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Abstract

We show a simple no-bubble theorem that applies to a wide range of deterministic sequential economies with infinitely lived agents. In particular, we show that asset bubbles never arise if there is at least one agent who can reduce his asset holdings permanently from some period onward. This is a substantial generalization of Kocherlakota's (1992, *Journal of Economic Theory* 57, 245–256) result on asset bubbles and short sales constraints. Our no-bubble theorem requires virtually no assumption except for the strict monotonicity of preferences.

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1 Introduction

Since the global financial crisis of 2007-2008, there has been a surge of interest in rational asset pricing bubbles, or simply "asset bubbles." Numerous economic mechanisms that give rise to asset bubbles are still being proposed, and the implications of asset bubbles on various economic issues are actively discussed in the current literature; we refer the reader to Miao (2014) for a short survey on recent developments.

In constructing models of asset bubbles, it is important to understand conditions under which asset bubbles exist or do not exist. While conditions for existence are mostly restricted to specific models, some general conditions for nonexistence are known. In fact, it is well known that asset bubbles never arise if the present value of the aggregate endowment is finite. This was shown by Santos and Woodford (1997) for a general equilibrium model with incomplete markets and possibly infinitely many agents each of whom may be finitely or infinitely lived. Wilson's (1981) result on the existence of a competitive equilibrium in a deterministic economy with infinitely many agents can be viewed as an earlier version of this no-bubble theorem. Huang and Werner (2000) showed a version of the no-bubble theorem applicable to an asset in zero net supply for a deterministic economy with finitely many agents. Werner (2014) extended Santos and Woodford's (1997) no-bubble theorem to a complete market economy with debt constraints (instead of borrowing constraints).

While these results are based on equilibrium prices and allocations, there are closely related results based mostly on the optimal behavior of a single agent. For example, in a deterministic economy with finitely many agents, Kocherlakota (1992) showed that in an equilibrium with a positive asset bubble, the short sales constraints of all agents must be asymptotically binding; equivalently, asset bubbles can be ruled out if there is at least one agent whose asset holdings can be lowered permanently from some period onward. A similar idea was used earlier by Obstfeld and Rogoff (1986) to rule out deflationary equilibria in a money-in-the-utility-function model.[1]

The results mentioned in the preceding paragraph rely on the necessity of a transversality condition,[2] and a general no-bubble result based on the necessity of a transversality condition was shown in Kamihigashi (2001, p.

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[1] See Kamihigashi (2008a, 2008b) for results on asset bubbles in related models.

[2] Various results on necessity of transversality conditions were established in Kamihigashi (2001, 2002, 2003, 2005).

1007) for deterministic representative-agent models. Essentially, this result only requires the differentiability and strict monotonicity of instantaneous utility functions; thus it can be used to rule out asset bubbles in various representative-agent models.

In this paper we establish a simple no-bubble theorem that can be used to rule out asset bubbles in a considerably broader range of deterministic models. More specifically, we consider the problem of a single agent facing a sequential budget constraint and having strictly monotone preferences. We show that asset bubbles never arise if the agent can reduce his asset holdings permanently from some period onward. This result uses the same idea as those based on transversality conditions mentioned above; the contribution of this paper is to show that the result holds true under extremely general conditions.

To demonstrate the applicability of our no-bubble theorem to general equilibrium models, we consider a general equilibrium model with multiple agents and multiple assets. Using our no-bubble theorem, we show substantial generalizations of Proposition 3 in Kocherlakotoa (1992).

The rest of the paper is organized as follows. In Section 2 we present a single agent's problem along with necessary assumptions, and formally define asset bubbles. In Section 3 we offer several examples satisfying our assumptions. In Section 4 we state our no-bubble theorem and show some immediate consequences. In Section 5 we present a general equilibrium model. In Section 6 we show several results on asset bubbles in general equilibrium. In Section 7 we offer some concluding comments.

2 Single-Agent/Single-Asset Framework

2.1 Feasibility and Optimality

Time is discrete and denoted by $t \in \mathbb{Z}_+$. In this section we assume that there are one consumption good and one asset that pays a dividend of d_t units of the consumption good in each period $t \in \mathbb{Z}_+$. Let p_t be the price of the asset in period $t \in \mathbb{Z}_+$. Consider an infinitely lived agent who faces the following constraints:

$$c_t + p_t s_t = y_t + (p_t + d_t)s_{t-1},\ c_t \geq 0, \quad \forall t \in \mathbb{Z}_+, \tag{2.1}$$

$$s \in \mathcal{S}(s_{-1}, y, p, d), \tag{2.2}$$

where c_t is consumption in period t, $y_t \in \mathbb{R}$ is (net) income in period t, s_t is asset holdings at the end of period t with s_{-1} historically given, and $\mathcal{S}(s_{-1}, y, p, d)$ is a set of sequences in $\mathbb{R}$ with $s = \{s_t\}_{t=0}^{\infty}$, $y = \{y_t\}_{t=0}^{\infty}$, $p = \{p_t\}_{t=0}^{\infty}$, and $d = \{d_t\}_{t=0}^{\infty}$, We present several examples of (2.2) in Section 3.

Although we consider a single agent's problem and assume that there is only one asset here, our results developed within this framework apply to a general equilibrium model with many agents and many assets, as shown in Section 6.

Let $\mathcal{C}$ be the set of sequences $\{c_t\}_{t=0}^{\infty}$ in $\mathbb{R}_+$. For any $c \in \mathcal{C}$, we let $\{c_t\}_{t=0}^{\infty}$ denote the sequence representation of c, and vice versa. In other words, we use c and $\{c_t\}_{t=0}^{\infty}$ interchangeably; likewise, we use s and $\{s_t\}_{t=0}^{\infty}$ interchangeably, and so on. We define the inequalities $<$ and $\leq$ on the set of sequences in $\mathbb{R}$ (which includes $\mathcal{C}$) as follows:

$$c \leq c' \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_t,\ c_t \leq c'_t, \tag{2.3}$$

$$c < c' \quad \Leftrightarrow \quad c \leq c' \text{ and } \exists t \in \mathbb{Z}_+, c_t < c'_t. \tag{2.4}$$

The agent's preferences are represented by a binary relation $\prec$ on $\mathcal{C}$. In particular, for any $c, c' \in \mathcal{C}$, the agent strictly prefers c' to c if and only if $c \prec c'$. The assumptions stated in this section are maintained until the end of Section 4 unless otherwise noted.

Assumption 2.1. $d_t \geq 0$ and $p_t \geq 0$ for all $t \in \mathbb{Z}_+$.

Assumption 2.2. $p_t > 0$ for all $t \in \mathbb{Z}_+$.

Although the second assumption is used for most of our results, there is an important case in which it cannot be used. In particular, if the asset is intrinsically useless, i.e., $d_t = 0$ for all $t \in \mathbb{Z}_+$, then it is more than natural to consider the possibility that $p_t = 0$ for all $t \in \mathbb{Z}_+$. One of our results deals with this particular case without assuming Assumption 2.2; see Proposition 4.2.

We say that a pair of sequences $c = \{c_t\}_{t=0}^{\infty}$ and $s = \{s_t\}_{t=0}^{\infty}$ in $\mathbb{R}$ is a *plan*; a plan (c, s) is *feasible* if it satisfies (2.1) and (2.2); and a feasible plan (c^*, s^*) is *optimal* if there exists no feasible plan (c, s) such that $c^* \prec c$. Whenever we take an optimal plan (c^*, s^*) as given, we assume the following.

Assumption 2.3. For any $c \in C$ with $c^* < c$, we have $c^* \prec c$.

This assumption is satisfied if $\prec$ is strictly monotone in the sense that for any $c, c' \in \mathcal{C}$ with $c < c'$, we have $c \prec c'$. Although this latter requirement may seem reasonable, there is an important case in which it is not satisfied. Such a case and other examples of preferences satisfying Assumption 2.3 are discussed in Subsection 3.2.

2.2 Asset Bubbles

In this subsection we define the fundamental value of the asset and the bubble component of the asset price by taking as given the implicit period 0 prices of all future consumption goods. To be specific, we let $q_0^0 = 1$, and let q_0^t be the implicit period 0 price of the consumption good in period $t \in \mathbb{N}$. Unless otherwise noted, we maintain the following assumption for the rest of the paper.

Assumption 2.4. For all $t \in \mathbb{N}$, we have $q_0^t > 0$.

We also maintain the following assumption until the end of Section 4 unless otherwise noted.

Assumption 2.5. We have

$$\forall t \in \mathbb{Z}_+, \quad \frac{p_t}{p_{t+1} + d_{t+1}} q_0^t \leq q_0^{t+1}. \tag{2.5}$$

Huang and Werner (2000, eq. (8)) use the equality version of (2.5):

$$\forall t \in \mathbb{Z}_+, \quad \frac{p_t}{p_{t+1} + d_{t+1}} q_0^t = q_0^{t+1}. \tag{2.6}$$

This can be shown to hold if there are Arrow-Debreu markets for future consumption goods and the agent has an interior optimal plan. We assume (2.5) instead to accommodate general constraints on asset holdings as well as the general equilibrium model with multiple assets introduced in Section 5. As shown below, the inequality in (2.5) is useful in ruling out strictly positive asset bubbles (see (2.19)); the reverse inequality is not useful for that purpose.

Instead of taking $\{q_0^t\}$ as exogenously given, one can also construct the sequence $\{q_0^t\}$ using (2.6) with $q_0^0 = 1$; in this case, Assumption 2.5 trivially holds.

For $t \in \mathbb{N}$, let $q_t^0 = 1$. For $t, i \in \mathbb{N}$ we define

$$q_t^i = q_0^{t+i} / q_0^t, \tag{2.7}$$

which can be interpreted as the period t price of the consumption good in period $t + i$. Note that

$$\forall i, j, t \in \mathbb{Z}_+, \quad q_t^i q_{t+i}^j = \frac{q_0^{t+i}}{q_0^t} \frac{q_0^{t+i+j}}{q_0^{t+i}} = q_t^{i+j} \tag{2.8}$$

Let $t \in \mathbb{Z}_+$. By (2.5) and (2.7) we have $p_t \leq q_t^1 (p_{t+1} + d_{t+1})$. By repeated application of this inequality and (2.8), we have

$$\begin{aligned} p_t &\leq q_t^1 d_{t+1} + q_t^1 p_{t+1} && (2.9) \\ &\leq q_t^1 d_{t+1} + q_t^1 q_{t+1}^1 (p_{t+2} + d_{t+2}) && (2.10) \\ &\leq q_t^1 d_{t+1} + q_t^2 d_{t+2} + q_t^2 p_{t+2} && (2.11) \\ &\vdots && (2.12) \\ &\leq \sum_{i=1}^{n} q_t^i d_{t+i} + q_t^n p_{t+n}, \quad \forall n \in \mathbb{N}. && (2.13) \end{aligned}$$

Since the above finite sum is increasing in $n \in \mathbb{N}$,[3] it follows that

$$p_t \leq \sum_{i=1}^{\infty} q_t^i d_{t+i} + \varliminf_{n \uparrow \infty} q_t^n p_{t+n}. \tag{2.14}$$

As usual, we define the *fundamental value* of the asset in period t as the present discounted value of the dividend stream from period $t + 1$ onward:

$$f_t = \sum_{i=1}^{\infty} q_t^i d_{t+i}. \tag{2.15}$$

The *bubble* component of the asset price in period t is the part of p_t that is not accounted for by the fundamental value:

$$b_t = p_t - f_t. \tag{2.16}$$

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[3] In this paper, "increasing" means "nondecreasing," and "decreasing" means "nonincreasing."

It follows from (2.14)–(2.16) that

$$b_t \leq \varliminf_{n\uparrow\infty} q_t^n p_{t+n}. \tag{2.17}$$

Using (2.8) we see that

$$q_0^t \varliminf_{n\uparrow\infty} q_t^n p_{t+n} = \varliminf_{n\uparrow\infty} q_0^{t+n} p_{t+n} = \varliminf_{i\uparrow\infty} q_0^i p_i. \tag{2.18}$$

Hence by (2.17) we have

$$\varliminf_{i\uparrow\infty} q_0^i p_i = 0 \quad \Rightarrow \quad \forall t \in \mathbb{Z}_+,\ b_t \leq 0. \tag{2.19}$$

Although the above implication may be useful in many cases, one would also need a condition to ensure that asset bubbles are exactly zero. Such a condition can be obtained by using (2.6) instead of (2.5). To see this, assume (2.6) for the moment. Let $t \in \mathbb{Z}_+$. Then inequalities (2.9)–(2.13) all hold with equality; thus

$$p_t = \sum_{i=1}^{n} q_t^i d_{t+i} + q_t^n p_{t+n}, \quad \forall n \in \mathbb{N}. \tag{2.20}$$

Since the above finite sum is increasing in $n \in \mathbb{N}$ and the left-hand side is independent of n, it follows that $q_t^n p_{t+n}$ is decreasing in n. Hence $\lim_{n\uparrow\infty} q_t^n p_{t+n}$ exists, which implies that

$$p_t = \sum_{i=1}^{\infty} q_t^i d_{t+i} + \lim_{n\uparrow\infty} q_t^n p_{t+n}. \tag{2.21}$$

From this and (2.16), we have $b_t = \lim_{n\uparrow\infty} q_t^n p_{t+n}$. In view of (2.18) we see that

$$\lim_{i\uparrow\infty} q_0^i p_i = 0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ b_t = 0. \tag{2.22}$$

3 Examples

In this section we present several examples of (2.2) as well as some examples of preferences that satisfy Assumption 2.3. Some of these examples are used in Section 4.

3.1 Constraints on Asset Holdings

The simplest example of (2.2) would be the following:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq 0. \tag{3.1}$$

This constraint is often used in representative-agent models; see, e.g., Lucas (1978) and Kamihigashi (1998).

Kocherlakota (1992) uses a more general version of (3.1):

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma, \tag{3.2}$$

where $\sigma \in \mathbb{R}$. If $\sigma < 0$, then the above constraint is called a short sales constraint.

The following constraint is even more general:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma_t, \tag{3.3}$$

where $\sigma_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Note that (3.2) is a special case of (3.3) with $\sigma_t = \sigma$ for all $t \in \mathbb{Z}_+$.

So far we have only considered inequality constraints on s_t, but other types of constraints are also covered by (2.2). For example, the right-hand side of the budget constraint in (2.1) is the agent's wealth at the beginning of period t; thus it may be reasonable to require it to be nonnegative:

$$\forall t \in \mathbb{N}, \quad (p_t + d_t)s_{t-1} + y_t \geq 0. \tag{3.4}$$

This is clearly an example of (2.2) as well as a special case of (3.3) with

$$\forall t \in \mathbb{Z}_+, \quad \sigma_t = -y_{t+1}/(p_{t+1} + d_{t+1}). \tag{3.5}$$

Santos and Woodford (1997, p. 24) consider a (state-dependent) borrowing constraint that reduces in our framework to

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t \geq -\xi_t, \tag{3.6}$$

where $\xi_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. This constraint is a special case of (3.3) with $\sigma_t = -\xi_t/p_t$.

The (state-dependent) debt constraint considered by Werner (2014) and LeRoy and Werner (2014, p. 313) can be written in our framework as

$$\forall t \in \mathbb{Z}_+, \quad (p_{t+1} + d_{t+1})s_t \geq -\xi_{t+1}. \tag{3.7}$$

This constraint is another special case of (3.3) with

$$\sigma_t = -\xi_{t+1}/(p_{t+1} + d_{t+1}). \tag{3.8}$$

In addition to (3.2), Kocherlakota (1992) considers the following wealth constraint:

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t + \sum_{i=1}^{\infty} q_t^i y_{t+i} \geq 0, \tag{3.9}$$

which is another example of (2.2). The left-hand side above is the period t value of the agent's current asset holdings and future income. Note that (3.9) is yet another special case of (3.3) with

$$\forall t \in \mathbb{Z}_+, \quad \sigma_t = -\sum_{i=1}^{\infty} q_t^i y_{t+i}/p_t. \tag{3.10}$$

See Wright (1987) and Huang and Werner (2000) for relations between different budget constraints.

3.2 Preferences

Example 3.1. A typical objective function in an agent's maximization problem takes the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{3.11}$$

where $\beta \in (0,1)$ and $u : \mathbb{R}_+ \to [-\infty, \infty)$ is a strictly increasing function. Suppose further that u is bounded, and define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad \sum_{t=0}^{\infty} \beta^t u(c_t) < \sum_{t=0}^{\infty} \beta^t u(c'_t). \tag{3.12}$$

Then $\prec$ clearly satisfies Assumption 2.3.

If u is unbounded below, i.e., if $u(0) = -\infty$, then the above definition of $\prec$ may not satisfy Assumption 2.3. In particular, given $c^*, c \in \mathcal{C}$ with $c^* < c$, we do not have $c^* \prec c$ if $c_t^* = c_t = 0$ for some $t \in \mathbb{Z}_+$ and if u is bounded above. Indeed, in this case,

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) = \sum_{t=0}^{\infty} \beta^t u(c_t) = -\infty. \tag{3.13}$$

Hence the inequality in (3.12) does not hold.

Example 3.2. The above problem with unbounded utility can be avoided by using an alternative optimality criterion. To be specific, let $u_t : \mathbb{R}_+ \to [-\infty, \infty)$ be a strictly increasing function for $t \in \mathbb{Z}_+$ as above. In this case, the infinite sum $\sum_{t=0}^{\infty} u_t(c_t)$ may not be well defined. Even if it is always well defined, it may not be strictly increasing, as discussed above. To deal with these problems, consider the binary relation $\prec$ defined by

$$c \prec c' \quad \Leftrightarrow \quad \varliminf_{n \uparrow \infty} \sum_{t=0}^{n} [u_t(c_t) - u_t(c'_t)] < 0, \tag{3.14}$$

where we follow the convention that $(-\infty) - (-\infty) = 0$; see Dana and Le Van (2006) for related optimality criteria. It is easy to see that the binary relation $\prec$ defined above satisfies Assumption 2.3.

Continuing with this example, suppose that (2.2) is given by (3.1). Suppose further that each u_t is differentiable on $\mathbb{R}_{++}$, and that there exists an optimal plan (c^*, s^*) such that

$$\forall t \in \mathbb{Z}_+, \quad c^*_t > 0, \; s^*_t = 1. \tag{3.15}$$

Then the standard Euler equation holds:

$$u'_t(c^*_t) p_t = u'_{t+1}(c^*_{t+1})(p_{t+1} + d_{t+1}), \quad \forall t \in \mathbb{Z}_+. \tag{3.16}$$

If the sequence $\{q^t_0\}$ of implicit prices satsifies

$$q^t_0 = \frac{u'_t(c^*_t)}{u'_0(c^*_0)}, \quad \forall t \in \mathbb{Z}_+, \tag{3.17}$$

then the Euler equation (3.16) implies (2.6), and the fundamental value f_t takes the familiar form:

$$f_t = \sum_{i=1}^{\infty} \frac{u'_{t+i}(c^*_{t+i})}{u'_t(c^*_t)} d_{t+i}, \quad \forall t \in \mathbb{Z}_+. \tag{3.18}$$

Example 3.3. Let $v : \mathcal{C} \to \mathbb{R}$ be a strictly increasing function. Define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad v(c_0, c_1, c_2, \ldots) < v(c'_0, c'_1, c'_2, \ldots). \tag{3.19}$$

Note that (3.19) satisfies Assumption 2.3 without any additional condition on v. For example, v can be a recursive utility function.

4 Implications of Feasibility and Optimality

4.1 No-Bubble Theorem

To state our no-bubble theorem, we need to introduce some notation. Given any sequence $\{s_t^*\}_{t=0}^{\infty}$ in $\mathbb{R}$, $\tau \in \mathbb{Z}_+$, and $\epsilon > 0$, let $\mathcal{S}_{\tau,\epsilon}(s^*)$ be the set of sequences $\{s_t\}_{t=0}^{\infty}$ in $\mathbb{R}$ such that

$$s_t \begin{cases} = s_t^* & \text{if } t < \tau, \\ \geq s_t^* - \epsilon & \text{if } t \geq \tau. \end{cases} \tag{4.1}$$

In other words, a sequence $\{s_t\}$ in $\mathcal{S}_{\tau,\epsilon}(s^*)$ coincides with $\{s_t^*\}$ up to period $\tau - 1$ and is only required to satisfy the lower bound $s_t^* - \epsilon$ from period τ onward. We are ready to state the main result of this paper.

Theorem 4.1. *Let (c^*, s^*) be an optimal plan. Suppose that there exist $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ such that*

$$\mathcal{S}_{\tau,\epsilon}(s^*) \subset \mathcal{S}(s_{-1}, y, p, d). \tag{4.2}$$

Then the following conclusions hold:

(a) *We have $b_t \leq 0$ for all $t \in \mathbb{Z}_+$.*

(b) *Under (2.6), we have $b_t = 0$ for all $t \in \mathbb{Z}_+$.*

Proof. See Appendix A. □

It seems remarkable that asset bubbles can be ruled out by a simple condition such as (4.2) alone. In particular, no explicit utility function is assumed, and the only requirement on the binary relation $\prec$ is Assumption 2.3, which merely requires strict monotonicity at the given optimal consumption plan c^*.

Conclusion (a) only shows that asset bubbles cannot be strictly positive under (2.5) and (4.2). One can also conclude that asset bubbles are exactly zero by strengthening (2.5) to (2.6), as shown in conclusion (b).

The idea of the proof of Theorem 4.1 is simple. In the proof, assuming that the equality in (2.19) is violated, we construct an alternative plan as follows. Let $\delta > 0$, and let $s_\tau = s_\tau^* - \delta$ and $c_\tau = c_\tau^* + p_\tau \delta$, where τ is given by the statement of the theorem. For $t \neq \tau$, let s_t be determined by the budget constraint (2.1) with $c_t = c_t^*$. This alternative plan gives the same consumption sequence except in period τ, where consumption is increased by $p_\tau \delta > 0$. Hence this plan is strictly preferred to the original plan (c^*, s^*). We derive a contradiction by showing that the alternative plan satisfies (4.2) for sufficiently small $\delta > 0$ provided that the equality in (2.19) is violated.

Similar constructions are used as "Ponzi schemes" by Huang and Werner (2000, Theorems 5.1, 6.1), but they are not directly linked to the nonexistence of asset bubbles.

4.2 Consequences of Theorem 4.1

In this subsection we provide fairly simple consequences of Theorem 4.1 in the current single-agent framework. Throughout this subsection we take an optimal plan (c^*, s^*) as given. We start with a simple result assuming that the feasibility constraint on asset holdings (2.2) is given by a sequence of constraints of the form (3.3). As discussed in Subsection 3.1, this simple form covers various constraints on borrowing, debt, and wealth,

Corollary 4.1. *Suppose that (2.2) is given by (3.3) with $\sigma_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Suppose further that*

$$\varliminf_{t \uparrow \infty} (s_t^* - \sigma_t) > 0. \tag{4.3}$$

Then conclusions (a) and (b) of Theorem 4.1 hold.

Proof. Assume (4.3). Let $\epsilon \in (0, \varliminf_{t \uparrow \infty} (s_t^* - \sigma_t))$. Then there exists $\tau \in \mathbb{Z}_+$ such that $s_t^* - \sigma_t \geq \epsilon$, or $s_t^* - \epsilon \geq \sigma_t$, for all $t \geq \tau$. This implies (4.2). Hence both conclusions of Theorem 4.1 hold. $\square$

If there is a constant lower bound on asset holdings s_t, the above result reduces to the following.

Corollary 4.2. *Suppose that (2.2) is given by (3.2) for some $\sigma \in \mathbb{R}$. Suppose further that $\varliminf_{t \uparrow \infty} s_t^* > \sigma$. Then conclusions (a) and (b) of Theorem 4.1 hold.*

In Section 6 we present some consequences of the above two results in the context of general equilibrium and discuss them in relation to Proposition 3 in Kocherlakota (1992).

Next we present two results that apply to representative-agent models.

Corollary 4.3. *Suppose that (2.2) is given by (3.1). Suppose that*

$$\forall t \in \mathbb{Z}_+, \quad s_t^* = 1. \tag{4.4}$$

Then conclusions (a) and (b) of Theorem 4.1 hold.

Proof. Note that (4.4) and (3.1) imply (4.2) with $\tau = 0$ and $\epsilon = 1$. Thus both conclusions of Theorem 4.1 hold. □

The following proposition is immediate from the above result and (3.18).

Proposition 4.1. *In the setup of Example 3.2 (including (3.17) and (3.18)), we have*

$$\forall t \in \mathbb{Z}_+, \quad p_t = \sum_{i=1}^{\infty} \frac{u'_{t+i}(c^*_{t+i})}{u'_t(c^*_t)} d_{t+i}. \tag{4.5}$$

A similar result is shown in Kamihigashi (2001, Section 4.2.1) for a continuous-time model with a nonlinear constraint. It is known that a stochastic version of Proposition 4.1 requires additional assumptions; see Kamihigashi (1998) and Montrucchio and Privileggi (2001).[4]

Finally we consider the case of fiat money, or an asset with no dividend payment. Since the fundamental value of fiat money is zero, its price must also be zero if there is no asset bubble. Hence the case of fiat money is not directly covered by Theorem 4.1, which requires Assumption 2.2,

Proposition 4.2. *Drop Assumptions 2.2, 2.4, and 2.5, but maintain Assumptions 2.1 and 2.3. Suppose that there exist $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ satisfying (4.2). Suppose further that*

$$\forall t \geq \tau + 1, \quad d_t = 0. \tag{4.6}$$

Then

$$\forall t \geq \tau, \quad p_t = 0. \tag{4.7}$$

Proof. See Appendix B. □

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[4] See Kamihigashi (2011) for sample-path properties of stochastic asset bubbles.

5 General Equilibrium with Multiple Agents and Multiple Assets

5.1 Feasibility, Optimality, and Equilibrium

Consider an exchange economy with countably many infinitely lived agents indexed by $a \in A$, where $A = \{1, 2, \dots, \overline{a}\}$ with $\overline{a} \in \mathbb{N} \cup \{\infty\}$. There are countably many assets indexed by $k \in K$, where $K = \{1, 2, \dots, \overline{k}\}$ with $\overline{k} \in \mathbb{N} \cup \{\infty\}$. Agent $a \in A$ faces the following constraints:

$$c_t^a + \sum_{k \in K} p_{k,t} s_{k,t}^a = y_t^a + \sum_{k \in K} (p_{k,t} + d_{k,t}) s_{k,t-1}^a, \; c_t^a \geq 0, \quad \forall t \in \mathbb{Z}_+, \tag{5.1}$$

$$s^a \in \mathcal{S}^a(s_{-1}^a, y^a, p, d), \tag{5.2}$$

where c_t^a and y_t^a are agent a's consumption and endowment in period t; for each $k \in K$, $s_{k,t}^a$ is agent a's holdings of asset k at the end of period t, $p_{k,t}$ is the price of asset k in period t, and $d_{k,t}$ is the dividend payment of asset k in period t. In (5.2), $s_{-1}^a = (s_{k,-1}^a)_{k \in K}$ is agent a's initial portfolio of all assets $k \in K$, which are historically given, and $\mathcal{S}^a(s_{-1}^a, y^a, p, d)$ is a set of sequences in $\mathbb{R}^{\overline{k}}$ with $s^a = \{(s_{k,t}^a)_{k \in K}\}_{t=0}^\infty$, $y^a = \{y_t^a\}_{t=0}^\infty$, $p = \{(p_{k,t})_{k \in K}\}_{t=0}^\infty$, and $d = \{(d_{k,t})_{k \in K}\}_{t=0}^\infty$.

The supply of each asset $k \in K$ is given by $\overline{s}_k \geq 0$ and is constant over time. We assume the following for the rest of the paper.

Assumption 5.1. For any $k \in K$ and $t \in \mathbb{Z}_+$, we have $d_{k,t} \geq 0$. Furthermore, for each $k \in K$ we have

$$\sum_{a \in A} s_{k,-1}^a = \overline{s}_k. \tag{5.3}$$

Agent a's preferences are represented by a binary relation $\prec^a$ on $\mathcal{C}$. We say that a pair of sequences $c^a = \{c_t^a\}_{t=0}^\infty$ and $s^a = \{(s_{k,t}^a)_{k \in K}\}_{t=0}^\infty$ in $\mathbb{R}$ and $\mathbb{R}^{\overline{k}}$, respectively, is a *plan*; a plan (c^a, s^a) is *feasible* for agent a if it satisfies (5.1) and (5.2); and a feasible plan $(\hat{c}^a, \hat{s}^a)$ is *optimal* for agent a if there exists no feasible plan (c^a, s^a) for agent a such that $\hat{c}^a \prec c^a$.

An *equilibrium* of this economy is a set of sequences $(p, \{c^a, s^a\}_{a \in A})$ such that (i) (c^a, s^a) is optimal for each agent $a \in A$, (ii) for each $k \in K$ and $t \in \mathbb{Z}_+$, we have $p_{k,t} \geq 0$, and (iii) the asset and good markets clear in all periods:

$$\sum_{a \in A} s^a_{k,t} = \overline{s}_k, \quad \forall k \in K, \forall t \in \mathbb{Z}_+, \tag{5.4}$$

$$\sum_{a \in A} c^a_t = \sum_{a \in A} y^a_t + \sum_{k \in K} \overline{s}_k d_{k,t}, \quad \forall t \in \mathbb{Z}_+. \tag{5.5}$$

Whenever we take an equilibrium $(p, \{c^a, s^a\}_{a \in A})$ as given, we assume the following.

Assumption 5.2. For any $a \in A$ and $\tilde{c}^a \in C$ with $c^a < \tilde{c}^a$, we have $c^a \prec \tilde{c}^a$.

5.2 Asset Bubbles

As in Subsection 2.2 we take as given the implicit period 0 prices of all future consumption goods; i.e., we let $q^0_0 = 1$ and let q^t_0 be the implicit period 0 price of the consumption good in period $t \in \mathbb{N}$. For the rest of the paper, we maintain Assumption 2.4; i.e., $q^t_0 > 0$ for all $t \in \mathbb{N}$. We also define q^i_t for $t, i \in \mathbb{N}$ by (2.7).[5]

The fundamental value of asset $k \in K$ and the bubble component of the price of asset k in period t are defined as in Subsection 2.2:

$$f_{k,t} = \sum_{i=1}^{\infty} q^i_t d_{k,t+i}, \tag{5.6}$$

$$b_{k,t} = p_{k,t} - f_{k,t}. \tag{5.7}$$

Consider an asset $k \in K$ such that

$$\forall t \in \mathbb{Z}_+, \quad p_{k,t} > 0. \tag{5.8}$$

If we assume that

$$\forall t \in \mathbb{Z}_+, \quad \frac{p_{k,t}}{p_{k,t+1} + d_{k,t+1}} q^t_0 \le q^{t+1}_0, \tag{5.9}$$

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[5] Although we take these implicit prices as given, they can easily be endogenized. For example, a sequence of implicit prices can be defined using (5.11) for each asset separately. Alternatively, a sequence of implicit prices satisfying (5.9) for all assets can be defined using (6.4).

then we can repeat the arguments for (2.9)–(2.19) to conclude that

$$\varliminf_{i\uparrow\infty} q_0^i p_{k,i} = 0 \quad \Rightarrow \quad \forall t \in \mathbb{Z}_+,\ b_{k,t} \le 0. \tag{5.10}$$

If we strengthen (5.9) to

$$\forall t \in \mathbb{Z}_+, \quad \frac{p_{k,t}}{p_{k,t+1} + d_{k,t+1}} q_0^t = q_0^{t+1}, \tag{5.11}$$

then we can repeat the arguments for (2.20)–(2.22) to conclude that

$$\lim_{i\uparrow\infty} q_0^i p_{k,i} = 0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ b_{k,t} = 0. \tag{5.12}$$

6 General Equilibrium Results

In this section we take an equilibrium $(p, \{c^a, s^a\}_{a\in A})$ as given, and consider conditions to rule out asset bubbles in the general equilibrium setting introduced in the previous section. We maintain the following assumption throughout this section.

Assumption 6.1. There exists an asset $h \in K$ satisfying (5.8) and (5.9) with $k = h$.

For the rest of this section we fix h as above. We start by extending Theorem 4.1 to the current general equilibrium setting. For this purpose, we need additional notation. Given any $s^a = \{(s^a_{k,t})_{k\in K}\}_{t=0}^\infty$ in $\mathbb{R}^{\overline{k}}$, $\tau \in \mathbb{Z}_+$, and $\epsilon > 0$, let $\mathcal{S}_{h,\tau,\epsilon}(s^a)$ be the set of sequences $\{(s_{k,t})_{k\in K}\}_{t=0}^\infty$ in $\mathbb{R}^{\overline{k}}$ such that

$$s_{k,t} = s^a_{k,t}, \quad \forall t \in \mathbb{Z}_+ \text{ if } k \neq h, \tag{6.1}$$

$$s_{h,t} \begin{cases} = s^a_{h,t} & \text{if } t < \tau, \\ \ge s^a_{h,t} - \epsilon & \text{if } t \ge \tau. \end{cases} \tag{6.2}$$

Note that (6.2) takes the same form as (4.1).

Theorem 6.1. *Suppose that there exists an agent $a \in A$ such that for some $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$, we have*

$$\mathcal{S}_{h,\tau,\epsilon}(s^a) \subset \mathcal{S}^a(s^a_{-1}, y^a, p, d). \tag{6.3}$$

Then the following conclusions hold:

(a) *We have* $b_{h,t} \leq 0$ *for all* $t \in \mathbb{Z}_+$.

(b) *Under (5.11) with* $k = h$, *we have* $b_{h,t} = 0$ *for all* $t \in \mathbb{Z}_+$.

Proof. Both conclusions follow from Theorem 4.1 because (c^a, s^a) is optimal for agent a and Assumptions 2.1–2.5 hold by Assumptions 2.4, 5.1, 5.2, and 6.1. $\square$

The above result applies to any asset $k \in K$ satisfying (5.8) and (5.9). Provided that the former condition holds for all assets $k \in K$, the latter condition can be guaranteed to hold for all assets if we construct $\{q_0^t\}$ using the following equation with $q_0^0 = 1$:

$$q_0^{t+1} = \sup_{k \in K} \frac{p_{k,t}}{p_{k,t+1} + d_{k,t+1}} q_0^t, \quad \forall t \in \mathbb{Z}_+. \tag{6.4}$$

This construction implies (5.9) for all $k \in K$ (provided that the above supremum is finite for all $t \in \mathbb{Z}_+$).

In what follows, we present some results that can be regarded as generalizations of Proposition 3 in Kocherlakota (1992). We discuss his and our results after showing our results. For the rest of this section, we maintain the following assumption.

Assumption 6.2. For each agent $a \in A$, there exists a sequence $\{(\sigma_{k,t}^a)_{k \in K}\}_{t=0}^{\infty}$ in $\mathbb{R}^{\bar{k}}$ such that given any sequence $s = \{(s_{k,t})_{k \in K}\}_{t=0}^{\infty}$ in $\mathbb{R}^{\bar{k}}$, we have

$$s_{k,t} \geq \sigma_{k,t}^a, \forall k \in K, \forall t \in \mathbb{Z}_+ \quad \Leftrightarrow \quad s \in \mathcal{S}^a(s_{-1}^a, y^a, p, d). \tag{6.5}$$

This assumption means that the feasibility constraint on asset holdings for each agent, (5.2), consists of sequences of constraints of the form (3.3) for all assets. As discussed in Subsection 3.1, various constraints on borrowing, debt, and wealth can be written in the form of (6.5).

The following result is a restatement of Corollary 4.1 in the current general equilibrium setting.

Proposition 6.1. *If there exists an agent* $a \in A$ *such that*

$$\varliminf_{t \uparrow \infty} (s_{h,t}^a - \sigma_{h,t}^a) > 0, \tag{6.6}$$

then conclusions (a) and (b) of Theorem 6.1 hold.

Proof. This follows from Corollary 4.1 applied to the given agent a. ☐

To state the next result, we define the following for $a \in A$ and $k \in K$:

$$\overline{\sigma}_k^a = \overline{\lim_{t \uparrow \infty}} \sigma_{k,t}^a. \tag{6.7}$$

Corollary 6.1. *If there exists an agent $a \in A$ such that*

$$\underline{\lim}_{t \uparrow \infty} s_{h,t}^a > \overline{\sigma}_k^a, \tag{6.8}$$

then conclusions (a) and (b) of Theorem 6.1 hold.

Proof. Let $a \in A$ satisfy (6.8). This strict inequality implies that

$$\underline{\lim}_{t \uparrow \infty} s_{h,t}^a > -\infty, \quad \overline{\sigma}_{h,t}^a < \infty. \tag{6.9}$$

Hence

$$\underline{\lim}_{t \uparrow \infty} (s_{h,t}^a - \sigma_{h,t}^a) \geq \underline{\lim}_{t \uparrow \infty} s_{h,t}^a - \overline{\sigma}_{h,t}^a > 0, \tag{6.10}$$

where the second inequality holds by (6.8). Thus both conclusions of Theorem 6.1 hold by Proposition 6.1. ☐

If there exists a constant lower bounded on the asset h for each agent $a \in A$, the above result reduces to the following.

Corollary 6.2. *Suppose that*

$$\forall a \in A, \exists \sigma_h^a \in \mathbb{R}, \forall t \in \mathbb{Z}_+, \quad \sigma_{h,t}^a = \sigma_h^a. \tag{6.11}$$

Suppose further that there exists an agent $a \in A$ such that

$$\underline{\lim}_{t \uparrow \infty} s_{h,t}^a > \sigma_h^a. \tag{6.12}$$

Then conclusions (a) and (b) of Theorem 6.1 hold.

Kocherlakota (1992, Proposition 3) in effect shows a special case of conclusion (b) in Corollary 6.2 under the following additional assumptions: (i) there is only one asset (i.,e $\overline{k} = 1$); (ii) the binary relation $\prec^a$ of each agent $a \in A$ is represented by (3.12), where u depends on $a \in A$ and is denoted as $u_a : \mathbb{R}_+ \to [-\infty, \infty)$ but β is common to all agents; (iii) for each $a \in A$, u_a is continuously differentiable on $\mathbb{R}_{++}$, strictly increasing, concave, and bounded above or below by zero; and (iv) the optimal plan (c^a, s^a) of each agent $a \in A$ satisfies

$$\forall t \in \mathbb{Z}_+, \quad c_t^a > 0, \tag{6.13}$$

$$\left| \sum_{t=0}^{\infty} \beta^t u_a(c_t^a) \right| < \infty. \tag{6.14}$$

Corollary 6.2 shows that none of Kocherlakota's additional assumptions is needed under Assumption 5.2, which is implied by his assumptions. Hence Corollary 6.2 is a substantial generalization of Proposition 3 in Kocherlakota (1992). He uses the extra assumptions mostly to derive a transversality condition, which is crucial to his approach. By contrast, our results are based on an elementary perturbation argument that fully exploits the structure of maximization problems subject to sequential budget constraints.

7 Concluding Comments

In this paper we showed a simple no-bubble theorem that applies to a wide range of deterministic economies with infinitely lived agents facing sequential budget constraints. In particular, we showed that asset bubbles can be ruled out if there is at least one agent who can reduce his asset holdings permanently from some period onward. This is a substantial generalization of Kocherlakota's (1992) result on asset bubbles and short sales constraints; our no-bubble theorem requires virtually no assumption except for the strict monotonicity of preferences.

Although we also developed some results on asset bubbles in a general equilibrium setting, all of them are solely based on the optimal behavior of a single agent. Additional results can be shown by using our results in conjunction with other arguments based on market-clearing and aggregation.

Appendix A Proof of Theorem 4.1

Let (c^*, s^*) be an optimal plan. Suppose that $f_0 = \infty$. Then $f_t = \infty$ for all $t \in \mathbb{N}$ (recall (2.15)). Hence conclusion (a) is immediate. On the other hand, conclusion (b) does not follow, but note from (2.20) that (2.6) implies that $f_t < \infty$ for all $t \in \mathbb{Z}_+$. Thus for both conclusions, it suffices to consider the case $f_0 < \infty$:

$$\sum_{i=1}^{\infty} q_0^i d_i = f_0 < \infty. \tag{A.1}$$

We assume the above for the rest of the proof.

To show conclusion (a), it suffices to verify that

$$\varliminf_{i \uparrow \infty} q_0^i p_i = 0, \tag{A.2}$$

which implies conclusion (a) by (2.19). Suppose by way of contradiction that

$$\varliminf_{i \uparrow \infty} q_0^i p_i > 0. \tag{A.3}$$

Then since $q_0^i p_i > 0$ for all $i \in \mathbb{Z}_+$ by Assumptions 2.2 and 2.4, it follows that there exists $\underline{b} > 0$ such that

$$\forall i \in \mathbb{Z}_+, \quad q_0^i p_i \geq \underline{b}. \tag{A.4}$$

Equivalently, we have $1/p_i \leq q_0^i / \underline{b}$ for all $i \in \mathbb{Z}_+$. We have

$$\sum_{i=1}^{\infty} \frac{d_i}{p_i} \leq \sum_{i=1}^{\infty} \frac{q_0^i d_i}{\underline{b}} = \frac{f_0}{\underline{b}} < \infty, \tag{A.5}$$

where the equality and the last inequality hold by (A.1).[6]

Let $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ be as given by (4.2). For $\delta \in (0, \epsilon)$ we construct an alternative plan (c^δ, s^δ) as follows:

$$c_t^\delta = \begin{cases} c_t^* & \text{if } t \neq \tau, \\ c_\tau^* + p_\tau \delta & \text{if } t = \tau, \end{cases} \tag{A.6}$$

$$s_t^\delta = \begin{cases} s_t^* & \text{if } t \leq \tau - 1, \\ s_\tau^* - \delta & \text{if } t = \tau, \\ [y_t + (p_t + d_t) s_{t-1}^\delta - c_t^*]/p_t & \text{if } \tau \geq \tau + 1. \end{cases} \tag{A.7}$$

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[6] An arguments similar to (A.5) is used by Montrucchio (2004, Theorem 2).

It suffices to show that (c^δ, s^δ) is feasible for $\delta > 0$ sufficiently small; for then, we have $c^* \prec c^\delta$ by (A.6) and Assumption 2.3, contradicting the optimality of (c^*, s^*).

Note that (c^δ, s^δ) satisfies (2.1) by construction. Hence by (2.1) we have

$$\forall t \geq \tau + 1, \quad p_t(s_t^* - s_t^\delta) = (p_t + d_t)(s_{t-1}^* - s_{t-1}^\delta). \tag{A.8}$$

For $t \geq \tau$ define

$$\delta_t = s_t^* - s_t^\delta. \tag{A.9}$$

Note that $\delta_\tau = \delta$ by (A.7). We have $p_t \delta_t = (p_t + d_t)\delta_{t-1}$ for all $t > \tau$ by (A.8). Thus for any $t > \tau$ we have

$$\delta_t = \frac{p_t + d_t}{p_t}\delta_{t-1} = \frac{p_t + d_t}{p_t}\frac{p_{t-1} + d_{t-1}}{p_{t-1}}\delta_{t-2} = \cdots \tag{A.10}$$

$$= \delta \prod_{i=\tau+1}^{t} \frac{p_i + d_i}{p_i} \leq \delta \prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}, \tag{A.11}$$

where the equality in (A.11) holds since $\delta_\tau = \delta$, and the inequality in (A.11) holds since $d_t \geq 0$ for all $t \in \mathbb{Z}_+$ by Assumption 2.1.[7]

To show that (c^δ, s^δ) is feasible, it suffices to verify that $\delta_t \leq \epsilon$ for all $t \geq \tau$; for then, we have $s \in \mathcal{S}(s_{-1}, y, p, d)$ by (4.2) and (A.9). For this purpose, note from (A.5) that

$$\frac{f_0}{\underline{b}} = \sum_{i=1}^{\infty} \frac{d_i}{p_i} \geq \sum_{i=1}^{\infty} \ln\left(1 + \frac{d_i}{p_i}\right) \tag{A.12}$$

$$= \sum_{i=1}^{\infty} \ln\left(\frac{p_i + d_i}{p_i}\right) = \ln\left(\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}\right). \tag{A.13}$$

It follows that

$$\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i} < \infty. \tag{A.14}$$

Using this and recalling (A.10)–(A.11), we can choose $\delta > 0$ small enough that $\delta_t \leq \epsilon$ for all $t \geq \tau$. For such δ, (c^δ, s^δ) is feasible, contradicting the

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[7] An argument similar to (A.10)–(A.11) is used by Bosi et al. (2014).

optimality of (c^*, s^*). We have verified (A.2), which implies conclusion (a) by (2.19).

To see conclusion (b), assume (2.6). Then by (2.21), $\lim_{i\uparrow\infty} q_0^i p_i$ exists. This together with (A.2) shows that $\lim_{i\uparrow\infty} q_0^i p_i = 0$, which implies conclusion (b) by (2.22).

Appendix B Proof of Proposition 4.2

Let $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ be as in (4.2). Suppose by way of contraction that $p_{\tau'} > 0$ for some $\tau' \geq \tau$. Without loss of generality, we assume that $\tau' = \tau = 0$; i.e., $p_0 > 0$.[8]

First suppose that

$$\forall t \in \mathbb{N}, \quad p_t > 0. \tag{B.1}$$

Then Assumption 2.2 holds. We construct $\{q_0^t\}_{t=0}^t$ by (2.6) with $q_0^0 = 1$. Then Assumptions 2.4 and 2.5 as well as (2.6) hold. Since Assumptions 2.1–2.5 and (2.6) hold now, we can apply conclusion (b) of Theorem 4.1. To do this, note from (4.6) and (2.15) that

$$\forall t \in \mathbb{Z}_+, \quad f_t = 0. \tag{B.2}$$

Hence by conclusion (b) of Theorem 4.1, we have $b_t = 0$, i.e., $p_t = f_t = 0$, for all $t \in \mathbb{Z}_+$. This contradicts (B.1).

We have shown that (B.1) cannot be true. In other words, there must be $t \in \mathbb{N}$ such that $p_t = 0$. Let T be the first $T \in \mathbb{Z}_+$ with

$$p_T > 0, \quad p_{T+1} = 0. \tag{B.3}$$

We construct an alternative plan (c, s) as follows:

$$c_t = \begin{cases} c_t^* & \text{if } t \neq T, \\ c_T^* + p_T\epsilon & \text{if } t = T, \end{cases} \tag{B.4}$$

$$s_t = \begin{cases} s_t^* & \text{if } t \neq T, \\ s_T^* - \epsilon & \text{if } t = T. \end{cases} \tag{B.5}$$

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[8] It is no loss of generality to assume that $\tau' = \tau = 0$ since we only consider variables in and after period τ'.

According to this plan, the agents sells the asset when its price is strictly positive, and buys it back when it is free. It is easy to see from (4.2), (2.1), and (B.3) that (c, s) is feasible. But we have $c^* \prec c$ by (B.4) and Assumption 2.3, contradicting the optimality of (c^*, s^*).

We have shown that we reach a contradiction whether (B.1) holds or not; thus we must have $p_0 = 0$. By the innocuous assumptions we made at the outset, this means that $p_t = 0$ for all $t \geq \tau$.

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