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Takashi KAMIHIGASHI

John STACHURSKI

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Research Institute for Economics and Business Administration

Kobe University

2-1 Rokkodai, Nada, Kobe 657-8501 JAPAN

The Nikodym Convergence Theorem for Countably Additive Set Functions on an Arbitrary Family of Sets

Takashi Kamihigashi* John Stachurski†

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Abstract

Let $\mathscr{S}$ be a semiring of subsets of a set $\mathbb{X}$, and let $\sigma(\mathscr{S})$ be the σ-algebra generated by $\mathscr{S}$. Let $\{\mu_n\}$ be a sequence of measures on $\sigma(\mathscr{S})$ such that for each countable collection $\{S_i\}$ of pairwise disjoint sets in $\mathscr{S}$, $\lim_n \sum_i \mu_n(S_i)$ exists in $\mathbb{R}_+$. Then there exists a measure $\overline{\mu}$ on $\sigma(\mathscr{S})$ such that $\overline{\mu}(S) = \lim_n \mu_n(S)$ for all $S \in \mathscr{S}$. To show this result, we first extend the Nikodym convergence theorem to a sequence of countably additive set functions on an arbitrary family of subsets of $\mathbb{X}$. Then we assume that the family is a semiring and apply the Carathéodory extension theorem.

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*RIEB, Kobe University, Rokkodai, Nada, Kobe 657-8501 Japan
Email: tkamihig@rieb.kobe-u.ac.jp
†Research School of Economics, Australian National University, Canberra, Australia

1 Introduction

The Nikodym convergence theorem states that if a sequence of measures on a common σ-algebra converges on every measurable set, then the setwise limit of the sequence is again a measure. This is among the most fundamental results in measure theory, and it has been extensively generalized (e.g., [8, 6]). Nevertheless, setwise convergence is often regarded as too strong a convergence criterion (e.g., [7, p. 123]). For example, weak convergence of a sequence of probability measures on the Borel σ-algebra of a metric space only implies convergence on "P-continuity" sets, where P is the weak limit of the sequence (e.g., [2]). In addition, the Nikodym convergence theorem is known to fail for algebras in general (e.g., [6]).

In this paper we present some elementary results that extend the Nikodym convergence theorem to less demanding convergence criteria. In particular, we consider a sequence of countably additive set functions on an arbitrary family of subsets of a set that converges setwise on this family. Under an additional condition that in effect implies "uniform countable additivity" of the sequence (e.g., [5, p. 160], [4, p. 87]), we show that its limit is a countably additive set function.

Although this is more of an observation for anyone who is familiar with the modern proofs of the Nikodym theorem (e.g., [3, 4, 7, 9]), it has a useful implication: Under the hypotheses of the above result, if the given family is a semiring, then the limit of the sequence is a measure on the semiring, which can then be extended to a measure on the σ-algebra generated by the semiring as a consequence of the Carathéodory extension theorem. Put differently, given a sequence of measures on the σ-algebra that is setwise convergent on the semiring and satisfies the additional condition mentioned above, there exists a measure to which the sequence converges setwise on the semiring but not necessarily on the σ-algebra.

2 Preliminaries

Throughout this paper, we fix $\mathbb{X}$ to be an arbitrary set. Let $\mathscr{A}$ be a family of subsets of $\mathbb{X}$. Let $M(\mathscr{A})$ be the set of set functions $\mu : \mathscr{A} \to \mathbb{R}_+$. Let $\mu \in M(\mathscr{A})$. We say that μ is *countably additive* if for each countable collection

$\{A_i\}_{i\in I}$, $I \subset \mathbb{N}$, of pairwise disjoint sets in $\mathscr{A}$ such that $\cup_{i\in I} A_i \in \mathscr{A}$, we have

$$\mu\left(\bigcup_{i\in I} A_i\right) = \sum_{i\in I} A_i. \tag{2.1}$$

Let $ca(\mathscr{A})$ be the set of countably additive functions in $M(\mathscr{A})$. Let $\sigma(\mathscr{A})$ be the σ-algebra generated by $\mathscr{A}$.

A family $\mathscr{S}$ of subsets of $\mathbb{X}$ is called a *semiring* if (i) $\emptyset \in \mathscr{S}$, (ii) for any $S_1, S_2 \in \mathscr{S}$, we have $S_1 \cap S_2 \in \mathscr{S}$, and (iii) for any $S_1, S_2 \in \mathscr{S}$, there exist pairwise disjoint sets $C_1, \ldots, C_n \in \mathscr{S}$, $n \in \mathbb{N}$, such that $S_1 \setminus S_2 = \cup_{i=1}^n C_i$. Note that a σ-algebra is a semiring.

If $\mathscr{S}$ is a semiring on $\mathbb{X}$, then a set function in $ca(\mathscr{S})$ is called a *measure* on $\mathscr{S}$. A measure μ on a semiring $\mathscr{S}$ is called *σ-finite* if there exists a sequence $\{S_i\}_{i\in\mathbb{N}}$ in $\mathscr{S}$ such that $\mathbb{X} = \cup_{i\in\mathbb{N}} S_i$ and $\mu(S_i) < \infty$ for each $i \in \mathbb{N}$. Given a measure μ on a semiring $\mathscr{S}$, a measure ν on $\sigma(\mathscr{S})$ such that $\nu(S) = \mu(S)$ for all $S \in \mathscr{S}$ is called an *extension* of μ to $\sigma(\mathscr{S})$.

Theorem 2.1 (Carathéodory)**.** *Any measure μ on a semiring $\mathscr{S}$ of subsets of $\mathbb{X}$ has an extension to $\sigma(\mathscr{S})$. The extension is unique if μ is σ-finite.*

Proof. See, e.g., [1, p. 382]. □

We say that a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $M(\mathscr{A})$ is *setwise convergent* if for each $A \in \mathscr{A}$, $\lim_{n\to\infty} \mu_n(A)$ exists in $\mathbb{R}_+$. For any setwise convergent sequence $\{\mu_n\}_{n\in\mathbb{N}} \in M(\mathscr{A})$, we define $\mu^* \in M(\mathscr{A})$ by

$$\mu^*(A) = \lim_{n\uparrow\infty} \mu_n(A), \quad \forall A \in \mathscr{A}. \tag{2.2}$$

Theorem 2.2 (Nikodym)**.** *Let $\mathscr{X}$ be a σ-algebra on $\mathbb{X}$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a setwise convergent sequence in $ca(\mathscr{X})$. Then $\mu^* \in ca(\mathscr{X})$; i.e., μ^* is a measure on $\mathscr{X}$.*

Proof. See, e.g., [3], [4, p. 90], [7, p. 31], or [9, p. 125]. □

This result is often called the Vitali-Hahn-Saks theorem (e.g., [7], [1]), which can also mean a related but different result.

3 Results

Given a family $\mathscr{A}$ of subsets of $\mathbb{X}$, we say that a sequence $\{\mu_n\}_{n\in\mathbb{N}}$ in $M(\mathscr{B})$ with $\mathscr{A} \subset \mathscr{B}$ satisfies Condition $\Lambda(\mathscr{A})$ if the following holds:

$\Lambda(\mathscr{A})$ For any countable collection $\{A_i\}_{i\in I}, I \subset \mathbb{N}$, of pairwise disjoint sets in $\mathscr{A}$, the following limit exists in $\mathbb{R}_+$:

$$\lim_{n\uparrow\infty} \sum_{i\in I} \mu_n(A_i) \in \mathbb{R}_+. \tag{3.1}$$

Under the hypotheses of the Nikodym convergence theorem, we have $\cup_i A_i \in \mathscr{X}$ for any countable collection $\{A_i\}$ of pairwise disjoint sets in $\mathscr{X}$, and $\sum_i \mu_n(A_i) = \mu_n(\cup_i A_i) \to \mu^*(\cup_i A_i)$ as $n \uparrow \infty$. Thus $\{\mu_n\}$ satisfies Condition $\Lambda(\mathscr{X})$. In fact, as long as $\mathscr{X}$ is a σ-algebra, setwise convergence on $\mathscr{X}$ is equivalent to Condition $\Lambda(\mathscr{X})$. Setwise convergence in this case implies that $\{\mu_n\}$ is "uniformly countably additive," which is often stated as one of the conclusions of the Nikodym convergence theorem (e.g., [4, 9]).

For an arbitrary family $\mathscr{A}$ of subsets of $\mathbb{X}$, Condition $\Lambda(\mathscr{A})$ implies setwise convergence on $\mathscr{A}$, but the converse is false because an arbitrary countable collection $\{A_i\}$ of pairwise disjoint sets in $\mathscr{A}$ may not satisfy $\cup_i A_i \in \mathscr{A}$. Condition $\Lambda(\mathscr{A})$ virtually forces setwise convergence on any such union as if it belonged to $\mathscr{A}$. In fact the condition is precisely what is required for the modern proofs of the Nikodym convergence theorem to go through without substantial modification (e.g., [3], [4, p. 90], [7, p. 31], [9, p. 125]). This is the idea of the following result.

Theorem 3.1. *Let $\mathscr{A}$ be a family of subsets of $\mathbb{X}$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $ca(\mathscr{A})$ satisfying Condition $\Lambda(\mathscr{A})$. Then $\mu^* \in ca(\mathscr{A})$.*

Proof. See Section 4. □

Corollary 3.1. *Let $\mathscr{S}$ be a semiring of subsets of $\mathbb{X}$. Let $\{\mu_n\}_{n\in\mathbb{N}}$ be a sequence in $ca(\mathscr{S})$ satisfying Condition $\Lambda(\mathscr{S})$. Then the following conclusions hold:*

(a) *$\mu^* \in ca(\mathscr{S})$; i.e., μ^* is a measure on $\mathscr{S}$.*

(b) *There exists an extension of μ^* to $\sigma(\mathscr{S})$.*

(c) *The extension is unique if there exists a sequence* $\{S_i\}_{i\in\mathbb{N}}$ *in* $\mathscr{S}$ *such that* $\mathbb{X} = \cup_{i\in\mathbb{N}} S_i$.

Proof. Conclusion (a) follows from Theorem 3.1. Conclusion (b) is immediate from (a) and the Carathéodory extension theorem. To see conclusion (c), note from the definition of $ca(\mathscr{S})$ that $\mu(S) < \infty$ for all $S \in \mathscr{S}$. This together with the condition of conclusion (c) implies that μ^* is σ-finite. Thus the conclusion holds by the Carathéodory extension theorem. □

The following result is an immediate consequence of Corollary 3.1.

Theorem 3.2. *Let* $\mathscr{S}$ *be a semiring of subsets of* $\mathbb{X}$. *Let* $\{\mu_n\}_{n\in\mathbb{N}}$ *be a sequence of measures on* $\sigma(\mathscr{S})$ *satisfying Condition* $\Lambda(\mathscr{S})$. *Then there exists a measure* $\overline{\mu}$ *on* $\sigma(\mathscr{S})$ *such that* $\overline{\mu}(S) = \lim_{n\uparrow\infty} \mu_n(S)$ *for all* $S \in \mathscr{S}$. *Furthermore,* $\overline{\mu}$ *is unique if there exists a sequence* $\{S_i\}_{i\in\mathbb{N}}$ *in* $\mathscr{S}$ *such that* $\mathbb{X} = \cup_{i\in\mathbb{N}} S_i$.

Note that if $\mathscr{S} = \sigma(\mathscr{S})$ in the above result, then we recover the Nikodym convergence theorem.

4 Proof of Theorem 3.1

The following proof closely parallels that of [9, Theorem 7.48]. We use the following classical result [9, Theorem 7.2].

Theorem 4.1 (Hahn)**.** *For* $n, i \in \mathbb{N}$, *let* $a_{n,i} \in \mathbb{R}_+$. *Assume the following:*

(a) *For each* $n \in \mathbb{N}$, *we have* $\sum_{i\in\mathbb{N}} a_{n,i} < \infty$.

(b) *For any* $I \subset \mathbb{N}$, $\lim_{n\uparrow\infty} \sum_{i\in I} a_{n,i}$ *exists in* $\mathbb{R}_+$.

For each $i \in \mathbb{N}$, *let* $a_i = \lim_{n\uparrow\infty} a_{n,i}$, *which exists in* $\mathbb{R}_+$ *by condition (b) above. Then for any* $I \subset \mathbb{N}$ *we have*

$$\lim_{n\uparrow\infty} \sum_{i\in I} a_{n,i} = \sum_{i\in I} a_i < \infty. \tag{4.1}$$

To prove Theorem 3.1, let $\{A_i\}_{i\in I}, I \subset \mathbb{N}$, be a countable collection of pairwise disjoint sets in $\mathscr{A}$ with $A \equiv \cup_{i\in I} A_i \in \mathscr{A}$. To show that $\mu^* \in ca(\mathscr{A})$, we need to verify that

$$\mu^*(A) = \sum_{i\in I} \mu(A_i). \tag{4.2}$$

If I is a finite set, then (4.2) is immediate. Suppose that I is infinite; without loss of generality, suppose that $I = \mathbb{N}$.

Since $\mu_n \in ca(\mathscr{A})$ for all $n \in \mathbb{N}$, we have

$$\forall n \in \mathbb{N}, \quad \infty > \mu_n(A) = \sum_{i \in \mathbb{N}} \mu_n(A_i). \tag{4.3}$$

For $n, i \in \mathbb{N}$, let $a_{n,i} = \mu_n(A_i)$. Then condition (a) of Theorem 4.1 is immediate from (4.3). Condition (b) is also immediate from Condition $\Lambda(\mathscr{A})$. Thus by (2.2), Condition $\Lambda(\mathscr{A})$, and Theorem 4.1 we have

$$\mu^*(A) = \lim_{n \uparrow \infty} \mu_n(A) = \lim_{n \uparrow \infty} \sum_{i \in \mathbb{N}} \mu_n(A_i) = \sum_{i \in \mathbb{N}} \mu^*(A_i). \tag{4.4}$$

Hence (4.2) holds. It follows that $\mu^* \in ca(\mathscr{A})$.

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