Discussion Paper Series
RIEB
Kobe University

DP2015-24

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June 23, 2015

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A Simple No-Bubble Theorem for Deterministic Dynamic Economies*

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Abstract

In this paper we show a simple no-bubble theorem that applies to a wide range of deterministic economies with infinitely lived agents. In particular, we show that asset bubbles are impossible if there is at least one agent who can reduce his asset holdings permanently from some period onward. This is a substantial generalization of Kocherlakota's (1992, *Journal of Economic Theory* 57, 245–256) result on asset bubbles and short sales constraints; our result requires virtually no assumption except for the strict monotonicity of preferences. We also provide a substantial generalization of his result on asset bubbles and the present value of a single agent's endowment. As a consequence of these results, we extend Huang and Werner's (2000, *Economic Theory* 15, 253–278) no-bubble theorem to an economy with multiple assets.

*I would like to thank an anonymous referee and the Guest Editor Jianjun Miao for helpful comments and suggestions. A substantial part of this paper has been written based on his suggestions. Financial support from the Japan Society for the Promotion of Science is gratefully acknowledges.

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1 Introduction

Since the global financial crisis of 2007-2008, there has been a surge of interest in rational asset pricing bubbles, or simply "asset bubbles." Numerous economic mechanisms that give rise to asset bubbles are still being proposed, and the implications of asset bubbles on various economic issues are actively discussed in the current literature; we refer the reader to Miao (2014) for a short survey on recent developments.

In constructing models of asset bubbles, it is important to understand conditions under which bubbles exist or do not exist. While conditions for existence are mostly restricted to specific models, some general conditions for nonexistence are known. In fact, it is well known that bubbles are impossible if the present value of the aggregate endowment is finite. This was shown by Santos and Woodford (1997) for a general equilibrium model with incomplete markets and possibly infinitely many agents each of whom may be finitely or infinitely lived. Wilson's (1981) result on the existence of a competitive equilibrium in a deterministic economy with infinitely many agents can be viewed as an earlier version of this no-bubble theorem. Huang and Werner (2000) showed a version of the no-bubble theorem applicable to an asset in zero net supply for a deterministic economy with finitely many agents. Werner (2014) extended Santos and Woodford's (1997) no-bubble theorem to a complete market economy with debt constraints (instead of borrowing constraints).

While these results are based on equilibrium prices and allocations, there are closely related results based mostly on the optimal behavior of a single agent. For example, in a deterministic economy with finitely many agents, Kocherlakota (1992) showed that in an equilibrium with a positive bubble, the short sales constraints of all agents must be asymptotically binding; equivalently, bubbles can be ruled out if there is at least one agent whose asset holdings can be lowered permanently from some period onward. A similar idea was used earlier by Obstfeld and Rogoff (1986) to rule out deflationary equilibria in a money-in-the-utility-function model.

The results mentioned in the preceding paragraph rely on the necessity of some transversality condition,[1] and a general no-bubble result based on the necessity of a transversality condition was shown in Kamihigashi (2001,

[1]Various results on necessity of transversality conditions were established in Kamihigashi (2001, 2002, 2003, 2005).

p. 1007) for deterministic representative agent models. Essentially, this result only requires the differentiability and strict monotonicity of instantaneous utility functions; thus it can be used to rule out bubbles in various representative-agent models.

In this paper, we establish a simple no-bubble theorem that can be used to rule out bubbles in a considerably broader range of deterministic models. More specifically, we consider the problem of a single agent facing a sequential budget constraint and having strictly monotone preferences. We show that bubbles are impossible if the agent can reduce his asset holdings permanently from some period onward. This result uses the same idea as those based on transversality conditions mentioned above; the contribution of this paper is to show that the result holds true under extremely general conditions.

In addition to our no-bubble theorem, we show a general version of the result shown by Kocherlakota (1992) on the relation between the existence of a bubble and the present value of an agent's endowment. Our result here is independent of our no-bubble theorem; it is shown merely as a consequence of the agent's sequential budget constraint. Hence the result is extremely general in that it requires no assumption on preferences and constraints except for the sequential budget constraint.

To clarify the relations between our theorems and related results in the literature, we consider a general equilibrium model with a finite number of infinitely many agents and a finite number of assets. Using our theorems mentioned above, we establish various results on bubbles in general equilibrium. In particular, we show substantial generalizations of Propositions 3 and 4 in Kocherlakotoa (1992), and an extension of Huang and Werner's (2000) no-bubble theorem to our economy with multiple assets. In Section 6 we fully discuss our results in relation to these and other results in the literature.

The rest of the paper is organized as follows. In Section 2 we present a single agent's problem along with necessary assumptions, and formally define asset bubbles. In Section 3 we offer several examples satisfying our assumptions. In Section 4 we state our no-bubble theorem and show some immediate consequences. We also show a general result on bubbles and the present value of an agent's endowment. In Section 5 we present a general equilibrium model. In Section 6 we show various results on bubbles in general equilibrium. In Section 7 we discuss how our general results can be extended to stochastic models. In Section 8 we offer some concluding comments.

2 Single-Agent/Single-Asset Framework

2.1 Feasibility and Optimality

Time is discrete and denoted by $t \in \mathbb{Z}_+$. In this section we assume that there are one consumption good (used as the numeraire) and one asset that pays a dividend of d_t units of the consumption good in each period $t \in \mathbb{Z}_+$. Let p_t be the price of the asset in period $t \in \mathbb{Z}_+$. Consider an infinitely lived agent who faces the following constraints:

$$c_t + p_t s_t = y_t + (p_t + d_t)s_{t-1},\ c_t \geq 0, \quad \forall t \in \mathbb{Z}_+, \tag{2.1}$$

$$s \in \mathcal{S}(s_{-1}, y, p, d), \tag{2.2}$$

where c_t is consumption in period t, $y_t \in \mathbb{R}$ is (net) income in period t, s_t is asset holdings at the end of period t with s_{-1} historically given, and $\mathcal{S}(s_{-1}, y, p, d)$ is a set of sequences in $\mathbb{R}$ with $s = \{s_t\}_{t=0}^\infty$, $y = \{y_t\}_{t=0}^\infty$, $p = \{p_t\}_{t=0}^\infty$, and $d = \{d_t\}_{t=0}^\infty$, We present several examples of (2.2) in Section 3.

Although we consider a single agent's problem and assume that there is only one asset here, we apply our results shown within this framework to a general equilibrium model with many agents and many assets in Section 5.

Let $\mathcal{C}$ be the set of sequences $\{c_t\}_{t=0}^\infty$ in $\mathbb{R}_+$. For any $c \in \mathcal{C}$, we let $\{c_t\}_{t=0}^\infty$ denote the sequence representation of c, and vice versa. In other words, we use c and $\{c_t\}_{t=0}^\infty$ interchangeably; likewise, we use s and $\{s_t\}_{t=0}^\infty$ interchangeably. We define the inequalities $<$ and $\leq$ on the set of sequences in $\mathbb{R}$ (which includes $\mathcal{C}$) as follows:

$$c \leq c' \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_t,\ c_t \leq c'_t, \tag{2.3}$$

$$c < c' \quad \Leftrightarrow \quad c \leq c' \text{ and } \exists t \in \mathbb{Z}_+, c_t < c'_t. \tag{2.4}$$

The agent's preferences are represented by a binary relation $\prec$ on $\mathcal{C}$. For any $c, c' \in \mathcal{C}$, the agent strictly prefers c' to c if and only if $c \prec c'$. In Sections 2–4 we maintain the following assumption.

Assumption 2.1. $d_t \geq 0$ and $p_t \geq 0$ for all $t \in \mathbb{Z}_+$.

Unless otherwise stated, we also assume the following.

Assumption 2.2. $p_t > 0$ for each $t \in \mathbb{Z}_+$.

This assumption is required for some of the variables introduced below to be well defined (see, e.g., (2.5)), but it is not always assumed. In particular, if the asset is intrinsically useless, i.e., $d_t = 0$ for all $t \in \mathbb{Z}_+$, then it is more than natural to consider the possibility that $p_t = 0$ for all $t \in \mathbb{Z}_+$. One of our results deals with this particular case without assuming Assumption 2.2; see Proposition 4.2.

We say that a pair of sequences $c = \{c_t\}_{t=0}^{\infty}$ and $s = \{s_t\}_{t=0}^{\infty}$ in $\mathbb{R}$ is a *plan*; a plan (c, s) is *feasible* if it satisfies (2.1) and (2.2); and a feasible plan (c^*, s^*) is *optimal* if there exists no feasible plan (c, s) such that $c^* \prec c$. Whenever we take an optimal plan (c^*, s^*) as given, we assume the following.

Assumption 2.3. For any $c \in C$ with $c^* < c$, we have $c^* \prec c$.

This assumption is satisfied if $\prec$ is strictly monotone in the sense that for any $c, c' \in \mathcal{C}$ with $c < c'$, we have $c \prec c'$. Although this latter requirement is also reasonable, there are important cases in which it is not satisfied; see Section 3. We present some examples of preferences satisfying Assumption 2.3 in Section 3.

2.2 Asset Bubbles

In this subsection we define the fundamental value and the bubble component of the asset by using only the price sequence $\{p_t\}$ and the dividend sequence $\{d_t\}$. For $t \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, we define

$$q_t^i = \prod_{j=t}^{t+i-1} \frac{p_j}{p_{j+1} + d_{j+1}}, \tag{2.5}$$

which can be interpreted as the period t price of one unit of consumption in period $t + i$. We also define $q_t^0 = 1$ for all $t \in \mathbb{Z}_+$. Note that for all $t, i, n \in \mathbb{Z}_+$, we have

$$q_t^i q_{t+i}^n = q_t^{i+n}. \tag{2.6}$$

Let $t \in \mathbb{Z}_+$. Note from (2.5) that $p_t = q_t^1(p_{t+1} + d_{t+1})$. By repeated application of this equation and (2.6), we have

$$p_t = q_t^1 d_{t+1} + q_t^1 p_{t+1} \tag{2.7}$$
$$= q_t^1 d_{t+1} + q_t^2 d_{t+2} + q_t^2 p_{t+2} \tag{2.8}$$
$$\vdots \tag{2.9}$$
$$= \sum_{i=1}^{n} q_t^i d_{t+i} + q_t^n p_{t+n} \quad (\forall n \in \mathbb{N}) \tag{2.10}$$
$$= \sum_{i=1}^{\infty} q_t^i d_{t+i} + \lim_{n\to\infty} q_t^n p_{t+n}. \tag{2.11}$$

To see that both the infinite sum and the limit above exist, note that the finite sum in (2.10) is increasing in $n \in \mathbb{N}$ (as a consequence of Assumptions 2.1 and 2.2). Since the right-hand side of (2.10) equals p_t for all $n \in \mathbb{N}$, it follows that

$$\forall n \in \mathbb{Z}_+, \quad q_t^n p_{t+n} \geq q_t^{n+1} p_{t+n+1}. \tag{2.12}$$

Thus the limits of both terms in (2.10) as $n \uparrow \infty$ exist in $\mathbb{R}$.

Using (2.11), we decompose p_t into two components:

$$p_t = f_t + b_t, \tag{2.13}$$

where f_t and b_t are called the *fundamental value* and the *bubble* component of the asset, which are defined, respectively, as follows:

$$f_t = \sum_{i=1}^{\infty} q_t^i d_{t+i}, \tag{2.14}$$
$$b_t = \lim_{n\uparrow\infty} q_t^n p_{t+n}. \tag{2.15}$$

Note from (2.15) and (2.6) that

$$b_0 = \lim_{i\uparrow\infty} q_0^i p_i = \lim_{i\uparrow\infty : i\geq t} q_0^t q_t^{i-t} p_i = q_0^t \lim_{n\uparrow\infty} q_t^n p_{t+n} = q_0^t b_t. \tag{2.16}$$

Therefore (under Assumptions 2.1 and 2.2)

$$b_0 = 0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ b_t = 0. \tag{2.17}$$

This together with (2.13) implies that

$$p_0 = f_0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ p_t = f_t. \tag{2.18}$$

3 Examples

In this section, we present several examples of (2.2) as well as some examples of preferences that satisfy Assumption 2.3. Many of these examples are used in subsequent sections.

3.1 Constraints on Asset Holdings

The simplest example of (2.2) would be the following:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq 0. \tag{3.1}$$

This constraint is often used in representative-agent models; see, e.g., Lucas (1978) and Kamihigashi (1998).

Kocherlakota (1992) uses a more general version of (3.1):

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma, \tag{3.2}$$

where $\sigma \in \mathbb{R}$. If $\sigma < 0$, then the above constraint is called a short sales constraint.

The following constraint is even more general:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma_t, \tag{3.3}$$

where $\sigma_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Note that (3.2) is a special case of (3.3) with $\sigma_t = \sigma$ for all $t \in \mathbb{Z}_+$.

So far we have only considered inequality constraints on s_t, but other types of constraints are also covered by (2.2). For example, the right-hand side of the budget constraint in (2.1) is the agent's wealth at the beginning of period t; thus it may be reasonable to require it to be nonnegative:

$$\forall t \in \mathbb{N}, \quad (p_t + d_t)s_{t-1} + y_t \geq 0. \tag{3.4}$$

This is clearly an example of (2.2); it is also a special case of (3.3) with

$$\forall t \in \mathbb{Z}_+, \quad \sigma_t = -y_{t+1}/(p_{t+1} + d_{t+1}). \tag{3.5}$$

In addition to (3.2), Kocherlakota (1992) considers the following "wealth constraint":

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t + \sum_{i=1}^{\infty} q_t^i y_{t+i} \geq 0, \tag{3.6}$$

which is another example of (2.2). The left-hand side above is the period t value of the agent's current asset holdings and future income. Note that (3.6) is a special case of (3.3) with

$$\forall t \in \mathbb{Z}_+, \quad \sigma_t = -\sum_{i=1}^{\infty} q_t^i y_{t+i}/p_t. \tag{3.7}$$

See, e.g., Wright (1987) and Huang and Werner (2000) for equivalence relations between different budget constraints.

3.2 Preferences

Example 3.1. A typical objective function in an agent's maximization problem takes the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{3.8}$$

where $\beta \in (0,1)$ and $u : \mathbb{R}_+ \to [-\infty, \infty)$ is a strictly increasing function. Suppose further that u is bounded, and define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad \sum_{t=0}^{\infty} \beta^t u(c_t) < \sum_{t=0}^{\infty} \beta^t u(c'_t). \tag{3.9}$$

Then $\prec$ clearly satisfies Assumption 2.3.

If u is unbounded, i.e., if $u(0) = -\infty$, then the above definition of $\prec$ may not satisfy Assumption 2.3. In particular, given $c^*, c \in \mathcal{C}$ with $c^* < c$, Assumption 2.3 does not hold if $c_t^* = c_t = 0$ for some $t \in \mathbb{Z}_+$ and if u is bounded above. In this case,

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) = \sum_{t=0}^{\infty} \beta^t u(c_t) = -\infty. \tag{3.10}$$

The next example considers an optimality criterion that handles this and other problems.

Example 3.2. For $t \in \mathbb{Z}_+$, let $u_t : \mathbb{R}_+ \to [-\infty, \infty)$ be a strictly increasing function. In this case, the infinite sum $\sum_{t=0}^{\infty} u_t(c_t)$ may not be well defined. Even if it is always well defined, it may not be strictly increasing, as discussed above. To deal with these problems, consider the binary relation $\prec$ defined by

$$c \prec c' \quad \Leftrightarrow \quad \varliminf_{T \uparrow \infty} \sum_{t=0}^{T} [u_t(c_t) - u(c'_t)] < 0, \tag{3.11}$$

where we follow the convention that $(-\infty)-(-\infty) = 0$; see Dana and Le Van (2006) for related optimality criteria. It is easy to see that $\prec$ here satisfies Assumption 2.3.

Continuing with this example, suppose that (2.2) is given by (3.1). Suppose further that each u_t is differentiable on $\mathbb{R}_{++}$, and that there exists an optimal plan (c^*, s^*) such that

$$\forall t \in \mathbb{Z}_+, \quad c^*_t > 0, \; s^*_t = 1. \tag{3.12}$$

Then the standard Euler equation holds for all $t \in \mathbb{Z}_+$:

$$u'_t(c^*_t)p_t = u'_{t+1}(c^*_{t+1})(p_{t+1} + d_{t+1}). \tag{3.13}$$

This together with (2.5) implies that

$$\forall t \in \mathbb{Z}_+, \forall i \in \mathbb{N}, \quad q^i_t = \frac{u'_{t+i}(c^*_{t+i})}{u'_t(c^*_t)}. \tag{3.14}$$

In this case, the fundamental value f_t takes the familiar form:

$$\forall t \in \mathbb{Z}_+, \quad f_t = \sum_{i=1}^{\infty} \frac{u'_{t+i}(c^*_{t+i})}{u'_t(c^*_t)} d_{t+i}. \tag{3.15}$$

Example 3.3. Let $v : \mathcal{C} \to \mathbb{R}$ be a strictly increasing function. Define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad v(c_0, c_1, c_2, \ldots) < v(c'_0, c'_1, c'_2, \ldots). \tag{3.16}$$

Note that (3.16) satisfies Assumption 2.3 without any additional condition on v. For example, v can be a recursive utility function.

4 Implications of Optimality and Feasibility

4.1 No-Bubble Theorem

Given any sequence $\{s_t^*\}_{t=0}^\infty$ in $\mathbb{R}$, $\tau \in \mathbb{Z}_+$, and $\epsilon > 0$, let $\mathcal{S}^{\tau,\epsilon}(s^*)$ be the set of sequences $\{s_t\}_{t=0}^\infty$ in $\mathbb{R}$ such that

$$s_t \begin{cases} = s_t^* & \text{if } t < \tau, \\ \geq s_t^* - \epsilon & \text{if } t \geq \tau. \end{cases} \tag{4.1}$$

We are ready to state our no-bubble theorem.

Theorem 4.1. *Let (c^*, s^*) be an optimal plan. Suppose that there exist $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ such that*

$$\mathcal{S}^{\tau,\epsilon}(s^*) \subset \mathcal{S}(s_{-1}, y, p, d). \tag{4.2}$$

Then $b_0 = 0$.

Proof. See Appendix A. □

It seems remarkable that asset bubbles can be ruled out by such a simple condition. In particular, no explicit utility function is assumed, and the only requirement on the binary relation $\prec$ is Assumption 2.3, which only requires strict monotonicity at the given optimal consumption plan c^*. In Section 5 we show some general equilibrium results using Theorem 4.1 and discuss related results in the literature,

The idea of the proof of Theorem 4.1 is simple. In the proof, we construct an alternative plan as follows. Let $\delta > 0$, and let $s_\tau = s_\tau^* - \delta$, where τ is given by the statement of the theorem. For $t > \tau$, let s_t be given by the budget constraint (2.1) with $c_t = c_t^*$. This alternative plan gives the same consumption sequence except in period τ, where consumption is increased by $p_\tau \delta > 0$. Hence this plan is strictly preferred to the original plan (c^*, s^*). We derive a contradiction by showing that the alternative plan is feasible for sufficiently small $\delta > 0$ provided that $b_0 > 0$.

Similar constructions are used as "Ponzi schemes" in the proofs of Huang and Werner (2000, Theorems 5.1, 6.1), but they are not directly linked to the nonexistence of bubbles. In Subsection 6.4 we show a general version of Huang and Werner's no-bubble theorem (Huang and Werner, 2000, Theorem 6.1) and discuss their result in some detail. We further discuss the proof of Theorem 4.1 in the following subsection.

4.2 Consequences of Theorem 4.1

In this subsection we provide fairly simple consequences of Theorem 4.1 in the current single-agent framework, partly to discuss related results in the literature. We start by presenting an intermediate result shown in the proof of Theorem 4.1. More specifically, it is shown ((A.1)–(A.3)) that if $b_0 > 0$, then

$$\sum_{t=1}^{\infty} \frac{d_t}{p_t} < \infty. \tag{4.3}$$

Since $d_0/p_0 < \infty$, we have the following implication:

$$b_0 > 0 \quad \Rightarrow \quad \sum_{t=0}^{\infty} \frac{d_t}{p_t} < \infty. \tag{4.4}$$

The contrapositive of this result is the following.

Proposition 4.1. *Suppose that*

$$\sum_{t=0}^{\infty} \frac{d_t}{p_t} = \infty. \tag{4.5}$$

Then $b_0 = 0$*.*

Essentially the same result is shown by Montrucchio (2004, Theorem 2) for a fairly general stochastic model using a martingale argument. A similar result is shown by Bosi et al. (2014) for "capital asset bubbles" in a production economy with heterogenous agents. We should emphasize that Proposition 4.1 is independent of the agent's behavior. The result depends only on the price and dividend sequences $\{p_t, d_t\}$ and the definition of b_0 based on them.

The following result assumes that the feasibility constraint on asset holdings (2.2) is given by a sequence of short sales constraints of the form (3.3).

Corollary 4.1. *Let* (c^*, s^*) *be an optimal plan. Suppose that (2.2) is given by (3.3) with* $\sigma_t \in \mathbb{R}$ *for all* $t \in \mathbb{Z}_+$*. Then the following equivalent conclusions hold:*

(a) *If* $\underline{\lim}_{t\uparrow\infty}(s_t^* - \sigma_t) > 0$*, then* $b_0 = 0$*.*

(b) *If* $b_0 > 0$, *then* $\underline{\lim}_{t\uparrow\infty}(s_t^* - \sigma_t) = 0$.

Proof. We verify conclusion (a). Suppose that $\underline{\lim}_{t\uparrow\infty}(s_t^* - \sigma_t) > 0$. Let $\epsilon \in (0, \underline{\lim}_{t\uparrow\infty}(s_t^* - \sigma_t))$. Then there exists $\tau \in \mathbb{Z}_+$ such that $s_t^* - \sigma_t \geq \epsilon$, or $s_t^* - \epsilon \geq \sigma_t$, for all $t \geq \tau$. This implies (4.2). Hence $b_0 = 0$ by Theorem 4.1. □

If there is a constant lower bound on asset holdings s_t, the above result reduces to the following.

Corollary 4.2. *Let* (c^*, s^*) *be an optimal plan. Suppose that (2.2) is given by (3.2) for some* $\sigma \in \mathbb{R}$. *Then the following equivalent conclusions hold:*

(a) *If* $\underline{\lim}_{t\uparrow\infty} s_t^* > \sigma$, *then* $b_0 = 0$.

(b) *If* $b_0 > 0$, *then* $\underline{\lim}_{t\uparrow\infty} s_t^* = \sigma$.

Proof. Both conclusions follow from those of Corollary 4.1 by setting $\sigma_t = \sigma$ for all $t \in \mathbb{Z}_+$. □

In Subsection 6.2 we present some consequences of the above two results in the context of general equilibrium and discuss them in relation to Proposition 3 in Kocherlakota (1992).

The next result considers the case of fiat money, or an asset with no dividend payment. Since the fundamental value of fiat money is zero, its price is also zero if there is no bubble. Hence the case of fiat money is not directly covered by Theorem 4.1, which requires Assumption 2.2,

Proposition 4.2. *Let* (c^*, s^*) *be an optimal plan. Drop Assumption 2.2 but maintain Assumptions 2.1 and 2.3. Suppose that there exist* $\tau \in \mathbb{Z}_+$ *and* $\epsilon > 0$ *satisfying (4.2). Suppose further that*

$$\forall t \geq \tau, \quad d_t = 0. \tag{4.6}$$

Then

$$\forall t \geq \tau, \quad p_t = 0. \tag{4.7}$$

Proof. See Appendix B. □

Finally, we present two results that apply to representative-agent models.

Corollary 4.3. *Suppose that (2.2) is given by (3.1). Let (c^*, s^*) be an optimal plan such that*

$$\forall t \in \mathbb{Z}_+, \quad s_t^* = 1. \tag{4.8}$$

Then $b_0 = 0$.

Proof. Note that (4.8) and (3.1) imply (4.2) with $\tau = 0$ and $\epsilon = 1$. Thus Theorem 4.1 applies. □

The following result is immediate from the above and (3.15).

Proposition 4.3. *In the setup of Example 3.2 (up to (3.15)), we have*

$$p_0 = \sum_{i=1}^{\infty} \frac{u_i'(c_t^*)}{u_0'(c_0^*)} d_t. \tag{4.9}$$

A similar result is shown in Kamihigashi (2001, Section 4.2.1) for a continuous-time model with a nonlinear constraint. It is known that a stochastic version of Corollary 4.3 requires additional assumptions; see Kamihigashi (1998) and Montrucchio and Privileggi (2001).

4.3 Bubbles and Individual Wealth

Kocherlakota (1992) shows that there is a close relation between the possibility of a bubble and an agent's life time wealth, or the present value of his endowment. In this subsection we establish some closely related results. In Subsection 6.3 we present general equilibrium versions of these results and discuss them in relation to Proposition 4 in Kocherlakota (1992).

Theorem 4.2. *Suppose that $b_0 > 0$. Suppose further that there exists a feasible plan (c, s) such that*

$$\varliminf_{t \uparrow \infty} s_t > -\infty. \tag{4.10}$$

Then the following equivalent conclusions hold:

(a) *If $\lim_{T\uparrow\infty} \sum_{t=0}^{T} q_0^t y_t$ exists in $\mathbb{R}$, then $\lim_{t\uparrow\infty} s_t$ exists in $\mathbb{R}$.*

(b) *If $\lim_{t\uparrow\infty} s_t$ does not exist in $\mathbb{R}$, then $\lim_{T\uparrow\infty} \sum_{t=0}^{T} q_0^t y_t$ does not exist in $\mathbb{R}$.*

Proof. See Appendix C. □

If y_t is taken to be the agent's endowment in period t, then it is reasonable to assume that

$$\forall t \in \mathbb{Z}_+, \quad y_t \geq 0. \tag{4.11}$$

This assumption is used in the following result.

Corollary 4.4. *Assume (4.11). Suppose further that $b_0 > 0$, and that there exists a feasible plan (c, s) satisfying (4.10). Then the following equivalent conclusions hold:*

(a) *If $\sum_{t=0}^{\infty} q_0^t y_t < \infty$, then $\lim_{t\uparrow\infty} s_t$ exists in $\mathbb{R}$.*

(b) *If $\lim_{t\uparrow\infty} s_t$ does not exist in $\mathbb{R}$, then $\sum_{t=0}^{\infty} q_0^t y_t < \infty$.*

Proof. Assume (4.11). Then $\lim_{T\uparrow\infty} \sum_{t=0}^{T} q_0^t y_t$ always exists in $[0, \infty]$ and can be written as $\sum_{t=0}^{\infty} q_0^t y_t$. Hence the result follows from Theorem 4.2. □

5 General Equilibrium with Multiple Agents and Multiple Assets

5.1 Feasibility, Optimality, and Equilibrium

Consider an exchange economy with a finite number of infinitely lived agents indexed by $a \in A$, where $A = \{1, 2, \ldots, \overline{a}\}$ with $\overline{a} \in \mathbb{N}$. There are a finite number of assets indexed by $k \in K$, where $K = \{1, 2, \ldots, \overline{k}\}$ with $\overline{k} \in \mathbb{N}$. Agent $a \in A$ faces the following constraints:

$$c_t^a + \sum_{k \in K} p_{k,t} s_{k,t}^a = y_t^a + \sum_{k \in K} (p_{k,t} + d_{k,t}) s_{k,t-1}^a, \; c_t^a \geq 0, \quad \forall t \in \mathbb{Z}_+, \tag{5.1}$$

$$s^a \in \mathcal{S}^a(s_{-1}^a, y^a, p, d), \tag{5.2}$$

where c_t^a and y_t^a are agent a's consumption and endowment in period t; for each $k \in K$, $s_{k,t}^a$ is agent a's holdings of asset k at the end of period t, $p_{k,t}$ is the price of asset k in period t, and $d_{k,t}$ is the dividend payment of asset k in period t. In (5.2), $s_{-1}^a = (s_{k,-1}^a)_{k \in K}$ is agent a's initial holdings of all assets $k \in K$, which are historically given, and $\mathcal{S}(s_{-1}^a, y^a, p, d)$ is a set of sequences in $\mathbb{R}^{\bar{k}}$ with $s^a = \{(s^a_{k,t})_{k\in K}\}_{t=0}^\infty$, $y^a = \{y^a_t\}_{t=0}^\infty$, $p = \{(p_{k,t})_{k\in K}\}_{t=0}^\infty$, and $d = \{(d_{k,t})_{k\in K}\}_{t=0}^\infty$.

The supply of each asset $k \in K$ is given by $\bar{s}_k$ and is constant over time. We assume the following for the rest of the paper.

Assumption 5.1. For each asset $k \in K$ we have

$$\sum_{a\in A} s^a_{k,-1} = \bar{s}_k \geq 0. \tag{5.3}$$

For any $k \in K$, $t \in \mathbb{Z}_+$, and $a \in A$, we have $d_{k,t} \geq 0$ and $y^a_t \geq 0$.

Agent a's preferences are represented by a binary relation $\prec^a$ on $\mathcal{C}$. We say that a pair of sequences $c^a = \{c^a_t\}_{t=0}^\infty$ and $s^a = \{(s^a_{k,t})_{k\in K}\}_{t=0}^\infty$ in $\mathbb{R}$ and $\mathbb{R}^{\bar{k}}$, respectively, is a *plan*; a plan (c, s) is *feasible* for agent a if it satisfies (5.1) and (5.2); and a feasible plan $(\hat{c}^a, \hat{s}^a)$ is *optimal* for agent a if there exists no feasible plan (c^a, s^a) for agent a such that $\hat{c}^a \prec c^a$.

An *equilibrium* of this economy is a set of sequences $(p, \{c^a, s^a\}_{a\in A})$ such that (i) (c^a, s^a) is optimal for each agent $a \in A$, (ii) for each $k \in K$ and $t \in \mathbb{Z}_+$, we have $p_{k,t} \geq 0$, and (iii) the asset and good markets clear in all periods:

$$\sum_{a\in A} s^a_{k,t} = \bar{s}_k, \quad \forall k \in K, \forall t \in \mathbb{Z}_+, \tag{5.4}$$

$$\sum_{a\in A} c^a_t = \bar{y}_t + \sum_{k\in K} \bar{s}_k d_{k,t}. \quad \forall t \in \mathbb{Z}_+, \tag{5.5}$$

where

$$\bar{y}_t = \sum_{a\in A} y^a_t. \tag{5.6}$$

Whenever we take an equilibrium $(p, \{c^a, s^a\}_{a\in A})$ as given, we assume the following.

Assumption 5.2. For any $a \in A$ and $\tilde{c}^a \in C$ with $c^a < \tilde{c}^a$, we have $c^a \prec \tilde{c}^a$.

In addition to equilibria, we often wish to consider allocations that need not be optimal from agents' point of view. For this purpose, we define a *quasi-equilibrium* as a set of sequences $(p, \{c^a, s^a\}_{a\in A})$ such that (a) (c^a, s^a) is feasible for each agent $a \in A$, and (b) conditions (ii) and (iii) above hold.

5.2 Asset Bubbles

We define the fundamental value and the bubble component of an asset $k \in K$ as in Subsection 2.2. For this purpose, fix $k \in K$ for the moment. We assume that

$$\forall t \in \mathbb{Z}_+, \quad p_{k,t} > 0. \tag{5.7}$$

Let $t \in \mathbb{Z}_+$. For $i \in \mathbb{N}$, we define

$$q_{k,t}^i = \prod_{j=t}^{t+i-1} \frac{p_{k,j}}{p_{k,j+1} + d_{k,j+1}}. \tag{5.8}$$

Given this equation, we define $f_{k,t}$ and $b_{k,t}$ for $t \in \mathbb{Z}_+$ as we defined f_t and b_t in Subsection 2.2:

$$f_{k,t} = \sum_{i=1}^{\infty} q_{k,t}^i d_{k,t+i}, \tag{5.9}$$

$$b_{k,t} = \lim_{n \uparrow \infty} q_{k,t}^n p_{k,t+n}. \tag{5.10}$$

The price $p_{k,t}$ of asset k can be decomposed into these two components:

$$p_{k,t} = f_{k,t} + b_{k,t}. \tag{5.11}$$

Although it is reasonable to expect that all assets have the same rate of return in each period (i.e., $q_{k,t}^1 = q_{k',t}^1$ for all $k, k' \in K$), this may or may not be the case depending on the exact specification of (5.2). For example, it is possible that agents may not have access to all assets, in which case the rates of return on assets may not be equalized.

6 General Equilibrium Results

6.1 Restatements of Theorems 4.1 and 4.2

To develop results on asset bubbles in general equilibrium, we start by restating Theorems 4.1 and 4.2 in the current general equilibrium setting. We discuss the results shown here in subsequent subsections.

Theorem 6.1. *Let $(p, \{c^a, s^a\}_{a\in A})$ be an equilibrium. Suppose that there exist an agent $a \in A$ and an asset $k \in K$ satisfying (5.7) such that for some $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$, we have*

$$\mathcal{S}^{\tau,\epsilon}(s^a) \subset \mathcal{S}^a(s^a_{-1}, y^a, p, d). \tag{6.1}$$

Then $b_{k,0} = 0$.

Proof. This follows from Theorem 4.1 because (c^a, s^a) is optimal for agent a and Assumptions 2.1–2.3 hold by Assumptions 5.1 and 5.2 and (5.7). □

To state the next result, we need to introduce an additional definition. Given a feasible plan (c^a, s^a) for agent $a \in A$, for each asset $k \in K$ and $t \in \mathbb{Z}_+$, we define

$$y^a_{k,t} = y^a_t + \sum_{h\in K:h\neq k} (p_{h,t} + d_{h,t}) s^a_{h,t-1} - \sum_{h\in K:h\neq k} p_{h,t} s^a_{h,t}. \tag{6.2}$$

Note that $y^a_{k,t}$ is agent a's net income in period t excluding the income generated by trading in asset k.

Theorem 6.2. *Let $(p, \{c^a, s^a\}_{a\in A})$ be a pseudo-equilibrium. Suppose that there exists an asset $k \in K$ satisfying (5.7) such that $b_{k,0} > 0$. Suppose further that*

$$\forall a \in A, \quad \varliminf_{t\uparrow\infty} s^a_{k,t} > -\infty. \tag{6.3}$$

Then the following equivalent conclusions hold:

(a) *For any $a \in A$ such that $\lim_{T\uparrow\infty} \sum_{t=0}^T q^t_{k,0} y^a_{k,t}$ exists in $\mathbb{R}$, $\lim_{t\uparrow\infty} s^a_{k,t}$ exists in $\mathbb{R}$.*

(b) *For any $a \in A$ such that $\lim_{t\uparrow\infty} s^a_{k,t}$ does not exist in $\mathbb{R}$, $\lim_{T\uparrow\infty} \sum_{t=0}^T q^t_{k,0} y^a_{k,t}$ does not exist in $\mathbb{R}$.*

Proof. This follows from Theorem 4.2 since Assumptions 2.1 and 2.2 hold by Assumption 5.1 and (5.7). □

6.2 Bubbles and Short Sales Constraints

In this subsection we present some results that can be regarded as generalizations of Proposition 3 in Kocherlakota (1992). We discuss his and our results after showing our results. For the rest of the paper we maintain the following assumption.

Assumption 6.1. For each agent $a \in A$, there exists a sequence $\{(\sigma^a_{k,t})_{k\in K}\}^{\infty}_{t=0}$ in $\mathbb{R}^{\bar{k}}$ such that given any sequence $s^a = \{(s^a_{k,t})_{k\in K}\}^{\infty}_{t=0}$ in $\mathbb{R}^{\bar{k}}$, we have

$$s^a_{k,t} \geq \sigma^a_{k,t}, \forall k \in K, \forall t \in \mathbb{Z}_+ \quad \Leftrightarrow \quad s^a \in \mathcal{S}(s^a_{-1}, y^a, p, d). \tag{6.4}$$

This assumption means that the feasibility constraint on asset holdings for each agent, (5.2), consists of sequences of short sales constraints of the form (3.3) for all assets. The following result is a restatement of Corollary 4.1 in the current general equilibrium setting.

Proposition 6.1. *Let $(p, \{c^a, s^a\}_{a\in A})$ be an equilibrium. Suppose that there exists an asset $k \in K$ satisfying (5.7). Then the following equivalent conclusions hold:*

(a) *If there exists an agent $a \in A$ such that $\underline{\lim}_{t\uparrow\infty}(s^a_{k,t} - \sigma^a_{k,t}) > 0$, then $b_{k,0} = 0$.*

(b) *If $b_{k,0} > 0$, then $\underline{\lim}_{t\uparrow\infty}(s^a_{k,t} - \sigma^a_{k,t}) = 0$ for all agents $a \in A$.*

Proof. Conclusion (a) follows from conclusion (a) of Corollary 4.1 applied to the given agent a. (Conclusion (b) follows from conclusion (b) of Corollary 4.1 applied to all agents $a \in A$.) □

To state the next result, we define the following for $a \in A$ and $k \in K$:

$$\overline{\sigma}^a_k = \overline{\lim_{t\uparrow\infty}}\, \sigma^a_{k,t}. \tag{6.5}$$

Corollary 6.1. *Let $(p, \{c^a, s^a\}_{a\in A})$ be an equilibrium. Suppose that there exists an asset $k \in K$ satisfying (5.7). Then the following equivalent conclusions hold:*

(a) *If there exists an agent $a \in A$ such that $\underline{\lim}_{t\uparrow\infty} s^a_{k,t} > \overline{\sigma}^a_{k,t}$, then $b_{k,0} = 0$.*

(b) *If $b_{k,0} > 0$, then $\underline{\lim}_{t\uparrow\infty} s^a_{k,t} \leq \overline{\sigma}^a_{k,t}$ for all agents $a \in A$.*

Proof. We verify conclusion (a). Let $a \in A$ be such that $\underline{\lim}_{t\uparrow\infty} s^a_{k,t} > \overline{\sigma}^a_{k,t}$. This strict inequality implies that $\underline{\lim}_{t\uparrow\infty} s^a_{k,t} > -\infty$ and $\overline{\sigma}^a_{k,t} < \infty$. Hence

$$\underline{\lim}_{t\uparrow\infty}(s^a_{k,t} - \sigma^a_{k,t}) \geq \underline{\lim}_{t\uparrow\infty} s^a_{k,t} - \overline{\sigma}^a_{k,t} > 0. \tag{6.6}$$

Thus $b_{k,0} = 0$ by conclusion (a) of Proposition 6.1. □

If there is a constant lower bounded on asset k for each agent $a \in A$, the above result reduces to the following.

Corollary 6.2. *Let $(p, \{c^a, s^a\}_{a\in A})$ be an equilibrium. Suppose that there exists an asset $k \in K$ satisfying (5.7) such that*

$$\forall a \in A, \exists \sigma^a_k \in \mathbb{R}, \forall t \in \mathbb{Z}_+, \quad \sigma^a_{k,t} = \sigma^a_k. \tag{6.7}$$

Then the following equivalent conclusions hold:

(a) *If there exists an agent $a \in A$ such that $\underline{\lim}_{t\uparrow\infty} s^a_{k,t} > \sigma^a_k$, then $b_{k,0} = 0$.*

(b) *If $b_{k,0} > 0$, then $\underline{\lim}_{t\uparrow\infty} s^a_{k,t} = \sigma^a_k$ for all agents $a \in A$.*

Proof. Under (6.7), both conclusions follow from the conclusions of Proposition 6.1. □

If there is only one asset, the above result further reduces to the following.

Corollary 6.3. *Suppose that $\overline{k} = 1$. Let $(p, \{c^a, s^a\}_{a\in A})$ be an equilibrium satisfying (5.7) and (6.7) with $k = 1$. Then the following equivalent conclusions hold:*

(a) *If there exists an agent $a \in A$ such that $\underline{\lim}_{t\uparrow\infty} s^a_{1,t} > \sigma^a_1$, then $b_{1,0} = 0$.*

(b) *If $b_{1,0} > 0$, then $\underline{\lim}_{t\uparrow\infty} s^a_{1,t} = \sigma^a_1$ for all agents $a \in A$.*

Proof. This result is immediate from Corollary 6.2. □

Kocherlakota (1992, Proposition 3) shows conclusion (b) above under the following additional assumptions: (i) the binary relation $\prec^a$ of each agent $a \in A$ is represented by (3.9) with β and u replaced by $\beta_a \in (0, 1)$ and $u_a : \mathbb{R}_+ \to [-\infty, \infty)$; (ii) u_a is C^1 on $\mathbb{R}_{++}$, strictly increasing, concave, and bounded above or below by zero; and (iii) the optimal plan (c^a, s^a) of each agent $a \in A$ satisfies

$$\forall t \in \mathbb{Z}_+, \quad c_t^a > 0, \tag{6.8}$$

$$\left| \sum_{t=0}^{\infty} (\beta_a)^t u_a(c_t^a) \right| < \infty. \tag{6.9}$$

Corollary 6.3 shows that none of Kocherlakota's additional assumptions is needed under Assumption 5.2, which is implied by his assumptions. Hence Corollary 6.3 is a substantial generalization of Proposition 3 in Kocherlakota (1992).

Corollary 6.2 extends Corollary 6.3 to the case of multiple assets. Corollary 6.1 generalizes Corollary 6.2 by relaxing (6.7). Proposition 6.1 strengthens Corollary 6.1 by offering somewhat sharper conclusions. We obtain these generalizations thanks to Theorem 4.1, which requires none of Kocherlakota's additional assumptions as long as Assumption 5.2 holds. Kocherlakota uses the extra assumptions mostly to derive a transversality condition, which is crucial to his approach but is not needed in ours.

6.3 Bubbles and Individual Wealth

In this subsection we present some results that can be regarded as generalizations of Proposition 4 in Kocherlakota (1992). We discuss his and our results after showing our results. We start by simplifying the conclusions of Theorem 6.2 under an additional assumption.

Corollary 6.4. *Let $(p, \{c^a, s^a\}_{a \in A})$ be a pseudo-equilibrium. Suppose that there exists an asset $k \in K$ satisfying (5.7) and (6.3) such that $b_{k,0} > 0$. Suppose further that*

$$\forall a \in A, \forall t \in \mathbb{Z}_+, \quad y_{k,t}^a \geq 0. \tag{6.10}$$

Then the following equivalent conclusions hold:

(a) *For any $a \in A$ such that $\sum_{t=0}^{\infty} q_{k,0}^t y_{k,t}^a < \infty$, $\lim_{t \uparrow \infty} s_{k,t}^a$ exists in $\mathbb{R}$.*

(b) *For any $a \in A$ such that $\lim_{t \uparrow \infty} s_{k,t}^a$ does not exist in $\mathbb{R}$, $\sum_{t=0}^{\infty} q_{k,0}^t y_{k,t}^a = \infty$.*

Proof. By (6.10), $\lim_{T\uparrow\infty}\sum_{t=0}^{T} q_{k,0}^{t} y_{k,t}^{a}$ exists in $[0,\infty]$ and can be written as $\sum_{t=0}^{\infty} q_{k,0}^{t} y_{k,t}^{a}$. Hence the result follows from Theorem 6.2.[2] □

If there are only one asset and a constant lower bound on each agent's asset holdings, the above result reduces to the following.

Corollary 6.5. *Suppose that* $\overline{k} = 1$. *Let* $(p, \{c^a, s^a\}_{a\in A})$ *be a pseudo-equilibrium satisfying (5.7) and (6.7) with* $k = 1$. *Suppose further that* $b_{1,0} > 0$. *Then the following equivalent conclusions hold:*

(a) *For any* $a \in A$ *such that* $\sum_{t=0}^{\infty} q_{1,0}^{t} y_{t}^{a} < \infty$, $\lim_{t\uparrow\infty} s_{1,t}^{a}$ *exists in* $\mathbb{R}$.

(b) *For any* $a \in A$ *such that* $\lim_{t\uparrow\infty} s_{1,t}^{a}$ *does not exist in* $\mathbb{R}$, $\sum_{t=0}^{\infty} q_{1,0}^{t} y_{t}^{a} = \infty$.

Proof. Since $\overline{k} = 1$, we have $y_{1,t}^{a} = y_{t}^{a} \geq 0$ for all $a \in A$ by Assumption 5.1. Thus (6.10) holds. Note that (6.7) implies (6.3). Hence the result follows from Corollary 6.4. □

Kocherlakota (1992, Proposition 4) shows conclusion (b) above as a property of an equilibrium $(p, \{c^a, s^a\}_{a\in A})$ (rather than a pseudo-equilibrium) under the additional assumptions specified after Corollary 6.3. Corollary 6.5 shows that none of his extra assumptions is needed. We emphasize that Corollary 6.5 does not even require Assumption 5.2.

Corollary 6.4 extends Corollary 6.5 to the case of multiple assets while relaxing (6.7). We obtain these generalizations thanks to Theorem 4.2, which holds for any feasible plan of an agent. In fact, Theorem 4.2 is merely an implication of the sequential budget constraint (2.1). Essentially, the result is shown by multiplying the budget constraint by q_0^t, summing over $t = 0$ to ∞, and taking limits. Hence Theorem 4.2 and its general equilibrium counterpart, Theorem 6.2, are applicable to any model with a similar sequential budget constraint.

6.4 Bubbles and Aggregate Wealth

In this subsection we present some results that can be regarded as generalizations of Theorem 6.1 in Huang and Werner (2000). We discuss their and our results after showing our results.

[2] This proof is essentially the same as that of Corollary 4.4. We include it here because it takes two lines.

Theorem 6.3. *Let $(p, \{c^a, s^a\}_{a \in A})$ be an equilibrium. Suppose that there exists an asset $k \in K$ satisfying (5.7) and (6.10) such that*

$$\sum_{t=0}^{\infty} q_{k,0}^t \left[\overline{y}_t + \sum_{h \in K : h \neq k} \overline{s}_h d_{h,t} \right] < \infty, \tag{6.11}$$

$$-\infty < \sum_{a \in A} \overline{\sigma}_k^a < \overline{s}_k. \tag{6.12}$$

Then $b_{k,0} = 0$.

Proof. See Appendix D. □

If there is only one asset, the above result reduces to the following.

Corollary 6.6. *Suppose that $\overline{k} = 1$. Let $(p, \{c^a, s^a\}_{a \in A})$ be an equilibrium satisfying (5.7) and (6.12) with $k = 1$. Suppose further that*

$$\sum_{t=0}^{\infty} q_{1,0}^t \overline{y}_t < \infty. \tag{6.13}$$

Then $b_{1,0} = 0$.

Proof. Suppose that $\overline{k} = 1$. Then (6.11) reduces to (6.13). In addition, $y_{k,t}^a = y_t^a \geq 0$ for all $a \in A$ and $t \in \mathbb{Z}_+$ by (6.2) and Assumption 5.1. Thus (6.10) holds. Now the result follows from Theorem 6.3. □

Huang and Werner (2000, Theorem 6.1) show the same result under the following additional assumptions: (i) the binary relation $\prec^a$ of each agent $a \in A$ is represented by (3.16) with $\prec$ and v replaced by $\prec^a$ and $v_a : \mathcal{C} \to \mathbb{R}$, respectively; (ii) for each agent $a \in A$, v_a is quasi-concave, nondecreasing, and nonsatiated; and (iii) $\overline{\sigma}_k^a \leq 0$ for all $a \in A$. Corollary 6.6 shows that none of these assumptions is necessary under Assumption 5.2. Furthermore, Theorem 6.3 extends Corollary 6.6 to the case of multiple assets.

The proof of Theorem 6.3 is rather similar to that of Theorem 6.1 in Huang and Werner (2000). Our proof can be outlined as follows: First, (6.11) implies that the present value of each agent's endowment is finite, which in turn implies that each agent's holdings of asset k converge in $\mathbb{R}$ by Corollary 6.4(a). This together with (6.12) implies that there must be at least one agent whose short sales constraint is not binding asymptotically. Then we obtain $b_{k,0} = 0$ by Corollary 6.1(a).

Santos and Woodford (1997, Theorem 3.1) show a closely related result for a general equilibrium model with incomplete markets and possibly infinitely many agents. In particular, they show that no bubble exists on any asset if the present value of the aggregate endowment is finite with respect to any possible state price process. The general structure of their model is considerably more general than ours, and their result is also general in that the preferences of agents are only required to be strictly monotone.[3] In addition, there is no condition corresponding to (6.10) in their result, which rules out bubbles on all assets simultaneously. In our result, we rule out a bubble on asset k alone by evaluating the present value of the aggregate endowment using the "state price process" associated with asset k.

A major advantage of our result over Santos and Woodford's no-bubble theorem is that while their theorem does not apply to an asset in zero net supply, ours easily applies to such an asset. This advantage is in fact inherited from Huang and Werner's (2000, Theorem 6.1) result. As mentioned above, while Huang and Werner's result requires that $\overline{\sigma}_k^a \leq 0$ for all $k \in K$ and $a \in A$, our result does not require this additional assumption.

7 Stochastic Extensions

In this section we discuss how the results in Section 4, especially Theorems 4.1 and 4.2, can be extended to stochastic economies. To be specific, let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Let $\{\mathscr{F}_t\}_{t=0}^\infty$ be a filtration, i.e., an increasing sequence of σ-fields with $\mathscr{F}_t \subset \mathscr{F}$ for all $t \in \mathbb{Z}_+$. We equip $\mathbb{R}$ with its Borel σ-field $\mathscr{B}$. In this section, all sequences that appear in (2.1) are assumed to be adapted to this filtration. This means that any variable with subscript $t \in \mathbb{Z}_+$ is a measurable function from $(\Omega, \mathscr{F}_t)$ to $(\mathbb{R}, \mathscr{B})$, i.e., the realization of the variable is known in period t.

For simplicity, we require the equality and the inequality in (2.1) to hold for each $\omega \in \Omega$ rather than with probability one; we could instead require them to hold with probability one, in which case we could weaken the feasibility requirement. We redefine $\mathcal{C}$ to be the set of nonnegative stochastic processes $\{c_t\}_{t=0}^\infty$ adapted to $\{\mathscr{F}_t\}_{t=0}^\infty$. Likewise, in (2.2), we let $\mathcal{S}(s_{-1}, p, d, y)$ be a set of stochastic processes $\{s_t\}_{t=0}^\infty$ in $\mathbb{R}$ adapted to $\{\mathscr{F}_t\}_{t=0}^\infty$. The defi-

[3] See Kamihigashi (1998, Proposition 3.1) for another result based on the strict monotonicity of preferences. Further results based on stronger assumptions on preferences are available in Santos and Woodford (1997), Huang and Werner (2000), and Werner (2014).

nitions of feasible and optimal plans as well as Assumptions 2.1–2.3 can be modified accordingly. In particular, we assume that $p_t(\omega) > 0$ for all $t \in \mathbb{Z}_+$ and $\omega \in \Omega$.

For $t \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, we define q_t^i, f_t, and b_t as random variables satisfying (2.5), (2.14), and (2.15). To be more precise, for all $\omega \in \Omega$ and $t \in \mathbb{Z}_+$, we define

$$q_t^i(\omega) = \prod_{j=t}^{t+i-1} \frac{p_j(\omega)}{p_{j+1}(\omega) + d_{j+1}(\omega)}, \tag{7.1}$$

$$f_t(\omega) = \sum_{i=1}^{\infty} q_t^i(\omega) d_{t+i}(\omega), \tag{7.2}$$

$$b_t(\omega) = \lim_{n \uparrow \infty} q_t^n(\omega) p_{t+n}(\omega). \tag{7.3}$$

With these definitions, all equations in Subsection 2.2 hold true for each fixed $\omega \in \Omega$. For example, the following equation is valid for each $\omega \in \Omega$ and $t \in \mathbb{Z}_+$.

$$p_t(\omega) = f_t(\omega) + b_t(\omega). \tag{7.4}$$

One can now replicate most of the arguments in the proof of Theorem 4.1 for each fixed $\omega \in \Omega$. In particular, one can construct an alternative plan exactly as in the proof of Theorem 4.1 (i.e., by using (A.4) and (A.5) for some fixed $\delta > 0$). The problem is that this plan may not be feasible. Hence it is necessary to introduce an additional condition to ensure that this alternative plan is feasible for all $\omega \in \Omega$ (for some fixed $\delta > 0$).

On the other hand, the proof of Theorem 4.2 does not use any alternative plan. Thus the conclusion of Theorem 4.2 can be regarded as a sample path property of any feasible plan satisfying (4.10) for each $\omega \in \Omega$.[4] In this sense, Theorem 4.2 can be extended to stochastic economies in a straightforward manner.

We should add however that the results suggested above must be interpreted with care even if they are valid under additional assumptions. This is because none of the variables defined in (7.1)–(7.3) is $\mathcal{F}_t$-measurable; i.e., they all depend on future information. This may not be a problem for $q_t^i(\omega)$ since it can be regarded as a state price of the consumption good in period

[4] See Kamihigashi (2011) for various results on sample path properties of bubbles.

$t+i$. With state prices given by (7.1), the fundamental value and the bubble component of the asset can be defined as the conditional expectations of $f_t(\omega)$ and $b_t(\omega)$ given $\mathcal{F}_t$. To be more specific, it follows from (7.4) that p_t can be decomposed into two components as follows:

$$p_t = \mathbb{E}[f_t|\mathscr{F}_t] + \mathbb{E}[b_t|\mathscr{F}_t], \tag{7.5}$$

where $\mathbb{E}[f_t|\mathscr{F}_t]$ and $\mathbb{E}[b_t|\mathscr{F}_t]$ are the conditional expectations of $f_t(\omega)$ and $b_t(\omega)$ given $\mathcal{F}_t$, respectively. The stochastic versions of Theorems 4.1 and 4.2 suggested above may be useful in obtaining properties of these conditional expectations.

8 Concluding Comments

[beigne] In this paper we showed a simple no-bubble theorem that applies to a wide range of deterministic economies with infinitely lived agents. In particular, we showed that asset bubbles are impossible if there is at least one agent who can reduce his asset holdings permanently from some period onward. This is a substantial generalization of Kocherlakota's (1992) result on asset bubbles and short sales constraints; our result requires virtually no assumption except for the strict monotonicity of preferences. We also provided a substantial generalization of his result on asset bubbles and the present value of a single agent's endowment. As a consequence of these results, we extended Huang and Werner's (2000) no-bubble theorem to an economy with multiple assets. As a possible extension for future research, we discussed how our general results can be extended to stochastic economies.

Although we developed numerous results on asset bubbles in a general equilibrium setting, many of them are solely based on the optimal behavior of a single agent. The exception is our generalization of Huang and Werner's no-bubble theorem, which utilizes market-clearing conditions as well. Most of the other results are implications of optimal behavior without equilibrium consideration. We believe that we demonstrated their usefulness by showing various general equilibrium results based on them.

Appendix A Proofs of Theorem 4.1

Let (c^*, s^*) be an optimal plan. Suppose by way of contradiction that

$$b_0 = \lim_{n\to\infty} q_0^n p_n > 0. \tag{A.1}$$

This together with (2.12) implies that

$$\forall n \in \mathbb{Z}_+, \quad q_0^n p_n \geq b_0. \tag{A.2}$$

Note from (2.11) and (A.1) that $\sum_{n=1}^\infty q_0^n d_n < p_0$. Since $1/p_n \leq q_0^n/b_0$ for all $n \in \mathbb{Z}_+$ by (A.2), it follows that

$$\sum_{n=1}^\infty \frac{d_n}{p_n} \leq \sum_{n=1}^\infty \frac{q_0^n d_n}{b_0} < \frac{p_0}{b_0}, \tag{A.3}$$

where the strict inequality holds by (2.11) and (A.1).

Let $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ be as given by (4.2). Let $\delta \in (0, \epsilon)$. We construct an alternative plan (c^δ, s^δ) as follows:

$$c_t^\delta = \begin{cases} c_t^* & \text{if } t \neq \tau, \\ c_\tau^* + p_\tau \delta & \text{if } t = \tau, \end{cases} \tag{A.4}$$

$$s_t^\delta = \begin{cases} s_t^* & \text{if } t \leq \tau - 1, \\ s_\tau^* - \delta & \text{if } t = \tau, \\ [y_t + (p_t + d_t)s_{t-1}^\delta - c_t^*]/p_t & \text{if } \tau \geq \tau + 1. \end{cases} \tag{A.5}$$

It suffices to show that (c^δ, s^δ) is feasible for $\delta > 0$ sufficiently small; for then, we have $c^* \prec c^\delta$ by (A.4) and Assumption 2.3, contradicting the optimality of (c^*, s^*).

For the rest of the proof, we only consider variables in periods $t \geq \tau$; thus for simplicity we assume without loss of generality that $\tau = 0$. For $t \geq \tau = 0$, define

$$\delta_t = s_t^* - s_t^\delta. \tag{A.6}$$

Note from (2.1), (A.5), and (A.6) that $p_t \delta_t = (p_t + d_t)\delta_{t-1}$ for all $t \in \mathbb{N}$; thus

$$\delta_t = \frac{p_t + d_t}{p_t}\delta_{t-1} = \frac{p_t + d_t}{p_t}\frac{p_{t-1} + d_{t-1}}{p_{t-1}}\delta_{t-2} = \cdots \tag{A.7}$$

$$= \delta \prod_{i=1}^t \frac{p_i + d_i}{p_i} \leq \delta \prod_{i=1}^\infty \frac{p_i + d_i}{p_i}, \tag{A.8}$$

where the equality in (A.8) holds since $\delta_0 = \delta$ by (A.5), and the inequality holds since $d_t \geq 0$ for all $t \in \mathbb{Z}_+$ by Assumption 2.1.

Since (c^δ, s^δ) satisfies (2.1) by construction, to show that (c^δ, s^δ) is feasible, it suffices to verify that $\delta_t \leq \epsilon$ for all $t \in \mathbb{Z}_+$; for then, we have $s \in \mathcal{S}(s_{-1}, y, p, d)$ by (4.2) and (A.6). To this end, note from (A.3) that

$$\frac{p_0}{b_0} > \sum_{i=1}^{\infty} \frac{d_i}{p_i} \geq \sum_{i=1}^{\infty} \ln\left(1 + \frac{d_i}{p_i}\right) \tag{A.9}$$

$$= \sum_{i=1}^{\infty} \ln\left(\frac{p_i + d_i}{p_i}\right) = \ln\left(\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}\right). \tag{A.10}$$

It follows that

$$\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i} < \infty. \tag{A.11}$$

Using this and recalling (A.7) and (A.8), we can choose $\delta > 0$ small enough that $\delta_t \leq \epsilon$ for all $t \in \mathbb{Z}_+$, as desired.

Appendix B Proof of Proposition 4.2

Let $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ be as in (4.2). Without loss of generality, we assume that $\tau = 0$. Suppose by way of contraction that $p_t > 0$ for some $t \geq \tau = 0$. Without loss of generality, we assume that $t = 0$; i.e., $p_0 > 0$.

First suppose that $p_t > 0$ for all $t \in \mathbb{Z}_+$. Then Assumption 2.2 holds. From (4.6) and (2.14) we have

$$\forall t \in \mathbb{Z}_+, \quad f_t = 0. \tag{B.1}$$

By Theorem 4.1 we have $b_0 = 0$. But by (B.1), we obtain $p_0 = 0$, a contradiction.

Next suppose that $p_t = 0$ for some $t \in \mathbb{N}$. Let T be the first $T \in \mathbb{Z}_+$ with

$$p_T > 0, \quad p_{T+1} = 0. \tag{B.2}$$

We construct an alternative plan (c, s) as follows:

$$c_t = \begin{cases} c_t^* & \text{if } t \neq T, \\ c_T^* + p_T \epsilon & \text{if } t = T, \end{cases} \tag{B.3}$$

$$s_t = \begin{cases} s_t^* & \text{if } t \neq T, \\ s_T^* - \epsilon & \text{if } t = T. \end{cases} \tag{B.4}$$

It is easy to see from (4.1), (2.1), and (B.2) that (c, s) is feasible. But we have $c^* \prec c$ by (B.3) and Assumption 2.3, contradicting the optimality of (c^*, s^*).

Appendix C Proof of Theorem 4.2

First we show two lemmas.

Lemma C.1. *For any feasible plan (c, s) and $T \in \mathbb{Z}_+$, we have*

$$\sum_{t=0}^{T} q_0^t c_t + q_0^T p_T s_T = (p_0 + d_0)s_{-1} + \sum_{t=0}^{T} q_0^t y_t. \tag{C.1}$$

Proof. Note from (2.5) that for any $t \in \mathbb{N}$ we have

$$q_0^t(p_t + d_t) = q_0^{t-1} p_{t-1}. \tag{C.2}$$

Let (c, s) be a feasible plan. For each $t \in \mathbb{N}$, multiplying (2.1) through by q_0^t and using (C.2), we have

$$q_0^t c_t + q_0^t p_t s_t = q_0^{t-1} p_{t-1} s_{t-1} + q_0^t y_t. \tag{C.3}$$

Summing over $t = 1, \dots, T$ and adding (2.1) with $t = 0$, we obtain (C.1). □

Lemma C.2. *Suppose that $b_0 > 0$. Let (c, s) be a feasible plan. Then for any subsequence $\{s_{t_i}\}_{i\in\mathbb{N}}$ of $\{s_t\}_{t=0}^\infty$ such that $\lim_{i\uparrow\infty} s_{t_i}$ exists in $\overline{\mathbb{R}}$, we have*

$$\lim_{i\uparrow\infty} q_0^{t_i} p_{t_i} s_{t_i} = b_0 \lim_{i\uparrow\infty} s_{t_i}. \tag{C.4}$$

Proof. Suppose that $b_0 > 0$. Let (c, s) be a feasible plan. Let $\{s_{t_i}\}_{i\in\mathbb{N}}$ be a subsequence of $\{s_t\}_{t=0}^\infty$ such that $\lim_{i\uparrow\infty} s_{t_i}$ exists in $\overline{\mathbb{R}}$. Note from (2.16) that

$$\lim_{i\uparrow\infty} q_0^{t_i} p_{t_i} = b_0 > 0. \tag{C.5}$$

Thus (C.4) is immediate if $\lim_{i\uparrow\infty} s_{t_i} \in \mathbb{R}$. If $\lim_{i\uparrow\infty} s_{t_i} = \infty$ (resp. $-\infty$), then $\lim_{i\uparrow\infty} q_0^{t_i} p_{t_i} s_{t_i} = \infty$ (resp. $-\infty$) by (C.5). Thus (C.4) holds in all cases. □

To prove Theorem 4.2, suppose that $b_0 > 0$, and let (c, s) be a feasible plan satisfying (4.10). Suppose that the following limit exists in $\mathbb{R}$:

$$\lim_{T\uparrow\infty} \sum_{t=0}^{T} q_0^t y_t \in \mathbb{R}. \tag{C.6}$$

Recall from (2.16) that $b_0 = \lim_{T\uparrow\infty} q_0^T p_T$. Let $\{s_{T_i}\}_{i\in\mathbb{N}}$ be any subsequence of $\{s_T\}_{T\in\mathbb{N}}$ that converges in $(-\infty, \infty]$. By Lemmas C.1 and C.2 and (C.6), we have

$$\sum_{t=0}^{\infty} q_0^t c_t + b_0 \lim_{i\uparrow\infty} s_{T_i} = (p_0 + d_0)s_{-1} + \sum_{t=0}^{\infty} q_0^t y_t, \tag{C.7}$$

where $\sum_{t=0}^{\infty} q_0^t y_t = \lim_{T\uparrow\infty} \sum_{t=0}^{T} q_0^t y_t$. Since the right-hand side of (C.7) is finite by (C.6), it follows from (C.7) and (4.10) that

$$\sum_{t=0}^{\infty} q_0^t c_t < \infty, \quad -\infty < \lim_{i\uparrow\infty} s_{T_i} < \infty. \tag{C.8}$$

From (C.7) and (C.8) we have

$$b_0 \lim_{i\uparrow\infty} s_{T_i} = (p_0 + d_0)s_{-1} + \sum_{t=0}^{\infty} q_0^t y_t - \sum_{t=0}^{\infty} q_0^t c_t. \tag{C.9}$$

Since the right-hand side does not depend on the subsequence $\{s_{T_i}\}_{i\in\mathbb{N}}$, it follows that $\lim_{T\uparrow\infty} s_T$ exists. This limit belongs to $\mathbb{R}$ by C.8.

D Proof of Theorem 6.3

Throughout the proof, we take an equilibrium $(p, \{c^a, s^a\}_{a\in A})$ as given, and assume that there exists an asset $k \in K$ satisfying (5.7), (6.10), and (6.11) such that $b_{k,0} > 0$. We show two lemmas.

Lemma D.1. *We have*

$$\forall a \in A, \quad \sum_{t=0}^{\infty} q_{k,0}^t y_{k,t}^a < \infty. \tag{D.1}$$

Proof. Note from (6.2) that for any $t \in \mathbb{Z}_+$ we have

$$\sum_{a\in A} y^a_{k,t} = \sum_{a\in A}\left[y^a_t + \sum_{h\in K:h\neq k}(p_{h,t}+d_{h,t})s^a_{h,t-1} - \sum_{h\in K:h\neq k} p_{h,t}s^a_{h,t}\right] \tag{D.2}$$

$$= \sum_{a\in A} y^a_t + \sum_{a\in A}\sum_{h\in K:h\neq k}(p_{h,t}+d_{h,t})s^a_{h,t-1} - \sum_{a\in A}\sum_{h\in K:h\neq k} p_{h,t}s^a_{h,t} \tag{D.3}$$

$$= \overline{y}_t + \sum_{h\in K:h\neq k}(p_{h,t}+d_{h,t})\sum_{a\in A} s^a_{h,t-1} - \sum_{h\in K:h\neq k} p_{h,t}\sum_{a\in A} s^a_{h,t} \tag{D.4}$$

$$= \overline{y}_t + \sum_{h\in K:h\neq k}(p_{h,t}+d_{h,t})\overline{s}_h - \sum_{h\in K:h\neq k} p_{h,t}\overline{s}_h \tag{D.5}$$

$$= \overline{y}_t + \sum_{h\in K:h\neq k} d_{h,t}\overline{s}_h, \tag{D.6}$$

where (D.5) uses (5.4). Multiplying through by $q^t_{k,0}$ and summing over $t = 0$ to T, we obtain

$$\sum_{t=0}^{T} q^t_{k,0}\left[\overline{y}_t + \sum_{h\in K:h\neq k} d_{h,t}\overline{s}_h\right] \tag{D.7}$$

$$= \sum_{t=0}^{T} q^t_{k,0}\sum_{a\in A} y^a_{k,t} = \sum_{t=0}^{T}\sum_{a\in A} q^t_{k,0}y^a_{k,t} = \sum_{a\in A}\sum_{t=0}^{T} q^t_{k,0}y^a_{k,t}. \tag{D.8}$$

Applying $\lim_{T\uparrow\infty}$ we have

$$\lim_{T\uparrow\infty}\sum_{t=0}^{T} q^t_{k,0}\left[\overline{y}_t + \sum_{h\in K:h\neq k} d_{h,t}\overline{s}_h\right] = \sum_{a\in A}\lim_{T\uparrow\infty}\sum_{t=0}^{T} q^t_{k,0}y^a_{k,t} \tag{D.9}$$

$$= \sum_{a\in A}\sum_{t=0}^{\infty} q^t_{k,0}y^a_{k,t}, \tag{D.10}$$

where the last equality holds by (6.10). Since the left-hand side of (D.9) is finite by (6.11), we obtain (D.1). □

Lemma D.2. *We have (6.3).*

Proof. Note from (5.9) and (5.11) that

$$\sum_{t=1}^{\infty} q^t_{k,0}d_{k,t} \le p_{k,0} < \infty. \tag{D.11}$$

From this and (6.11), we have

$$\infty > \sum_{t=0}^{\infty} q_{k,0}^t \left[\overline{y}_t + \sum_{h \in K : h \neq k} \overline{s}_h d_{h,t} \right] + \sum_{t=0}^{\infty} q_{k,0}^t \overline{s}_k d_{k,t} \tag{D.12}$$

$$= \sum_{t=0}^{\infty} q_{k,0}^t \left[\overline{y}_t + \sum_{h \in K} \overline{s}_h d_{h,t} \right] = \sum_{t=0}^{\infty} q_{k,0}^t \sum_{a \in A} c_t^a \tag{D.13}$$

$$= \sum_{t=0}^{\infty} \sum_{a \in A} q_{k,0}^t c_t^a = \sum_{a \in A} \sum_{t=0}^{\infty} q_{k,0}^t c_t^a, \tag{D.14}$$

where the second equality in (D.13) uses (5.5). It follows that

$$\forall a \in A, \quad \sum_{t=0}^{\infty} q_{k,0}^t c_t^a < \infty. \tag{D.15}$$

By Lemmas C.1 and C.2, we have

$$b_{k,0} \varliminf_{T \uparrow \infty} s_{k,T} = (p_{k,0} + d_{k,0}) s_{k,-1} + \sum_{t=0}^{\infty} q_{k,0}^t y_t^a - \sum_{t=0}^{\infty} q_{k,0}^t c_t^a. \tag{D.16}$$

Since the right-hand side is finite by Lemma D.1 and (D.15), we obtain (6.3). □

To complete the proof of Theorem 6.3, assume (6.12). By Lemma D.2 and Corollary 6.4(a) (or by the proof of Lemma D.2), $\lim_{t \uparrow \infty} s_{k,t}^a$ exists in $\mathbb{R}$ for each $a \in A$. Note from (6.12) and (5.4) that

$$\sum_{a \in A} \overline{\sigma}_k^a < \overline{s}_k = \lim_{t \uparrow \infty} \sum_{a \in A} s_{k,t}^a = \sum_{a \in A} \lim_{t \uparrow \infty} s_{k,t}^a. \tag{D.17}$$

Hence there exists at least one agent $a \in A$ with $\lim_{t \uparrow \infty} s_{k,t}^a > \overline{\sigma}_k^a$. Thus by Corollary 6.1(a), we obtain $b_{k,0} = 0$.

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