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Abstract

A policy change that involves a redistribution of income or wealth is typically controversial, affecting some people positively but others negatively. In this paper we extend the "robust comparative statics" result for large aggregative games established by Acemoglu and Jensen (2010, *49th IEEE Conference on Decision and Control*, 3133–3139) to possibly controversial policy changes. In particular, we show that both the smallest and the largest equilibrium values of an aggregate variable increase in response to a policy change to which individuals' reactions may be mixed but the overall aggregate response is positive. We provide sufficient conditions for such a policy change in terms of distributional changes in parameters.

Keywords: Large aggregative games; robust comparative statics; positive shocks; stochastic dominance; mean-preserving spreads

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1 Introduction

Recently, Acemoglu and Jensen (2010, 2015) developed new comparative statics techniques for large aggregative games, where there are a continuum of individuals interacting with each other only through an aggregate variable (which integrates all individuals' actions). The surprising insight of their analysis is that in such games, one can obtain a "robust comparative statics" result without considering the interaction between the aggregate variable and individuals' actions. In particular, Acemoglu and Jensen (2010) defined a positive shock as a positive parameter change (a set of simultaneous increases in individuals' parameters) that positively affects each individual's action for each value of the aggregate variable. They showed that both the smallest and the largest equilibrium values of the aggregate variable increase in response to a positive shock.

Although positive shocks are common in economic models, many important policy changes in reality tend to be controversial, affecting some individuals positively but others negatively. For example, a policy change that involves a redistribution of income necessarily affects some individuals' income positively but others' negatively. Such policy changes of practical importance cannot be positive shocks.

The purpose of this paper is to show that Acemoglu and Jensen's (2010, 2015) analysis can in fact be extended to such policy changes. Using Acemoglu and Jensen's (2010) static framework, we consider possibly controversial policy changes by defining an "overall positive shock" to be a parameter change to which individuals' reactions may be mixed but the overall aggregate response is positive for each value of the aggregate variable. We show that both the smallest and the largest equilibrium values of the aggregate variable increase in response to an overall positive shock. Then we provide sufficient conditions for an overall positive shock in terms of distributional changes in parameters. These conditions enable one to deal with various policy changes, including ones that involve a redistribution of income.

The concept of overall positive shocks is closely related not only to that of positive shocks but also to Acemoglu and Jensen's (2013) concept of "shocks that hit the aggregator," which were defined as parameter changes that directly affect the "aggregator" positively along with additional restrictions. Such parameter changes are not considered in this paper, but they can easily be incorporated by slightly extending our framework.

This paper is not the first to study comparative statics for distributional changes. In a general dynamic stochastic model with a continuum of individuals, Acemoglu and Jensen (2015) considered robust comparative statics for changes in the stationary distributions of individuals' idiosyncratic shocks, but their analysis was restricted to positive shocks in the above sense.[1] Jensen (2015) and Nocetti (2015) studied comparative statics for more general distributional changes, but neither of them considered robust comparative statics. This paper bridges the gap between robust comparative statics and distributional comparative statics.

The rest of the paper is organized as follows. In Section 2 we provide a simple example of income redistribution and aggregate labor supply to illustrate how Acemoglu and Jensens's (2010) analysis may fail to apply to a policy change of practical importance. In Section 3 we present our general framework along with basic assumptions, and show the existence of a pure-strategy Nash equilibrium. In Section 4 we formally define overall positive shocks. We also introduce a more general definition of "overall monotone shocks." We then present our robust comparative statics result. In Section 5 we provide sufficient conditions for an overall monotone shock in terms of distributional changes in parameters based on first-order stochastic dominance and mean-preserving spreads. In Section 6 we apply our results to the example of income redistribution. In Section 7 we conclude the paper.

2 A Simple Model of Income Redistribution

Consider an economy with a continuum of agents indexed by $i \in [0, 1]$. Agent i solves the following maximization problem:

$$\max_{c_i, x_i \geq 0} \quad u(c_i) - x_i \tag{2.1}$$

$$\text{s.t.} \quad c_i = wx_i + e_i + s_i, \tag{2.2}$$

where $u : \mathbb{R}_+ \to \mathbb{R}$ is strictly increasing, strictly concave, and twice continuously differentiable, w is the wage rate, s_i is a lump-sum transfer to agent i, and c_i, x_i, and e_i are agent i's consumption, labor supply, and endowment, respectively. We assume that $e_i + s_i \geq 0$ for all $i \in [0, 1]$. If $s_i < 0$, agent i pays a lump-sum tax of $-s_i$. For simplicity, we assume that the upper

[1] See Balbus et al. (2015) for monotone comparative statics results on distributional Bayesian Nash equilibria with strategic complementarities.

bound on x_i is never binding for relevant values of w and is thus not explicitly imposed. This simply means that no agent works 24 hours a day, 7 days a week. The government has no external revenue and satisfies

$$\int_{i \in I} s_i di = 0. \tag{2.3}$$

Aggregate demand for labor is given by a demand function $D(w)$ such that $D(0) < \infty$, $D(\overline{w}) = 0$ for some $\overline{w} > 0$, and $D : [0, \overline{w}] \to \mathbb{R}_+$ is continuous and strictly decreasing. The market-clearing condition is

$$D(w) = \int_{i \in I} x_i di. \tag{2.4}$$

Given (2.3), any change in the profile of s_i affects some agents' income positively but others' negatively. Hence it cannot be a positive shock in the sense of Acemoglu and Jensen's (2010). However, one may still ask, how does a policy change that widens income inequality affect aggregate labor supply and the wage rate? This may seem a difficult question to answer without additional assumptions. It turns out that our results help one answer this type of question in a robust way.

3 Large Aggregative Games

We consider a large aggregative game as defined by Acemoglu and Jensen (2010, Sections II, III). There are a continuum of players indexed by $i \in I \equiv [0, 1]$. Player i's action and action space are denoted by x_i and $X_i \subset \mathbb{R}$, respectively. The assumption made in this section are maintained throughout the paper.

Assumption 3.1. *For each $i \in I$, X_i is nonempty and compact. Furthermore, there exists a compact set $K \subset \mathbb{R}$ such that $X_i \subset K$ for all $i \in I$.*

Let $X = \prod_{i \in I} X_i$. Let $\mathscr{X}$ be the set of action profiles $x \in X$ such that the mapping $i \in I \mapsto x_i$ is measurable.[2] Let H be a function from K to a subset Ω of $\mathbb{R}$. We define $G : \mathscr{X} \to \Omega$, called the *aggregator*, by

$$G(x) = H\left(\int_{i \in I} x_i di\right). \tag{3.1}$$

[2]Unless otherwise specified, measurability means Lebesgue measurability.

Assumption 3.2. The set Ω is compact and convex, and $H : K \to \Omega$ is continuous.[3]

Each player i's payoff depends on his own action $x_i \in X_i$, the entire action profile $x \in X$ through the aggregate $G(x)$, and a parameter t_i specific to player i. In other words, player i's payoff takes the form $\pi_i(x_i, G(x), t_i)$. Any additional information on x_j and t_j with $j \neq i$ is irrelevant to his decision-making.

Let T_i be the underlying space for t_i for each $i \in I$; i.e., $t_i \in T_i$. Let $T \subset \prod_{i \in I} T_i$. We regard T as a set of well-behaved parameter profiles; for example, T can be a set of measurable functions from I to $\mathbb{R}$. We only consider parameter profiles t in T.

Assumption 3.3. For each $i \in I$, player i's payoff function π_i maps each $(k, Q, \tau) \in K \times \Omega \times T_i$ into $\mathbb{R}$.[4] Furthermore, for each $t \in T$, $\pi_i(\cdot, \cdot, t_i)$ is continuous on $K \times \Omega$, and for each $(k, Q) \in K \times \Omega$, $\pi_i(k, Q, t_i)$ is measurable in $i \in I$.

The game here is aggregative in the sense that each player's payoff is affected by other players' actions only through the aggregate $G(x)$. Accordingly, each player i's best response correspondence depends on other players' actions only through $Q = G(x)$. Let $R_i(Q, t_i)$ denote player i's best response correspondence:

$$R_i(Q, t_i) = \arg\max_{x_i \in X_i} \pi_i(x_i, Q, t_i). \tag{3.2}$$

The following technical assumption ensures that given any $Q \in \Omega$, we can find a measurable action profile $x \in \mathscr{X}$ such that $x_i \in R_i(Q, t_i)$ for all $i \in I$.[5]

Assumption 3.4. For each open subset U of K, the set $\{i \in I : X_i \cap U \neq \emptyset\}$ is measurable.

Throughout the paper, we restrict attention to pure-strategy Nash equilibria, which we simply call equilibria. To be more precise, given $t \in T$, an *equilibrium* of this game is an action profile $x \in \mathscr{X}$ such that $x_i \in R_i(G(x), t_i)$

[3]Given the assumptions on H and K, the properties of Ω here can be assumed without loss of generality.

[4]If π_i is initially defined only on $X_i \times \Omega \times T_i$, then this means that π_i can be extended to $K \times \Omega \times T_i$ in such a way as to satisfy Assumption 3.3.

[5]See the paragraph below (A.1) for details.

for all $i \in I$. Given $t \in T$, we define an *equilibrium aggregate* as $Q(t) \in \Omega$ such that $Q(t) = G(x)$ for some equilibrium x. We define $\underline{Q}(t)$ and $\overline{Q}(t)$ as the smallest and largest equilibrium aggregates, respectively, provided that they exist. The following result shows that an equilibrium as well as the smallest and largest equilibrium aggregates exist.

Theorem 3.1. *For any $t \in T$, an equilibrium exists. Furthermore, the set of equilibrium aggregates is compact (and nonempty). Thus the smallest and largest equilibrium aggregates $\underline{Q}(t)$ and $\overline{Q}(t)$ exist.*

Proof. See Appendix A. □

Following Acemoglu and Jensen (2010, Theorem 1), we prove the above result using Kakutani's fixed point theorem and Aumann's (1965, 1976) results on the integral of a correspondence. Our result differs from Acemoglu and Jensen's in that we assume a continuum of player types rather than a finite number of player types.[6]

The existence of an equilibrium can alternatively be shown by using Theorem 3.4.1 in Balder (1995) under slightly different assumptions. There are other general existence results for games with a continuum of players in the literature (e.g., Khan et all. 1997; Carmona and Podczeck, 2009). However, these results do not directly apply here since they assume a common strategy space for all players and do not explicitly establish the existence of the smallest and largest equilibrium aggregates.

4 Overall Monotone Shocks

By a *parameter change*, we mean a change in $t \in T$ from one profile to another. Given $\underline{t}, \overline{t} \in T$, the *parameter change from $\underline{t}$ to $\overline{t}$* means the change in t from $\underline{t}$ to $\overline{t}$. The following definitions take $\underline{t}, \overline{t} \in T$ as given.

Definition 4.1 (Acemoglu and Jensen, 2010)**.** The parameter change from $\underline{t}$ to $\overline{t}$ is a *positive shock* if (a) T is equipped with a partial order $\prec$, (b) $G(\cdot)$ is an increasing function,[7] (c) $\underline{t} \prec \overline{t}$, and (d) for each $Q \in \Omega$ and $i \in I$, the following properties hold:

[6] Acemoglu and Jensen (2015) allow for a continuum of player types by using the Pettis integral in (3.1).

[7] In this paper, "increasing" means "nondecreasing," and "decreasing" means "nonincreasing."

(i) For each $\underline{x}_i \in R_i(Q, \underline{t}_i)$ there exists $\overline{x}_i \in R_i(Q, \overline{t}_i)$ such that $\underline{x}_i \leq \overline{x}_i$.

(ii) For each $\overline{y}_i \in R_i(Q, \overline{t}_i)$ there exists $\underline{y}_i \in R_i(Q, \underline{t}_i)$ such that $\underline{y}_i \leq \overline{y}_i$.

For comparison purposes, Acemoglu and Jensen's (2010) key assumptions are included in the above definition. We introduce additional definitions.

Definition 4.2. The parameter change from $\underline{t}$ to $\overline{t}$ is a *negative shock* if the parameter change from $\overline{t}$ to $\underline{t}$ is a positive shock. A parameter change is a *monotone shock* if it is a positive shock or a negative shock.

Acemoglu and Jensen (2010, Theorem 2) show that if the parameter change from $\underline{t}$ to $\overline{t}$ is a positive shock, then the following inequalities hold:

$$\underline{Q}(\underline{t}) \leq \underline{Q}(\overline{t}), \quad \overline{Q}(\underline{t}) \leq \overline{Q}(\overline{t}). \tag{4.1}$$

In this section we show that these inequalities hold for a substantially larger class of parameter changes. To this end, for $Q \in \Omega$ and $t \in T$, we define

$$\mathcal{G}(Q, t) = \{G(x) : x \in \mathscr{X}, \forall i \in I, x_i \in R_i(Q, t_i)\}. \tag{4.2}$$

The following definitions, which do not require T to be partially ordered and $G(\cdot)$ to be increasing, play a central role in our comparative statics results.

Definition 4.3. The parameter change from $\underline{t}$ to $\overline{t}$ is an *overall positive shock* if for each $Q \in \Omega$ the following properties hold:

(i) For each $\underline{q} \in \mathcal{G}(Q, \underline{t})$ there exists $\overline{q} \in \mathcal{G}(Q, \overline{t})$ such that $\underline{q} \leq \overline{q}$.

(ii) For each $\overline{r} \in \mathcal{G}(Q, \overline{t})$ there exists $\underline{r} \in \mathcal{G}(Q, \underline{t})$ such that $\underline{r} \leq \overline{r}$.

Definition 4.4. The parameter change from $\underline{t}$ to $\overline{t}$ is an *overall negative shock* if the parameter change from $\overline{t}$ to $\underline{t}$ is an overall positive shock. A parameter change is an *overall monotone shock* if it is an overall positive shock or an overall negative shock.

It is easy to see that a positive shock (which requires $G(\cdot)$ to be increasing) is an overall positive shock. We are ready to state our result on robust comparative statics:

Theorem 4.1. *Let $\underline{t}, \overline{t} \in T$. Suppose that the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive shock. Then both inequalities in (4.1) hold. The reserve inequalities hold if the parameter change is an overall negative shock.*

Proof. See Appendix B. □

The proof of this result closely follows that of Theorem 2 in Acemoglu and Jensen (2010). The latter result is immediate from Theorem 4.1 under our assumptions since a positive shock is an overall positive shock.[8]

5 Sufficient Conditions

In this section we provide sufficient conditions for overall monotone shocks by assuming that players differ only in their parameters t_i. This assumption by itself is innocuous since it can be ensured by redefining t_i as (t_i, i). More specifically, we assume the following for the rest of the paper.

Assumption 5.1. There exists a Borel-measurable convex set $\mathcal{T} \subset \mathbb{R}^n$ (equipped with the usual partial order) with $n \in \mathbb{N}$ such that $T_i \subset \mathcal{T}$ for all $i \in I$. There exists a convex-valued correspondence $\mathcal{X} : \mathcal{T} \to 2^{\mathcal{T}}$ such that $X_i = \mathcal{X}(t_i)$ for all $i \in I$ and $t_i \in T_i$. Moreover, there exists a function $\pi : K \times \Omega \times \mathcal{T} \to \mathbb{R}$ such that

$$\forall i \in I, \forall (k, Q, \tau) \in K \times \Omega \times \mathcal{T}, \quad \pi_i(k, Q, \tau) = \pi(k, Q, \tau). \tag{5.1}$$

This assumption implies that player i's best response correspondence $R_i(Q, \tau)$ does not directly depend on i; we denote this correspondence by $R(Q, \tau)$. For $(Q, \tau) \in \Omega \times \mathcal{T}$, we define

$$\underline{R}(Q, \tau) = \min R(Q, \tau), \quad \overline{R}(Q, \tau) = \max R(Q, \tau). \tag{5.2}$$

Both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are well-defined since $R(Q, \tau)$ is a compact set for each $(Q, \tau) \in (\Omega, \mathcal{T})$ (see Lemma A.1). To consider distributional changes in $t \in T$, we assume the following for the rest of the paper.

Assumption 5.2. T is a set of measurable functions from I to $\mathcal{T}$, and $H : K \to \Omega$ is an increasing function.

For any $t \in T$, let $F_t : \mathbb{R}^n \to I$ denote the distribution function of t:

$$F_t(z) = \int_{i \in I} 1\{t_i \leq z\} di, \tag{5.3}$$

where $1\{\cdot\}$ is the indicator function; i.e., $1\{t_i \leq z\} = 1$ if $t_i \leq z$, and $= 0$ otherwise. Note that $F_t(z)$ is the proportion of players $i \in I$ with $t_i \leq z$.

[8] Acemoglu and Jensen (2015, Theorem 5) establish a dynamic version of their result based on positive shocks. Their dynamic result can also be extended to overall monotone shocks in a similar way.

5.1 First-Order Stochastic Dominance

Given a pair of distributions $\underline{F}$ and $\overline{F}$, $\overline{F}$ is said to *(first-order) stochastically dominate* $\underline{F}$ if

$$\int \phi(z)\, d\underline{F}(z) \leq \int \phi(z)\, d\overline{F}(z) \tag{5.4}$$

for any increasing bounded Borel-measurable function $\phi : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}^n$ is equipped with the usual partial order $\leq$. It is well known (e.g., Müller and Stoyan, 2002, Section 1) that in case $n = 1$, $\overline{F}$ stochastically dominates $\underline{F}$ if and only if

$$\forall z \in \mathbb{R}, \quad \underline{F}(z) \geq \overline{F}(z). \tag{5.5}$$

The following result provides a sufficient condition for an overall monotone shock based on stochastic dominance.

Theorem 5.1. *Let $\underline{t}, \overline{t} \in T$. Suppose that $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. Suppose that both $\underline{R}(Q,\tau)$ and $\overline{R}(Q,\tau)$ are increasing (resp. decreasing) Borel-measurable functions of $\tau \in \mathcal{T}$ for each $Q \in \Omega$. Then the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive (resp. negative) shock.*

Proof. We only consider the increasing case; the decreasing case is symmetric. Let $\underline{q} \in \mathcal{G}(Q, \underline{t})$. Then there exists $x \in \mathscr{X}$ such that $\underline{q} = H(\int_{i\in I} x_i di)$ and $x_i \in R(Q, \underline{t}_i)$ for all $i \in I$. Since $x_i \leq \overline{R}(Q, \underline{t}_i)$ for all $i \in I$ by (5.2), and since H is an increasing function by Assumption 5.2, we have

$$\underline{q} \leq H\left(\int_{i\in I} \overline{R}(Q, \underline{t}_i) di\right) = H\left(\int \overline{R}(Q, z) dF_{\underline{t}}(z)\right) \tag{5.6}$$

$$\leq H\left(\int \overline{R}(Q, z) dF_{\overline{t}}(z)\right) = H\left(\int_{i\in I} \overline{R}(Q, \overline{t}_i) di\right) \in \mathcal{G}(Q, \overline{t}), \tag{5.7}$$

where the inequality in (5.7) holds since $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$ and $\overline{R}(Q, \cdot)$ is an increasing function. It follows that condition (i) of Definition 4.3 holds. By a similar argument, condition (ii) also holds. Hence the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive shock. □

If the parameter change from $\underline{t}$ to $\overline{t}$ is a positive shock, then it is easy to see from (5.3) and (5.5) that $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. However, there are many other ways in which $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. Figure 1 shows a simple example. In this example, the parameter change from $\underline{t}$ to $\overline{t}$ is not

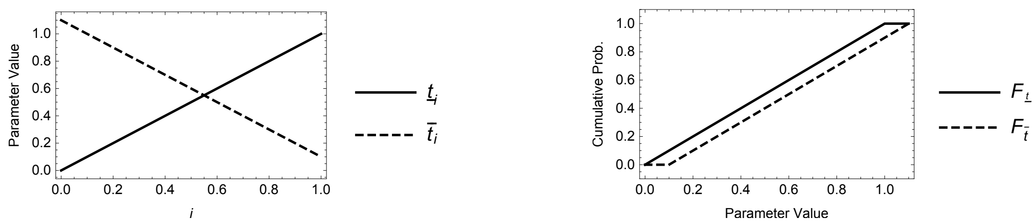

Figure 1: The parameter change from $\underline{t}$ to $\overline{t}$ is not a monotone shock (left panel), but $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$ (right panel).

a monotone shock, but $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$ by (5.5). Thus the parameter change here is an overall positive shock by Theorem 5.1 if both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are increasing in τ.

There are well known sufficient conditions for both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ to be increasing or decreasing; see Milgrom and Shannon (1994, Theorem 4), Topkis (1998, Theorem 2.8.3), Vives (1999, P. 35), Amir (2005, Theorems 1, 2), and Roy and Sabarwal (2010, Theorem 2). Any of the conditions can be combined with Theorem 5.1 to replace the assumption that both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are increasing or decreasing. Here we state a simple result using the well-known conditions shown in Amir (2005, Lemma 1, Theorems 1, 2).

Corollary 5.1. *Let $\underline{t}, \overline{t} \in T$. Suppose that $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. Suppose that $\mathcal{T} \subset \mathbb{R}$ and the upper and lower boundaries of $\mathcal{X}(\tau)$ are increasing (resp. decreasing) functions of $\tau \in \mathcal{T}$. Suppose further that for each $Q \in \Omega$, $\pi(k, Q, \tau)$ is twice continuously differentiable in $(k, \tau) \in K \times \mathcal{T}$ and $\partial^2 \pi(k, Q, \tau)/\partial k \partial \tau \geq 0$ (resp. ≤ 0) for all $(k, \tau) \in K \times \mathcal{T}$. Then the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive (resp. negative) shock.*

5.2 Mean-Preserving Spreads

Following Acemoglu and Jensen (2015), we say that $F_{\overline{t}}$ is a *mean-preserving spread* of $F_{\underline{t}}$ if (5.4) holds for any Borel-measurable convex function $\phi : \mathcal{T} \to \mathbb{R}$.[9] Rothschild and Stiglitz (1970, p. 231) and Machina and Pratt (1997,

[9] Our approach differs from that of Acemoglu and Jensen (2015) in that while they consider positive shocks induced by applying a mean-preserving spread to the stationary distribution of each player's idiosyncratic shock, we consider non-monotone shocks induced by applying a mean-preserving spread to the entire distribution of parameters.

Theorem 3) show that in case $n = 1$, $F_{\bar{t}}$ is a mean-preserving spread of $F_{\underline{t}}$ if

$$\int F_{\underline{t}}(z)dz = \int F_{\bar{t}}(z)dz, \tag{5.8}$$

and if there exists $\tilde{z} \in \mathbb{R}$ such that

$$F_{\underline{t}}(z) - F_{\bar{t}}(z) \begin{cases} \leq 0 & \text{if } z \leq \tilde{z}, \\ \geq 0 & \text{if } z > \tilde{z}. \end{cases} \tag{5.9}$$

The following result provides a sufficient condition for an overall monotone shock based on mean-preserving spreads.

Theorem 5.2. *Let $\underline{t}, \bar{t} \in T$. Suppose that $F_{\bar{t}}$ is a mean-preserving spread of $F_{\underline{t}}$. Suppose that both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are Borel-measurable convex (resp. concave) functions of $\tau \in \mathcal{T}$ for each $Q \in \Omega$. Then the parameter change from $\underline{t}$ to $\bar{t}$ is an overall positive (resp. negative) shock.*

Proof. The proof is essentially the same as that of Theorem 5.1 except that the inequality in (5.7) holds since $F_{\bar{t}}$ is a mean-preserving spread of $F_{\underline{t}}$ and $\overline{R}(Q, \tau)$ is convex in τ. □

Figure 2 shows a simple example of a mean-preserving spread. As can be seen in the left panel, the parameter change from $\underline{t}$ to $\bar{t}$ is not a monotone shock. However, it is a mean-preserving spread by (5.8) and (5.9), as can be seen in the right panel. Thus the parameter change here is an overall positive shock by Theorem 5.2 if both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are convex in $\tau \in \mathcal{T}$.

Sufficient conditions for $\underline{R}(Q, \tau)$ or $\overline{R}(Q, \tau)$ to be convex or concave are established by Jensen (2015). Here we state a simple result using his conditions (Lemmas 1, 3, Theorem 2, Corollary 2).

Corollary 5.2. *Let $\underline{t}, \bar{t} \in T$. Suppose that $F_{\bar{t}}$ is a mean-preserving spread of $F_{\underline{t}}$. Suppose that the upper and lower boundaries of $\mathcal{X}(\tau)$ are convex (resp. concave), continuous functions of $\tau \in \mathcal{T}$. For each $(Q, \tau) \in \Omega \times \mathcal{T}$, suppose that $\pi(k, Q, \tau)$ is strictly quasi-concave and continuously differentiable in $k \in K$, and that $\overline{R}(Q, \tau) < \max \mathcal{X}(\tau)$ (resp. $\underline{R}(Q, \tau) > \min \mathcal{X}(\tau)$). For each $Q \in \Omega$, suppose that $\partial\pi(k, Q, \tau)/\partial k$ is quasi-convex (resp. quasi-concave) in $(k, \tau) \in K \times \mathcal{T}$. Then the parameter change from $\underline{t}$ to $\bar{t}$ is an overall positive (resp. negative) shock.*

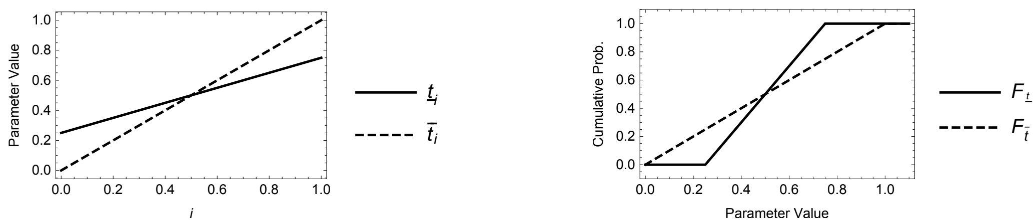

Figure 2: The parameter change from $\underline{t}$ to $\overline{t}$ is not a monotone shock (left panel), but $F_{\overline{t}}$ is a mean-preserving spread of $F_{\underline{t}}$ (right panel).

6 Applications

Recall the model of Section 2. Let $t_i = e_i + s_i$ for $i \in I$. The first-order condition for the maximization problem (2.1)-(2.2) is written as

$$u'(wx_i + t_i)w \begin{cases} \leq 1 & \text{if } x_i = 0, \\ = 1 & \text{if } x_i > 0. \end{cases} \tag{6.1}$$

Let $x(w, t_i)$ denote the solution for x_i as a function of w and t_i.

To see that this model is a large aggregative game, let $Q = \int_{i \in I} x(w, t_i) di$. Then (2.4) implies that $w = D^{-1}(Q)$. Let $\overline{\tau} > 0$ and $\mathcal{T} = [0, \overline{\tau}]$. The model here is a special case of the game in Section 5 with

$$\pi(k, Q, \tau) = u(D^{-1}(Q)k + \tau) - k, \quad \mathcal{X}(\tau) = K = \Omega = [0, \overline{k}], \tag{6.2}$$

where $\overline{k}$ is a constant satisfying $\overline{k} > \max_{(w,\tau) \in [0,\overline{w}] \times \mathcal{T}} x(w, \tau)$.

First suppose that $s_i = 0$ and $t_i = e_i$ for all $i \in I$. Let $\underline{t}_i = \underline{e}_i$ and $\overline{t}_i = \overline{e}_i$ be as in Figure 1. Then the parameter change from $\underline{t}$ to $\overline{t}$ is not a monotone shock. However, it is straightforward to verify the conditions of Corollary 5.1 to conclude that the parameter change is an overall negative shock. Hence the smallest and largest equilibrium values of aggregate labor supply decrease in response to this parameter change, which implies that the smallest and largest equilibrium values of the wage rate increase.

Now suppose that $e_i = e$ and $t_i = s_i$ for all $i \in I$ for some $e > 0$. Let $\underline{t}_i = e + \underline{s}_i$ and $\overline{t}_i = e + \overline{s}_i$ be as in Figure 2. Then $F_{\overline{t}}$ is a mean-preserving spread of $F_{\underline{t}}$. The parameter change from $\underline{t}$ to $\overline{t}$ widens income inequality, and is not a monotone shock. However, it is straightforward to verify the conditions of Corollary 5.2 to conclude that the parameter change is an overall positive shock. Hence the smallest and largest equilibrium values

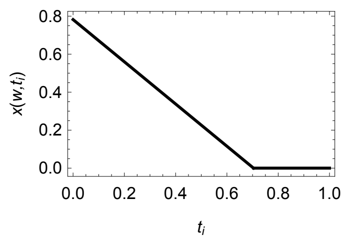

Figure 3: Individual labor supply as a function of t_i with $u(c) = c^{0.7}/0.7$ and $w = 0.9$.

of aggregate labor supply increase in response to this parameter change, which implies that the smallest and largest equilibrium values of the wage rate decrease.

We can confirm the above results by solving (6.1) for $x_i = x(w, t_i)$:

$$x(w, t_i) = \begin{cases} \max\{[u'^{-1}(1/w) - t_i]/w, 0\} & \text{if } w > 0, \\ 0 & \text{if } w = 0. \end{cases} \tag{6.3}$$

This function is decreasing, piecewise linear, and convex in t_i; see Figure 3. Hence the above comparative statics results directly follow from Theorems 5.1 and 5.2.

7 Concluding Comments

Acemoglu and Jensen (2010) established that the smallest and largest equilibrium aggregates of a large aggregative game both increase in response to a positive shock, which is a positive parameter change that positively affects each player's action for each value of the aggregate variable. In this paper we have extended their result to an overall positive shock, to which individuals' reactions may be mixed but the overall aggregate response is positive for each value of the aggregate variable. We have provided sufficient conditions for an overall positive shock in terms of distributional changes in parameters based on stochastic dominance and mean-preserving spreads.

Although we have considered only one-dimensional distributions in our figures and example, Theorems 5.1 and 5.2 do not require one-dimensional distributions. Therefore more interesting or complex policy changes can be analyzed using our results.

Appendix A Proof of Theorem 3.1

Since t is fixed here, we suppress the dependence of π_i, R_i, and $\mathcal{G}$ on t and t_i throughout the proof. For $i \in I$, define $\mu(i, Q) = R_i(Q)(= R_i(Q, t))$.

Lemma A.1. *For any $Q \in \Omega$, the correspondence $\mu(\cdot, Q)$ from $i \in I$ to $R_i(Q) \subset K$ has nonempty compact values, and admits a measurable selection.*

Proof. Fix $Q \in \Omega$. We show this lemma by applying the measurable maximum theorem (Aliprantis and Border 2006, p. 605) to player i's maximization problem with $i \in I$ taken as a parameter:

$$\max_{x_i \in X_i} \pi_i(x_i, Q). \tag{A.1}$$

Let ϕ denote the correspondence $i \in I \mapsto X_i \subset K$. By Assumption 3.1, ϕ has nonempty compact values. Note that the set $\{i \in I : X_i \cap U \neq \emptyset\}$ in Assumption 3.4 is the lower inverse of U under ϕ; see Aliprantis and Border (2006, p. 557). Thus Assumption 3.4 means that ϕ is weakly measurable; see Aliprantis and Border (2006, p. 592). Assumption 3.3 means that the mapping $(i, k) \in I \times K \mapsto \pi_i(k, Q) \in \mathbb{R}$ is a Carathéodory function.

It follows that the measurable maximum theorem applies to (A.1); thus the correspondence $i \mapsto \mu(i, Q)$ has nonempty compact values, and admits a measurable selection. □

Lemma A.2. *For each $Q \in \Omega$, the set $\mathcal{G}(Q)$ is nonempty and convex, where $\mathcal{G}$ is defined in (4.2).*

Proof. Fix $Q \in \Omega$. Since $\mu(\cdot, Q)$ admits a measurable selection by Lemma A.1, $\mathcal{G}(Q)$ is nonempty. To see that $\mathcal{G}(Q)$ is convex, note from Aumann (1965, Theorem 1) that the set

$$\left\{ \int_{i \in I} x_i d_i : x \in \mathscr{X}, \forall i \in I, x_i \in \mu(i, Q) \right\} \tag{A.2}$$

is convex. The image of this convex set under H is convex since H is continuous and real-valued.[10] Recalling (4.2) we see that $\mathcal{G}(Q)$ is convex. □

Lemma A.3. *The correspondence $\mathcal{G}(\cdot)$ has compact values and a closed (in fact, compact) graph.*

[10]The image may not be convex if the range of H is not one-dimensional.

Proof. Fix $i \in I$. Note that X_i does not depend on Q; thus the correspondence $Q \mapsto X_i$ is continuous in a trivial way. By Assumption 3.3, $\pi_i(k, Q)$ is continuous in $(k, Q) \in X_i \times \Omega$. Hence by the Berge maximum theorem (Aliprantis and Border, 2006, p. 570) and the closed graph theorem (Aliprantis and Border, 2006, p. 561), the correspondence $\mu(i, \cdot)$ has a closed graph. In other words,

$$F(i) \equiv \{(k, Q) \in X_i \times \Omega : k \in \mu(i, Q)\} \text{ is closed.} \tag{A.3}$$

Let $\mathscr{G}$ be the graph of the correspondence $\mathcal{G}(\cdot)$:

$$\mathscr{G} = \{(Q, S) \in \Omega \times \Omega : S \in \mathcal{G}(Q)\}. \tag{A.4}$$

To verify that $\mathscr{G}$ is closed, it suffices to show that $\mathscr{G}$ contains the limit of any sequence $\{(Q^j, S^j)\}_{j \in \mathbb{N}}$ in $\mathscr{G}$ that converges in Ω^2. For this purpose, let $\{(Q^j, S^j)\}_{j \in \mathbb{N}}$ be a sequence in $\mathscr{G}$ that converges to some $(Q^*, S^*) \in \Omega^2$.

For each $j \in \mathbb{N}$ we have $S^j \in \mathcal{G}(Q^j)$; thus there exists a measurable selection $x^j \in \mathscr{X}$ of $\mu(\cdot, Q^j)$ such that $S^j = H(\int_{i \in I} x_i^j di)$. Taking a subsequence of $\{S^j\}$, we can assume that $\xi^j \equiv \int_{i \in I} x_i^j di$ converges to some $\xi^* \in K$ as $j \uparrow \infty$. Since H is continuous by Assumption 3.2, it follows that $S^* = H(\xi^*)$.

Recalling that x^j is a selection of $\mu(\cdot, Q^j)$ for all $j \in \mathbb{N}$, we see from (A.3) that $(x_i^j, Q^j) \in F(i)$ for all $i \in I$ and $j \in \mathbb{N}$. Since $F(i)$ is closed, any convergent subsequence of $\{(x_i^j, Q^j)\}_{j \in \mathbb{N}}$ converges in $F(i)$. Since $Q^j \to Q^*$ as $j \uparrow \infty$, it follows that any limit point y_i of $\{x_i^j\}_{j \in \mathbb{N}}$ satisfies $(y_i, Q^*) \in F(i)$; i.e., $y_i \in \mu(i, Q^*)$. In addition, $|x_i^j| \leq \max\{|k| : k \in K\}$ for all $i \in I$ and $j \in \mathbb{N}$. Hence by Aumann (1976, Lemma), there exists $x^* \in \mathscr{X}$ such that $x_i^* \in \mu(i, Q^*)$ for all $i \in I$ and $\int_{i \in I} x_i^* di = \xi^*$. Thus $S^* = H(\int_{i \in I} x_i^* di)$, and $S^* \in \mathcal{G}(Q^*)$; i.e., $(Q^*, S^*) \in \mathscr{G}$. It follows that $\mathscr{G}$ is closed.

Since $\mathscr{G} \subset \Omega \times \Omega$, which is compact, it follows that $\mathscr{G}$ is compact. Hence $\mathcal{G}(\cdot)$ has compact values. □

Now by Kakutani's fixed point theorem (Aliprantis and Border, 2006, p. 583), Assumption 3.2, and Lemmas A.2 and A.3, the set of fixed points of the correspondence $\mathcal{G}(\cdot)$ is nonempty and compact. Let $Q \in \Omega$ be a fixed point. Then there exists a measurable selection $x \in \mathscr{X}$ of $\mu(\cdot, Q)$ such that $Q = H(\int_{i \in I} x_i di)$; i.e., x is an equilibrium. Hence an equilibrium exists.

The preceding argument shows that any fixed point of $\mathcal{G}(\cdot)$ is an equilibrium aggregate. Since the set of fixed points of $\mathcal{G}(\cdot)$ is nonempty and compact, it follows that the set of equilibrium aggregates is also nonempty and compact; thus the smallest and largest equilibrium aggregates exist.

Appendix B Proof of Theorem 4.1

Since the correspondence $Q \mapsto \mathcal{G}(Q,t)$ has compact values by Lemma A.3, the minimum and the maximum of $\mathcal{G}(Q,t)$ exist for each $Q \in \Omega$. Define $\underline{\mathcal{G}}(Q,t) = \min \mathcal{G}(Q,t)$ $\overline{\mathcal{G}}(Q,t) = \max \mathcal{G}(Q,t)$. Since $\mathcal{G}(\cdot,t)$ has convex values by Lemma A.2, we have

$$\forall Q \in \Omega, \quad \mathcal{G}(Q,t) = [\underline{\mathcal{G}}(Q,t), \overline{\mathcal{G}}(Q,t)]. \tag{B.1}$$

By Lemma A.3, $\mathcal{G}(\cdot,t)$ has a compact graph. Hence it is easy to see that $\mathcal{G}(\cdot,t)$ is "continuous but for upward jumps"; see Milgrom and Roberts (1994, p. 447). To conclude both inequalities in (4.1) from Milgrom and Roberts (1994, Corollary 2), it remains to show that for all $Q \in \Omega$ we have

$$\underline{\mathcal{G}}(Q,\underline{t}) \leq \underline{\mathcal{G}}(Q,\bar{t}), \quad \overline{\mathcal{G}}(Q,\underline{t}) \leq \overline{\mathcal{G}}(Q,\bar{t}). \tag{B.2}$$

To see the first inequality in (B.2), let $\bar{r} = \underline{\mathcal{G}}(Q,\bar{t})$. By Definition 4.3(ii), there exists $\underline{r} \in \mathcal{G}(Q,\underline{t})$ with $\underline{r} \leq \bar{r}$. Since $\underline{\mathcal{G}}(Q,\underline{t}) \leq \underline{r}$, the desired inequality follows. The second inequality in (B.2) can be verified in a similar way.

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