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Takashi KAMIHIGASHI

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Research Institute for Economics and Business Administration

Kobe University

2-1 Rokkodai, Nada, Kobe 657-8501 JAPAN

Multiple Interior Steady States in the Ramsey Model with Elastic Labor Supply*

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Abstract

In this paper, we show that multiple interior steady states are possible in the Ramsey model with elastic labor supply. In particular, we establish the following three results: (i) for any discount factor and production function, there is a utility function such that a continuum of interior steady states exist; (ii) the number of interior steady states can also be any finite number; and (iii) for any discount factor and production function, there is a utility function such that there is no interior steady state. Some numerical examples are provided.

Keywords: multiple steady states, Ramsey model, elastic labor supply, neoclassical growth

JEL Classification: C61, C62, E13, O41

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†RIEB, Kobe University, Rokkodai, Nada, Kobe 657-8501 Japan.
Email: tkamihig@rieb.kobe-u.ac.jp

‡IPAG Business School, 184 Bd Saint Germain, 75006 Paris, France.

1 Introduction

One-sector neoclassical optimal growth models have been used extensively to address two major issues in macroeconomics: long-run growth and short-run fluctuations. A special case of this class of models with inelastic labor supply is known as the Ramsey model, and one of its important properties is that all optimal paths converge to a unique steady state. This convergence property has various cross-country implications, on which there is a large empirical literature (e.g., Barro and Sala-i-Martin, 2004). Stochastic versions of the Ramsey model with elastic labor supply are known as RBC (real business cycle) models (e.g., King et al., 1988; Christiano and Eichenbaum, 1992), which are in turn considered to be prototype DSGE (dynamic stochastic general equilibrium) models. These models are often solved by linear approximation around a steady state, which is typically assumed to be unique.

In this paper, we study the possibility of multiple steady states in the Ramsey model with elastic labor supply. Despite the importance of this model, to our knowledge, there has been no systematic analysis of this problem in the literature though there are various parametric models whose steady states are solved explicitly.[1]

The literature on multiple steady states in one-sector neoclassical growth models dates back to Diamond (1965), who considered an overlapping generations model; see Galor and Ryder (1989) for subsequent results. More closely related to this paper is the analysis of multiple steady states in the context of optimal growth, dating back to Kurz (1968). He showed that multiple steady states are possible if utility is a function of consumption and capital.

In this paper, we show that multiple interior steady states are possible if utility is a function of consumption and leisure, and if leisure is not a normal good. In particular, we establish the following three results: (i) for any discount rate and production function (satisfying standard conditions), there is a utility function such that a continuum of interior steady states exist; (ii) the number of interior steady states can also be any finite number;

[1] In the case of inelastic labor supply, the aforementioned convergence property (which implies the existence of a unique steady state) was established by Cass (1965) and Koopmans (1965) for the continuous time case, and by Brock and Mirman (1972, Section 2) for the discrete time case. Le Van et al. (2007) showed a convergence property for the case of elastic labor supply with heterogenous agents. Various convergence results on general optimal growth models are also available; see Yano (1999, 2012) and the references therein.

and (iii) for any discount factor and production function, there is a utility function such that there is no interior steady state.

In a wider context, the connection between multiplicity of steady states and non-normality of goods is already well recognized; see, e.g., De Hek (1998) and Bond et al. (2012, 2014). In particular, De Hek's work is closest to ours in the literature; he showed through numerical examples that multiple steady states and non-monotone dynamics are possible in a closely related model. His model differs from ours in that current output is assumed to be a function of the levels of capital and labor chosen in the previous period.[2] However, the difference is insignificant as far as steady states are concerned; thus our results apply to his model with minor modifications. Although it may seem reasonable to rule out multiple steady states by assuming that leisure is a normal good (e.g., Bosi et al., 2005), there is some empirical evidence that suggests backward bending labor supply (e.g., Keane, 2011), which implies that leisure is not entirely a normal good.

The rest of the paper is organized as follows. In the next section, we present the model. In Section 3, we show some preliminary results. In Section 4, we establish our main results. In Section 5, we offer some numerical examples. All proofs appear in the Appendix unless otherwise indicated.

2 The Model

Consider the following one-sector optimal growth model:

$$
\begin{aligned}
\max_{\{c_t, n_t, k_{t+1}\}_{t=0}^{\infty}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, n_t) && (1)\\
\text{s.t. } \forall t \in \mathbb{Z}_+, \quad & c_t + k_{t+1} = f(k_t, n_t), && (2)\\
& 0 \le n_t \le 1, && (3)\\
& c_t, k_{t+1} \ge 0, && (4)\\
& k_0 > 0 \text{ given}, && (5)
\end{aligned}
$$

where c_t is consumption, n_t is labor ($1-n_t$ is leisure), k_t is capital, $\beta \in (0,1)$ is the discount factor, u is the utility function, and f is the production function. We state our assumptions after introducing some definitions.

[2] In our notation, this means that the right-hand side of (2) is $f(k_t, n_{t-1})$. See Cai et al. (2014) for a stochastic version of this specification.

A *feasible plan* (*from* $\overline{k}_0$) is a set of paths $\{c_t, n_t, k_t\}_{t=0}^{\infty}$ satisfying (2)–(4) (with $k_0 = \overline{k}_0$). An *interior plan* is a feasible plan such that none of the inequality constraints in (3) and (4) is binding for any $t \in \mathbb{Z}_+$. An *optimal plan* is a feasible plan that solves the maximization problem (1)–(5). An *interior optimal plan* is an optimal plan that is also an interior plan.

Throughout the paper, we maintain the following assumptions.

Assumption 1. (f1) $f : \mathbb{R}_+^2 \to \mathbb{R}_+$ is C^1 on $\mathbb{R}_{++}^2$, concave, strictly concave in the first argument, and linearly homogeneous. (f2) For all $(k, n) \in \mathbb{R}_{++}^2$, we have

$$f_1(k, n) > 0, \quad f_2(k, n) > 0. \tag{6}$$

(f3) There exists $\overline{k} > 0$ such that

$$f(\overline{k}, 1) = \overline{k}. \tag{7}$$

(f4) We have $k_0 \in (0, \overline{k}]$. (f5) We have

$$\lim_{k \downarrow 0} f_1(k, 1) > 1/\beta > \lim_{k \uparrow \infty} f_1(k, 1). \tag{8}$$

Since $f(k, 1)$ is strictly concave in k by (f1), condition (f4) implies that for any feasible plan $\{c_t, n_t, k_t\}$ from $k_0 \le \overline{k}$, we have

$$\forall t \in \mathbb{Z}_+, \quad c_t, k_t \le \overline{k}. \tag{9}$$

We restrict the domain of u accordingly:

Assumption 2. (u1) There exists $\overline{c} > \overline{k}$ such that $u : [0, \overline{c}] \times [0, 1] \to \mathbb{R} \cup \{-\infty\}$ is C^1 on $(0, \overline{c}) \times (0, 1)$ and strictly concave. (u2) For all $(c, n) \in (0, \overline{c}) \times (0, 1)$, we have

$$u_1(c, n) > 0, \quad u_2(c, n) < 0. \tag{10}$$

It is easy to see that any interior optimal plan satisfies the following conditions for all $t \in \mathbb{Z}_+$:

$$-u_1(f(k_t, n_t) - k_{t+1}, n_t) + \beta u_1(f(k_{t+1}, n_{t+1}) - k_{t+2}, n_{t+1}) f_1(k_{t+1}, n_{t+1}) = 0, \tag{11}$$

$$u_1(f(k_t, n_t) - k_{t+1}, n_t) f_2(k_t, n_t) + u_2(f(k_t, n_t) - k_{t+1}, n_t) = 0. \tag{12}$$

Equation (11) is the Euler equation for k_{t+1}, and equation (12) is the first-order condition for n_t. The transversality condition is

$$\lim_{t\uparrow\infty} \beta^t u_1(f(k_t, n_t) - k_{t+1}, n_t)k_{t+1} = 0. \tag{13}$$

Since u and f are concave, (11)–(13) are sufficient for optimality.[3]

We define an *interior steady state* as a pair $(k, n) \in (0, \overline{k}) \times (0, 1)$ such that

$$\beta f_1(k, n) = 1, \tag{14}$$

$$u_1(f(k, n) - k, n)f_2(k, n) + u_2(f(k, n) - k, n) = 0, \tag{15}$$

$$f(k, n) - k > 0. \tag{16}$$

For any interior steady state (k, n), a feasible plan $\{c_t, n_t, k_t\}$ with $k_t = k$ and $n_t = n$ for all $t \in \mathbb{Z}_+$ is an interior optimal plan since it satisfies (11)–(13) with $c_t = f(k, n) - k > 0$.

3 Preliminary Results

In this section, we define some constants, and provide simple characterizations of interior steady states.

Lemma 1. *There exits a unique constant $\gamma > 0$ such that for any $(k, n) \in \mathbb{R}^2_{++}$, we have*

$$\beta f_1(k, n) = 1 \quad \Longleftrightarrow \quad k = \gamma n. \tag{17}$$

Lemma 1 implies that any interior steady state (k, n) satisfies $k = \gamma n$; i.e., the capital-labor ratio is constant across all possible interior steady states. This allows us to focus on levels of labor in characterizing interior steady states. We say that $n \in (0, 1)$ is a *steady state level of labor* if the pair (k, n) with $k = \gamma n$ is an interior steady state.

We define the following constants:

$$\mu = f_2(\gamma, 1) > 0, \tag{18}$$

$$\lambda = f(\gamma, 1) - \gamma, \tag{19}$$

$$\rho = \frac{1}{\beta} - 1 > 0. \tag{20}$$

[3] See Kamihigashi (2005) for discussion on the transversality condition.

The inequality in (18) is immediate from (6). By linear homogeneity, for all $n > 0$, we have

$$\mu = f_2(\gamma n, n), \tag{21}$$

$$\lambda n = f(\gamma n, n) - \gamma n. \tag{22}$$

Thus μ and λ are the marginal product of labor and the consumption-labor ratio, respectively, in any interior steady state. Differentiating (22) with respect to n and using (17) and (22), we obtain[4]

$$\lambda = \mu + \rho\gamma > 0. \tag{23}$$

The following result shows that the steady state levels of labor are characterized by a single equation.

Lemma 2. *Let $n \in (0, 1)$. Then n is a steady state level of labor if and only if*

$$g(n) \equiv u_1(\lambda n, n)\mu + u_2(\lambda n, n) = 0. \tag{24}$$

The proofs of the following two corollaries are straightforward and thus omitted.

Corollary 1. *There exists an interior steady state if (25) or (26) below holds:*

$$\lim_{n \downarrow 0} g(n) < 0 < \lim_{n \uparrow 1} g(n), \tag{25}$$

$$\lim_{n \downarrow 0} g(n) > 0 > \lim_{n \uparrow 1} g(n). \tag{26}$$

Corollary 2. *Suppose that there exist C^1 functions $v : (0, \bar{c}) \to \mathbb{R} \cup \{\infty\}$ and $w : (0, 1) \to \mathbb{R} \cup \{-\infty\}$ such that*

$$u(c, n) = v(c) + w(n). \tag{27}$$

Then there can be at most one interior steady state.

To obtain another useful characterization of interior steady states, consider the following maximization problem:

$$\max_{c \geq 0, n \in [0,1]} u(c, n) \quad \text{s.t.} \;\; c - \mu n = y, \tag{28}$$

[4] $\lambda = f_1(\gamma n, n)\gamma + f_2(\gamma n, n) - \gamma = \gamma/\beta + \mu - \gamma = \rho\gamma + \mu$.

where $y \in (0, \overline{k})$. Since u is strictly concave, this problem has a unique solution, which we denote by $(c^*(y), n^*(y))$. Provided that $n \in (0,1)$, we have $n = n^*(y)$ if and only if

$$u_1(y + \mu n, n)\mu + u_2(y + \mu n, n) = 0. \tag{29}$$

Exploring the similarity between (24) and (29), we obtain the following:

Lemma 3. *Let $n \in (0,1)$. Then n is a steady state level of labor if and only if*

$$n = n^*(\rho\gamma n). \tag{30}$$

Since μ is the marginal product of labor in any interior steady state, μ can be interpreted as a steady state wage rate. Hence the constraint in (28) can be viewed as a budget constraint under this wage rate given non-labor income y. We say that *consumption is a normal (inferior) good* if $c^*(y)$ is strictly increasing (decreasing) in y whenever $n^*(y) \in (0,1)$. We say that *leisure is a normal (inferior) good* if $n^*(y)$ is strictly decreasing (increasing) in y whenever $n^*(y) \in (0,1)$.

4 Main Results

The properties of consumption and leisure defined above are closely related to the possibility of multiple interior steady states, as we now see:

Proposition 1. *If leisure is a normal good, then there can be at most one interior steady state.*

If consumption is an inferior good, then leisure must be a normal good. Thus the following result is immediate from Proposition 1.

Corollary 3. *If consumption is an inferior good, then there can be at most one interior steady state.*

Proposition 1 implies that there can be multiple interior steady states only if leisure is not a normal good. The following result shows that there can even be a continuum of interior steady states.

Proposition 2. *For any discount factor $\beta \in (0,1)$ and production function f satisfying Assumption 1, there exists a utility function u satisfying Assumption 2 such that any $n \in (0,1)$ is a steady state level of labor.*

The number of interior steady states can also be any finite number:

Proposition 3. *Let $m \in \mathbb{N}$ and $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$. Then for any discount factor $\beta \in (0,1)$ and production function f satisfying Assumption 1, there exists a utility function u satisfying Assumption 2 such that $n \in (0,1)$ is a steady state level of labor if and only if $n \in \{\eta_1, \ldots, \eta_m\}$.*

Finally, it is even possible that no interior steady state exists:

Proposition 4. *For any discount factor $\beta \in (0,1)$ and production function f satisfying Assumptions 1, there exists a utility function u satisfying Assumptions 2 such that there exists no interior steady state.*

5 Numerical Examples

In this section, we provide some numerical examples to illustrate our results. We assume that u is bounded below, so that we can rely on dynamic programming without technical concerns. The Bellman equation for the maximization problem (1)–(5) can be written as

$$v(k) = \max_{c,n,x} \{u(c,n) + \beta v(x) : c + x = f(k,n), n \in [0,1], c, x \geq 0\}, \tag{31}$$

where $k \in [0, \overline{k}]$ and $v : [0, \overline{k}] \to \mathbb{R}$ is the value function. We fix f and β as follows:

$$f(k,n) = k^{\alpha} n^{1-\alpha}, \quad \alpha = 0.4, \quad \beta = 0.95. \tag{32}$$

Under this specification, we have

$$\gamma = (\alpha\beta)^{1/(1-\alpha)} \approx 0.245,\ \mu = (1-\alpha)\gamma^{\alpha} \approx 0.172,\ \lambda \approx 0.185,\ \overline{k} = 1. \tag{33}$$

In what follows, the value function and the associated policy functions are computed for three different utility functions.[5] Although we can use $g(n)$ in (24) to compute interior steady states, we wish to confirm our results by using a method that does not use $g(n)$. In addition, dynamic programming helps us understand other aspects of the model.

[5] All of these functions are computed by modified policy iteration (e.g., Puterman, 2005) with 10,000 equally spaced grid points.

The first utility function that we consider is

$$u(c,n) = c^{0.4} - 0.2n^{1.2}. \tag{34}$$

Figure 1(a) depicts this utility function as a function of consumption and leisure $(1-n)$. By Corollary 2, there can be at most one interior steady state under (34). This is confirmed by Figure 1(c), which shows that there is a unique interior steady state.

The second utility function that we consider is

$$u(c,n) = \left[-\frac{(c-\sigma)^2}{2} + \nu\left(c - \frac{\sigma\mu}{\nu}\right)n - \frac{\tau^2}{2}n^2 - \zeta\right]^{1/2}, \tag{35}$$

where

$$\tau = \frac{\mu+\lambda}{2} \approx 0.179, \quad \nu = \frac{\tau^2+\lambda\mu}{\lambda+\mu} \approx 0.178, \quad \sigma = \frac{1.2\overline{k}\nu}{\mu} \approx 1.244, \tag{36}$$

and ζ is chosen so that $u(0,1)=0$. This function satisfies Assumption 2, as can be seen from Figure 2(a). The utility function constructed in the proof of Proposition 2 is the expression inside the square brackets in (35). As in the proof, the parameter values are chosen so that any $n \in (0,1)$ solves (24); i.e., any $n \in (0,1)$ is a steady state level of labor. This is confirmed by Figure 2(c),(e).[6]

The last utility function that we consider is

$$u(c,n) = \left[-\frac{(c-\sigma)^2}{2} + \nu\left(c - \frac{\sigma\mu}{\nu}\right)^{*} n - \frac{\tau^2}{2}n^2 - \zeta\right]^{1/2} - \epsilon(n-1), \tag{37}$$

where $\epsilon = 0.01$. This is a perturbed version of (35); all parameter values are as above. Since (24) holds for all $n \in (0,1)$ under (35), there is no $n \in (0,1)$ satisfying (24) under (37); i.e., there is no interior steady state. This is confirmed by Figure 3(c).

[6]Numerical results based on (35) and the proof of Proposition 3 are not reported here since they are visually almost indistinguishable from those in Figure 2.

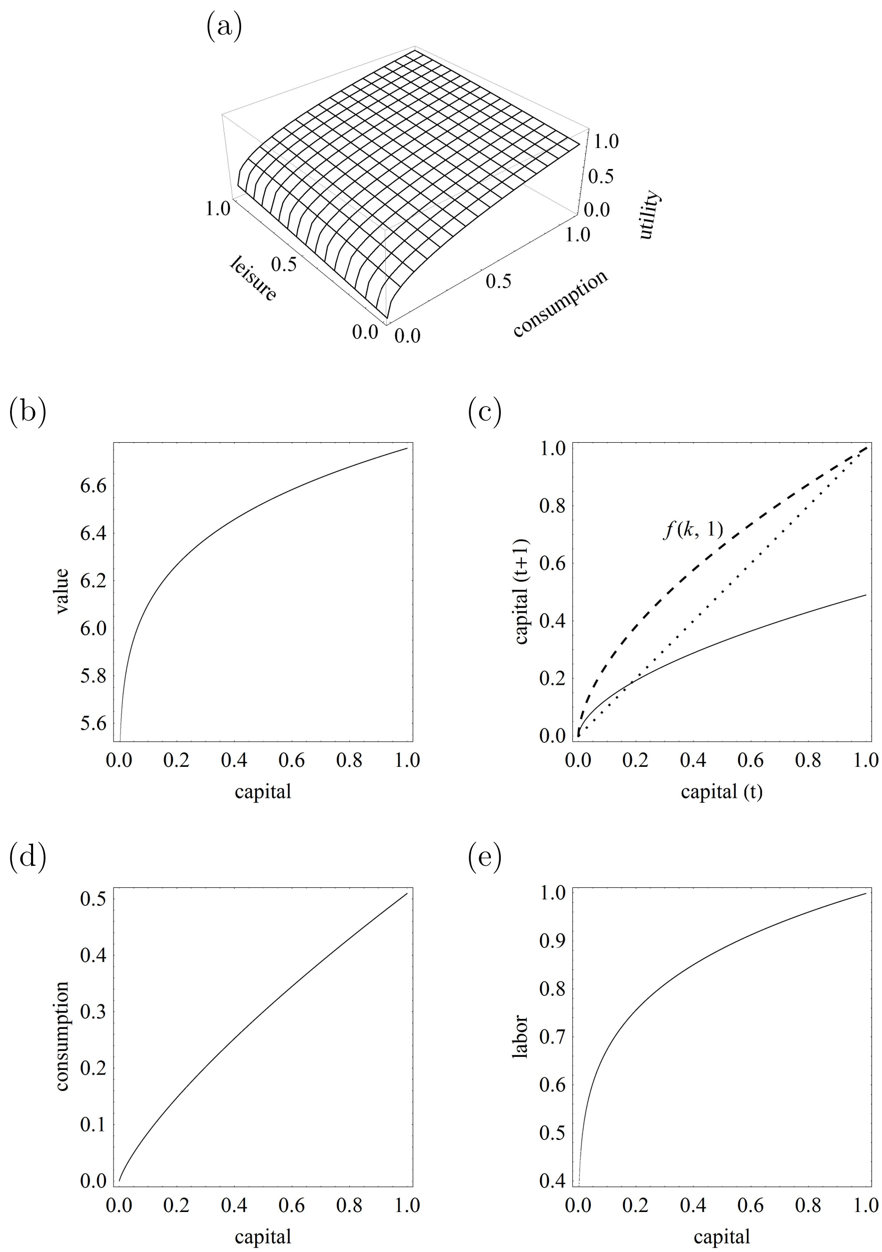

Figure 1: Functions computed under (34): (a) utility function $u(c, 1-l)$; (b) value function $v(k)$; and optimal policies for (c) capital (k_{t+1}), (d) consumption (c_t), and (e) labor (n_t) as functions of current capital k_t

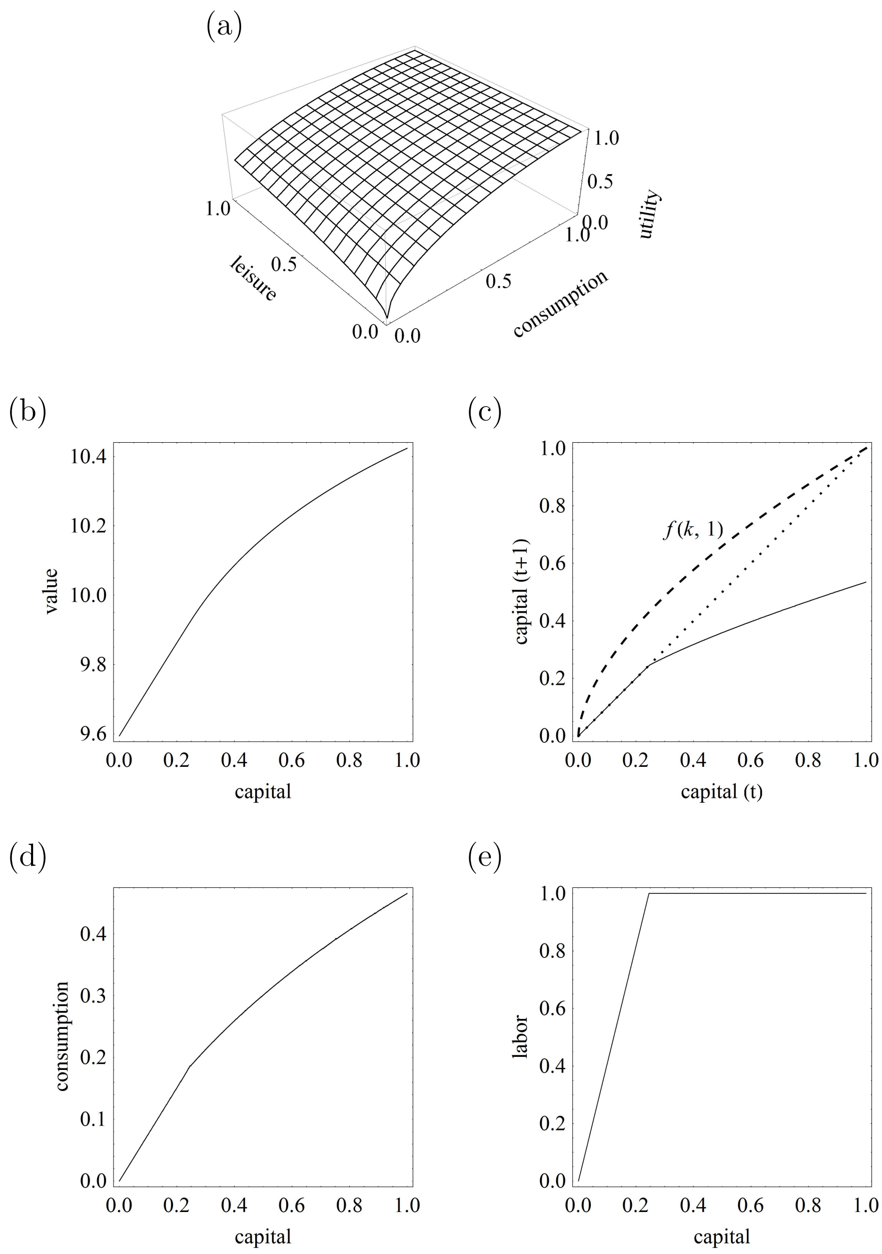

Figure 2: Functions computed under (35): (a) utility function $u(c, 1-l)$; (b) value function $v(k)$; and optimal policies for (c) capital (k_{t+1}), (d) consumption (c_t), and (e) labor (n_t) as functions of current captial k_t

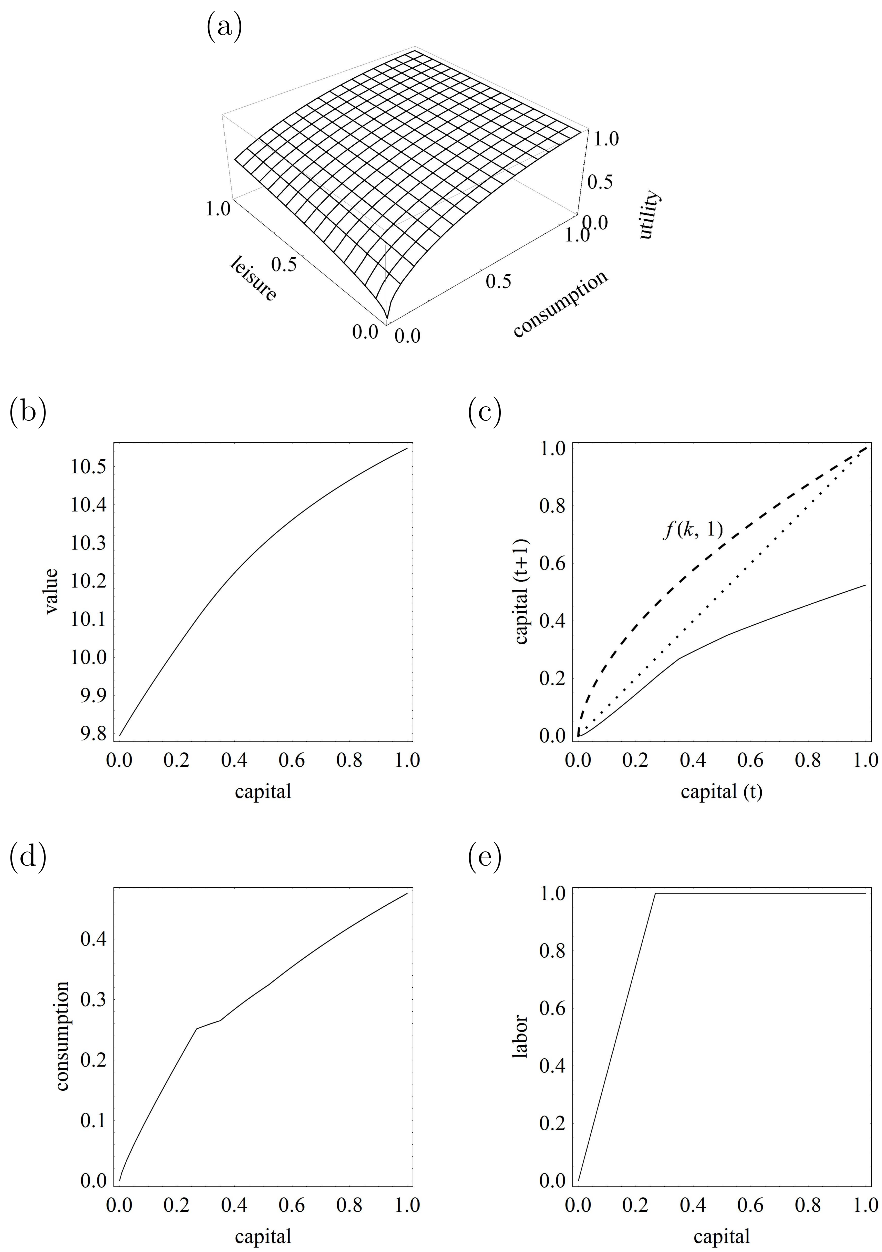

Figure 3: Functions computed under (37): (a) utility function $u(c, 1-l)$; (b) value function $v(k)$; and optimal policies for (c) capital (k_{t+1}), (d) consumption (c_t), and (e) labor (n_t) as functions of current captial k_t

A Appendix: Proofs

A.1 Proof of Lemma 1

By Assumption 1(f5), there exists $\gamma > 0$ such that $\beta f_1(\gamma, 1) = 1$. By linear homogeneity, we have

$$\forall n > 0, \qquad \beta f_1(\gamma n, n) = 1. \tag{38}$$

We have shown the "$\Leftarrow$" implication in (17).

To verify the reverse implication, let $(k', n') \gg 0$ be such that $\beta f_1(k', n') = 1$. Let $\gamma' = k'/n'$. To show that $\gamma' = \gamma$, define $m(\tilde{\gamma}) = f_1(\tilde{\gamma}n', n')$ for $\tilde{\gamma} > 0$. Then $m(\gamma') = f_1(k', n') = 1/\beta = m(\gamma)$, where the last equality holds by (38). Since f is strictly concave in the first argument, $m(\cdot)$ is strictly decreasing. Hence $\gamma' = \gamma$.

A.2 Proof of Lemma 2

Let $n \in (0, 1)$ satisfy (24). Let $k = \gamma n > 0$. Then (14) holds by Lemma 1. By (22), we have

$$f(k, n) - k = \lambda n > 0. \tag{39}$$

Thus (16) holds. Since $f(k, 1) > f(k, n) > k$ by (39), we have $k < \overline{k}$. We obtain (15) from (39) and (21). Hence n is a steady state level of labor.

Conversely, let $(k, n) \in (0, \overline{k}) \times (0, 1)$ satisfy (14)–(16). Then $k = \gamma n$ by Lemma 1. Thus (39) holds. Substituting (39) and (21) into (15), we obtain (24).

A.3 Proof of Lemma 3

Let $n \in (0, 1)$. Suppose that $n = n^*(\rho\gamma n)$. It follows from (29) that

$$u_1(\rho\gamma n + \mu n, n)\mu + u_2(\rho\gamma n + \mu n, n) = 0. \tag{40}$$

Since $\rho\gamma + \mu = \lambda$ by (23), we obtain (24). Conversely, assume (24). Substituting (23) into (24), we obtain (40), which implies that $n = n^*(\rho\gamma n)$.

A.4 Proof of Proposition 1

Suppose that leisure is a normal good. Then $n^*(\cdot)$ is strictly decreasing. Hence there can be at most one solution n to (30). Thus by Lemma 3, there can be at most one interior steady state.

A.5 Proof of Proposition 2

Define $\phi : \mathbb{R}^2 \to \mathbb{R}$ by

$$\phi(c,n) = -\frac{1}{2}(c-\sigma)^2 + \nu\left(c - \frac{\sigma\mu}{\nu}\right)n - \frac{\tau^2}{2}n^2, \tag{41}$$

where $\sigma, \tau > 0$ are parameters and ν is defined by

$$\nu = \frac{\tau^2 + \lambda\mu}{\lambda + \mu}. \tag{42}$$

For all $c, n \in \mathbb{R}$, we have

$$\phi_1(c,n) = (\sigma - c) + \nu n, \qquad \phi_2(c,n) = \nu\left(c - \frac{\sigma\mu}{\nu}\right) - \tau^2 n. \tag{43}$$

Direct computation shows that for all $n \in \mathbb{R}$, we have[7]

$$\phi_1(\lambda n, n)\mu + \phi_2(\lambda n, n) = 0. \tag{44}$$

It thus follows by Lemma 2 that if $u = \phi$ on $[0,\bar{c}] \times [0,1]$ for some $\bar{c} > \overline{k}$, then any $n \in (0,1)$ is a steady state level of labor. Hence it suffices to show that ϕ satisfies Assumption 2 on $[0,\bar{c}] \times [0,1]$ for some $\sigma, \tau > 0$ and $\bar{c} > \overline{k}$. To this end, we need two lemmas:

Lemma 4. *Let*

$$\hat{c} = \frac{\sigma\mu}{\nu}. \tag{45}$$

Then

$$\forall (c,n) \in [0,\hat{c}) \times [0,1], \qquad \phi_1(c,n) > 0, \quad \phi_2(c,n) < 0. \tag{46}$$

Proof. Let $(c,n) \in [0,\hat{c}) \times [0,1]$. Since $\hat{c} < \sigma$, we have $\phi_1(c,n) > 0$ by (43). Note that

$$\phi_2(c,n) < \nu\left(\hat{c} - \frac{\sigma\mu}{\nu}\right) - \tau^2 n = -\tau^2 n < 0. \tag{47}$$

This completes the proof. □

Lemma 5. *The function ϕ is strictly concave if and only if*

$$\mu < \tau < \lambda. \tag{48}$$

[7] $\phi_1(\lambda n, n)\mu + \phi_2(\lambda n, n) = [(\sigma - \lambda n) + \nu n]\mu + \nu(\lambda n - \sigma\mu/\nu) - \tau^2 n = [\nu(\mu + \lambda) - (\lambda\mu + \tau^2)]n + \sigma\mu - \sigma\mu = 0$.

Proof. Note from (43) that the Hessian matrix of ϕ at any (c, n) is

$$\begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} = \begin{bmatrix} -1 & \nu \\ \nu & -\tau^2 \end{bmatrix}. \tag{49}$$

Thus ϕ is strictly concave if and only if $\nu^2 < \tau^2$. Since $\nu, \tau > 0$, it follows that ϕ is strictly concave if and only if $\nu < \tau$. Recalling (42), we see that $\nu < \tau \Leftrightarrow \tau^2 + \lambda\mu < (\lambda + \mu)\tau \Leftrightarrow \tau^2 - (\lambda + \mu)\tau + \lambda\mu < 0 \Leftrightarrow (\tau - \lambda)(\tau - \mu) < 0$. The last inequality is equivalent to (48) since $\mu < \lambda$ by (23). □

To complete the proof of Proposition 2, fix $\tau \in (\mu, \lambda)$ and $\overline{c} > \overline{k}$. Let $\sigma > 0$ be such that $\hat{c} > \overline{c}$, where $\hat{c}$ and ν are given by (45) and (42), respectively. Suppose that $u(c, n) = \phi(c, n)$ for all $(c, n) \in [0, \overline{c}] \times [0, 1]$. Then Lemmas 4 and 5 imply Assumption 2. Thus by (44) and Lemma 2, any $n \in (0, 1)$ is a steady state level of labor.

A.6 Proof of Proposition 3

Let $m \in \mathbb{N}$ and $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$. It suffices to show that there exists a utility function u satisfying Assumption 2 such that $n \in (0, 1)$ satisfies (24) if and only if $n \in \{\eta_1, \ldots, \eta_m\}$. To this end, let $J = \{1, \ldots, m\}$, and define $h : \mathbb{R}_+ \to \mathbb{R}$ by

$$h(n) = \int_0^n \prod_{j \in J} (x - \eta_j) dx. \tag{50}$$

Since $h'(n) = \prod_{j \in J} (n - \eta_j)$, we have $h'(n) = 0$ if and only if $n \in \{\eta_1, \ldots, \eta_m\}$.

Let $\overline{c}$ and ϕ be as in the proof of Proposition 2. Let $\epsilon \in \mathbb{R}$. Define $u(c, n) = \phi(c, n) + \epsilon h(n)$ for $(c, n) \in [0, \overline{c}] \times [0, 1]$. Then for any $n \in (0, 1)$, we have $g(n) = \epsilon h'(n)$ by (44) and (24); thus (24) holds if and only if $n \in \{\eta_1, \ldots, \eta_m\}$. It remains to show that u satisfies Assumption 2 for some $\epsilon \in \mathbb{R}$.

Recalling (46) and (49), we see that if $\epsilon = 0$, then $u_1(c, n) > 0, u_2(c, n) > 0$, and the Hessian matrix of u at (c, n) is negative definite for all $(c, n) \in [0, \overline{c}] \times [0, 1]$. Since these properties are preserved for ϵ sufficiently close to 0, it follows that u satisfies Assumption 2 for such ϵ, as desired.

A.7 Proof of Proposition 4

Let $\bar{c}$ and ϕ be as in the proof of Proposition 2. Let $\epsilon > 0$. Suppose that $u(c, n) = \phi(c, n) - \epsilon n$ for all $(c, n) \in [0, \bar{c}] \times [0, 1]$. It is easy to see that u satisfies Assumption 2. By (44) and (24), we have $g(n) = -\epsilon$ for all $n \in (0, 1)$. Hence there is no interior steady state by Lemma 2.

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