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Infinite-Horizon Deterministic Dynamic Programming in Discrete Time: A Monotone Convergence Principle and a Penalty Method*

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Abstract

We consider infinite-horizon deterministic dynamic programming problems in discrete time. We show that the value function of such a problem is always a fixed point of a modified version of the Bellman operator. We also show that value iteration converges increasingly to the value function if the initial function is dominated by the value function, is mapped upward by the modified Bellman operator, and satisfies a transversality-like condition. These results require no assumption except for the general framework of infinite-horizon deterministic dynamic programming. As an application, we show that the value function can be approximated by computing the value function of an unconstrained version of the problem with the constraint replaced by a penalty function.

Keywords: Dynamic programming, Bellman operator, fixed point, value iteration.

AMS Subject Classifications: 90C39; 47N10

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1 Introduction

Infinite-horizon dynamic programming in discrete time is a major tool in various areas of engineering, operations research, and economics. Of particular importance in dynamic programming is the convergence of the value iteration algorithm to the (true) value function. While this convergence property is fairly easy to establish for models with bounded returns [1], unbounded returns are common in practice, especially in economic models. Accordingly, various results on the convergence of value iteration have been established for such models under numerous technical—especially topological—assumptions; see [2–7] for deterministic problems. Stochastic models require additional assumptions concerning measurability (e.g., [8, 9]); it is beyond the scope of this paper to discuss stochastic models in detail.

Recently, an order-theoretic approach that does not require topology was developed and applied to deterministic dynamic programming [10–12]. This approach can be viewed as an extension of the earlier order-theoretic approach of [13, Chapter 5]. One of the results based on the new approach is the following [10, Theorem 2.2]: value iteration converges increasingly to the value function if the initial function is dominated by the value function, is mapped upward by the Bellman operator, and satisfies a transversality-like condition.

This result requires only two assumptions in addition to the general framework of infinite-horizon deterministic dynamic programming. First, the constraint correspondence is nonempty-valued. Second, the value function never equals $+\infty$. The second assumption ensures that the Bellman operator is well defined for any function dominated by the value function, but can be nontrivial to verify since the value function is a priori unknown.

In this paper we establish a more general result that does not require even the two assumptions above. We call this result a *monotone convergence principle* since it requires no assumption except for the general framework itself. To show this principle, we follow the approach of [14] in modifying the Bellman operator in such a way that it is well defined for any function. We show that the value function is a fixed point of this modified Bellman operator. The monotone convergence principle is that value iteration converges increasingly to the value function if the initial function is dominated by the value function, is mapped upward by the modified Bellman operator, and satisfies the same transversality-like condition as in the result of [10, Theorem 2.2].

As an application of this result, we consider an unconstrained problem with the constraint replaced by a penalty function, which penalizes violations of the constraint. We apply the monotone convergence principle to the unconstrained problem, showing that value iteration converges to the value function of the unconstrained problem under the same conditions of the monotone convergence principle mentioned above. Then we show that the value function of the unconstrained problem converges to that of the original constrained problem as the penalty function decreases to $-\infty$ outside the constraint set. This result facilitates applications of penalty methods to dynamic programming problems.

The result seems significant since, to our knowledge, there have been very few applications of penalty methods to dynamic programming in the literature; among the exceptions are [18, 19]. Another related study is [20], where a dynamic programming approach was used to solve a problem with penalty functions. See [21, 22] for general discussion of penalty methods.

The rest of the paper is organized as follows. In the next section we set out the general framework, and show that the value function is always a fixed point of this operator. In Section 3 we present the monotone convergence principle discussed above. In Section 4 we comment on our assumption that the feasibility correspondence is allowed to be empty-valued. In Section 5 we consider an unconstrained problem with a penalty function. We prove our main resutls in Section 6.

2 Dynamic Programming

Our setup closely follows those of [10, 14]. Let X be a set, and let Γ be a correspondence from X to X. Let D be the graph of Γ:

$$D = \{(x, y) \in X \times X : y \in \Gamma(x)\}. \tag{1}$$

Let $u : D \to [-\infty, \infty)$. A sequence $\{x_t\}_{t=0}^{\infty}$ in X is called a *feasible path* if $x_{t+1} \in \Gamma(x_t)$ for all $t \in \mathbb{Z}_+$. A sequence $\{x_t\}_{t=1}^{\infty}$ in X is called a *feasible path from* x_0 if the sequence $\{x_t\}_{t=0}^{\infty}$ is feasible. Let Π and $\Pi(x_0)$ denote the set of feasible paths and that of feasible paths from x_0, respectively:

$$\Pi = \{\{x_t\}_{t=0}^{\infty} \in X^{\infty} : \forall t \in \mathbb{Z}_+, x_{t+1} \in \Gamma(x_t)\}, \tag{2}$$

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X^{\infty} : \{x_t\}_{t=0}^{\infty} \in \Pi\}, \qquad x_0 \in X. \tag{3}$$

Throughout the paper, we follow the convention that

$$\sup \emptyset = -\infty. \tag{4}$$

Let $\beta \geq 0$. The *value function* $v^* : X \to [-\infty, \infty]$ is defined by

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \underset{T\uparrow\infty}{\mathrm{L}} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}), \qquad x_0 \in X, \tag{5}$$

where $\mathrm{L} \in \{\underline{\lim}, \overline{\lim}\}$ with $\underline{\lim} = \liminf$ and $\overline{\lim} = \limsup$. Though L can be $\underline{\lim}$ or $\overline{\lim}$, its definition is fixed for the rest of the paper. Since $u(x,y) < \infty$ for all $(x,y) \in D$, the right-hand side of (5) is well defined for any feasible path. This together with (4) means that v^* is always well defined.

Let W be the set of functions from X to $[-\infty, \infty]$. Let $V = \{v \in W : \forall x \in X, v(x) < \infty\}$. The *Bellman operator* B on V is defined by

$$(Bv)(x) = \sup_{y \in \Gamma(x)} \{u(x,y) + \beta v(y)\}, \quad x \in X. \tag{6}$$

Although Bv is well defined for any function $v \in V$, it may not be well defined for all functions in W. This is because the right-hand side of (6) is not well defined if $u(x,y) = -\infty$ and $v(y) = \infty$ for some $(x,y) \in D$. This problem and its consequences are discussed in [14].

Following [14] we avoid the above problem by slightly modifying the right-hand side of (6). For this purpose, we define

$$\check{\Gamma}(x) = \{y \in \Gamma(x) \colon u(x,y) > -\infty\}, \quad x \in X, \tag{7}$$

$$\check{\Pi} = \left\{ \{x_t\}_{t=0}^{\infty} \in \Pi : \underset{T\uparrow\infty}{\mathrm{L}} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}) > -\infty \right\}, \tag{8}$$

$$\check{\Pi}(x_0) = \{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0) : \{x_t\}_{t=0}^{T} \in \check{\Pi}\}, \qquad x_0 \in X. \tag{9}$$

Recalling (4) we see that

$$\forall x_0 \in X, \quad v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)} \underset{T\uparrow\infty}{\mathrm{L}} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}). \tag{10}$$

We define the *modified Bellman operator* $\check{B}$ on W by

$$(\check{B}v)(x) = \sup_{y \in \check{\Gamma}(x)} \{u(x,y) + \beta v(y)\}, \quad x \in X. \tag{11}$$

The right-hand side above is well defined for any $v \in W$ and $x \in X$ since for any $y \in \check{\Gamma}(x)$ we have $u(x, y) \in (-\infty, \infty)$, which implies that the sum $u(x, y) + \beta v(y)$ is well defined even if $v(y) = -\infty$ or $+\infty$. The following result shows that $\check{B}$ is an extension of B to W.

Lemma 2.1. *For any $v \in V$ we have $\check{B}v = Bv$.*

Proof. Let $v \in V$ and $x \in X$. We claim that

$$\forall y \in \Gamma(x) \setminus \check{\Gamma}(x), \quad u(x, y) + \beta v(y) = -\infty. \tag{12}$$

To see this, let $y \in \Gamma(x) \setminus \check{\Gamma}(x)$. Then $u(x, y) = -\infty$. Since $v \in V$, we have $v(y) < -\infty$. Hence $u(x, y) + \beta v(y) = -\infty$; thus (12) follows.

To simplify notation, let $g(x, y) = u(x, y) + \beta v(y)$ for $y \in \Gamma(x)$. We have

$$(Bv)(x) = \max \left\{ \sup_{y \in \check{\Gamma}(x)} g(x, y), \sup_{y \in \Gamma(x) \setminus \check{\Gamma}(x)} g(x, y) \right\} \tag{13}$$

$$= \max \left\{ \sup_{y \in \check{\Gamma}(x)} g(x, y), -\infty \right\} \tag{14}$$

$$= \sup_{y \in \check{\Gamma}(x)} g(x, y) = (\check{B}v)(x), \tag{15}$$

where (14) uses (12). Since x was arbitrary, it follows that $Bv = \check{B}v$. $\square$

A function $v \in W$ satisfying $\check{B}v = v$ is called a *fixed point* of $\check{B}$. A fixed point of B is defined similarly. We have the following result.

Theorem 2.1. *Any fixed point of B in V is a fixed point of $\check{B}$. Furthermore, v^* is a fixed point of $\check{B}$; i.e.,*

$$\forall x \in X, \quad v^*(x) = \sup_{y \in \check{\Gamma}(x)} \{u(x, y) + \beta v^*(y)\}. \tag{16}$$

Proof. See Subsection 6.2. $\square$

The first statement above is immediate from Lemma 2.1. The second statement uses the argument of [14, Theorem 1]. We call (16) the *modified optimality equation*. Since β is only required to be nonnegative, Theorem 2.1 applies to undiscounted problems of the type studied by [15, 16].

3 A Monotone Convergence Principle

We define the partial order $\leq$ on W as follows:

$$v \leq w \quad \Leftrightarrow \quad \forall x \in X,\ v(x) \leq w(x). \tag{17}$$

It is easy to see that $\check{B}$ is order-preserving in the sense that for any $v, w \in W$,

$$v \leq w \quad \Rightarrow \quad \check{B}v \leq \check{B}w. \tag{18}$$

We are ready to sate what we call a *monotone convergence principle*:

Theorem 3.1. *Let $\underline{v} \in W$ satisfy*

$$\underline{v} \leq v^*, \tag{19}$$

$$\underline{v} \leq \check{B}\underline{v}. \tag{20}$$

Then the sequence $\{\check{B}^n \underline{v}\}_{n \in \mathbb{N}}$ converges increasingly to a fixed point $\underline{v}^*$ of $\check{B}$ pointwise. Furthermore, if

$$\forall \{x_t\}_{t=0}^{\infty} \in \check{\Pi}, \quad \varliminf_{t \uparrow \infty} \beta^t \underline{v}(x_t) \geq 0, \tag{21}$$

then $\underline{v}^* = v^*$; i.e., $\{\check{B}^n \underline{v}\}_{n \in \mathbb{N}}$ converges increasingly to v^* pointwise.

Proof. See Subsection 6.3. □

The results of [10, Theorem 2.2] and [12, Theorems 2, 3] easily follow from the above result; see [10, 12] for discussion of other related results in the literature.

In Section 6 we prove Theorem 3.1 by extending the proof of [12, Theorem 3]. Unlike the latter proof, we directly show the first conclusion of Theorem 3.1 without using Kleene's fixed point theorem. It is worth emphasizing that Theorem 3.1 requires no additional assumption; thus it can be regarded as a principle in deterministic dynamic programming.

4 Comments on Possible Emptiness of $\Gamma(x)$

Possible emptiness of $\Gamma(x)$ is useful even if it is known that $\Gamma(x) \neq \emptyset$ for all $x \in X$. For example, suppose that X is a subset of some space S. If we extend Γ to S by setting $\Gamma(x) = \emptyset$ for all $x \in S \setminus X$, then the modified optimality equation still holds on S:

$$\forall x \in S, \quad v^*(x) = \sup_{y \in \check{\Gamma}(x)} \{u(x,y) + \beta v^*(y)\}, \tag{22}$$

where we apply (4) before evaluating $u(x,y) + \beta v^*(y)$. This trivial extension is possible exactly because our approach allows for emptiness of $\Gamma(x)$. Extensions like (22) are useful when one views the original modified optimality equation (16) as a special or limiting case of an optimality equation holding on a larger space.

5 A Penalty Method

A penalty method is an approach to solving a constrained optimization problem through an approximating sequence of unconstrained problems with "penalty functions," which penalize violations of the constraint. Since penalty functions are often unbounded, our results are useful even for problems with bounded returns. To highlight this point we assume that u is bounded in this section.

Assumption 5.1. There exists $\mu > 0$ such that $0 \leq u(x,y) \leq \mu$ for all $(x,y) \in D$. Furthermore, $\beta \in (0,1)$.

Note that the existence of such μ can be assumed without loss of generality as long as u is bounded.

Let us first consider a trivial example. Let S be some superset of X; i.e., $X \subset S$. We define $\rho : S \times S \to \{0, -\infty\}$ as follows:

$$\rho(x,y) = \begin{cases} 0 & \text{if } (x,y) \in D, \\ -\infty & \text{otherwise.} \end{cases} \tag{23}$$

This is a simple penalty function that can be added to the return function u to remove the constraints from (5) and (6). To add ρ to u, however, we need to extend u to $S \times S$. These two steps can be accomplished in one step by extending u to $S \times S$ by setting

$$u(x,y) = -\infty, \quad \forall (x,y) \in (S \times S) \setminus D. \tag{24}$$

Now we can remove the constraint $\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)$ from (5):

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in S \times S \times \cdots} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}), \qquad x_0 \in X. \tag{25}$$

The above infinite sum exists since u is bounded above here.

For the rest of this section we assume the following.

Assumption 5.2. There exists $\underline{v} \in W$ satisfying (19)–(21).

In what follows we take such $\underline{v}$ as given. We extend $\underline{v}$ and v^* to S by setting

$$\underline{v}(x) = v^*(x) = -\infty, \quad \forall x \in S \setminus X. \tag{26}$$

We extend the modified Bellman operator $\check{B}$ to the functions from S to $[-\infty, \infty]$:

$$(\check{B}v)(x) = \sup_{y \in \check{\Gamma}(x)} \{u(x,y) + \beta v(y)\}, \quad x \in S. \tag{27}$$

Then (19) and (20) hold on S; more precisely

$$\forall x \in S, \qquad \underline{v}(x) \leq v^*(x), \quad \underline{v}(x) \leq (\check{B}\underline{v})(x). \tag{28}$$

To see this, note that if $x \in X$, both inequalities directly follow from (19) and (20). If $x \in S \setminus X$, both inequalities are immediate since $\underline{v}(x) = -\infty$ by (26). Since (21) is unaffected by the extensions defined in (24) and (26), the conclusion of Theorem 3.1 holds for the extended versions of $\underline{v}$ and v^*. For the rest of the section, $u, \underline{v}$, and v^* are understood as the extended versions given by (24) and (26).

To consider a more interesting case, we make more specific assumptions in addition to Assumptions 5.1 and 5.2:

Assumption 5.3. (i) $S = \mathbb{R}^N$. (ii) X is closed (in $\mathbb{R}^N$). (iii) D is closed. (iv) For each $x \in X$, $\Gamma(x)$ is nonempty and bounded (thus compact). (v) $u : D \to \mathbb{R}$ is upper semicontinuous.

We define $\tilde{u} : \mathbb{R}^N \times \mathbb{R}^N \to [0, \mu]$ by

$$\tilde{u}(x,y) = \begin{cases} u(x,y) & \text{if } (x,y) \in D, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

Note that $\tilde{u}$ is upper semicontinuous and is an extension of u to $\mathbb{R}^N \times \mathbb{R}^N$.

Let $d : (\mathbb{R}^N \times \mathbb{R}^N)^2 \to \mathbb{R}_+$ be the Euclidean distance on $\mathbb{R}^N \times \mathbb{R}^N$. For $x, y \in \mathbb{R}^N$ define

$$p(x, y) = \inf_{(x',y') \in D} \{d((x, y), (x', y'))\}^2. \tag{30}$$

Given any $x \in X$ and $y \in \mathbb{R}^N$, since $\Gamma(x)$ is compact, we have $y \in \Gamma(x)$ if and only if $p(x, y) = 0$. This function can be thought of as a quadratic penalty function. We assume (30) merely for concreteness; indeed, we can use any strictly increasing function of any equivalent norm or metric on $\mathbb{R}^N \times \mathbb{R}^N$.

For $i \in \mathbb{N}$ and $x, y \in \mathbb{R}^N$, define

$$u_i(x, y) = \tilde{u}(x, y) - ip(x, y). \tag{31}$$

It is easy to see that $\{u_i\}_{i \in \mathbb{N}}$ is a decreasing sequence satisfying $u \leq u_i$ for all $i \in \mathbb{N}$. Since $ip(x, y)$ tends to $\rho(x, y)$ as $i \uparrow \infty$ for each $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$, it follows that $\{u_i\}$ converges decreasingly to u pointwise.

For $i \in \mathbb{N}$, let v_i^* be the value function corresponding to u_i:

$$v_i^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \mathbb{R}^N \times \mathbb{R}^N \times \cdots} \sum_{t=0}^{\infty} \beta^t u_i(x_t, x_{t+1}), \qquad x_0 \in \mathbb{R}^N. \tag{32}$$

Let B_i be the Bellman operator corresponding to u_i:

$$(B_i v)(x) = \sup_{y \in \mathbb{R}^N} \{u_i(x, y) + \beta v(y)\}, \quad x \in \mathbb{R}^N. \tag{33}$$

We are ready to state the main result of this section:

Proposition 5.1. (i) *For each $i \in \mathbb{N}$, the sequence $\{(B_i)^n \underline{v}\}_{n \in \mathbb{N}}$ converges increasingly to v_i^* pointwise.* (ii) *The sequence $\{v_i^*\}_{i \in \mathbb{N}}$ converges decreasingly to v^* pointwise.*

Proof. For $i \in \mathbb{N}$, let $\check{B}_i$ be the modified Bellman operator corresponding to u_i. Since $u_i(x, y) > -\infty$ for all $x, y \in \mathbb{R}^N$, we have $B_i = \check{B}_i$. Since $u \leq u_i$ for all $i \in \mathbb{N}$, for any $i \in \mathbb{N}$ we have

$$\underline{v} \leq v^* \leq v_i^*, \tag{34}$$

$$\underline{v} \leq \check{B}\underline{v} \leq \check{B}_i \underline{v}. \tag{35}$$

By Assumption 5.2, $\underline{v}$ also satisfies (21), which is independent of i. Thus part (i) holds by Theorem 3.1. See Subsection 6.4 for the proof of part (ii). $\square$

Part (ii) of the above result shows that the value function v^* of the original problem (5) can be approximated by the value function of the unconstrained problem (32) with the penalty function sufficiently large in absolute value. Part (i) shows that the value function of the unconstrained problem can be computed by the monotone convergence principle.

6 Proofs

6.1 Preliminary Result

In this subsection we state an elementary result shown in [14]. Recall from (4) that $\sup A$ is well defined for any $A \subset \overline{\mathbb{R}}$. We emphasize that none of the sets in the following result is required to be nonempty.

Lemma 6.1. *Let Y and Z be sets. Let $\Omega \subset Y \times Z$, and let $f : \Omega \to \overline{\mathbb{R}}$. For $y \in Y$ and $z \in Z$, define*

$$\Omega_y = \{z \in Z : (y, z) \in \Omega\}, \tag{36}$$
$$\Omega_z = \{y \in Y : (y, z) \in \Omega\}. \tag{37}$$

Then

$$\sup_{(y,z)\in\Omega} f(y, z) = \sup_{y\in Y} \sup_{z\in\Omega_y} f(y, z) = \sup_{z\in Z} \sup_{y\in\Omega_z} f(y, z). \tag{38}$$

Proof. See [14, Lemma 1]. □

Essentially the same result is shown in [17, Lemma 3.2] under the additional assumption that the sets Y, Z, and Ω are all nonempty.

6.2 Proof of Theorem 2.1

Let $v \in V$ be a fixed point of B. Then $v = Bv = \check{B}v$ by Lemma 2.1. Hence v is a fixed point of $\check{B}$.

To show that v^* is a fixed point of $\check{B}$, let $x_0 \in X$. Note that $\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)$ if and only if

$$u(x_0, x_1) > -\infty, \quad \underset{T\uparrow\infty}{\mathrm{L}} \sum_{t=1}^{T} \beta^t u(x_t, x_{t+1}) > -\infty. \tag{39}$$

Therefore

$$\check{\Pi}(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X \times X \times \cdots : x_1 \in \check{\Gamma}(x_0), \{x_t\}_{t=2}^{\infty} \in \check{\Pi}(x_1)\}. \tag{40}$$

We apply Lemma 6.1 with $y = x_1$, $z = \{x_t\}_{t=2}^{\infty}$, $\Omega = \check{\Pi}(x_0), Y = \check{\Gamma}(x_0)$, $Z = X \times X \times \cdots$, and $\Omega_y = \check{\Pi}(x_1)$. Note from (10) that

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)} \left\{ u(x_0, x_1) + \operatorname*{L}_{T\uparrow\infty} \sum_{t=1}^{T} \beta^t u(x_t, x_{t+1}) \right\} \tag{41}$$

$$= \sup_{x_1 \in \check{\Gamma}(x_0)} \sup_{\{x_t\}_{t=2}^{\infty} \in \check{\Pi}(x_1)} \left\{ u(x_0, x_1) + \operatorname*{L}_{T\uparrow\infty} \sum_{t=1}^{T} \beta^t u(x_t, x_{t+1}) \right\} \tag{42}$$

$$= \sup_{x_1 \in \check{\Gamma}(x_0)} \left\{ u(x_0, x_1) + \sup_{\{x_t\}_{t=2}^{\infty} \in \check{\Pi}(x_1)} \operatorname*{L}_{T\uparrow\infty} \sum_{t=1}^{T} \beta^t u(x_t, x_{t+1}) \right\} \tag{43}$$

$$= \sup_{x_1 \in \check{\Gamma}(x_0)} \{u(x_0, x_1) + \beta v^*(x_1)\} = (\check{B}v^*)(x_0), \tag{44}$$

where (42) uses Lemma 6.1 and (40). Since x_0 was arbitrary, it follows that $\check{B}v^* = v^*$.

6.3 Proof of Theorem 3.1

We first prove two lemmas.

Lemma 6.2. *Suppose that there exists $\underline{v} \in W$ satisfying (19) and (20). Define $\underline{v}^* = \sup_{n\in\mathbb{N}}(\check{B}^n \underline{v})$, where the supremum is taken pointwise. Then $\{\check{B}^n \underline{v}\}_{n\in\mathbb{N}}$ converges increasingly to $\underline{v}^*$ pointwise. Furthermore, $\underline{v}^*$ is a fixed point of $\check{B}$.*

Proof. For $n \in \mathbb{N}$, let $v_n = \check{B}^n \underline{v}$. It follows from (18) and (20) that $\{v_n\}_{n\in\mathbb{N}}$ is an increasing sequence. Hence $\{v_n\}_{n\in\mathbb{N}}$ converges increasingly to $\underline{v}^*$ pointwise. To see that $\underline{v}^*$ is a fixed point of $\check{B}$, fix $x \in X$. Note that

$$\underline{v}^*(x) = \sup_{n\in\mathbb{N}}(\check{B}v_n)(x) = \sup_{n\in\mathbb{N}} \sup_{y\in\check{\Gamma}(x)} \{u(x, y) + \beta v_n(y)\} \tag{45}$$

$$= \sup_{y\in\check{\Gamma}(x)} \sup_{n\in\mathbb{N}} \{u(x, y) + \beta v_n(y)\} \tag{46}$$

$$= \sup_{y\in\check{\Gamma}(x)} \{u(x, y) + \beta \underline{v}^*(y)\} = (\check{B}\underline{v}^*)(x), \tag{47}$$

where (46) uses Lemma 6.1. Since $x \in X$ was arbitrary, it follows that $\underline{v}^* = \check{B}\underline{v}^*$; i.e., $\underline{v}^*$ is a fixed point of $\check{B}$. $\square$

Lemma 6.3. *Let $\underline{v} \in W$ satisfy (21). Let $v \in W$ be a fixed point of $\check{B}$ with $\underline{v} \leq v$. Then $v^* \leq v$.*

Proof. Let $v \in W$ be a fixed point of $\check{B}$ with $\underline{v} \leq v$. Let $x_0 \in X$. If $\check{\Pi}(x_0) = \emptyset$, then $v^*(x_0) = -\infty \leq v(x_0)$. For the rest of the proof, suppose that $\check{\Pi}(x_0) \neq \emptyset$. Let $\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)$. Then $x_{t+1} \in \check{\Gamma}(x_t)$ for all $t \in \mathbb{Z}_+$. We have

$$\begin{aligned}
v(x_0) &= \sup_{y \in \check{\Gamma}(x_0)} \{u(x_0, y) + \beta v(y)\} && (48)\\
&\geq u(x_0, x_1) + \beta v(x_1) && (49)\\
&\geq u(x_0, x_1) + \beta u(x_1, x_2) + \beta^2 v(x_2) && (50)\\
&\ \ \vdots && (51)\\
&\geq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T) && (52)\\
&\geq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T). && (53)
\end{aligned}$$

Let $\delta > 0$. By (21) we have $\beta^T \underline{v}(x_T) \geq -\delta$ for sufficiently large T. For such T we have

$$v(x_0) \geq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) - \delta. \tag{54}$$

Hence we have

$$v(x_0) \geq \mathop{\mathrm{L}}_{T \uparrow \infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) - \delta. \tag{55}$$

Since this holds for any $\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)$, applying $\sup_{\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)}$ to the right-hand side of (55) and recalling (10), we have $v(x_0) \geq v^*(x_0) - \delta$. As δ was arbitrary, we obtain $v(x_0) \geq v^*(x_0)$. Because this is true for any $x_0 \in X$, we have $v \geq v^*$. $\square$

To complete the proof of Theorem 3.1, let $\underline{v} \in W$ satisfy (19) and (20). Then by Lemma 6.2, $\{\check{B}^n \underline{v}\}_{n\in\mathbb{N}}$ converges increasingly to $\underline{v}^*$ pointwise, and $\underline{v}^*$ is a fixed point of $\check{B}$. Assume (21). We have $\underline{v}^* \leq v^*$ since $\underline{v} \leq v^*$ by (19), $\check{B}$ is order-preserving, and v^* is a fixed point of $\check{B}$ by Theorem 2.1. We also have $\underline{v}^* \geq v^*$ by Lemma 6.3. Hence $\underline{v}^* = v^*$.

6.4 Proof of Proposition 5.1(ii)

Note that $\{v_i^*\}_{i\in\mathbb{N}}$ is a decreasing sequence since $\{u_i\}_{i\in\mathbb{N}}$ is a decreasing sequence. It remains to show that $\{v_i^*\}$ converges to v^* pointwise.

Fix $x_0 \in \mathbb{R}^N$. We show that $v_i^*(x_0) \to v^*(x_0)$ as $i \uparrow \infty$. Let $\epsilon > 0$. For each $i \in \mathbb{N}$, by the definition of v_i^* in (32), there exists $\{x_t^i\}_{t=1}^\infty \in \mathbb{R}^N \times \mathbb{R}^N \cdots$ such that

$$v_i^*(x_0) - \epsilon \leq \sum_{t=0}^\infty \beta^t u_i(x_t^i, x_{t+1}^i) \leq \frac{\mu}{1-\beta} - i \sum_{t=0}^\infty \beta^t p(x_t, x_{t+1}), \qquad (56)$$

where the second inequality holds by Assumption 5.1 and (31).

Suppose that $x_0 \notin X$. Then x_0 belongs to the open set $\mathbb{R}^N \setminus X$. Recalling (30) we see that there exists $\delta > 0$ such that $p(x_0, y) \geq \delta$ for all $y \in \mathbb{R}^N$. This together with (56) implies that $v_i^*(x_0) \to -\infty = v^*(x_0)$ as $i \uparrow \infty$.

Suppose that $x_0 \in X$. Since $v_i^* \geq v^* \geq 0$ on X for all $i \in \mathbb{N}$, it follows from (56) that

$$\sum_{t=0}^\infty \beta^t p(x_t^i, x_{t+1}^i) \leq \frac{1}{i}\left[\frac{\mu}{1-\beta} + \epsilon\right]. \qquad (57)$$

Since the right-hand side tends to 0 as $i \uparrow \infty$, we have $p(x_t^i, x_{t+1}^i) \to 0$ as $i \uparrow \infty$ for each $t \in \mathbb{Z}_+$. This together with Assumption 5.3 implies that there exist a subsequence of $\{\{x_t^i\}_{t=1}^\infty\}_{i\in\mathbb{N}}$, again denoted by $\{\{x_t^i\}_{t=1}^\infty\}_{i\in\mathbb{N}}$, and a feasible path $\{x_t^*\}_{t=1}^\infty \in \Pi(x_0)$ such that $x_t^i \to x_t^*$ as $i \uparrow \infty$ for each $t \in \mathbb{N}$.

Note that for any $i \in \mathbb{N}$ we have

$$\sum_{t=0}^\infty \beta^t u_i(x_t^i, x_{t+1}^i) \leq \sum_{t=0}^\infty \beta^t \tilde{u}(x_t^i, x_{t+1}^i). \qquad (58)$$

Since $\tilde{u}$ is bounded, by (56), (58), and the dominated convergence theorem, we have

$$\lim_{i\uparrow\infty} v_i^*(x_0) - \epsilon \le \sum_{t=0}^{\infty} \beta^t \lim_{i\uparrow\infty} \tilde{u}(x_t^i, x_{t+1}^i) \le \sum_{t=0}^{\infty} \beta^t \tilde{u}(x_t^*, x_{t+1}^*) \tag{59}$$

$$= \sum_{t=0}^{\infty} \beta^t u(x_t^*, x_{t+1}^*) \le v^*(x_0), \tag{60}$$

where the second inequality in (59) holds by upper semicontinuity of $\tilde{u}$, and the equality in (60) holds since $\{x_t^*\} \in \Pi(x_0)$. Since $\epsilon > 0$ was arbitrary, we have $\lim_{i\uparrow\infty} v_i^*(x_0) \le v^*(x_0)$. Recalling (34) we conclude that $\lim_{i\uparrow\infty} v_i^*(x_0) = v^*(x_0)$.

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