Discussion Paper Series
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DP2015-25

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June 24, 2015

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Robust Comparative Statics of Non-monotone Shocks in Large Aggregative Games*

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June 9, 2015

Abstract

A policy change that involves redistribution of income or wealth is typically controversial, affecting some people positively but others negatively. In this paper we extend the "robust comparative statics" result on large aggregative games established by Acemoglu and Jensen (2010, *49th IEEE Conference on Decision and Control*, 3133–3139) to possibly controversial policy changes. In particular, we show that both the smallest and the largest equilibrium values of an aggregate variable increase in response to a policy change to which individuals' reactions may be mixed but the overall aggregate response is positive. We provide sufficient conditions for such a policy change in terms of distributional changes in parameters.

Keywords: Large aggregative games; robust comparative statics; positive shocks; stochastic dominance; mean-preserving spreads

*Financial support from the Japan Society for the Promotion of Science is gratefully acknowledged.

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1 Introduction

Recently, Acemoglu and Jensen (2010, 2014) developed new comparative statics techniques for large aggregative games, where there are a continuum of individuals interacting with each other only through an aggregate variable (which integrates all individuals' actions). The surprising insight of their analysis is that in such games, one can obtain a "robust comparative statics" result without considering the interaction between the aggregate variable and individuals' actions. In particular, Acemoglu and Jensen (2010) defined a positive shock as a policy change that positively affects each individual's action for each value of the aggregate variable. Then they showed that both the smallest and the largest equilibrium values of the aggregate variable increase in response to a positive shock.[1]

Although positive shocks are common in economic models, many important policy changes in reality tend to be non-positive or controversial, affecting some people positively but others negatively. For example, a policy change that involves redistribution of income typically affects some people's income positively but others' negatively. There are many such practical policy changes that cannot be expressed as positive shocks.

The purpose of this paper is to show that Acemoglu and Jensen's (2010, 2014) analysis can in fact be extended to such policy changes to a large extent. This is a significant extension since many important policy changes in reality tend to be controversial, as mentioned above. To accommodate possibly controversial policy changes, we define an "overall positive shock" as a policy change to which individuals' reactions may be mixed but the overall aggregate response is positive for each value of the aggregate variable. Clearly, a positive shock is an overall positive shock, but not vice versa. Following Acemoglu and Jensen's (2010) argument, we show that both the smallest and the largest equilibrium values of the aggregate variable increase in response to an overall positive shock. Then we provide sufficient conditions for an overall positive shock in terms of distributional changes in parameters. When used with these sufficient conditions, our robust comparative statics result becomes particularly powerful.

The concept of overall positive shocks is closely related not only to that of positive shocks but also to Acemoglu and Jensen's (2013) concept of "shocks

[1]While Acemoglu and Jensen (2014) considered the stationary states of dynamic stochastic models, we focus on static models in this paper; see Martimort and Stole (2012) for recent developments on static large aggregative games.

that hit the aggregator," which were defined as policy changes that directly affect the "aggregator" positively along with additional restrictions. Such policy changes are not considered in this paper, but they can easily be incorporated by slightly extending our framework.

This paper is not the first to study the comparative statics of distributional changes. In a general dynamic stochastic model with a continuum of individuals, Acemoglu and Jensen (2014) considered the comparative statics of changes in the stationary distributions of individuals' idiosyncratic shocks, but their analysis was restricted to positive shocks in the above sense. Jensen (2013) and Nocetti (2015) studied the comparative statics of more general distributional changes, but neither of them considered the comparative statics of the smallest and the largest equilibrium values of an aggregate variable. The novelty of this paper is to show some useful robust comparative statics results on distributional changes in parameters for large aggregative games.

The rest of the paper is organized as follows. In Section 2 we present our general framework along with basic assumptions, and prove the existence of a pure-strategy Nash equilibrium. In Section 3 we formally define overall positive shocks. We also introduce a more general definition of "overall monotone shocks." We then present our robust comparative statics result. In Section 4 we provide sufficient conditions for an overall monotone shock in terms of distributional changes in parameters based on first-order stochastic dominance and mean-preserving spreads. In Section 5 we provide two applications. The first is a contest game; the second is a partial-equilibrium model of aggregate labor supply. In Section 6 we conclude the paper.

2 Large Aggregative Games

We consider a large aggregative game as defined by Acemoglu and Jensen (2010, Sections II, III). There are a continuum of players indexed by $i \in I \equiv [0, 1]$, and player i's action space is denoted by $X_i \subset \mathbb{R}$. The assumption made in this section are maintained throughout the paper.

Assumption 2.1. For each $i \in I$, X_i is nonempty and compact. Furthermore, there exists a compact set $K \subset \mathbb{R}$ such that $X_i \subset K$ for all $i \in I$.

Let $X = \prod_{i \in I} X_i$. Let H be a function from K to a subset Ω of $\mathbb{R}$. We define $G : X \to \Omega$, called the *aggregator*, by

$$G(x) = H\left(\int_{i\in I} x_i di\right). \tag{2.1}$$

Assumption 2.2. The set Ω is compact and convex, and $H : K \to \Omega$ is continuous.

Each player i's payoff depends on his own action $x_i \in X_i$, the entire action profile $x \in X$ through the aggregate $G(x)$, and a parameter t_i specific to player i. In other words, player i's payoff takes the form

$$\pi_i(x_i, G(x), t_i). \tag{2.2}$$

Let T_i be the underlying space for t_i for each $i \in I$; i.e., $t_i \in T_i$. Let $T \subset \prod_{i\in I} T_i$. We regard T as a set of well-behaved parameter profiles; for example, T can be a set of measurable functions from I to $\mathbb{R}$. We only consider parameter profiles t in T.

Assumption 2.3. For each $i \in I$, player i's payoff function π_i maps each $(k, Q, \tau) \in K \times \Omega \times T_i$ into $\mathbb{R}$.[2] Furthermore, for each $t \in T$, $\pi_i(\cdot, \cdot, t_i)$ is continuous on $K \times \Omega$, and for each $(k, Q) \in K \times \Omega$, $\pi_i(k, Q, t_i)$ is measurable in $i \in I$.[3]

Instead of assuming that $\pi_i(k, Q, \tau)$ is continuous in k, Acemoglu and Jensen (2010) assume that it is upper semicontinuous in k under the additional assumption that there are only a finite number of player types. On the other hand, Acemoglu and Jensen (2014) essentially assume a continuum of player types by using the Pettis integral in (2.1). We assume a continuum of player types while using the Lebesgue integral in (2.1).

The game here is aggregative in the sense that each player's payoff is affected by other players' actions only through the aggregate $G(x)$. Accordingly, each player i's best response correspondence depends on other players' actions only through $Q = G(x)$:

$$R_i(Q, t_i) = \arg\max_{x_i \in X_i} \pi_i(x_i, Q, t_i). \tag{2.3}$$

[2]If π_i is defined only on $X_i \times \Omega \times T_i$, then this means that π_i can be extended to $K \times \Omega \times T_i$ in such a way as to satisfy Assumption 2.3.

[3]Unless otherwise specified, measurability means Lebesgue measurability.

Let $\mathscr{X}$ be the set of action profiles $x \in X$ such that the mapping $i \in I \mapsto x_i$ is measurable. We need to restrict attention to measurable action profiles so that the equilibrium aggregate $G(x)$ can be computed through (2.1). The following technical assumption ensures that given any $Q \in \Omega$, we can find a measurable action profile $x \in \mathscr{X}$ such that $x_i \in R_i(Q, t_i)$ for all $i \in I$.

Assumption 2.4. For each open subset U of K, the set $\{i \in I : X_i \cap U \neq \emptyset\}$ is measurable.

Throughout the paper, we restrict attention to pure-strategy Nash equilibria, which we simply call equilibria. To be more precise, given $t \in T$, an *equilibrium* of this game is an action profile $x \in \mathscr{X}$ such that $x_i \in R_i(G(x), t_i)$ for all $i \in I$. Given $t \in T$, we define an *equilibrium aggregate* as $Q(t) \in \Omega$ such that $Q(t) = G(x)$ for some equilibrium x. We define $\underline{Q}(t)$ and $\overline{Q}(t)$ as the smallest and the largest equilibrium aggregates, respectively, provided that they exist.

The following result shows that an equilibrium as well as the smallest and the largest equilibrium aggregates exit.

Theorem 2.1. *For any $t \in T$, an equilibrium exists. Furthermore, the set of equilibrium aggregates is nonempty and compact. Thus the smallest and the largest equilibrium aggregates $\underline{Q}(t)$ and $\overline{Q}(t)$ exist.*

Proof. See Appendix A. □

Following Acemoglu and Jensen (2010, Theorem 1), we prove the above result using Kakutani's fixed point theorem and Aumann's (1965, 1976) results on the integral of a correspondence. Our result differs from Acemoglu and Jensen's in that we assume a continuum of player types rather than a finite number of player types, as mentioned above.

3 Overall Monotone Shocks

By a *parameter change*, we mean a change in $t \in T$ from one value to another. Given $\underline{t}, \overline{t} \in T$, the *parameter change from $\underline{t}$ to $\overline{t}$* means the change in t from $\underline{t}$ to $\overline{t}$. The following definitions take $\underline{t}, \overline{t} \in T$ as given.

Definition 3.1 (Acemoglu and Jensen, 2010)**.** The parameter change from $\underline{t}$ to $\overline{t}$ is a *positive shock* if for each $Q \in \Omega$ the following properties hold:

(i) For each $\underline{x}_i \in R_i(Q, \underline{t}_i)$ there exists $\overline{x}_i \in R_i(Q, \overline{t}_i)$ such that $\underline{x}_i \leq \overline{x}_i$.

(ii) For each $\overline{y}_i \in R_i(Q, \overline{t}_i)$ there exists $\underline{y}_i \in R_i(Q, \underline{t}_i)$ such that $\underline{y}_i \leq \overline{y}_i$.

For comparison purposes, it is useful to define "negative shocks" and more general "monotone shocks."

Definition 3.2. The parameter change from $\underline{t}$ to $\overline{t}$ is a *negative shock* if the parameter change from $\overline{t}$ to $\underline{t}$ is a positive shock. A parameter change is a *monotone shock* if it is either a positive shock or a negative shock.

Acemoglu and Jensen (2010, Theorem 2) show that if the parameter change from $\underline{t}$ to $\overline{t}$ is a positive shock, then the following inequalities hold:

$$\underline{Q}(\underline{t}) \leq \underline{Q}(\overline{t}), \quad \overline{Q}(\underline{t}) \leq \overline{Q}(\overline{t}). \tag{3.1}$$

In this section we show that these inequalities hold for a substantially larger class of parameter changes. To this end, for $Q \in \Omega$ and $t \in T$, we define

$$\mathcal{G}(Q, t) = \{G(x) : x \in \mathscr{X}, \forall i \in I, x_i \in R_i(Q, t_i)\}. \tag{3.2}$$

The following definitions play a central role in our comparative statics results.

Definition 3.3. The parameter change from $\underline{t}$ to $\overline{t}$ is an *overall positive shock* if for each $Q \in \Omega$ the following properties hold:

(i) For each $\underline{q} \in \mathcal{G}(Q, \underline{t})$ there exists $\overline{q} \in \mathcal{G}(Q, \overline{t})$ such that $\underline{q} \leq \overline{q}$.

(ii) For each $\overline{r} \in \mathcal{G}(Q, \overline{t})$ there exists $\underline{r} \in \mathcal{G}(Q, \underline{t})$ such that $\underline{r} \leq \overline{r}$.

Definition 3.4. The parameter change from $\underline{t}$ to $\overline{t}$ is an *overall negative shock* if the parameter change from $\overline{t}$ to $\underline{t}$ is an overall positive shock. A parameter change is an *overall monotone shock* if it is either an overall positive shock or an overall negative shock.

We are ready to state our result on robust comparative statics:

Theorem 3.1. *Let $\underline{t}, \overline{t} \in T$. Suppose that the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive shock. Then both inequalities in (3.1) hold. The reserve inequalities hold if the parameter change is an overall negative shock.*

Proof. See Appendix B. □

Since a positive shock is an overall positive shock, Acemoglu and Jensen's (2010, Theorem 2) result mentioned above immediately follows under our assumptions. On the other hand, the proof of Theorem 3.1 closely follows that of their result, and Definitions 3.3 and 3.4 may not be easy to verify directly. As we see in Section 5, however, Theorem 3.1 becomes particularly powerful when used with sufficient conditions for an overall monotone shock such as those in the next section.

4 Sufficient Conditions

In this section we assume that players differ only in their parameters t_i. This by itself entails no loss of generality since t_i can include i as one of its components. More specifically, we assume the following for the rest of the paper.

Assumption 4.1. There exists a convex set $\mathcal{T} \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ such that $T_i \subset \mathcal{T}$ for all $i \in I$. There exists a correspondence $\mathcal{X} : \mathcal{T} \to 2^{\mathcal{T}}$ such that $X_i = \mathcal{X}(t_i)$ for all $i \in I$ and $t_i \in T_i$. Moreover, there exists a function $\pi : K \times \Omega \times \mathcal{T} \to \mathbb{R}$ such that

$$\forall i \in I, \forall (k, Q, \tau) \in K \times \Omega \times \mathcal{T}, \quad \pi_i(k, Q, \tau) = \pi(k, Q, \tau). \tag{4.1}$$

This assumption implies that player i's best response correspondence $R_i(Q, \tau)$ does not directly depend on i; we denote this correspondence by $R(Q, \tau)$. For $(Q, \tau) \in \Omega \times \mathcal{T}$, we define

$$\underline{R}(Q, \tau) = \min R(Q, \tau), \quad \overline{R}(Q, \tau) = \max R(Q, \tau). \tag{4.2}$$

Both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are well-defined since $R(Q, \tau)$ is a compact set for each $(Q, \tau) \in (\Omega, \mathcal{T})$ (see Lemma A.1). To consider distributional changes in $t \in T$, we assume the following for the rest of the paper.

Assumption 4.2. T is a set of measurable functions from I to $\mathcal{T}$, and $H : K \to \Omega$ is a nondecreasing function.

For any $t \in T$, let $F_t : \mathbb{R}^n \to I$ denote the distribution function of t:

$$F_t(z) = \int_{i \in I} 1\{t_i \leq z\} di, \tag{4.3}$$

where $1\{\cdot\}$ is the indicator function; i.e., $1\{t_i \leq z\} = 1$ if $t_i \leq z$, and $= 0$ otherwise. Note that $F_t(z)$ is the proportion of players $i \in I$ with $t_i \leq z$.

4.1 First-Order Stochastic Dominance

Given a pair of distributions $\underline{F}$ and $\overline{F}$, $\overline{F}$ is said to *(first-order) stochastically dominate* $\underline{F}$ if $\int \phi d\underline{F} \leq \int \phi d\overline{F}$ for any nondecreasing bounded Borel-measurable function $\phi : \mathbb{R}^n \to \mathbb{R}$, where $\mathbb{R}^n$ is equipped with the usual partial order $\leq$.[4] It is well known (e.g., Müller and Stoyan, 2002, Section 1) that in case $n = 1$, $\overline{F}$ stochastically dominates $\underline{F}$ if and only if

$$\forall z \in \mathbb{R}, \quad \underline{F}(z) \geq \overline{F}(z). \tag{4.4}$$

The following result provides a sufficient condition for an overall monotone shock based on stochastic dominance.

Theorem 4.1. *Let $\underline{t}, \overline{t} \in T$. Suppose that $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. Suppose that both $\underline{R}(Q,\tau)$ and $\overline{R}(Q,\tau)$ are nondecreasing (resp. nonincreasing) Borel-measurable functions of $\tau \in \mathcal{T}$ for each $Q \in \Omega$. Then the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive (resp. negative) shock.*

Proof. We only consider the case in which both $\underline{R}(Q,\tau)$ and $\overline{R}(Q,\tau)$ are nondecreasing in $\tau \in \mathcal{T}$ since the other case is symmetric. Let $\underline{q} \in \mathcal{G}(Q,\underline{t})$. Then there exists $x \in \mathscr{X}$ such that $x_i \in R(Q,\underline{t}_i)$ for all $i \in I$ and $\underline{q} = H(\int_{i\in I} x_i di)$. Since $x_i \leq \overline{R}(Q,\underline{t}_i)$ for all $i \in I$ by (4.2), and since H is a nondecreasing function by Assumption 4.2, we have

$$\begin{aligned} \underline{q} &\leq H\left(\int_{i\in I} \overline{R}(Q,\underline{t}_i)di\right) = H\left(\int \overline{R}(Q,z)dF_{\underline{t}}(z)\right) \qquad (4.5)\\ &\leq H\left(\int \overline{R}(Q,z)dF_{\overline{t}}(z)\right) = H\left(\int_{i\in I} \overline{R}(Q,\overline{t}_i)\right) \in \mathcal{G}(Q,\overline{t}), \qquad (4.6)\end{aligned}$$

where the inequality in (4.6) holds since $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$ and $\overline{R}(Q,\cdot)$ is a nondecreasing function. It follows that condition (i) of Definition 3.3 holds. By a similar argument, condition (ii) also holds. Hence the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive shock. □

If the parameter change from $\underline{t}$ to $\overline{t}$ is a positive shock, then it is easy to see from (4.3) that $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. However, there are many other ways in which $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. Figure 1 shows a simple example. In this example, the parameter change from $\underline{t}$ to $\overline{t}$ is clearly not

[4] To be precise, given $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n$, we write $a \leq b$ if $a_j \leq b_j$ for all $j = 1, \ldots, n$.

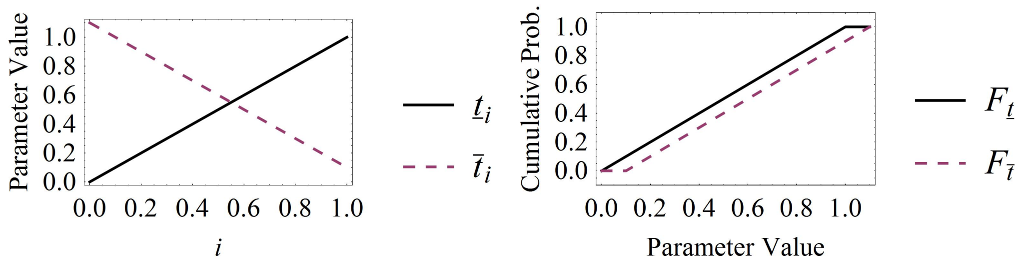

Figure 1: The parameter change from $\underline{t}$ to $\overline{t}$ is not a monotone shock (left panel), but $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$ (right panel).

a monotone shock, but $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$ by (4.4). Thus the parameter change here is an overall positive shock by Theorem 4.1 if both $\underline{R}(Q,\tau)$ and $\overline{R}(Q,\tau)$ are nondecreasing in τ.

There are well known sufficient conditions for both $\underline{R}(Q,\tau)$ and $\overline{R}(Q,\tau)$ to be nondecreasing or nonincreasing; see Milgrom and Shannon (1994, Theorem 4), Topkis (1998, Theorem 2.8.3), Amir (2005, Theorems 1, 2), and Roy and Sabarwal (2010, Theorem 2). Any of those conditions can be combined with Theorem 4.1 to replace the assumption that both $\underline{R}(Q,\tau)$ and $\overline{R}(Q,\tau)$ are nondecreasing or nonincreasing. Here we state a simple result based on Amir (2005, Theorems 1, 2).

Corollary 4.1. *Suppose that $\mathcal{T} \subset \mathbb{R}$. Let $\underline{t}, \overline{t} \in T$. Suppose that $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$. If $\pi(k,Q,\tau)$ has increasing (resp. decreasing) differences in $(k,\tau) \in K \times \mathcal{T}$ for each $Q \in \Omega$, then the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive (resp. negative) shock.*

4.2 Mean-Preserving Spreads

Following Acemoglu and Jensen (2014), we say that $F_{\overline{t}}$ is a *mean-preserving spread* of $F_{\underline{t}}$ if $\int \phi dF_{\underline{t}} \leq \int \phi dF_{\overline{t}}$ for any convex Borel-measurable function $\phi : \mathcal{T} \to \mathbb{R}$. Rothschild and Stiglitz (1970, p. 231) and Machina and Pratt (1997, Theorem 3) show that in case $n = 1$, $F_{\overline{t}}$ is a mean-preserving spread of $F_{\underline{t}}$ if

$$\int F_{\underline{t}}(z)dz = \int F_{\overline{t}}(z)dz, \tag{4.7}$$

and if there exists $\tilde{z} \in \mathbb{R}$ such that

$$F_{\underline{t}}(z) - F_{\overline{t}}(z) \begin{cases} \leq 0 & \text{if } z \leq \tilde{z}, \\ \geq 0 & \text{if } z > \tilde{z}. \end{cases} \tag{4.8}$$

The following result provides a sufficient condition for an overall monotone shock based on mean-preserving spreads.

Theorem 4.2. *Let $\underline{t}, \overline{t} \in T$. Suppose that $F_{\overline{t}}$ is a mean-preserving spread of $F_{\underline{t}}$. Suppose that both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are convex (resp. concave) Borel-measurable functions of $\tau \in \mathcal{T}$ for each $Q \in \Omega$. Then the parameter change from $\underline{t}$ to $\overline{t}$ is an overall positive (resp. negative) shock.*

Proof. The proof is essentially the same as that of Theorem 4.1 except that the inequality in (4.6) holds since $F_{\overline{t}}$ is a mean-preserving spread of $F_{\underline{t}}$ and $\overline{R}(Q, \tau)$ is convex in τ. □

Since a mean-preserving spread of the distribution of parameters forces a tradeoff across players, it can *never* be a monotone shock. Our approach differs from that of Acemoglu and Jensen (2014) in that while they consider positive shocks induced by applying a mean-preserving spread to the stationary distribution of each player's idiosyncratic shock, we consider non-monotone shocks induced by applying a mean-preserving spread to the entire distribution of parameters.

Figure 2 shows a simple example of a mean-preserving spread. In this example, the parameter change from $\underline{t}$ to $\overline{t}$ is clearly not a monotone shock, but it is a mean-preserving spread by (4.7) and (4.8). Thus the parameter change here is an overall positive shock by Theorem 4.2 if both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are convex in $\tau \in \mathcal{T}$. Various conditions related to such convexity (or concavity) properties can be found in Jensen (2013).

5 Applications

5.1 Contest Games

We consider a simplified variation on the contest games studied by Acemoglu and Jensen (2010, 2013). There exist a continuum of players $i \in I$. Each player i exerts costly effort $x_i \in K \equiv [0, \overline{k}]$ with $\overline{k} > 0$ in order to increase his chance of winning a prize $V > 0$. We assume that the probability that player

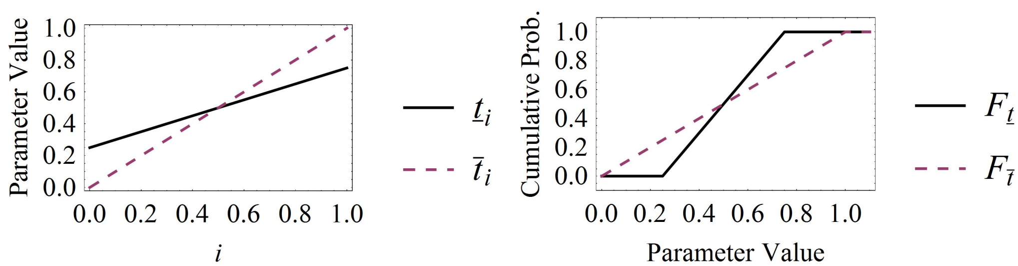

Figure 2: The parameter change from $\underline{t}$ to $\overline{t}$ is not a monotone shock (left panel), but $F_{\overline{t}}$ is a mean-preserving spread of $F_{\underline{t}}$ (right panel).

i wins the prize is given by $f(x_i)/H(\int_0^1 x_i di)$, where $f, H : [0, \overline{k}] \to \mathbb{R}_{++}$ are nondecreasing continuous functions. Let

$$Q = H\left(\int_{i\in I} x_i di\right). \tag{5.1}$$

Let $\mathcal{T} = [0, 1]$. We assume that for any $t \in T$, the expected payoff of player i is given by

$$\pi(x_i, Q, t_i) = \frac{f(x_i)}{Q}V - c(x_i)t_i, \tag{5.2}$$

where $c : [0, \overline{k}] \to \mathbb{R}_+$ is a nondecreasing cost function, and t_i is a parameter specific to player i. It is easy to see that $\pi(k, Q, \tau)$ has decreasing differences in (k, τ). Thus by Corollary 4.1, for any $\underline{t}, \overline{t} \in T$ such that $F_{\overline{t}}$ stochastically dominates $F_{\underline{t}}$, the parameter change from $\underline{t}$ to $\overline{t}$ is an overall negative shock.

Figure 1 provides a simple example. Once again, since the parameter change in this example is not a monotone shock, Acemoglu and Jensen's (2010) analysis does not apply. By contrast, one can easily conclude from Theorem 3.1 in conjunction with Corollary 4.1 that the smallest and the largest equilibrium aggregates decrease in response to the parameter change in Figure 1.

5.2 Aggregate Labor Supply

Consider an economy with a continuum of agents indexed by $i \in I$. Agent i solves the following maximization problem:

$$\max_{c_i, x_i \geq 0} \quad u(c_i) - x_i \tag{5.3}$$

$$\text{s.t.} \quad c_i = wx_i + e_i + s_i, \tag{5.4}$$

where $u : \mathbb{R}_+ \to \mathbb{R} \cup \{-\infty\}$ is strictly increasing, strictly concave, and continuously differentiable, w is the wage rate, s_i is a lump-sum subsidy to agent i. and c_i, x_i, and e_i are agent i's consumption, labor supply, and endowment, respectively. If $s_i < 0$, then agent i pays a lump-sum tax of $-s_i$. The government's budget constraint is

$$\int_{i \in I} s_i di = 0. \tag{5.5}$$

For simplicity, we do not explicitly impose an upper bound on x_i; i.e., we assume that the upper bound on x_i is never binding for relevant values of w (to be specified below). This simply means that no agent works 24 hours a day, 7 days a week.

Let $t_i = e_i + s_i$ for $i \in I$. Suppose that there exists $\overline{\tau} > 0$ such that $t_i \in \mathcal{T} \equiv [0, \overline{\tau}]$ for all $i \in I$. The first-order condition for the above maximization problem is written as

$$u'(wx_i + t_i)w \begin{cases} \leq 1 & \text{if } x_i = 0, \\ = 1 & \text{if } x_i > 0. \end{cases} \tag{5.6}$$

Let $x(w, t_i)$ denote the solution for x_i as a function of w and t_i. Then aggregate labor supply is given by $\int_{i \in I} x(w, t_i) di$.

We assume that aggregate demand for labor is given by a demand function $D(w)$ such that $D(0) < \infty$, $D(\overline{w}) = 0$ for some $\overline{w} > 0$, and $D : [0, \overline{w}] \to \mathbb{R}_+$ is continuous and strictly decreasing. The market-clearing condition is

$$D(w) = \int_{i \in I} x(w, t_i) di. \tag{5.7}$$

To see that this model is a large aggregative game, let $Q = \int_{i \in I} x_i di$. Then (5.7) implies that

$$w = D^{-1}(Q). \tag{5.8}$$

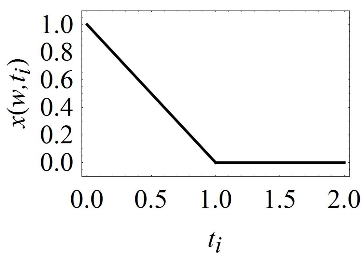

Figure 3: Individual labor supply as a function of t_i ($w > 0$).

Thus the model here is a special case of the game in Section 4 with

$$\pi(k, Q, \tau) = u(D^{-1}(Q)k + \tau) - k, \quad K = \Omega = [0, \overline{k}], \tag{5.9}$$

where $\overline{k} = \max_{(w,\tau)\in[0,\overline{w}]\times\mathcal{T}} x(w, \tau)$.

Since $t_i = e_i + s_i$ for all $i \in I$, it follows from (5.5) that

$$\int_{i\in I} t_i di = \int_{i\in I} e_i di. \tag{5.10}$$

Thus as long as the right-hand side is constant, a parameter change can never be a monotone shock. Hence Acemoglu and Jensen's (2010) analysis never applies here. On the other hand, note from (5.6) that for any $i \in I$ we have

$$x(w, t_i) = \begin{cases} \max\{[u'^{-1}(1/w) - t_i]/w, 0\} & \text{if } w > 0, \\ 0 & \text{if } w = 0. \end{cases} \tag{5.11}$$

This function is piecewise linear and convex in t_i; see Figure 3. Since $\underline{R}(Q, \tau) = \overline{R}(Q, \tau) = x(D^{-1}(Q), \tau)$, it follows that both $\underline{R}(Q, \tau)$ and $\overline{R}(Q, \tau)$ are convex in $\tau \in \mathcal{T}$.

Let $\underline{t}, \overline{t} \in T$ be such that $F_{\overline{t}}$ is a mean-preserving spread of $F_{\underline{t}}$. Figure 2 provides a simple example, in which the parameter change from $\underline{t}$ to $\overline{t}$ is clearly not a monotone shock. However, it is an overall positive shock by Theorem 4.2. Thus by Theorem 3.1, the smallest and the largest equilibrium aggregate labor quantities increase in response to this parameter change. This together with (5.8) implies that the highest and the lowest equilibrium wage rates decrease in response to the same parameter change.

6 Concluding Comments

Acemoglu and Jensen (2010) established that the smallest and the largest equilibrium aggregates of a large aggregative game are nondecreasing in a positive shock, which is a parameter change that affects each player's action positively for each value of the aggregate variable. In this paper we extended their result by showing that the smallest and the largest equilibrium aggregates are nondecreasing in an overall positive shock, which is a parameter change to which individuals' reactions may be mixed but the overall aggregate response is positive for each value of the aggregate variable. We provided sufficient conditions for an overall positive shock in terms of distributional changes in parameters based on stochastic dominance and mean-preserving spreads. These conditions clarified that positive shocks are not necessary for robust comparative statics.

Although we considered only one-dimensional distributions in our applications, Theorems 4.1 and 4.2 in fact assume multidimensional distributions. Examples with multidimensional distributions are easy to construct at least at the informal level if each player's parameter consists of n components, and if the distributions of these components over all players are independent. In this case, one can apply stochastic dominance and mean-preserving spreads componentwise. Furthermore, Theorems 4.1 and 4.2 can be modified to better fit this particular setting.

On the other hand, as soon as one treats parameters as random variables, one may face technical difficulties concerning a continuum of random variables and the law of large numbers. In this paper we avoided such difficulties entirely, but they may be overcome by following Acemoglu and Jensen (2014) in using an alternative definition of the integral of random variables.

Appendix A Proof of Theorem 2.1

Since t is fixed here, we suppress the dependence of π_i, R_i, and $\mathcal{G}$ on t and t_i throughout the proof. For $i \in I$, define $\mu(i, Q) = R_i(Q)(= R_i(Q, t))$.

Lemma A.1. *For any $Q \in \Omega$, the correspondence $\mu(\cdot, Q)$ from $i \in I$ to $R_i(Q) \subset K$ has nonempty compact values, and admits a measurable selection.*

Proof. Fix $Q \in \Omega$. We show this lemma by applying the measurable maximum theorem (Aliprantis and Border 2006, p. 605) to player i's maximization problem with $i \in I$ taken as a parameter:

$$\max_{x_i \in X_i} \pi_i(x_i, Q). \tag{A.1}$$

Let ϕ denote the correspondence $i \in I \mapsto X_i \subset K$. By Assumption 2.1, ϕ has nonempty compact values. Note that the set $\{i \in I : X_i \cap U \neq \emptyset\}$ in Assumption 2.4 is the lower inverse of U under ϕ; see Aliprantis and Border (2006, p. 557). Thus Assumption 2.4 means that ϕ is weakly measurable; see Aliprantis and Border (2006, p. 592). Assumption 2.3 means that the mapping $(i, k) \in I \times K \mapsto \pi_i(k, Q) \in \mathbb{R}$ is a Carathéodory function.

It follows that the measurable maximum theorem applies to (A.1); thus the correspondence $i \mapsto \mu(i, Q)$ has nonempty compact values, and admits a measurable selection. □

Lemma A.2. *For each $Q \in \Omega$, the set $\mathcal{G}(Q)$ is nonempty and convex, where $\mathcal{G}$ is defined in (3.2).*

Proof. Fix $Q \in \Omega$. Since $\mu(\cdot, Q)$ admits a measurable selection by Lemma A.1, $\mathcal{G}(Q)$ is nonempty. To see that $\mathcal{G}(Q)$ is convex, note from Aumann (1965, Theorem 1) that the set

$$\left\{ \int_{i \in I} x_i d_i : x \in \mathscr{X}, \forall i \in I, x_i \in \mu(i, Q) \right\} \tag{A.2}$$

is convex. The image of this convex set under H is convex since H is continuous and real-valued.[5] Recalling (3.2) we see that $\mathcal{G}(Q)$ is convex. □

Lemma A.3. *The correspondence $\mathcal{G}(\cdot)$ has compact values and a closed (in fact, compact) graph.*

Proof. Fix $i \in I$. Note that X_i does not depend on Q; thus the correspondence $Q \mapsto X_i$ is continuous in a trivial way. By Assumption 2.3, $\pi_i(k, Q)$ is continuous in $(k, Q) \in X_i \times \Omega$. Hence by the Berge maximum theorem (Aliprantis and Border, 2006, p. 570) and the closed graph theorem (Aliprantis and Border, 2006, p. 561), the correspondence $\mu(i, \cdot)$ has a closed graph. In other words,

$$F(i) \equiv \{(k, Q) \in X_i \times \Omega : k \in \mu(i, Q)\} \text{ is closed.} \tag{A.3}$$

[5] The image may not be convex if the range of H is not one-dimensional.

Let $\mathscr{G}$ be the graph of the correspondence $\mathcal{G}(\cdot)$:

$$\mathscr{G} = \{(Q, S) \in \Omega \times \Omega : S \in \mathcal{G}(Q)\}. \tag{A.4}$$

To verify that $\mathscr{G}$ is closed, it suffices to show that $\mathscr{G}$ contains the limit of any sequence $\{(Q^j, S^j)\}_{j\in\mathbb{N}}$ in $\mathscr{G}$ that converges in Ω^2. For this purpose, let $\{(Q^j, S^j)\}_{j\in\mathbb{N}}$ be a sequence in $\mathscr{G}$ that converges to some $(Q^*, S^*) \in \Omega^2$. Then for each $j \in \mathbb{N}$ we have $S^j \in \mathcal{G}(Q^j)$; thus there exists a measurable selection $x^j \in \mathscr{X}$ of $\mu(\cdot, Q^j)$ such that

$$S^j = H\left(\int_{i\in I} x_i^j di\right). \tag{A.5}$$

Taking a subsequence of $\{S^j\}$, we can assume that $\xi^j \equiv \int_{i\in I} x_i^j di$ converges to some $\xi^* \in K$ as $j \uparrow \infty$. Since H is continuous by Assumption 2.2, it follows that

$$S^* = H(\xi^*). \tag{A.6}$$

Since x^j is a selection of $\mu(\cdot, Q^j)$ for all $j \in \mathbb{N}$, recalling the definition of $F(i)$ we have

$$\forall i \in I, \forall j \in \mathbb{N}, \quad (x_i^j, Q^j) \in F(i). \tag{A.7}$$

Since $F(i)$ is closed, any convergent subsequence of $\{(x_i^j, Q^j)\}_{j\in\mathbb{N}}$ converges in $F(i)$. Since $Q^j \to Q^*$ as $j \uparrow \infty$, it follows that any limit point y_i of $\{x_i^j\}_{j\in\mathbb{N}}$ satisfies $(y_i, Q^*) \in F(i)$; i.e., $y_i \in \mu(i, Q^*)$. In addition, $|x_i^j| \le \max\{|k| : k \in K\}$ for all $i \in I$ and $j \in \mathbb{N}$. Hence by Aumann (1976, Lemma), there exists $x^* \in \mathscr{X}$ such that $x_i^* \in \mu(i, Q^*)$ for all $i \in I$ and $\int_{i\in I} x_i^* di = \xi^*$. Since $S^* = H(\int_{i\in I} x_i^* di)$ by (A.6), we have $S^* \in \mathcal{G}(Q^*)$; i.e., $(Q^*, S^*) \in \mathscr{G}$. It follows that $\mathscr{G}$ is closed.

Since $\mathscr{G} \subset \Omega \times \Omega$, which is compact, it follows that $\mathscr{G}$ is compact. Hence $\mathcal{G}(\cdot)$ has compact values. □

Now by Kakutani's fixed point theorem (Aliprantis and Border, 2006, p. 583), Assumption 2.2, and Lemmas A.2 and A.3, the set of fixed points of the correspondence $\mathcal{G}(\cdot)$ is nonempty and compact. Let $Q \in \Omega$ be a fixed point. Then there exists a measurable selection $x \in \mathscr{X}$ of $\mu(\cdot, Q)$ such that $Q = H(\int_{i\in I} x_i di)$; i.e., x is an equilibrium. Hence an equilibrium exists.

The preceding argument shows that any fixed point of $\mathcal{G}(\cdot)$ is an equilibrium aggregate. Since the set of fixed points of $\mathcal{G}(\cdot)$ is nonempty and compact, it follows that the set of equilibrium aggregates is also nonempty and compact; thus the smallest and the largest equilibrium aggregates exist.

Appendix B Proof of Theorem 3.1

Since the correspondence $Q \mapsto \mathcal{G}(Q,t)$ has compact values by Lemma A.3, the minimum and the maximum of $\mathcal{G}(Q,t)$ exit for each $Q \in \Omega$. We define

$$\underline{\mathcal{G}}(Q,t) = \min \mathcal{G}(Q,t), \quad \overline{\mathcal{G}}(Q,t) = \max \mathcal{G}(Q,t). \tag{B.1}$$

Since $\mathcal{G}(\cdot,t)$ has convex values by Lemma A.2, we have

$$\forall Q \in \Omega, \quad \mathcal{G}(Q,t) = [\underline{\mathcal{G}}(Q,t), \overline{\mathcal{G}}(Q,t)]. \tag{B.2}$$

Since $\mathcal{G}(\cdot,t)$ has a compact graph by Lemma A.3, it is easy to see that $\mathcal{G}(\cdot,t)$ is "continuous but for upward jumps"; see Milgrom and Roberts (1994, p. 447). To conclude both inequalities in (3.1) from Milgrom and Roberts (1994, Corollary 2), it remains to show that for all $Q \in \Omega$ we have

$$\underline{\mathcal{G}}(Q,\underline{t}) \leq \underline{\mathcal{G}}(Q,\overline{t}), \quad \overline{\mathcal{G}}(Q,\underline{t}) \leq \overline{\mathcal{G}}(Q,\overline{t}). \tag{B.3}$$

To see the first inequality in (B.3), let $\overline{r} = \underline{\mathcal{G}}(Q,\overline{t})$. Then by Definition 3.3(ii), there exists $\underline{r} \in \mathcal{G}(Q,\underline{t})$ such that $\underline{r} \leq \overline{r}$. Since $\underline{\mathcal{G}}(Q,\underline{t}) \leq \underline{r}$, the desired inequality follows. The second inequality in (B.3) can be verified in a similar way.

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