l)$ is strictly increasing in $c \in \mathbb{R}_+$ for any $l \in [0, 1)$, and that $r_t - \delta \geq 0$ and $w_t \geq 0$ for all $t \in \mathbb{Z}_+$. Given a sequence $\{l_{j,t}\}_{t=0}^{\infty}$ of labor supply, we define the binary relation $\prec$ on $\mathcal{C}$ by (3.11). We also define $q_{i,t}$ for $i, t \in \mathbb{Z}_+$ by (2.5)–(2.7) with $p_t = 1$ and $d_t = r_t - \delta$. Then both Assumptions 2.1 and 2.2 hold under (5.5) below. The following result is essentially a restatement of Corollary 4.2 with $\sigma = 0$.

Proposition 5.1. *Suppose that there exists an agent $j \in J$ with an optimal plan $\{c_{j,t}, l_{j,t}, k_{j,t+1}\}_{t=0}^{\infty}$ such that*

$$\forall t \geq \mathbb{Z}_+, \quad l_{j,t} \in [0, 1), \tag{5.5}$$

$$\varliminf_{t \uparrow \infty} k_{j,t} > 0. \tag{5.6}$$

Then

$$\lim_{i\uparrow\infty} q_{0,i} = 0, \tag{5.7}$$

$$\forall t \in \mathbb{Z}_+, \quad \sum_{i=1}^{\infty} q_{t,t+i}(r_t - \delta) = 1. \tag{5.8}$$

Becker et al. (2014) show the following under their assumptions:

$$\lim_{i\uparrow\infty} q_{0,i}(1-\delta)^i = 0. \tag{5.9}$$

Note that (5.7) implies (5.9).

One can show the existence of an agent $j \in J$ satisfying (5.5) and (5.6) under appropriate assumptions on the production function and the utility functions; see Bosi and Seegmuller (2010). Proposition 5.1 can be combined with such assumptions to conclude (5.7) and (5.8) as equilibrium properties.

6 Concluding Comments

In this paper we established a simple no-bubble theorem that applies to a wide range of deterministic economies with infinitely lived agents facing sequential budget constraints. In particular, we showed that asset bubbles can be ruled out if at least one agent can reduce his asset holdings permanently from some period onward. This is a substantial generalization of Kocherlakota's (1992) result on asset bubbles and short sales constraints.

Our no-bubble theorem is based on the optimal behavior of a single agent, requiring virtually no assumption beyond the strict monotonicity of preferences. One of the useful consequences of the theorem is the asset-pricing formula (4.8) for a representative-agent economy with a non-time-additive, non-recursive utility function. Additional results can be shown by using the results presented in this paper in conjunction with other arguments based on market-clearing and aggregation.

Appendix A Proof of Theorem 4.1

Let (c^*, s^*) be an optimal plan. Recalling (2.20), it suffices to verify that

$$\lim_{i\uparrow\infty} q_{0,i} p_i = 0. \tag{A.1}$$

Suppose by way of contradiction that

$$\lim_{i\uparrow\infty} q_{0,i}p_i > 0. \tag{A.2}$$

Then since $q_{0,i}p_i > 0$ for all $i \in \mathbb{Z}_+$ by Assumption 2.1 and (2.6), it follows that there exists $\underline{b} > 0$ such that

$$\forall i \in \mathbb{Z}_+, \quad q_{0,i}p_i \geq \underline{b}. \tag{A.3}$$

Equivalently, we have $1/p_i \leq q_{0,i}/\underline{b}$ for all $i \in \mathbb{Z}_+$. Multiplying through by d_i and summing over $i \in \mathbb{N}$, we obtain

$$\sum_{i=1}^{\infty} \frac{d_i}{p_i} \leq \sum_{i=1}^{\infty} \frac{q_{0,i}d_i}{\underline{b}} = \frac{f_0}{\underline{b}} < \infty, \tag{A.4}$$

where the equality uses (2.16)[7] and the last inequality holds by (2.17).

Let $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ be given by (4.2). For each $\delta \in (0, \epsilon)$ we construct an alternative plan (c^δ, s^δ) as follows:

$$c_t^\delta = \begin{cases} c_t^* & \text{if } t \neq \tau, \\ c_\tau^* + p_\tau \delta & \text{if } t = \tau, \end{cases} \tag{A.5}$$

$$s_t^\delta = \begin{cases} s_t^* & \text{if } t \leq \tau - 1, \\ s_\tau^* - \delta & \text{if } t = \tau, \\ [y_t + (p_t + d_t)s_{t-1}^\delta - c_t^*]/p_t & \text{if } \tau \geq \tau + 1. \end{cases} \tag{A.6}$$

It suffices to show that (c^δ, s^δ) is feasible for $\delta > 0$ sufficiently small; for then, we have $c^* \prec c^\delta$ by (A.5) and Assumption 2.2, contradicting the optimality of (c^*, s^*).

Note that (c^δ, s^δ) satisfies (2.1) by construction. By (2.1) and (A.5) we have

$$\forall t \geq \tau + 1, \quad p_t(s_t^* - s_t^\delta) = (p_t + d_t)(s_{t-1}^* - s_{t-1}^\delta). \tag{A.7}$$

For $t \geq \tau$ define

$$\delta_t = s_t^* - s_t^\delta. \tag{A.8}$$

[7] An arguments similar to (A.4) is used by Montrucchio (2004, Theorem 2).

Note that $\delta_\tau = \delta$ by (A.6). We have $p_t \delta_t = (p_t + d_t)\delta_{t-1}$ for all $t > \tau$ by (A.7). Thus for any $t > \tau$ we have

$$0 \le \delta_t = \frac{p_t + d_t}{p_t}\delta_{t-1} = \frac{p_t + d_t}{p_t}\frac{p_{t-1} + d_{t-1}}{p_{t-1}}\delta_{t-2} = \cdots \tag{A.9}$$

$$= \delta \prod_{i=\tau+1}^{t} \frac{p_i + d_i}{p_i} \le \delta \prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}, \tag{A.10}$$

where the equality in (A.10) holds since $\delta_\tau = \delta$, and the inequality in (A.10) holds since $d_t \ge 0$ for all $t \in \mathbb{Z}_+$ by Assumption 2.1.[8]

To show that (c^δ, s^δ) is feasible, it suffices to verify that $\delta_t \le \epsilon$ for all $t \ge \tau$; for then, we have $s \in \mathcal{S}(s_{-1}, y, p, d)$ by (4.2), (A.9), and (A.8). For this purpose, note from (A.4) that

$$\frac{f_0}{\underline{b}} \ge \sum_{i=1}^{\infty} \frac{d_i}{p_i} \ge \sum_{i=1}^{\infty} \ln\left(1 + \frac{d_i}{p_i}\right) \tag{A.11}$$

$$= \sum_{i=1}^{\infty} \ln\left(\frac{p_i + d_i}{p_i}\right) = \ln\left(\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}\right). \tag{A.12}$$

It follows that

$$\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i} < \infty. \tag{A.13}$$

Using this and recalling (A.9)–(A.10), we can choose $\delta > 0$ small enough that $0 \le \delta_t \le \epsilon$ for all $t \ge \tau$. For such δ, (c^δ, s^δ) is feasible, contradicting the optimality of (c^*, s^*). We have verified (A.1), which implies the conclusion of the theorem.

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[8] An argument similar to (A.9)–(A.10) is used by Bosi et al. (2014).

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