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Regime-Switching Sunspot Equilibria in a One-Sector Growth Model with Aggregate Decreasing Returns and Small Externalities

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Abstract

This paper shows that regime-switching sunspot equilibria easily arise in a one-sector growth model with aggregate decreasing returns and arbitrarily small externalities. We construct a regime-switching sunspot equilibrium under the assumption that the utility function of consumption is linear. We also construct a stochastic optimal growth model whose optimal process turns out to be a regime-switching sunspot equilibrium of the original economy under the assumption that there is no capital externality. We illustrate our results with numerical examples.

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1 Introduction

In macroeconomics, sunspot equilibria are often associated with local indeterminacy, or the existence of a locally stable steady state. In the context of growth models, the phenomenon of local indeterminacy has been well known since Benhabib and Farmer (1994) and Farmer and Guo (1994). While earlier results required unduly large degrees of increasing returns and externalities,[1] local indeterminacy has been established for various settings under less objectionable assumptions, such as decreasing returns to labor (e.g., Pelloni and Waldmann, 1998), moderate externalities (Dufourt et al., 2015), aggregate constant returns to scale (e.g., Benhabib et al., 2000; Mino, 2001), and arbitrarily small increasing returns and externalities (e.g., Kamihigashi, 2002; Pintus, 2006).

This paper seeks to point out the possibility of an additional mechanism that gives rise to sunspot equilibria in an economy with aggregate decreasing returns to scale and arbitrarily small externalities. The combination on which we focus, aggregate decreasing returns to scale in tandem with arbitrarily small externalities, sets a stage almost indistinguishable from the standard neoclassical setting, posing a challenge to proponents of sunspot equilibria.

Instead of small fluctuations around a locally indeterminate steady state, we consider large fluctuations caused by a regime-switching sunspot process. Assuming that the sunspot process is a two-state Markov chain, we construct regime-switching sunspot equilibria in which labor supply is positive in one state and zero in the other.

Although this type of regime-switching sunspot equilibrium is rather extreme and may not match many of the empirical regularities discussed in the local indeterminacy literature, there are merits to studying such equilibria in addition to local sunspot equilibria driven by local indeterminacy. First, local sunspot equilibria can explain only small fluctuations around a steady state, whereas economic events of great magnitude—such as the Great Depression of the early 20th century, Japan's "lost decades" since the early 1990s, and the Global Financial Crisis of 2007–2008—are characterized by large downfalls. Our model can at least generate large and sudden downfalls from a steady state. Second, our analysis suggests that regime-switching sunspot equilibria of the type considered in this paper are widespread in models with

[1] See Benhabib and Farmer (1999) for a survey of earlier results. See Grandmont (1989, 1991) for a discussion of the relations between the local stability properties of a steady state and the possibility of sunspot equilibria.

externalities, and can even coexist with local sunspot equilibria. Our approach thus complements, rather than substitutes, the more common local indeterminacy approach. It would be possible, in fact, to construct a model in which two types of sunspot shocks are present and both small and large fluctuations are endogenously generated. While the construction of such a model would be beyond the scope of this paper, our analysis can form a basis for further development in this direction.

We present two main results in this paper. First, assuming that the utility function of consumption is linear, we establish the existence of a regime-switching sunspot equilibrium by recursively solving the Euler condition for capital and the first-order condition for labor supply, and then verifying the associated transversality condition. These conditions are easy to verify especially when the utility of consumption is linear, as none of the three aforementioned conditions depend on consumption in such a scenario. Second, assuming an absence of capital externalities, we establish the existence of a regime-switching sunspot equilibrium by constructing a stochastic optimal growth model whose optimal process turns out to be a sunspot equilibrium of the original economy. This latter result is somewhat similar to the observational equivalence result shown by Kamihigashi (1996). In contrast to Kamihigashi (1996), however, we only use the three aforementioned conditions to verify that an optimal process for the stochastic optimal growth model can be interpreted as a sunspot equilibrium. Both results are illustrated with numerical examples.

In addition to the results on local indeterminacy previously mentioned, this paper is also related to results on global indeterminacy (e.g., Drugeon and Venditti, 2001; Coury and Wen, 2009) and regime-switching sunspot equilibria (e.g., Drugeon and Wigniolle, 1996; Dos Santos Ferreira and Lloyd-Braga, 2008).[2] While our findings show the existence of regime-switching sunspot equilibria based on global indeterminacy, they differ from the existing literature in that our model deviates only slightly from the standard neoclassical setting under our assumptions of aggregate decreasing returns and small externalities.[3] Our model can also be viewed as a variant of the Farmer-Guo (1994) model with aggregate decreasing returns and small externalities.

[2]See Clain-Chamosset-Yvrard and Kamihigashi (2015) for an example of a regime-switching sunspot equilibrium in a two-country model with asset bubbles.

[3]Kamihigashi (2015) shows that multiple steady states are possible even without externalities.

The remainder of this paper is organized as follows. In the next section we present the model along with basic definitions and assumptions. In Section 3 we show a standard result that offers a sufficient set of conditions for a feasible process to be an equilibrium. In Section 4 we present our main results along with numerical examples. In Section 5 we conclude the paper by discussing possible extensions. Longer proofs are relegated to the appendices.

2 The Model

We consider an economy with many agents, each of whom solves the following maximization problem:

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t [u(c_t) - w(n_t)] \tag{2.1}$$

$$\text{s.t. } \forall t \in \mathbb{Z}_+, \quad c_t + k_{t+1} = f(k_t, n_t, K_t, N_t) + (1-\delta)k_t, \tag{2.2}$$

$$c_t, k_{t+1} \geqslant 0, \quad n_t \in [0,1], \tag{2.3}$$

where c_t is consumption, n_t is labor supply, k_t is the capital stock at the beginning of period t, N_t is aggregate labor supply, and K_t is the aggregate capital stock. The utility function u of consumption, the disutility function w of labor supply, and the production function f are specified below. The discount factor β and the depreciation rate δ satisfy

$$\beta, \delta \in (0,1). \tag{2.4}$$

In the above maximization problem, the initial capital stock $k_0 > 0$ and the stochastic processes $\{K_t\}_{t=0}^{\infty}$ and $\{N_t\}_{t=0}^{\infty}$ are taken as given. In equilibrium, however, we have

$$\forall t \in \mathbb{Z}_+, \quad K_t = k_t, \quad N_t = n_t. \tag{2.5}$$

To formally define an equilibrium of this economy, we first define a *pre-equilibrium* as a five-dimensional stochastic process $\{c_t, n_t, k_t, N_t, K_t\}_{t=0}^{\infty}$ such that $\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}$ solves the maximization problem (2.1)–(2.3) given $k_0 > 0$ and $\{N_t, K_t\}_{t=0}^{\infty}$. We define an *equilibrium* as a three-dimensional stochastic process $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ such that the five-dimensional stochastic process $\{c_t, n_t, k_t, n_t, k_t\}_{t=0}^{\infty}$ is a pre-equilibrium. We also define a *feasible process* as a three-dimensional stochastic process $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ satisfying (2.2), (2.3), and (2.5).

We specify the functions u, w, and f as follows:

$$u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \tag{2.6}$$

$$w(n) = \eta \frac{n^{\gamma+1}}{\gamma+1}, \tag{2.7}$$

$$f(k, n, K, N) = \theta k^{\alpha} n^{\rho} K^{\overline{\alpha}} N^{\overline{\rho}}. \tag{2.8}$$

We impose the following restrictions on the parameters:

$$\sigma \in [0, 1], \tag{2.9}$$

$$\theta, \alpha, \rho, \eta > 0, \tag{2.10}$$

$$\overline{\alpha}, \overline{\rho}, \gamma \geqslant 0, \tag{2.11}$$

$$\overline{\alpha} + \alpha + \overline{\rho} + \rho \leqslant 1. \tag{2.12}$$

If $\sigma = 1$, then it is understood that $u(c) = \ln c$. Since $\sigma \in [0, 1]$ by (2.9), u is bounded below unless $\sigma = 1$. The inequality in (2.12) means that the production function exhibits decreasing returns to scale at the aggregate level.[4] In what follows, we use the nonparametric forms u, w, and f and the parametric forms given by (2.6)–(2.8) above interchangeably.

Let $\widehat{k}$ be the unique strictly positive capital stock $k > 0$ such that $\theta k^{\alpha+\overline{\alpha}} = \delta k$. The capital stock $\widehat{k}$ is the maximum sustainable capital stock. It has the property that for any feasible process $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ we have

$$\forall k_t \in \mathbb{Z}_+, \quad k_t \leqslant \max\{k_0, \widehat{k}\}. \tag{2.13}$$

All equilibria are therefore bounded.

3 Sufficient Optimality Conditions

It follows from (2.8) that

$$\forall k, n > 0, \quad f_1(k, n, k, 0) = f_2(k, n, k, 0) = 0, \tag{3.1}$$

[4] In the decentralized version of the model, profits are given to consumers, who are the owners of the firms.

where $f_i(\cdot,\cdot,\cdot,\cdot)$ is the derivative of f with respect to the ith argument. To simplify notation, we define the following for $i = 1, 2$:

$$f_i(k,n) = f_i(k,n,k,n). \tag{3.2}$$

For $k, n \geqslant 0$ we also define

$$g(k,n) = f(k,n,k,n) + \zeta k, \tag{3.3}$$

where $\zeta = 1 - \delta$.

The first-order condition for labor supply n_t in period t is given by

$$u'(c_t)f_2(k_t,n_t) - w'(n_t) \begin{cases} = 0 & \text{if } n_t \in (0,1), \\ \geqslant 0 & \text{if } n_t = 1, \\ \leqslant 0 & \text{if } n_t = 0. \end{cases} \tag{3.4}$$

Note from (3.1) that $n_t = 0$ is always a solution to (3.4). This observation forms the basis for our construction of sunspot equilibria. On the other hand, as long as $k_t, K_t, N_t > 0$, it is always optimal to choose a strictly positive labor supply since $f_2(k, 0, K, N) = \infty$ for any $k, K, N > 0$.

The stochastic Euler condition for the capital stock k_{t+1} at the beginning of period $t+1$ can be written as

$$-u'(c_t) + \beta E_t u'(c_{t+1})[f_1(k_{t+1}, n_{t+1}) + \zeta] \tag{3.5}$$

$$\begin{cases} = 0 & \text{if } k_{t+1} \in (0, g(k_t,n_t)), \\ \geqslant & \text{if } k_{t+1} = g(k_t,n_t), \\ \leqslant & \text{if } k_{t+1} = 0. \end{cases} \tag{3.6}$$

We also need to consider corner solutions since one of the results shown in the next section assumes that the utility function of consumption is linear. Yet as long as labor supply in period $t+1$ is strictly positive with strictly positive probability, there is a solution to (3.5) with $k_{t+1} > 0$, because $\lim_{k \downarrow 0} f_1(k,n) = \infty$ for any $n > 0$.

The transversality condition is

$$\lim_{T\to\infty} \beta^T E u'(c_T) k_{T+1} = 0. \tag{3.7}$$

Kamihigashi (2003, 2005) provides more details on transversality conditions for stochastic problems.

We see from the following result that the above first-order conditions in conjunction with the transversality condition are sufficient for a feasible process to be an equilibrium. The proof is a stochastic version of the standard sufficiency argument; see Brock (1982) for a similar stochastic argument.

Lemma 3.1. *A feasible process* $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ *is an equilibrium if it satisfies (3.4) and (3.5) for all* $t \in \mathbb{Z}_+$ *and (3.7).*

Proof. See Appendix A. □

4 Sunspot Equilibria

4.1 Common Structure

We consider a special type of sunspot equilibrium by taking a regime-switching sunspot process $\{s_t\}$ as given. In particular, we assume that there are two sunspot states, 0 and 1, and that $\{s_t\}$ is a two-state Markov chain with transition matrix

$$\begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix}, \tag{4.1}$$

where p_{ij} is the probability that $s_{t+1} = j$ given $s_t = i$ for $i, j \in \{0, 1\}$. To simplify the analysis, we assume that $p_{ij} > 0$ for all $i, j \in \{0, 1\}$. Since (4.1) is a transition matrix, we have

$$p_{00} + p_{01} = p_{10} + p_{11} = 1. \tag{4.2}$$

In what follows, all stochastic processes (sequences) are assumed to be adapted to the σ-field generated by the Markov chain $\{s_t\}_{t=0}^{\infty}$. This simply means that any variable indexed by t is a function of the history of sunspot states $s_0, s_1, \ldots, s_t$ up to period t. Since $\{s_t\}$ is a sunspot process, it has no direct influence on the fundamentals of the economy. An equilibrium $\{c_t, n_t, k_t\}$ is a *sunspot equilibrium* if it depends on the sunspot process $\{s_t\}$ in a nontrivial way.

To see the possibility of a sunspot equilibrium, suppose that we have the following in the maximization problem (2.1)–(2.3):

$$N_t \begin{cases} > 0 & \text{if } s_t = 1, \\ = 0 & \text{if } s_t = 0. \end{cases} \tag{4.3}$$

Then, provided that $k_t = K_t > 0$ for all $t \in \mathbb{Z}_+$, we must have

$$n_t \begin{cases} > 0 & \text{if } s_t = 1, \\ = 0 & \text{if } s_t = 0. \end{cases} \tag{4.4}$$

For the rest of the paper, we assume that $k_t = K_t > 0$ for all $t \in \mathbb{Z}_+$, focusing on regime-switching sunspot equilibria satisfying (4.3) and (4.4).

Under (4.3) the first-order condition (3.4) for n_t can be written as

$$s_t = 1 \quad \Rightarrow \quad u'(c_t) f_2(k_t, n_t) - w'(n_t) \begin{cases} = 0 & \text{if } n_t \in (0,1), \\ \geqslant 0 & \text{if } n_t = 1, \end{cases} \tag{4.5}$$

$$s_t = 0 \quad \Rightarrow \quad n_t = 0. \tag{4.6}$$

In (4.5) we have no need to consider the case $n_t = 0$ since $f_2(k, 0, k, N) = \infty$ for any $k, N > 0$ by (2.8), (2.10), and (2.12), as mentioned above. If, on the other hand, $s_t = 0$, then $N_t = 0$ by (4.3), and $n_t = 0$ since $f_2(k, n, k, 0) = 0$ for any $k > 0$ and $n \geqslant 0$.

4.2 Linear Utility of Consumption

One way to show the existence of a sunspot equilibrium is to use Lemma 3.1 to explicitly construct a sunspot equilibrium $\{c_t, n_t, k_t\}$ satisfying (4.4). To do so, we need to verify the Euler condition (3.5) and the transversality condition (3.7) in addition to (4.5) and (4.6). While this is not easy to do in general, we can explicitly construct a sunspot equilibrium using these conditions if we assume that the utility function of consumption is linear, and hence that none of the conditions depend on consumption (except for feasibility). We consider this special case in the following result.

Proposition 4.1. *If $\sigma = 0$, then a sunspot equilibrium satisfying (4.4) exists.*

Proof. See Appendix B. □

The sunspot equilibrium constructed in the proof of Proposition 4.1 is generated by the following system of equations:

$$n_t = m(k_t, s_t), \tag{4.7}$$

$$k_{t+1} = \min\{q(p_{s_t 1}), g(k_t, n_t)\}, \tag{4.8}$$

$$c_t = g(k_t, n_t) - k_{t+1}, \tag{4.9}$$

where $m(\cdot,\cdot)$ and $q(\cdot)$ are given by (B.5) and (B.14), respectively, in Appendix B, and $p_{s_t1} = p_{01}$ or p_{11} depending on $s_t = 0$ or 1. Given $k_t > 0$ and $s_t \in \{0,1\}$, n_t is determined by (4.7), k_{t+1} is determined by (4.8), and c_t is determined by (4.9). With a new sunspot variable s_{t+1} drawn according to (4.1), n_{t+1} is determined by (4.7) again, and so one.

Figure 1 depicts the functions in (4.7)–(4.9) with the following parameter values:

$$\beta = 0.9, \quad \eta = 1, \quad \gamma = 0.1, \quad p_{01} = 0.2, \quad p_{11} = 0.8, \tag{4.10}$$

$$\delta = 0.05, \quad \theta = 3, \quad \rho = 0.55, \quad \overline{\rho} = 0.03, \tag{4.11}$$

$$\sigma = 0, \quad \alpha = 0.35, \quad \overline{\alpha} = 0.02. \tag{4.12}$$

Figure 2 shows sample paths for sunspot states, capital, labor, and consumption generated by (4.7)–(4.9). The sample path for labor supply n_t closely follows the pattern of sunspot states s_t, as expected from (4.4). The sample paths for capital and consumption inherit the same pattern to a large extent.

We also find a feature specific to consumption, which rises to its highest level when the sunspot state changes from 1 to 0 after remaining in the state of 1 for a few periods. This is expected from the consumption function in Figure 1. Note that this function is increasing in k_t but unlike the labor and capital functions, decreasing in s_t in the sense that consumption is higher when $s_t = 0$ than when $s_t = 1$.

4.3 Stochastic Optimal Growth

One can conjecture that the foregoing feature of the consumption function may not necessarily arise from the presence of externalities, but rather as a consequence of optimal behavior. In this subsection we consider a stochastic optimal growth model without externalities and with regime-switching productivity coefficients. We show that the dynamics of this model are very similar to those of the previous model. The purpose of this subsection is to suggest the presence of a close connection between the two types of models to facilitate the analysis of regime-switching sunspot equilibria in the next subsection.

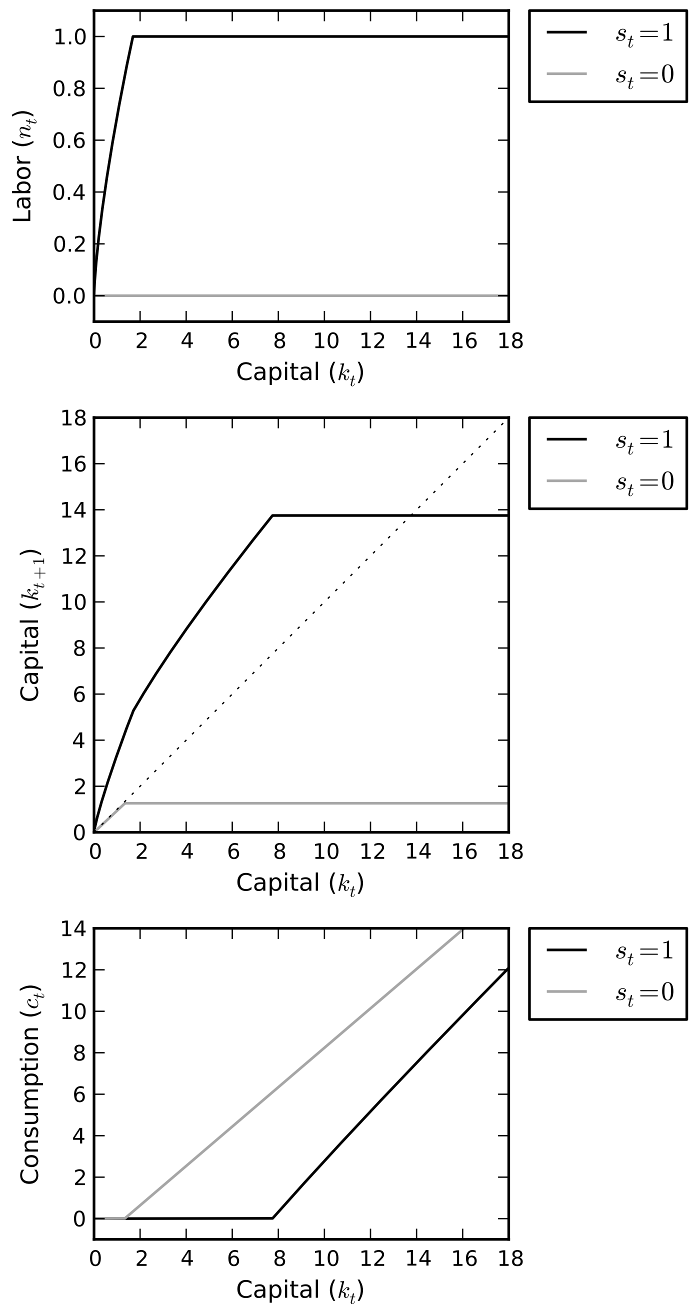

Figure 1: Regime-switching sunspot equilibria under (4.7)–(4.9) with parameter values given by (4.10)–(4.12)

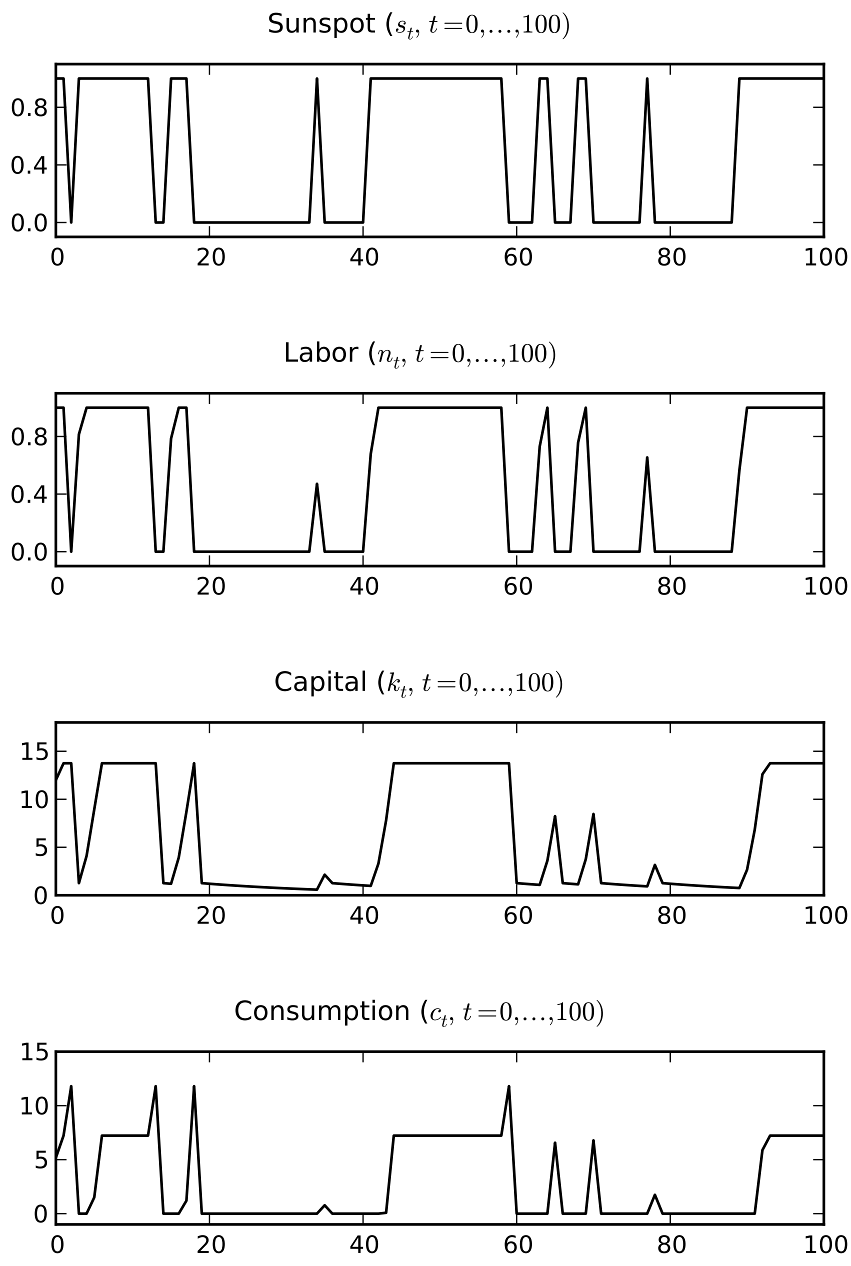

Figure 2: Sample paths for sunspot states, capital, labor, and consumption generated by (4.7)–(4.9) with parameter values given by (4.10)–(4.12).

Consider the following stochastic optimal growth model:

$$\max_{\{c_t,n_t,k_{t+1}\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t[u(c_t)-w(n_t)] \tag{4.13}$$

$$\text{s.t. } \forall t\in\mathbb{Z}_+,\quad c_t+k_{t+1}=s_t(k_t)^{\alpha+\overline{\alpha}}(n_t)^{\rho+\overline{\rho}}+(1-\delta)k_t, \tag{4.14}$$

$$c_t,k_{t+1}\geqslant 0,\quad n_t\in[0,1], \tag{4.15}$$

where $\{s_t\}$ is the same two-state Markov process following (4.1). In this subsection, we define s_t not as a sunspot shock, but as a stochastic productivity coefficient that directly affects production. When $s_t = 1$, the aggregate production function in the maximization problem above is unchanged from that in the previous subsection, but the externalities are internalized here. Since output is zero whenever $s_t = 0$, the problem inherits the pattern of (4.3).

Figure 3 depicts the optimal policy functions for the stochastic optimal growth model (4.13)–(4.15) under (4.10)–(4.12). We obtain these functions from (4.7)–(4.9) by setting

$$\alpha=0.37,\quad \rho=0.58,\quad \overline{\alpha}=\overline{\rho}=0 \tag{4.16}$$

in (4.7)–(4.9).

Note that the consumption function in Figure 3 is decreasing in s_t like the consumption function in Figure 1; in fact, consumption with $s_t = 1$ is even lower than in Figure 1. We can explain this by referring to the capital function in Figure 3, which shows that more capital is accumulated in the stochastic optimal growth model (4.13)–(4.15) than in the original economy with externalities under (4.10)–(4.12). The functions in Figure 3 serve as an example of a stochastic optimal growth model in which consumption is *decreasing* in productivity, while capital and labor are increasing in productivity. This contrasts sharply with the Brock-Mirman (1972) model with i.i.d. productivity shocks, where consumption is always increasing in productivity; see Kamihigashi (2008, Theorem 2.1).

Figure 4 shows sample paths generated by these optimal policy functions with productivity states identical to the sunspot states in Figure 2. The capital and consumption paths are also similar to those in Figure 2, but the capital path and peaks in consumption are both overall higher.

4.4 No Capital Externality

Our analysis in the previous two subsections suggests that the sunspot equilibria of the original economy are closely connected to the optimal process

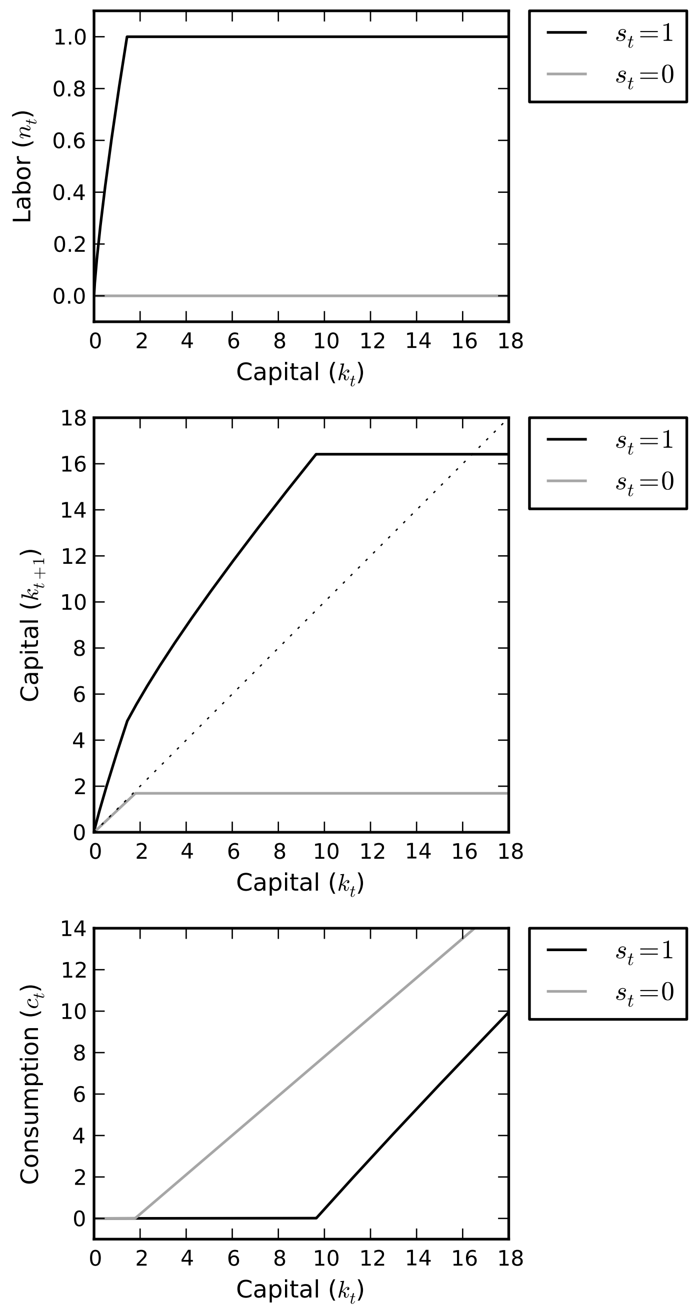

Figure 3: Optimal policy functions for (4.13)–(4.15)

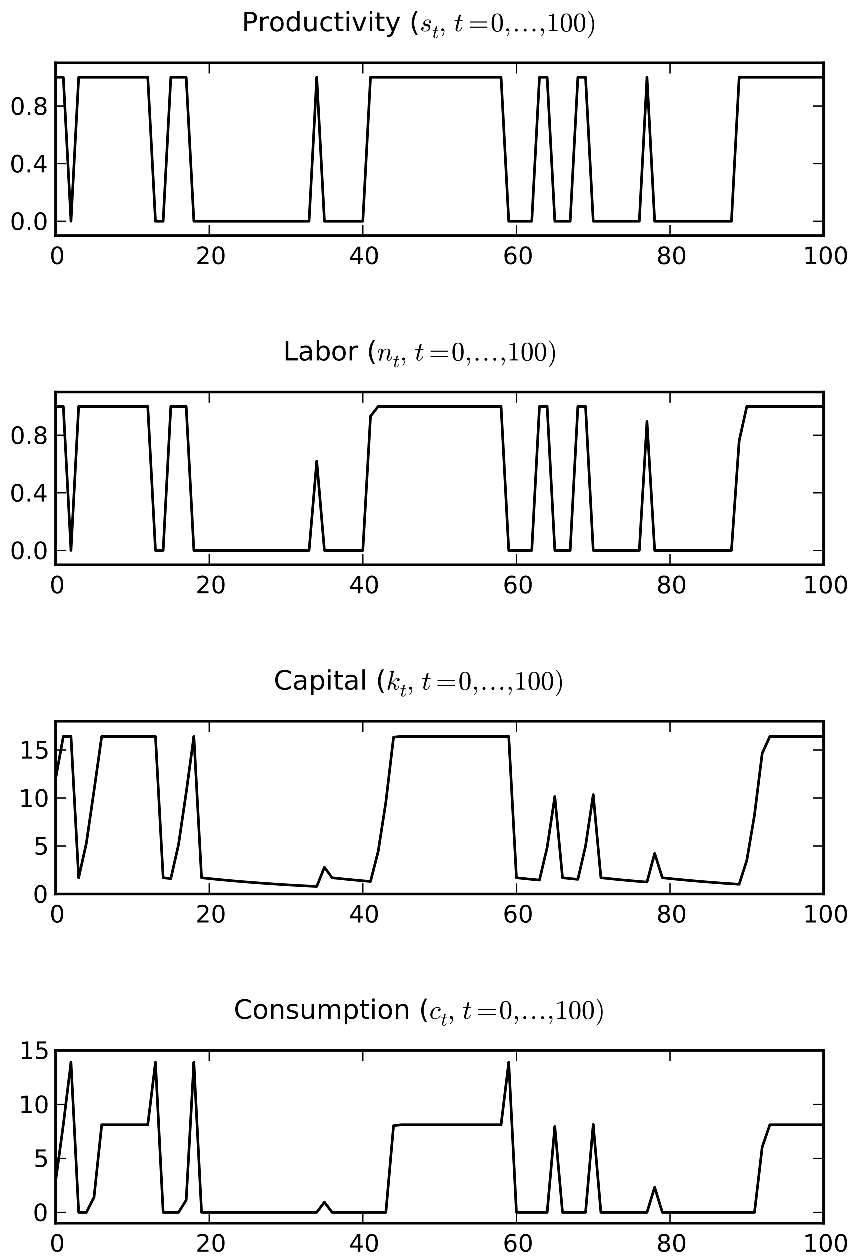

Figure 4: Sample paths for productivity states, capital, labor, and consumption

of a stochastic optimal growth model. While this connection between the models is not trivial to show in general, we can establish it fairly easily in the absence of capital externalities. The proof of the following result utilizes the connection.

Proposition 4.2. *If* $\overline{\alpha} = 0$*, then a sunspot equilibrium satisfying (4.4) exists.*

Proof. See Appendix C □

The condition $\overline{\alpha} = 0$ in the above proposition means that there is no capital externality. In the proof of Proposition 4.2, we consider the following stochastic optimal growth model:

$$\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} E \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - \frac{\rho + \overline{\rho}}{\rho} w(n_t) \right] \tag{4.17}$$

$$\text{s.t.} \quad \forall t \geqslant 0, \quad c_t + k_{t+1} = s_t \theta (k_t)^{\alpha} (n_t)^{\rho + \overline{\rho}} + \zeta k_t, \tag{4.18}$$

$$c_t, k_{t+1} \geqslant 0, \quad n_t \in [0, 1], \tag{4.19}$$

where $\{s_t\}_{t=0}^{\infty}$ is the same two-state Markov chain following (4.1). In the proof, we show that the Euler condition for k_{t+1}, the first order condition for n_t, and the transversality condition are necessary for optimality, and are equivalent to the sufficient optimality conditions for the original economy (2.1)–(2.3) with $\overline{\alpha} = 0$. We can thus establish the existence of a sunspot equilibrium by showing the existence of an optimal process for the above stochastic optimal growth model.

The Bellman equation for (4.17)–(4.19) can be written as

$$v(k_t) = \max_{c_t, n_t, k_{t+1}} \left\{ u(c_t) - \frac{\rho + \overline{\rho}}{\rho} w(n_t) + \beta E_t v(k_{t+1}) \right\} \tag{4.20}$$

$$\text{s.t.} \quad c_t + k_{t+1} = s_t \theta (k_t)^{\alpha} (n_t)^{\rho + \overline{\rho}} + \zeta k_t, \tag{4.21}$$

$$c_t, k_{t+1} \geqslant 0, \quad n_t \in [0, 1]. \tag{4.22}$$

From the proof of Proposition 4.2, we can interpret the optimal policy functions for the above Bellman equation as constituting a regime-switching sunspot equilibrium. To illustrate these functions, we use the same parameter values used in (4.10) and (4.11), but replace the values of σ, α, and $\overline{\alpha}$ as follows:

$$\sigma = 0.99, \quad \alpha = 0.37, \quad \overline{\alpha} = 0. \tag{4.23}$$

With the above parameter values, the utility function u is almost logarithmic, and the aggregate production function $f(k, n, k, n)$ remains unchanged from that in (4.11).

Figure 5 shows the optimal policy functions for the Bellman equation (4.20)–(4.22) under the above parameter values. These functions are computed by numerically solving the Bellman equation using modified policy iteration (e.g., Puterman, 2005) with 5,000 equally spaced grid points.

Figure 6 shows sample paths for sunspot states, labor, capital, and consumption generated by the functions in Figure 5. Compared to those in Figure 2, the sample paths in Figure 6 appear less extreme due to the concavity of the utility function u. Capital accumulates while the sunspot state is 1, and decumulates while it is 0. Consumption follows almost exactly the same pattern as capital, while labor supply moves in the opposite directions when $s_t = 1$, as expected from the labor and consumption functions in Figure 5.

5 Concluding Comments

In this paper we have shown that regime-switching sunspot equilibria easily arise in a one-sector growth model with aggregate decreasing returns and arbitrarily small externalities. We have explicitly constructed a regime-switching sunspot equilibrium under the assumption that the utility function of consumption is linear. We have also constructed a stochastic optimal growth model whose optimal process turns out to be a regime-switching sunspot equilibrium of the original economy under the assumption that there is no capital externality.

Although our results assume aggregate decreasing returns to scale, the existence of regime-switching sunspot equilibria can easily be shown for models with increasing returns and large externalities, at least when the utility function is linear, as in Subsection 4.2. On the other hand, the proof of Proposition 4.2 relies on the assumption of decreasing returns; it could be non-trivial to extend the proof to the case of increasing returns.

To conclude this paper, we discuss possible ways to extend our analysis. First, we can use a similar approach to construct more realistic sunspot equilibria than the rather extreme equilibria considered in this paper, where labor supply and output are both zero when the sunspot state is zero. Consider, for example, a one-sector growth model with externalities in which the

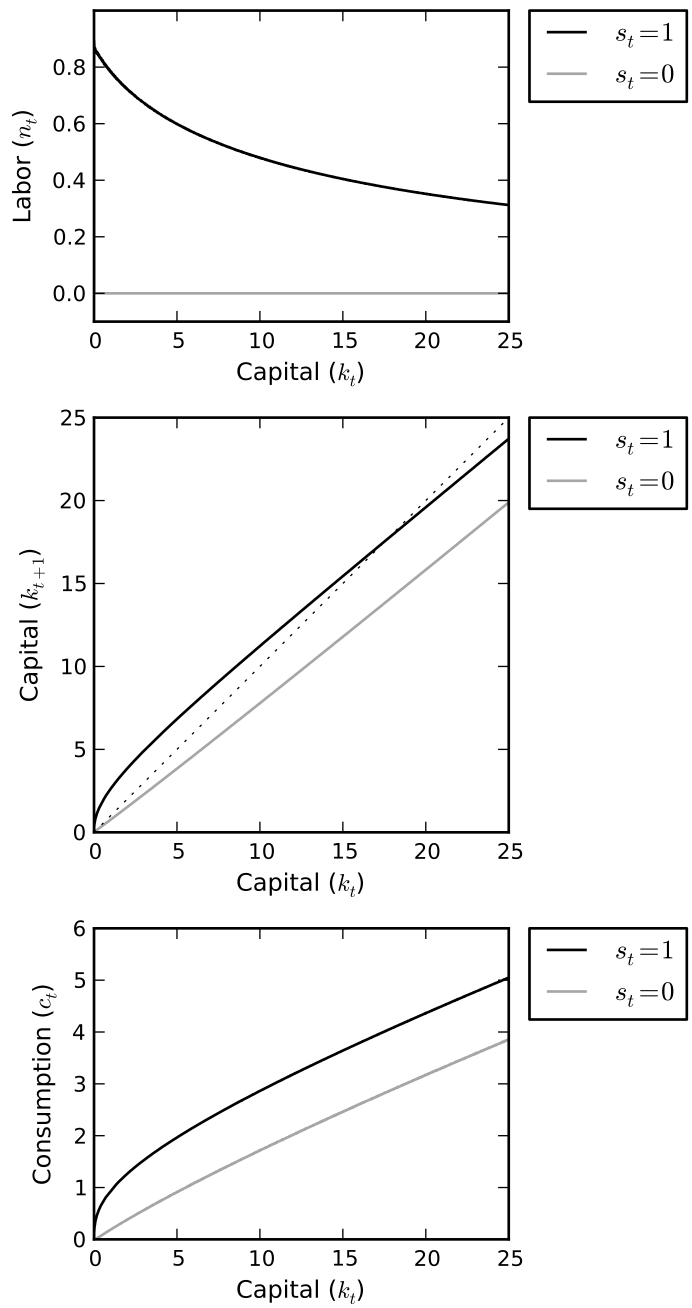

Figure 5: Optimal policy functions for (4.20)–(4.22) and regime-switching sunspot equilibria for (2.1)–(2.3) under (4.10), (4.11), and (4.23)

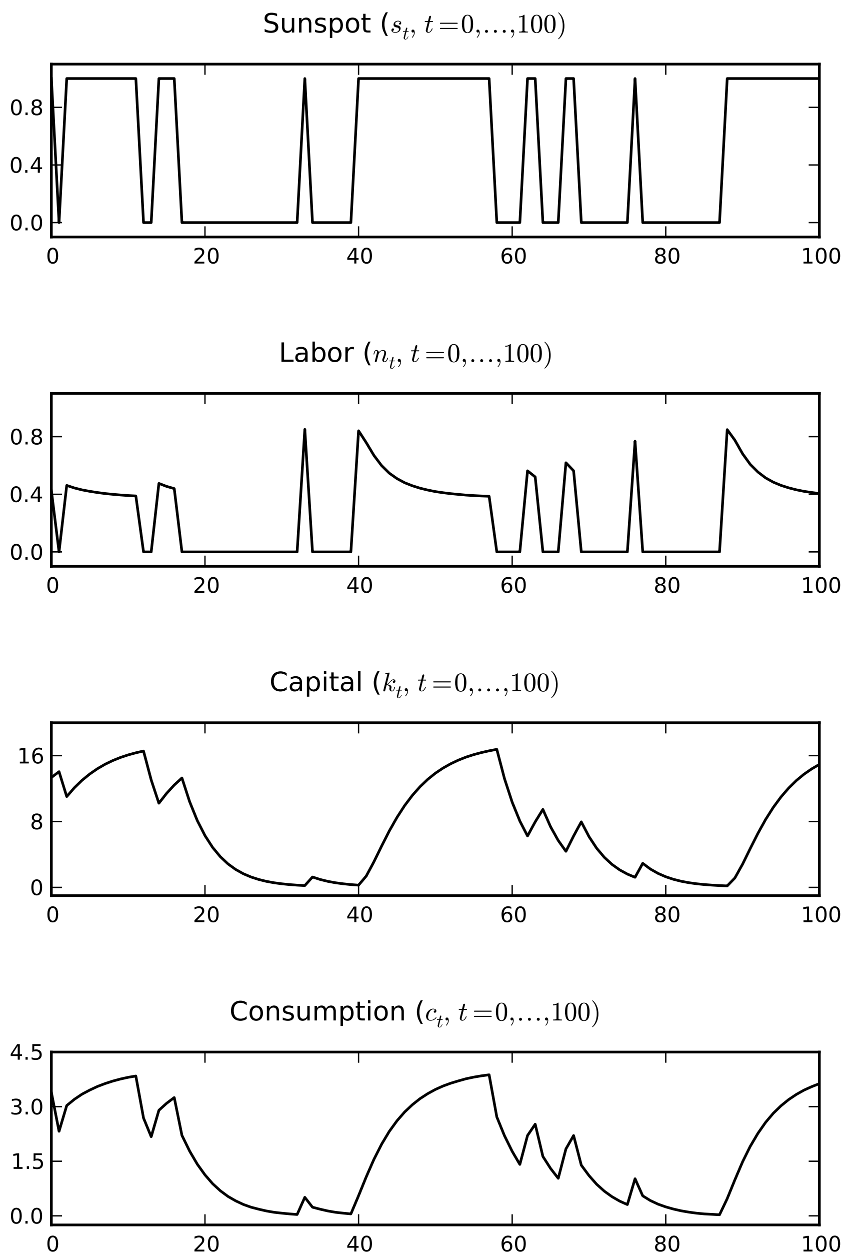

Figure 6: Sample paths for sunspot states, capital, labor, and consumption under (4.10), (4.11), and (4.23)

first-order condition for labor supply has multiple solutions. Such a model can easily be produced if we are allowed to assume externalities of a general form. With such a model, we can construct a sunspot equilibrium that switches between the multiple solutions of the first-order condition for labor supply according to the sunspot state.

Second, although we have focused on sunspot equilibria, we can also construct deterministic equilibria that exhibit chaotic dynamics. We can take, for example, a deterministic sequence of states s_t each assigned a value of either 0 or 1, and solve the deterministic version of the maximization problem (4.17)–(4.19). The resulting optimal path then follows the pattern of the sequence $\{s_t\}$. We can think of this as an example of symbolic dynamics; see Kamihigashi (1999) for economic applications of symbolic dynamics.

Finally, in Proposition 4.2 we have only considered the case without capital externality. While it seems difficult to extend the same approach to models with capital externalities (as long as the capital depreciation rate is less than one), there is a way to deal with such models. In particular, if we allow for nonlinear discounting along the lines of Kamihigashi (2002), we can construct a stochastic optimal growth model whose optimal process turns out to be a sunspot equilibrium of the original economy.

Appendix A Proof of Lemma 3.1

Let $\{c_t^*, n_t^*, k_t^*\}_{t=0}^\infty$ be a feasible process satisfying (3.4)–(3.7) (with c_t^*, n_t^*, k_t^* replacing c_t, n_t, k_t). To simplify notation, for $t \in \mathbb{Z}_+$ and $i = 1, 2$ we define

$$f(t) = f(k_t^*, n_t^*, k_t^*, n_t^*), \tag{A.1}$$

$$f_i(t) = f_i(k_t^*, n_t^*, k_t^*, n_t^*). \tag{A.2}$$

We are to show that for any feasible process $\{c_t, n_t, k_t\}_{t=0}^\infty$, we have

$$E\sum_{t=0}^\infty \beta^t[u(c_t) - w(n_t)] - E\sum_{t=0}^\infty \beta^t[u(c_t^*) - w(n_t^*)] \leqslant 0. \tag{A.3}$$

To this end, let $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ be a feasible process. Fix $T \in \mathbb{N}_+$ for the moment. Let

$$\Delta_T = E\sum_{t=0}^{T} \beta^t[u(c_t) - w(n_t)] - E\sum_{t=0}^{T} \beta^t[u(c_t^*) - w(n_t^*)] \tag{A.4}$$

$$\leqslant E\sum_{t=0}^{T} \beta^t\{u'(c_t^*)(c_t - c_t^*) - w'(n_t^*)(n_t - n_t^*))\}, \tag{A.5}$$

where $u'(c_t^*)$ is the right derivative of u at 0 if $c_t^* = 0$, and similarly for $w'(n_t^*)$. We have

$$\begin{aligned}\Delta_T \leqslant{}& E\sum_{t=0}^{T} \beta^t\{u'(c_t^*)[f(k_t, n_t, k_t^*, n_t^*) - f(t)\\ &+ \zeta(k_t - k_t^*) - (k_{t+1} - k_{t+1}^*)] - w'(n_t^*)(n_t - n_t^*))\} \qquad \text{(A.6)}\\ \leqslant{}& E\sum_{t=0}^{T} \beta^t[u'(c_t^*)(f_1(t) + \zeta)(k_t - k_t^*)\\ &+ \{u'(c_t^*)f_2(t) - w'(n_t^*)\}(n_t - n_t^*) - u'(c_t^*)(k_{t+1} - k_{t+1}^*)]. \qquad \text{(A.7)}\end{aligned}$$

Recalling the first-order condition (3.4) for n_t, we see that for all $t \in \mathbb{Z}_+$,

$$\{u'(c_t^*)f_2(t) - w'(n_t^*)\}(n_t - n_t^*) \leqslant 0. \tag{A.8}$$

Substituting into (A.7) we obtain

$$\Delta_T \leqslant E\sum_{t=0}^{T} \beta^t[u'(c_t^*)(f_1(t) + \zeta)(k_t - k_t^*) - u'(c_t^*)(k_{t+1} - k_{t+1}^*)] \tag{A.9}$$

$$= E\sum_{t=0}^{T-1} \beta^t[-u'(c_t^*) + \beta u'(c_{t+1}^*)(f_1(t+1) + \zeta)](k_{t+1} - k_{t+1}^*) \tag{A.10}$$

$$- \beta^T E u'(c_T^*)(k_{T+1} - k_{T+1}^*) \tag{A.11}$$

$$= E\sum_{t=0}^{T-1} \beta^t[-u'(c_t^*) + \beta E_t u'(c_{t+1}^*)(f_1(t+1) + \zeta)](k_{t+1} - k_{t+1}^*) \tag{A.12}$$

$$- \beta^T E u'(c_T^*)(k_{T+1} - k_{T+1}^*), \tag{A.13}$$

where the last equality holds by the law of iterated expectations. Recalling the Euler condition (3.5) for k_{t+1}, we see that for all $t \in \mathbb{Z}_+$,

$$[-u'(c_t^*) + \beta E_t u'(c_{t+1}^*)(f_1(t+1) + \zeta)](k_{t+1} - k_{t+1}^*) \leqslant 0. \tag{A.14}$$

Substituting into (A.13) we obtain

$$\Delta_T \leqslant -\beta^T E u'(c_T^*)(k_{T+1} - k_{T+1}^*) \tag{A.15}$$

$$\leqslant \beta^T E u'(c_T^*) k_{T+1}^* \to 0, \tag{A.16}$$

where the second inequality holds since $k_{T+1} \geqslant 0$, and the convergence holds by the transversality condition (3.7). This completes the proof of Lemma 3.1.

Appendix B Proof of Proposition 4.1

Suppose that $\sigma = 0$. Then conditions (4.5) and (4.6) can be written as

$$s_t = 1 \quad \Rightarrow \quad \rho\theta(k_t)^{\alpha+\overline{\alpha}}(n_t)^{\rho+\overline{\rho}-1} - \eta(n_t)^\gamma \begin{cases} = 0 & \text{if } n_t \in (0,1), \\ \geqslant 0 & \text{if } n_t = 1, \end{cases} \tag{B.1}$$

$$s_t = 0 \quad \Rightarrow \quad n_t = 0. \tag{B.2}$$

The Euler condition for k_{t+1}, (3.5), can be written as

$$\beta E_t[\alpha\theta(k_{t+1})^{\alpha+\overline{\alpha}-1}(n_{t+1})^{\rho+\overline{\rho}} + \zeta] \begin{cases} = 1 & \text{if } k_{t+1} \in (0, g(k_t, n_t)), \\ \geqslant 1 & \text{if } k_{t+1} = g(k_t, n_t), \\ \leqslant 1 & \text{if } k_{t+1} = 0. \end{cases} \tag{B.3}$$

The transversality condition (3.7) reduces to

$$\lim_{T\to\infty} \beta^T E k_{T+1} = 0. \tag{B.4}$$

Note that (B.1) and (B.2) can be combined into

$$n_t = m(k_t, s_t) \equiv s_t \min\left\{ \left[\frac{\rho\theta}{\eta}(k_t)^{\alpha+\overline{\alpha}}\right]^{\frac{1}{\gamma+1-\rho-\overline{\rho}}}, 1 \right\}. \tag{B.5}$$

Substituting into the left-hand side of (B.3) we obtain

$$E_t[\alpha\theta(k_{t+1})^{\alpha+\overline{\alpha}-1}(n_{t+1})^{\rho+\overline{\rho}} + \zeta] \tag{B.6}$$

$$= E_t[\alpha\theta(k_{t+1})^{\alpha+\overline{\alpha}-1} m(k_{t+1}, s_{t+1})^{\rho+\overline{\rho}} + \zeta] \tag{B.7}$$

$$= \begin{cases} p_{01} h(k_{t+1}) + \zeta & \text{if } s_t = 0, \\ p_{11} h(k_{t+1}) + \zeta & \text{if } s_t = 1, \end{cases} \tag{B.8}$$

where

$$h(k) = \alpha\theta k^{\alpha+\overline{\alpha}-1} m(k,1)^{\rho+\overline{\rho}} \tag{B.9}$$

$$= \min\left\{\alpha\theta\left[\frac{\rho\theta}{\eta}\right]^{\frac{\rho+\overline{\rho}}{\gamma+1-\rho-\overline{\rho}}} k^{\frac{(\alpha+\overline{\alpha}-1)(\gamma+1)+\rho+\overline{\rho}}{\gamma+1-\rho-\overline{\rho}}}, \alpha\theta k^{\alpha+\overline{\alpha}-1}\right\}. \tag{B.10}$$

Both expressions in the curly brackets are strictly decreasing in k by (2.10) and (2.12) (note that $(\alpha+\overline{\alpha}-1)(\gamma+1)+\rho+\overline{\rho} < \alpha+\overline{\alpha}-1+\rho+\overline{\rho} \leqslant 1$ by (2.12)). Thus $h(\cdot)$ is strictly decreasing, which implies that the inverse $h^{-1}(\cdot)$ exists. Indeed, for $z > 0$ we have

$$h^{-1}(z) = \min\left\{\left[\frac{z}{\alpha\theta}\left[\frac{\eta}{\rho\theta}\right]^{\frac{\rho+\overline{\rho}}{\gamma+1-\rho-\overline{\rho}}}\right]^{\frac{\gamma+1-\rho-\overline{\rho}}{(\alpha+\overline{\alpha}-1)(\gamma+1)+\rho+\overline{\rho}}}, \left[\frac{z}{\alpha\theta}\right]^{\frac{1}{\alpha+\overline{\alpha}-1}}\right\}. \tag{B.11}$$

Note that

$$\lim_{k\downarrow 0} h(k) = \infty. \tag{B.12}$$

Substituting (B.6)–(B.8) into (B.3) we obtain

$$\beta[p_{s_t 1} h(k_{t+1}) + \zeta] \begin{cases} = 1 & \text{if } k_{t+1} \in (0, g(k_t, n_t)), \\ \geqslant 1 & \text{if } k_{t+1} = g(k_t, n_t), \\ \leqslant 1 & \text{if } k_{t+1} = 0, \end{cases} \tag{B.13}$$

where $p_{s_t 1} = p_{01}$ or p_{11} depending on $s_t = 0$ or 1. For $p > 0$ define

$$q(p) = h^{-1}\left(\frac{1-\beta\zeta}{\beta p}\right). \tag{B.14}$$

Note from (B.12) that we can rule out the case $k_{t+1} = 0$ in (B.13). Hence we can write (B.13) as

$$k_{t+1} = \min\{q(p_{s_t 1}), g(k_t, n_t)\}. \tag{B.15}$$

We construct a process $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ recursively as follows: given $k_t > 0$ and $s_t \in \{0, 1\}$, let

$$n_t = m(k_t, s_t). \tag{B.16}$$

Determine k_{t+1} by (B.15). Let

$$c_t = g(k_t, n_t) - k_{t+1}. \tag{B.17}$$

Draw s_{t+1} according to (4.1). Determine n_{t+1} by (B.16), and so on. By construction, this process is feasible and satisfies (B.1)–(B.3). It also satisfies (B.4) by (2.13). Thus it is a sunspot equilibrium. The conclusion of the proposition now follows.

C Proof of Proposition 4.2

Suppose that $\overline{\alpha} = 0$. Consider the stochastic optimal growth model (4.17)–(4.19). The Euler condition for k_{t+1} is written as

$$\begin{aligned} & -u'(c_t) + \beta E_t u'(c_{t+1})[s_{t+1}\alpha\theta(k_{t+1})^{\alpha-1}(n_{t+1})^{\rho+\overline{\rho}} + \zeta] \\ & \begin{cases} = 0 & \text{if } k_{t+1} \in (0, g(k_t, n_t)), \\ \geqslant & \text{if } k_{t+1} = g(k_t, n_t), \\ \leqslant & \text{if } k_{t+1} = 0. \end{cases} \end{aligned} \tag{C.1}$$

This is equivalent to the equilibrium Euler condition (3.5) for k_{t+1} for the original economy (2.1)–(2.3) with $\overline{\alpha} = 0$ and (4.3). The first-order condition for n_t for the above stochastic optimal growth model is given by

$$u'(c_t)s_t(\rho+\overline{\rho})\theta(k_t)^{\alpha}(n_t)^{\rho+\overline{\rho}-1} - \frac{\rho+\overline{\rho}}{\rho}w'(n_t) \begin{cases} = 0 & \text{if } n_t \in (0,1), \\ \geqslant 0 & \text{if } n_t = 1, \\ \leqslant 0 & \text{if } n_t = 0, \end{cases} \tag{C.2}$$

which simplifies to

$$u'(c_t)s_t\rho\theta(k_t)^{\alpha}(n_t)^{\rho+\overline{\rho}-1} - w'(n_t) \begin{cases} = 0 & \text{if } n_t \in (0,1), \\ \geqslant 0 & \text{if } n_t = 1, \\ \leqslant 0 & \text{if } n_t = 0. \end{cases} \tag{C.3}$$

This is equivalent to (3.4) with $\overline{\alpha} = 0$ and (4.3). The transversality condition for the above problem is identical to (3.7).

Conditions (C.1) and (C.3) are necessary for optimality by standard arguments. The transversality condition (3.7) is also necessary by the argument of Kamihigashi (2005, Section 6).[5] Given that the sunspot variable s_t is discrete, we can easily establish the existence of an optimal process for the optimal stochastic growth model (4.17)–(4.19) by a standard argument (e.g., Ekeland and Sheinkman, 1986). Let $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ be an optimal process for (4.17)–(4.19). Then by the above argument, the process satisfies (3.4)–(3.7). Thus by Lemma 3.1, the process is an equilibrium of the original economy (2.1)–(2.3). Since it depends on s_t in a nontrivial way, it is a sunspot equilibrium.

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[5] The condition $\sigma \in [0, 1]$ is needed here.

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