Discussion Paper Series

RIEB

Kobe University

DP2014-03

On the (de)Stabilizing Effect of Public Debt In a Ramsey Model with Heterogeneous Agents

Kazuo NISHIMURA
Carine NOURRY
Thomas SEEGMULLER
Alain VENDITTI

February 12, 2014

Research Institute for Economics and Business Administration
Kobe University
2-1 Rokkodai, Nada, Kobe 657-8501 JAPAN

On the (de)stabilizing effect of public debt in a Ramsey model with heterogeneous agents*

Kazuo NISHIMURA

RIEB, Kobe University & KIER, Kyoto University

Carine NOURRY

Aix-Marseille University (Aix-Marseille School of Economics), CNRS-GREQAM, EHESS & Institut Universitaire de France

Thomas SEEGMULLER

Aix-Marseille University (Aix-Marseille School of Economics), CNRS-GREQAM & EHESS

and

Alain VENDITTI

Aix-Marseille University (Aix-Marseille School of Economics), CNRS-GREQAM, EHESS & EDHEC

First version: *September 2013*; Revised: *January 2014*

Abstract: *We introduce public debt in a Ramsey model with heterogenous agents and a public spending externality affecting utility which is financed by income tax and public debt. We show that public debt considered as a fixed portion of GDP can have a stabilizing or destabilizing effect depending on some fundamental elasticities. When the public spending externality is weak and the elasticity of capital labor substitution is low enough, public debt can only be destabilizing, generating damped or persistent macroeconomic fluctuations. Whereas when the public spending externality and the elasticity of capital labor substitution are strong enough, public debt can be stabilizing, driving to monotone convergence an economy experiencing damped or persistent fluctuations without debt.*

Keywords: *Endogenous cycles, heterogeneous agents, public spending, public debt, borrowing constraint*

Journal of Economic Literature Classification Numbers: C62, E32, H23.

*This work has been carried out thanks to the support of the A*MIDEX project (n° ANR-11-IDEX-0001-02) funded by the "Investissements d'Avenir" French Government program, managed by the French National Research Agency (ANR). We thank Hippolyte d'Albis, Emmanuelle Augeraud-Véron, Stefano Bosi, Jean-Pierre Drugeon, Mouez Fodha and Emmanuel Thibault for useful comments and suggestions.

1 Introduction

The last financial crisis has shed the light on the problem of large public debt in developed countries, in particular in Europe. In many advanced countries, debt levels have increased dramatically during the two last decades, now reaching extremely high amounts. The control of the growth rate of public spendings has became a major concern for economists and policy-makers while public deficits digging. A heavily indebted country may appear as fragile, for many reasons, among which solvency, or simply because it is unlikely to raise sufficient funds to deal with a large negative shock on its economy. The Maastricht treaty introduced a rule on the maximal amount a country may contract, limiting the debt to 60% of GDP, but this limit has been exceeded by almost all european countries. Indeed, most advanced countries are characterized both by large amounts of public debt and large fluctuations of GDP. Given this fact, two types of questions become central. The first concerns the relationship between the level of debt and growth, the second focuses on the relationship between macroeconomic stability and debt level. But whereas the literature has focused recently on the first question, little attention has been paid to the second.

While subject to a recent controversy,[1] the paper of Reinhart and Rogoff (2010) indeed shows that a gross public debt exceeding 90% of nominal GDP on a sustained basis may have a significant negative impact on the growth rate.[2] On the basis of this type of result, the IMF has strongly advised European countries over the last years to decrease their debt. The main objective was to boost growth but also to stabilize the economies. Indeed, since 2008, most advanced economies have been characterized by large fluctuations of GDP.

In a OECD Economics Department Policy note, Sutherland, Hoeller, Merola and Ziemann (2012) argue that the level of government debt has a significative impact on business cycle characteristics. They identify the characteristics of a "low debt" business cycle and a "high debt" business cycle aggregating the countries according to their level of debt. In countries with high debt, the cycle is more pronounced, with phases of expansions longer and larger and recessions also more pronounced. The arguments for such differences usually rely on the "vulnerability" of high public debt economies. Government then have less latitude to run the appropriate fiscal policy in case of negative shocks. Moreover, Ilzetzki, Mendoza and Vègh (2013) show that the

[1]See Herndon, Ash and Pollin (2013).

[2]See also Reinhart, Reinhart and Rogoff (2012).

impact of governments expenditure shocks depends crucially on public indebtness, as the fiscal multipliers seems to be lower in high debt countries.

This paper proposes to study the question of the relationship between debt and fluctuations in the simple framework of the neoclassical Ramsey model (1928). Our aim is to precisely discuss the effect of public debt on the macroeconomic stability of one country in the optimal growth model. As we focus on business cycle properties, we will consider a model with heterogeneous agents, which allows the emergence of endogenous fluctuations. Actually, in the standard Ramsey (1928) model with one representative agent, one sector and usual assumptions, the economy monotonically converges to the steady state. With many agents, conclusions may differ: Introducing borrowing constraints, the Ramsey conjecture holds at the steady state, i.e. the most patient holds all the capital (Becker (1980)). Dynamics can be non-monotone and endogenous cycles can occur around the steady state for a weak elasticity of capital-labor substitution, i.e. when capital income monotonicity fails (Becker and Foias (1987, 1994)).[3]

We focus on government intervention as a source of macroeconomic fluctuations when government spending is financed through taxes on income and public debt. Public spending is useful because it improves households' utility of consumption as an externality. Even with linear income taxation, endogenous fluctuations (flip bifurcation) become compatible with plausible values of the elasticity of capital-labor substitution in the economy without debt.[4] When public debt is introduced as a fixed proportion of GDP, we show that it can have a stabilizing or destabilizing effect depending on the value of the elasticity of capital-labor substitution.

If the elasticity of capital labor substitution is low, and the marginal utility of consumption with respect to public spendings is weak, i.e. the variation of individuals' welfare is not very sensitive to variations of the available public good, we prove that public debt can have a destabilizing effect. Indeed, economies characterized by a high level of the debt-output ratio are more subject to fluctuations. They can actually experience endogenous oscillations whereas the same economies are characterized by saddle-point stability if their level of debt is low.

Conversely, when the elasticity of capital labor substitution is high enough, and the elasticity of the marginal utility with respect to public spendings is large, i.e.

[3]See also Sorger (1994) for the existence of more complex dynamics.

[4]In Nishimura *et al.* (2013), the existence of endogenous fluctuations through the occurrence of local indeterminacy is analyzed in a Ramsey model with government spending, financed from a constant income tax only, and endogenous labor.

the variation of individuals' welfare is very sensitive to variations of the available public good, we prove that public debt has a strong stabilizing effect. Indeed a large enough amount of public debt can stabilize the economy by guaranteeing monotone convergence toward the steady state.

The destabilizing effect of public debt associated to cases where public spending does not matter a lot for agents decisions can be explained following an intuition which is closely related to the intuition given by Becker and Foias (1987, 1994). Without debt, endogenous fluctuations indeed occur only when the elasticity of capital-labor substitution is low enough, meaning that the "capital income monotonicity" fails. Capital income is then a decreasing function of capital and a high capital at one period is compatible with a lower investment at the next period. For indebted countries, as now the most patient agents own both assets, capital and debt, endogenous fluctuations occur when the "asset income monotonicity" fails, which is less restrictive and thus true for larger sets of values for the elasticity of capital labor substitution. The set of economies subject to these fluctuations is then larger if the economies are sufficiently indebted.

The stabilizing effect of public debt associated to cases where public spending matters a lot for agents decisions can be explained as follows. Assume that the elasticity of capital labor substitution is large enough and that at a given period, capital is high, meaning income is high. Fluctuations occur if the consumer's intertemporal trade-off is compatible with a decrease of capital at the following period, associated to a higher return to capital.

Without debt, public spending is procyclical, following mechanically the capital labor ratio through taxes on income, and endogenous fluctuations are compatible with the intertemporal trade-off. But when public spending is also related to the repayment of a previous debt and the contraction of a new one, it can be countercyclical, and the fluctuations cannot occur anymore as they become incompatible with the intertemporal trade-off. Thus when the welfare associated to consumption depends strongly of the level of public spendings, a high enough amount of debt can prevent the existence of endogenous fluctuations.

We finally show that that, depending on the value of the elasticity of capital-labor substitution, public debt can be used at the same time to stabilize the economy and to affect the degree of inequalities. Indeed, when capital and labor are weakly substitutable, a low debt can stabilize and decrease the inequalities as capital is growing. On the contrary, when the elasticity of capital-labor substitution is larger, a large debt can stabilize and decrease or increase the inequalities as capital is growing depending on whether the elasticity has intermediary or large values.

This paper is organized as follows. The model is presented in the next section. Section 3 is devoted to the steady state analysis. In section 4, we study the occurrence of endogenous business cycles and economic interpretations. Section 5 provides some concluding remarks. Proof and technical details are provided in the Appendix.

2 The model

We consider a discrete time economy ($t = 0, 1, ..., \infty$), with three types of agents, heterogeneous households, firms and a government.

2.1 Households

There are H heterogeneous infinitely lived households, indexed by $i = 1, \ldots, H$ with $H \geqslant 2$, who supply inelastically labor and face borrowing constraints. They have heterogeneous capital and debt endowments ($k_{i0}, b_{i0} \geqslant 0$), and heterogeneous preferences, i.e. different discount factors and different instantaneous utilities in consumption. Households are ranked according to their discount factors: $0 \leq \beta_H \leq \ldots \leq \beta_2 < \beta_1 < 1$.

Household i derives utility for consumption c_{it} at period t. Moreover, we assume that public spending G_t affects welfare, as an externality on utility for consumption. Utility is non separable between consumption and public spending at each period, but separable over time:

$$\sum_{t=0}^{+\infty} \beta_i^t u_i\left(c_{it}, G_t\right) \tag{1}$$

Each household derives income from wage, capital and government bonds that allow to finance public debt. Denote r_t the real interest rate on physical capital, $\bar{r}_t$ the return of government bonds, w_t the real wage and $\delta \in (0, 1)$ the rate of depreciation of capital. In addition, each household pays taxes on labor income, capital income and on the remuneration of bonds' holding, at a constant rate $\tau \in (0, 1)$.[5] Any household i maximizes (1) facing the budget constraint:

$$c_{it} + k_{it+1} + b_{it+1} = (1-\tau)[r_t k_{it} + w_t] + (1-\tau)\bar{r}_t b_{it} + (1-\delta)k_{it}, \tag{2}$$

and the borrowing constraint on individual capital and government bonds holding $k_{it}, b_{it} \geq 0$. The utility function satisfies the following assumption:

[5] We could assume different tax rates on labor income, capital income and remuneration of bonds, but this does not alter our results.

Assumption 1. *$u_i(c_i, G)$ is a continuous function defined on $[0, +\infty) \times [0, +\infty)$, and C^2 on $(0, +\infty) \times (0, +\infty)$. $u_i(c_i, G)$ is strictly increasing ($u_{ic}(c_i, G) > 0$), strictly concave ($u_{icc}(c_i, G) < 0$) with respect to its first argument, and the marginal utility of consumption increases with respect to public spending ($u_{icG}(c_i, G) \geqslant 0$).[6] For further reference, we introduce the following elasticities:[7]*

$$\epsilon_{icc} \equiv -u_{icc}c_i/u_{ic} > 0, \;\; \epsilon_{icG} \equiv u_{icG}G/u_{ic} \geqslant 0 \tag{3}$$

In addition, the Inada condition $\lim_{c_i \to 0} u_{ic}(c_i, G) = +\infty$ is satisfied.

Utility maximization gives:

$$\frac{u_{ic}(c_{it}, G_t)}{u_{ic}(c_{it+1}, G_{t+1})} \geq \beta_i R_{t+1}, \;\; \textit{with equality when } k_{it+1} > 0 \tag{4}$$

$$\frac{u_{ic}(c_{it}, G_t)}{u_{ic}(c_{it+1}, G_{t+1})} \geq \beta_i (1-\tau)\bar{r}_{t+1}, \;\; \textit{with equality when } b_{it+1} > 0 \tag{5}$$

with $R_{t+1} = (1-\tau)r_{t+1} + 1 - \delta$ and the transversality conditions

$$\lim_{t \to +\infty} \beta_i^t u_{ic}(c_{it}, G_t)\, k_{it+1} = 0 \quad \text{and} \quad \lim_{t \to +\infty} \beta_i^t u_{ic}(c_{it}, G_t)\, b_{it+1} = 0 \tag{6}$$

Note that for all $i = 1, \ldots, H$, c_{it} are forward variables while k_{it} and b_{it} are predetermined variables.

2.2 Firms

A representative firm produces the final good y_t, using capital k_t and labor l_t under a constant returns to scale technology $y_t = F(k_t, l_t)$. As there are $H \geq 1$ households who supply one unit of inelastic labor, it follows that $l_t = H$ and we denote $F(k_t, H) \equiv f(k_t)$. The production function $f(k)$ satisfies:

Assumption 2. *$f(k)$ is a continuous function defined on $[0, +\infty)$ and C^2 on $(0, +\infty)$, strictly increasing ($f'(k) > 0$) and strictly concave ($f''(k) < 0$). In addition, the conditions $\lim_{k \to 0} f'(k) = +\infty$ and $\lim_{k \to +\infty} f'(k) < \theta/\beta_1(1-\tau)$, with $\theta \equiv 1 - \beta_1(1-\delta)$, are satisfied.*

Profit maximization gives:

$$r_t = f'(k_t) \equiv r(k_t) \;\text{ and }\; Hw_t = f(k_t) - k_t f'(k_t) \equiv \Omega(k_t) \tag{7}$$

Due to constant returns to scale and perfect competition, profits are zero, *i.e.*,

[6] We denote $u_{ix_j}(x_1, x_2) = \partial u_i(x_1, x_2)/\partial x_j$ and $u_{ix_jx_h}(x_1, x_2) = \partial^2 u_i(x_1, x_2)/\partial x_j \partial x_h$.

[7] For simplicity, we omit the arguments of the functions.

$$f(k_t) = w_t H + r_t k_t \tag{8}$$

In the following, we denote by $s(k) \equiv kf'(k)/f(k) \in (0,1)$ the capital share in total income and $\sigma(k) \equiv [s(k) - 1] f'(k) / [kf''(k)] \geq 0$ the elasticity of capital-labor substitution. We derive the following useful relationships:

$$r'(k)k/r(k) \equiv -(1 - s(k))/\sigma(k) \quad \text{and} \quad \Omega'(k)k/\Omega(k) \equiv s(k)/\sigma(k) \tag{9}$$

2.3 Government

Public spending G_t is financed by total income taxation and debt. Since $y_t = \sum_{i=1}^{H} r_t k_{it} + H w_t$, the budget constraint faced by the government at period t writes:

$$G_t + \bar{r}_t b_t = \tau(y_t + \bar{r}_t b_t) + b_{t+1} \tag{10}$$

where $\tau \in (0,1)$ is the constant proportional tax rate on households' total income.

Total public expenditures, that are the sum of public spendings G_t and the reimbursement of debt contracted the previous period $\bar{r}_t b_t$, are financed by the new issue of debt b_{t+1} and taxation of (capital and labor) income and of remuneration of bonds. In order to match the constraint imposed by the Maastricht treaty, we will assume that public debt cannot exceed a fixed proportion $\alpha \geqslant 0$ of GDP, namely

$$b_t \leq \alpha y_t$$

We will focus in the following on equilibria where this constraint is binding, i.e. $b_t = \alpha y_t$. The case without debt is of course obtained when $\alpha = 0$ and α is defined as the debt-output ratio. Note that when the constraint is binding, α can also be interpreted as a policy parameter that allows to manage public debt excluding its explosive path (see de la Croix and Michel (2002), p.230-233). The parameter α will allow us to discuss the stability properties of the economy according to the level of the debt-output ratio.

2.4 Intertemporal equilibrium

An intertemporal equilibrium can be defined as follows:

Definition 1. *Under Assumptions 1 and 2, an equilibrium is a sequence* $(r_t, \bar{r}_t, w_t, k_t, b_t, G_t, (k_{it}, b_{it}, c_{it})_{i=1}^{H})_{t=0}^{+\infty}$ *satisfying the optimal behavior of households (2), (4), (5) and (6), profit maximization (7), the government constraint (10) and the equilibrium conditions on the asset markets* $k_t = \sum_{i=1}^{H} k_{it}$ *and* $b_t = \sum_{i=1}^{H} b_{it} = \alpha f(k_t)$*, the equilibrium on the labor market being satisfied since* $l_t = H$.

The existence of the intertemporal equilibrium is an issue that we do not address in this paper. The interested reader can refer to Becker *et al.* (1991), Bosi and Seegmuller (2010) or Becker *et al.* (2013). In the next section, we show the existence of a steady state. Since we focus on local dynamics around such an equilibrium, we consider that, by continuity, an intertemporal equilibrium exists in a neighborhood of the steady state.

3 Steady state analysis

Since the tax rate on income is constant, we can derive the existence of a steady state along which the equality $R = (1-\tau)\bar{r}$ holds as physical capital k and governments bonds b are perfectly substitutable saving assets.

Proposition 1. *Under Assumptions 1 and 2, let* $\alpha \in [0, \hat{\alpha})$ *with* $\hat{\alpha} = \tau\beta_1/(1-\beta_1)$. *Then there exists a steady state defined by the following properties:*

(S1) $r = f'(k) = \theta/\beta_1(1-\tau)$, $\bar{r} = R/(1-\tau) = f'(k) + (1-\delta)/(1-\tau)$ *and* $w = [f(k) - kf'(k)]/H$ *are constant;*

(S2) $R = 1/\beta_1 < 1/\beta_2 \leq \ldots \leq 1/\beta_H$*;*

(S3) $k = k_1 > 0$, $b = b_1 = \alpha f(k_1) > 0$ *and* $k_i = b_i = 0$ *for* $i \geq 2$*;*

(S4) $c_1 = (k_1 + b_1)(R-1) + (1-\tau)w$ *and* $c_i = (1-\tau)w$ *for* $i \geq 2$*;*

(S5) $G = \tau(rk_1 + Hw) + (1-R)b_1 \equiv \Delta f(k_1)$ *with* $\Delta = \tau - \frac{\alpha(1-\beta_1)}{\beta_1}$.

Proof: See Appendix 6.1.

This proposition shows that because of the borrowing constraints, there exists a steady state. In accordance with the so-called Ramsey (1928) conjecture and the seminal contribution of Becker (1980), the most patient household holds the whole capital stock and government debt. Note that the debt-output ratio has to be lower than $\widehat{\alpha}$, because otherwise, the public debt burden is too heavy and is no more compatible with a positive government spending.

4 Endogenous business cycles under public spending externalities

In the neighborhood of the steady state exhibited in Proposition 1, the intertemporal equilibrium can be summarized by a two-dimensional dynamical system given by the patient household's trade-off between present and future consumption and his budget constraint. Indeed, using Definition 1, an intertemporal equilibrium can be redefined as a sequence $(c_{1t}, k_t)_{t=0}^{\infty}$, satisfying:

$$\frac{u_{1c}(c_{1t}, G_t)}{u_{1c}(c_{1t+1}, G_{t+1})} = \beta_1 R(k_{t+1}) \tag{11}$$

$$k_{t+1} + \alpha f(k_{t+1}) = R(k_t)[k_t + \alpha f(k_t)] + (1-\tau)\frac{\Omega(k_t)}{H} - c_{1t} \tag{12}$$

with $R(k_{t+1}) = (1-\tau)f'(k_{t+1}) + 1 - \delta$, $G_t = f(k_t)[\tau - \alpha R(k_t)] + \alpha f(k_{t+1})$, and where c_{1t} is a forward variable and k_t is the only predetermined variable.

We characterize the stability properties of the steady state and the occurrence of local bifurcations by linearizing the dynamic system (11)-(12) around the steady state (c_1, k) and computing the Jacobian matrix J, evaluated at this steady state.

Lemma 1. *Under Assumptions 1 and 2, the characteristic polynomial is given by* $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$, *where:*

$$D = \frac{B_2(\alpha) - \frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}} B_3(\alpha) B_1(\alpha)}{1 + \frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}} \alpha s B_1(\alpha)} \equiv D(\alpha), \quad T = 1 + D(\alpha) + \frac{\frac{(1-s)\theta B_1(\alpha)}{\sigma \epsilon_{1cc}}}{1 + \frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}} \alpha s B_1(\alpha)} \equiv T(\alpha) \tag{13}$$

with

$$\begin{aligned}
B_1(\alpha) &= \tfrac{1-\tau}{\beta_1(1-\tau)+\alpha\theta}\left[(1-\beta_1)\left(1 + \tfrac{\alpha\theta}{\beta_1(1-\tau)s}\right) + \tfrac{(1-s)\theta}{sH}\right] > 0 \\
B_2(\alpha) &= \tfrac{1}{\beta_1}\left[1 - \tfrac{\beta_1\theta(1-s)(1-\tau)}{\sigma[\beta_1(1-\tau)+\alpha\theta]}\left(1 - \tfrac{1}{H} + \tfrac{\alpha\theta}{\beta_1(1-\tau)s}\right)\right] \\
B_3(\alpha) &= \tfrac{s(\tau\beta_1 - \alpha)}{\beta_1} + \tfrac{\alpha(1-s)\theta}{\sigma\beta_1}
\end{aligned} \tag{14}$$

Proof: See Appendix 6.2.

As shown by Becker and Foias (1987, 1994), the existence of endogenous fluctuations in a standard Ramsey model with heterogeneous agents can be obtained only if the *capital income monotonicity* assumption is not satisfied. Actually, *capital income monotonicity* holds if $f'(k)k$ is an increasing function of k. At the steady state, it can be easily shown that the *capital income monotonicity* holds if

$$\sigma > 1 - s \equiv \sigma_{CIM} \tag{15}$$

Recent papers have explored the empirical value of the elasticity of capital-labor substitution and questioned the empirical relevance of the Cobb-Douglas specification which is widely used in growth theory. They find that capital and labor have an elasticity of substitution significantly different than unity. However, empirical evidences for both gross substitutability (elasticity above one) and gross complementarity (elasticity below one) of capital and labor are obtained in the literature.

For instance, Duffy and Papageorgiou (2000) report robust estimates that are contained in $[1.24, 3.24]$ and Krusell *et al.* (2007) find an elasticity of substitution between unskilled labor and equipment of 1.67. On the contrary, Chirinko (2008), Klump *et al.* (2007) and León-Ledesma *et al.* (2010) provide robust estimates in the range $[0.4, 0.6]$. When $s = 0.3$, we get $\sigma_{CIM} = 0.7$ a value which is precisely in between all the available empirical estimates. We then need to study both cases $\sigma < \sigma_{CIM}$ and $\sigma > \sigma_{CIM}$.

We start by considering an economy without debt. Our aim is then to show that, since public spending externalities affect utility of consumption, endogenous business cycles can occur in the Ramsey model with heterogeneous agents for any value of the elasticity of capital-labor substitution, and in particular even in the case where capital income monotonicity holds. Nishimura *et al.* (2013) have recently exhibited the existence of endogenous fluctuations through the occurrence of local indeterminacy in a similar Ramsey model with government spending financed by linear income taxes but without public debt and augmented to include endogenous labor.

4.1 The Ramsey economy without debt

The equilibrium without debt is obviously obtained when $\alpha = 0$. We then get the following Proposition:

Proposition 2. *Under Assumptions 1-2, let $\alpha = 0$. Then there exist $\bar{\epsilon}_{1cG} > \underline{\epsilon}_{1cG} \geq 0$, $\hat{\sigma} < \tilde{\sigma} < \sigma_{CIM}$ and $\bar{\epsilon}_{1cc} > 0$ such that the following results hold:*

i) if $\sigma > \tilde{\sigma}$, the steady state is saddle-point stable with monotone convergence when $\epsilon_{1cG} \in (0, \underline{\epsilon}_{1cG})$, saddle-point stable with damped oscillations when $\epsilon_{1cG} \in (\underline{\epsilon}_{1cG}, \bar{\epsilon}_{1cG})$, undergoes a flip bifurcation for $\epsilon_{1cG} = \bar{\epsilon}_{1cG}$, and becomes locally unstable when $\epsilon_{1cG} > \bar{\epsilon}_{1cG}$;

ii) if $\sigma \in (\hat{\sigma}, \tilde{\sigma})$, the steady state is saddle-point stable with damped oscillations when $\epsilon_{1cG} \in (0, \bar{\epsilon}_{1cG})$, undergoes a flip bifurcation for $\epsilon_{1cG} = \bar{\epsilon}_{1cG}$, and becomes locally unstable when $\epsilon_{1cG} > \bar{\epsilon}_{1cG}$;

iii) if $\sigma \in (0, \hat{\sigma})$ and $\epsilon_{1cc} < \bar{\epsilon}_{1cc}$, the steady state is saddle-point stable with damped oscillations when $\epsilon_{1cG} \in (0, \bar{\epsilon}_{1cG})$, undergoes a flip bifurcation for $\epsilon_{1cG} = \bar{\epsilon}_{1cG}$, and becomes locally unstable when $\epsilon_{1cG} > \bar{\epsilon}_{1cG}$.

In all cases, saddle-point stable (locally unstable) period-two cycles occur in a right (left) neighborhood of $\bar{\epsilon}_{1cG}$.

Proof: See Appendix 6.3.

This proposition shows that endogenous business cycles emerge through a flip bifurcation even when the capital income monotonicity assumption is satisfied, provided the government spending externality is large enough. Indeed, if $\sigma > \sigma_{CIM}$, case i) of Proposition 2 applies. This is at odds with the non monotonicity of capital income required to get period-two cycles in the model without public spending externalities (Becker and Foias (1987, 1994)).

Of course, when there is no externality, i.e. $\epsilon_{1cG} = 0$, we get as a Corollary the same conclusions as in Becker and Foias (1987, 1994):

Corollary 1. *Under Assumptions 1-2, let $\alpha = 0$ and $\epsilon_{1cG} = 0$. Then there exist $\hat{\sigma} < \tilde{\sigma} < \sigma_{CIM}$ and $\bar{\epsilon}_{1cc} > 0$ such that the following results hold:*

i) if $\sigma > \tilde{\sigma}$, the steady state is saddle-point stable with monotone convergence for any $\epsilon_{1cc} > 0$;

ii) if $\sigma \in (\hat{\sigma}, \tilde{\sigma})$, the steady state is saddle-point stable with damped oscillations for any $\epsilon_{1cc} > 0$;

iii) if $\sigma \in (0, \hat{\sigma})$, the steady state is saddle-point stable with damped oscillations when $\epsilon_{1cc} \in (0, \bar{\epsilon}_{1cc})$, undergoes a flip bifurcation for $\epsilon_{1cc} = \bar{\epsilon}_{1cc}$, and becomes locally unstable when $\epsilon_{1cc} > \bar{\epsilon}_{1cc}$. Moreover, saddle-point stable (locally unstable) period-two cycles occur in a right (left) neighborhood of $\bar{\epsilon}_{1cc}$.

4.2 The Ramsey economy with debt

Let us now consider the case with debt assuming $\alpha \in (0, \hat{\alpha})$. Our aim is to check whether debt has a stabilizing or a destabilizing effect on the economy. Put differently, we are looking for some conditions on the share α of debt over GDP that allow to generate or rule out endogenous fluctuations. Building on the results derived in the no-debt case, we show that the conclusions strongly depend on the value of the elasticity of capital-labor substitution.

We start by analyzing the standard formulation of Becker and Foias (1987, 1994) without public spending externality in the utility function. Indeed, it is worth noting from Lemma 1 that when $\epsilon_{1cG} = 0$, the Determinant and Trace of the characteristic polynomial are linear functions of the share α. This monotonicity property allows to derive the following clear-cut conclusions:

Proposition 3. *Under Assumptions 1-2, let $\epsilon_{1cG} = 0$. Then there exist $\underline{\beta}_1 \in (0, 1)$, $\bar{\sigma} > \underline{\sigma} > \tilde{\sigma} > \hat{\sigma}$, $\bar{\epsilon}_{1cc} > \tilde{\epsilon}_{1cc}$ and $\hat{\alpha} \geq \bar{\alpha} > \underline{\alpha} \geq 0$ such that when $\beta_1 \in (\underline{\beta}_1, 1)$ the following results hold:*

i) Public debt does not have any impact on the local stability properties of the steady state when $\sigma \geq \bar{\sigma}$, or $\sigma \in (\hat{\sigma}, \tilde{\sigma})$ and $\epsilon_{1cc} \leq \tilde{\epsilon}_{1cc}$, or $\sigma \in (0, \hat{\sigma})$ and $\epsilon_{1cc} \in (0, \tilde{\epsilon}_{1cc}] \cup (\bar{\epsilon}_{1cc}, +\infty)$.

ii) ***Public debt has a destabilizing effect*** *in the following cases:*

- when $\sigma \in (\underline{\sigma}, \bar{\sigma})$, or $\sigma \in (\tilde{\sigma}, \underline{\sigma})$ and $\epsilon_{1cc} \leq \tilde{\epsilon}_{1cc}$, as the steady state is saddle-point stable with monotone convergence when $\alpha \in [0, \underline{\alpha})$ and saddle-point stable with damped fluctuations when $\alpha \in (\underline{\alpha}, \hat{\alpha})$.

- when $\sigma \in (\tilde{\sigma}, \underline{\sigma})$ and $\epsilon_{1cc} > \tilde{\epsilon}_{1cc}$ as the steady state is saddle-point stable with monotone convergence when $\alpha \in [0, \underline{\alpha})$, saddle-point stable with damped fluctuations when $\alpha \in (\underline{\alpha}, \bar{\alpha})$, undergoes a flip bifurcation when $\alpha = \bar{\alpha}$ and becomes locally unstable with oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$.

- when $\sigma \in (\hat{\sigma}, \tilde{\sigma})$ and $\epsilon_{1cc} > \tilde{\epsilon}_{1cc}$ as the steady state is saddle-point stable with damped fluctuations when $\alpha \in [0, \bar{\alpha})$, undergoes a flip bifurcation when $\alpha = \bar{\alpha}$ and becomes locally unstable with oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$.

- when $\sigma \in (0, \hat{\sigma})$ and $\epsilon_{1cc} \in (\tilde{\epsilon}_{1cc}, \bar{\epsilon}_{1cc})$ as the steady state is saddle-point stable with damped fluctuations when $\alpha \in [0, \bar{\alpha})$, undergoes a flip bifurcation when $\alpha = \bar{\alpha}$ and becomes locally unstable with oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$.

Moreover, in all cases where $\bar{\alpha}$ exists, there are saddle-point stable (locally unstable) period-two cycles in a right (left) neighborhood of $\bar{\alpha}$.

Proof: See Appendix 6.4

Proposition 3 provides a complete picture of the impact of public debt on the local stability properties of the steady state when there is no public spending externality in preferences, i.e. $\epsilon_{1cG} = 0$. We have then clearly shown that when σ is low enough and ϵ_{1cc} is sufficiently large, public debt has a destabilizing effect as it may create damped and/or persistent macroeconomic fluctuations.

Let us finally consider the formulation with public spending externality in utility in order to check whether the destabilizing effect of debt is robust. When $\epsilon_{1cG} > 0$ the analysis becomes much more complex as the Determinant and Trace of the characteristic polynomial are no longer linear functions of α. We may however provide some results which show that depending on the values of σ and ϵ_{1cG}, the previous conclusion does not necessarily hold as a sufficiently large public debt may have a stabilizing or destabilizing effect.

Proposition 4. *Under Assumptions 1-2, let $\delta < s < 1/2$ and $\beta_1 > 1/(1 + s - \delta)$. Then there exist $\bar{\sigma}, \underline{\sigma} > 0$, satisfying $\sigma_{CIM} > \bar{\sigma} > \underline{\sigma} > \tilde{\sigma} > \hat{\sigma}$, $\bar{\epsilon}_{1cG} > \underline{\epsilon}_{1cG} > 0$, $\bar{\epsilon}_{1cc} > 0$ and $\bar{\alpha} \in [0, \hat{\alpha})$ such that the following results hold:*

*1- **Public debt has a stabilizing effect** in the following cases:*

i) when $\sigma > \bar{\sigma}$, as the steady state is saddle-point stable with monotone convergence when $\alpha \in (\bar{\alpha}, \hat{\alpha})$ no matter what are the stability properties of the steady state in the economy without debt (i.e. for any $\epsilon_{1cG} > 0$);

ii) when $\sigma \in (\underline{\sigma}, \bar{\sigma})$ and $\epsilon_{1cG} > \bar{\epsilon}_{1cG}$, as the steady state is saddle-point stable with damped oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$ while in the economy without debt it is unstable with possible persistent fluctuations (period-two cycles) in the left neighborhood of $\bar{\alpha}$.

*2- **Public debt has a destabilizing effect** in the following cases:*

i) when $\sigma \in (\underline{\sigma}, \bar{\sigma})$ and $\epsilon_{1cG} \in (0, \underline{\epsilon}_{1cG})$, as the steady state is saddle-point stable with damped oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$ while in the economy without debt it is saddle-point stable with monotone convergence;

ii) when $\sigma \in (\tilde{\sigma}, \underline{\sigma})$ and $\epsilon_{1cG} \in (0, \bar{\epsilon}_{1cG})$, as the steady state is totally unstable with oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$ and there exist saddle-point stable (locally unstable) period-two cycles in a right (left) neighborhood of $\bar{\alpha}$, while in the economy without debt it is saddle-point stable with monotone convergence (when $\epsilon_{1cG} \in (0, \underline{\epsilon}_{1cG})$) or damped oscillations (when $\epsilon_{1cG} \in (\underline{\epsilon}_{1cG}, \bar{\epsilon}_{1cG})$);

iii) when $\epsilon_{1cG} \in (0, \underline{\epsilon}_{1cG})$ and either $\sigma \in (\hat{\sigma}, \tilde{\sigma})$, or $\sigma \in (0, \hat{\sigma})$ and $\epsilon_{1cc} \in (0, \bar{\epsilon}_{1cc})$, as the steady state is totally unstable with oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$ and there exist saddle-point stable (locally unstable) period-two cycles in a right (left) neighborhood of $\bar{\alpha}$, while in the economy without debt it is saddle-point stable with damped oscillations.

Proof: See Appendix 6.5

Case 1-i) in Proposition 4 covers the configuration in which the capital income monotonicity (i.e. inequality (15)) holds. This result is particularly interesting in the case where $\epsilon_{1cG} > \underline{\epsilon}_{1cG}$ in which there exist damped or persistent fluctuations in the economy without debt. There is indeed a level of public debt $\bar{\alpha} > 0$ above which the economy does not fluctuate anymore and converges monotonically toward its steady state. Public debt has here a strong stabilizing effect. In case 1-ii) with a large public spending externality, i.e. $\epsilon_{1cG} > \bar{\epsilon}_{1cG}$, public debt also has a stabilizing effect by ruling out persistent fluctuations.

Cases 2-i) and ii) on the contrary imply that when the capital income monotonicity is not satisfied and the public spending externality is not too large, a large enough level of public debt with respect to GDP may destabilize the economy by generating endogenous fluctuations while the economy without debt is characterized by monotone convergence towards the steady state. Public debt has now a strong destabilizing effect.

In case 2-iii), public debt still has a destabilizing effect but which is less radical as it amplifies fluctuations by generating persistent cycles while the economy without debt is characterized by damped fluctuations.

To summarize, Propositions 3 and 4 show that public debt has a stabilizing effect for large values of σ and ϵ_{1cG}, while it has a destabilizing effect for low values of σ and ϵ_{1cG}. We then need to provide some economic intuitions for these results. But before that we can derive some conclusions on the consequences of the stabilization effect of public debt on inequalities.

4.3 The impact on inequalities

As initially shown by Becker (1980), inequalities occur in Ramsey models as the most patient holds all the capital and thus receives capital and labor incomes, while all the other agents only receive labor income. A possible measure of inequalities can then be provided by the ratio of patient over impatient agents' incomes which is proportional to the following expression

$$\mathcal{I}(k) = \frac{R(k)\,[k+\alpha f(k)]}{\Omega(k)}$$

Straightforward computations show that

$$\mathcal{I}'(k) \gtrless 0 \quad \Leftrightarrow \quad \sigma \gtrless \frac{1+\frac{\alpha\theta}{\beta_1(1-\tau)}}{\left(1+\frac{\alpha\theta}{\beta_1 s(1-\tau)}\right)[\theta(1-s)+s]} \equiv \sigma_{\mathcal{I}} \tag{16}$$

and we derive the following result:

Lemma 2. *There exist $\underline{\beta}_1 \in (0,1)$ and $\bar{\delta} \in (0,1)$ such that when $\delta \in (0,\bar{\delta})$ and $\beta_1 \in (\underline{\beta}_1, 1)$, then $\sigma_{\mathcal{I}} > 1$ for any $\alpha \in [0,\hat{\alpha})$.*

Proof: See Appendix 6.6

Depending on whether σ is larger or lower than $\sigma_{\mathcal{I}}$, inequalities increase or decrease along an optimal growth path where capital is monotonically growing. Considering this result together with Propositions 3 and 4, we can then derive some conclusions on the impact of public debt on inequalities when the proportion α is used as a policy instrument to stabilize the economy, i.e. that leads to a saddle-point steady state with monotone convergence.

Corollary 2. *Under Assumptions 1-2, let $\delta \in (0,\bar{\delta})$ and $\beta_1 \in (\underline{\beta}_1, 1)$ with $\underline{\beta}_1$ and $\bar{\delta}$ as given by Lemma 2. Consider also the bounds $\bar{\sigma} > \underline{\sigma} > \tilde{\sigma}$ and $\underline{\epsilon}_{1cG}$ as given by Proposition 4. Then there exists $\alpha_{\mathcal{I}} \in (0,\hat{\alpha})$ such that the following results hold:*

1- When $\sigma > \sigma_{\mathcal{I}}$, the steady state is saddle-point stable with monotone convergence and inequalities are pro-cyclical if $\alpha \in (\alpha_{\mathcal{I}}, \hat{\alpha})$. A large enough public debt stabilizes but increases inequalities.

2- When $\sigma \in (\bar{\sigma}, \sigma_{\mathcal{I}})$, the steady state is saddle-point stable with monotone convergence and inequalities are counter-cyclical if $\alpha \in (\alpha_{\mathcal{I}}, \hat{\alpha})$. A large enough public debt stabilizes and decreases inequalities.

3- When $\sigma \in (\tilde{\sigma}, \bar{\sigma})$ and $\epsilon_{1cG} \in (0, \underline{\epsilon}_{1cG})$, the steady state is saddle-point stable with monotone convergence and inequalities are counter-cyclical if $\alpha \in [0, \alpha_{\mathcal{I}})$. A low enough public debt stabilizes and decreases inequalities.

Note that when σ is lower than $\tilde{\sigma}$, a low enough public debt leads to a saddle-point steady state but with damped fluctuations. In such a case, the impact on inequalities is less clear as inequalities, being counter-cyclical, will successively increase and decrease along the fluctuations.

4.4 Economic interpretation

To give an economic intuition of our previous results, we recall that using the utility function $u_1(c_{1t}, G_t) = c_{1t}^{1-\epsilon_{1cc}} G_t^{\epsilon_{1cG}}/(1-\epsilon_{1cc})$, the two dynamic equations that govern the dynamics can be written:

$$\begin{aligned} c_{1t} + k_{t+1} + b_{t+1} &= R(k_t)(k_t + b_t) + (1-\tau)w(k_t)/H \equiv I_t \qquad (17) \\ \left(\frac{c_{1t+1}}{c_{1t}}\right)^{\epsilon_{1cc}} \left(\frac{G_t}{G_{t+1}}\right)^{\epsilon_{1cG}} &= \beta_1 R(k_{t+1}) \qquad (18) \end{aligned}$$

with $b_t = \alpha f(k_t)$, $G_t = f(k_t)(\tau - \alpha R(k_t)) + \alpha f(k_{t+1})$ and $R(k_t) = 1 - \delta + (1-\tau) f'(k_t)$.

As shown by Propositions 3 and 4, when ϵ_{1cG} is weak enough, public debt can only have a destabilizing effect. To understand this property, assume for simplicity that there is no public spending externality ($\epsilon_{1cG} = 0$) and consider first that there is no debt ($\alpha = 0$). As stressed by Becker and Foias (1994), a necessary condition to have oscillations and cycles of period 2 is that capital income monotonicity fails, i.e. capital income is decreasing in capital. More precisely, as shown in Corollary 1, we need that $\sigma < \theta(1-s)(1-1/H) \equiv \tilde{\sigma}$. In this case, following an increase of k_t, the income I_t decreases, implying a decrease of k_{t+1} and explaining non-monotone dynamic paths and endogenous cycles. Note that ϵ_{1cc} needs to be large enough to prevent any intertemporal arbitrage.

Consider now the case with debt $\alpha > 0$. Since the patient household holds two assets, capital and public debt, we argue that oscillations and instability are explained by the same mechanism than before, except that it requires now the lack of asset income monotonicity. Asset income $R(k_t)(k_t + b_t) = R(k_t)(k_t + \alpha f(k_t))$ is decreasing in k_t if $\sigma < \theta(1-s)(1+b/k)/(1+sb/k)$ with $b/k = \alpha\theta/[s\beta_1(1-\tau)]$. It follows therefore that an increase of k_t will be followed by a decrease of k_{t+1} if the income I_t is again a decreasing function of k_t, namely if

$$\sigma < \frac{\theta(1-s)(1-1/H+b/k)}{1+sb/k} \equiv \tilde{\sigma}_\alpha$$

with $\tilde{\sigma}_\alpha$ an increasing function of α. Assume then that when $\alpha = 0$, the economy is not subject to fluctuations with $\sigma \in (\tilde{\sigma}, \tilde{\sigma}_{\hat{\alpha}})$.[8] Then there necessarily exists a level of debt-output ratio above which $\sigma < \tilde{\sigma}_\alpha$ and endogenous fluctuations occur. This explains that public debt has a destabilizing effect since when α raises, the range of input substitutions for saddle-point stability with monotone convergence reduces.

Consider now the case where ϵ_{1cG} is large enough. Proposition 4 shows that when σ is large enough (in particular if $\sigma > \bar{\sigma}(> \tilde{\sigma})$), public debt has on the contrary a stabilizing effect. To understand this property, assume for simplicity that $\epsilon_{1cc} = 0$ and consider first that there is no debt ($\alpha = 0$). Since $\sigma > \bar{\sigma}$, following an increase of k_t, the income I_t increases. There are oscillations and cycles of period 2 if k_{t+1} decreases. This implies an increase of $R(k_{t+1})$, but requires also a strong increase of c_{1t}. When ϵ_{1cG} is equal to 0 or is sufficiently low, this is not compatible with the Euler equation (18), since the right-hand side is increasing and the left-hand side is almost constant. On the contrary, when ϵ_{1cG} is sufficiently large, since $G_t = \tau f(k_t)$, the dominant effect on the left-hand side of (18) comes from $G_t/G_{t+1} = f(k_t)/f(k_{t+1})$. When k_t increases and k_{t+1} decreases, this ratio increases, meaning that the left-hand side of the Euler equation becomes compatible with the raise of $R(k_{t+1})$.

Consider now the case with debt $\alpha > 0$. We have:

$$\frac{dG_t}{G} = \frac{f(k)}{\beta_1 G}\left[s(\tau\beta_1 - \alpha) + \frac{\theta(1-s)\alpha}{\sigma}\right]\frac{dk_t}{k} + s\frac{b}{G}\frac{dk_{t+1}}{k} \tag{19}$$

This means that $dG_t/dk_{t+1} > 0$ but $dG_t/dk_t < 0$ if α is sufficiently large (in any case larger than $\tau\beta_1$). Consider a sequence of capital stock with oscillations, i.e. k_t larger, k_{t+1} lower and k_{t+2} larger again. When α is sufficiently large, G_t is lower, G_{t+1} larger, meaning that the ratio G_t/G_{t+1} is lower (i.e. procyclical with respect to k_{t+1}). By direct inspection of equation (18), this is not compatible with a larger $R(k_{t+1})$, explaining that when inputs are high substitutes, public debt has a stabilizing effect since a sufficiently large α promotes saddle-point stability with monotone convergence.

[8] The upper bound $\tilde{\sigma}_{\hat{\alpha}}$ is given by

$$\tilde{\sigma}_{\hat{\alpha}} = \frac{\theta(1-s)[1-1/H)s(1-\tau)(1-\beta_1)+\tau\theta]}{(1-\tau)(1-\beta_1)+\tau\theta}$$

and occurs as the destabilizing effect of debt requires a low enough elasticity of capital-labor substitution.

5 Conclusion

Public debt is introduced in a Ramsey model with heterogenous agents and a public spending externality affecting utility. Public spending is financed by income tax and debt, which is assumed to be a fixed proportion of GDP. We show that depending on the size of the externality and the value of the elasticity of capital-labor substitution, public debt can have a stabilizing or destabilizing effect by ruling out, or promoting the occurrence of endogenous fluctuations. When the public spending externality is weak, government debt can only be destabilizing, by creating damped or persistent macroeconomic fluctuations when the elasticity of capital labor substitution is low enough. But when the public spending externality is strong enough, public debt can also be stabilizing for large values of the elasticity of capital labor substitution, driving to saddle-point stability, and thus monotone convergence, an economy experiencing damped or persistent fluctuations without debt. We also show that when the ratio of public debt over GDP is used as a policy instrument to stabilize the economy, it can also decrease or increase the degree of inequalities depending on whether the elasticity of capital-labor substitution is large or low.

6 Appendix

6.1 Proof of Proposition 1

The proof of this proposition consists in three steps.

Step 1. For $i = 1$, (S2)-(S4) satisfy the optimality conditions (2), (4) and (5). Moreover, since c_1, k_1, b_1 and G are constant and $0 < \beta_1 < 1$, the transversality conditions $\lim_{t\to+\infty} \beta_1^t u_{1c}(c_1, G)\, k_1 = 0$ and $\lim_{t\to+\infty} \beta_1^t u_{1c}(c_1, G)\, b_1 = 0$ hold.

Step 2. For $i \geq 2$, given G, consider the feasible sequence $\left(\tilde{k}_{it}, \tilde{b}_{it}, \tilde{c}_{it}\right)$, starting from $\tilde{k}_{i0} = \tilde{b}_{i0} = 0$. We now compare this path with the stationary solution c_i, such that $k_i = b_i = 0$ and $c_i = (1-\tau)w$, and show that the stationary solution is optimal.

$$
\begin{aligned}
& \sum_{t=0}^{+\infty} \beta_i^t \left[u_i(c_i, G) - u_i(\tilde{c}_{it}, G)\right] \\
\geq\ & \sum_{t=0}^{+\infty} \beta_i^t u_{ic}((1-\tau)w, G)\left[(1-\tau)w - \tilde{c}_{it}\right]
\end{aligned}
$$

$$
\begin{aligned}
&= u_{ic}\left((1-\tau)w, G\right) \lim_{T\to+\infty}\left[\beta_i^T(\tilde{k}_{iT+1}+\tilde{b}_{iT+1}) + (1/\beta_i - 1/\beta_1)\sum_{t=1}^{T}\beta_i^t(\tilde{k}_{it}+\tilde{b}_{it})\right.\\
&- \left.(\tilde{k}_{i0}+\tilde{b}_{i0})/\beta_1\right] \geq -u_{ic}\left((1-\tau)w, G\right)(\tilde{k}_{i0}+\tilde{b}_{i0})/\beta_1 = 0
\end{aligned}
$$

Step 3. Under Assumption 2, there is a unique finite and strictly positive value of k such that $f'(k) = \theta/\beta_1(1-\tau)$. We further note that:

1. If $R > \frac{1}{\beta_1}$, i.e. $f'(k) > \theta/\beta_1(1-\tau)$, then it is optimal for the most patient household to increase capital. This cannot be a stationary solution because of decreasing returns in capital.

2. If $R < 1/\beta_1 < 1/\beta_2 \leq \ldots \leq 1/\beta_H$, i.e. $f'(k) < \theta/(1-\tau)\beta_1$, each household decumulates to zero. Then k_t tends to 0 and $f'(k_t)$ to $+\infty$, violating stationarity.

It follows that, if $\alpha \in [0, \hat{\alpha})$ with $\hat{\alpha} = \tau\beta_1/(1-\beta_1)$, then $k_1 = k$, $b_1 = \alpha f(k_1)$, $G = \tau(rk_1 + Hw) + (1-R)b_1 \equiv \Delta f(k_1)$ with $\Delta = \tau - \frac{\alpha(1-\beta_1)}{\beta_1} > 0$. $\square$

6.2 Proof of Lemma 1

Linearizing the dynamic system (11)-(12) around the steady state, we obtain:

$$
\begin{aligned}
\frac{dk_{t+1}}{k} &= B_2(\alpha)\frac{dk_t}{k} - B_1(\alpha)\frac{dc_{1t}}{c_1}\\
\left[1+\frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}}\alpha s B_1(\alpha)\right]\frac{dc_{1t+1}}{c_1} - \frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}}\left[B_3(\alpha) - \alpha s - \frac{(1-s)\theta\Delta}{\sigma\epsilon_{1cG}} + \alpha s B_2(\alpha)\right]\frac{dk_{t+1}}{k} &= \frac{dc_{1t}}{c_1} - \frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}}B_3(\alpha)\frac{dk_t}{k}
\end{aligned}
$$

with $B_1(\alpha)$, $B_2(\alpha)$ and $B_3(\alpha)$ given by (14). We then derive the following linear system

$$
\begin{pmatrix} \frac{dc_{1t+1}}{c_1} \\ \frac{dk_{t+1}}{k} \end{pmatrix} = J \begin{pmatrix} \frac{dc_{1t}}{c_1} \\ \frac{dk_t}{k} \end{pmatrix}
$$

with

$$
J = \begin{pmatrix} \frac{1-\frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}}\left[B_3(\alpha)-\alpha s-\frac{(1-s)\theta\Delta}{\sigma\epsilon_{1cG}}+\alpha s B_2(\alpha)\right]B_1(\alpha)}{1+\frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}}\alpha s B_1(\alpha)} & \frac{\frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}}\left[B_2(\alpha)\left[B_3(\alpha)-\alpha s-\frac{(1-s)\theta\Delta}{\sigma\epsilon_{1cG}}+\alpha s B_2(\alpha)\right]-B_3(\alpha)\right]}{1+\frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}}\alpha s B_1(\alpha)} \\ -B_1(\alpha) & B_2(\alpha) \end{pmatrix}
$$

Since T and D represent respectively the trace and the determinant of J, the result follows after straightforward simplifications.

$\square$

6.3 Proof of Proposition 2

We start by stating a property that applies for any $\alpha \in [0, \hat{\alpha})$.

Lemma 6.1 *For any $\alpha \in [0, \hat{\alpha})$, there exists one root $\lambda_1 > 1$ solution of the characteristic polynomial $P(\lambda) \equiv \lambda^2 - T\lambda + D = 0$.*

Proof. Straightforward computations from Proposition 1 give

$$P(1) = 1 - T + D = -\frac{\frac{(1-s)\theta B_1(\alpha)}{\sigma\epsilon_{1cc}}}{1 + \frac{\epsilon_{1cG}}{\Delta\epsilon_{1cc}}\alpha s B_1(\alpha)} < 0$$

Since $\lim_{\lambda\to\pm\infty} P(\lambda) = +\infty$, the result follows. $\square$

Let $\alpha = 0$. We get from Proposition 1:

$$D(0) = \frac{1}{\beta_1}\left[1 - \frac{(1-s)\theta}{\sigma}\left(1 - \frac{1}{H}\right) - \frac{\epsilon_{1cG}}{\epsilon_{1cc}} s\left(1 - \beta_1 + \frac{(1-s)\theta}{sH}\right)\right]$$

and

$$P(-1) = \frac{2}{\beta_1}\left\{1 + \beta_1 - \frac{(1-s)\theta}{\sigma}\left[1 - \frac{1}{H} - \frac{1}{2\epsilon_{1cc}}\left(1 - \beta_1 + \frac{(1-s)\theta}{sH}\right)\right] - \frac{\epsilon_{1cG}}{\epsilon_{1cc}} s\left(1 - \beta_1 + \frac{(1-s)\theta}{sH}\right)\right\}$$

Let us introduce the following bound $\tilde{\sigma} \equiv (1-s)\theta\left(1 - \frac{1}{H}\right)$ such that

$$1 - \frac{(1-s)\theta}{\sigma}\left(1 - \frac{1}{H}\right) \gtreqless 0 \quad \Leftrightarrow \quad \sigma \gtreqless \tilde{\sigma}$$

It follows that when $\sigma > \tilde{\sigma}$, $D(0) \gtreqless 0$ if and only if

$$\epsilon_{1cG} \lesseqgtr \frac{\left[1 - \frac{(1-s)\theta}{\sigma}\left(1 - \frac{1}{H}\right)\right]\epsilon_{1cc}}{s\left(1 - \beta_1 + \frac{(1-s)\theta}{sH}\right)} \equiv \underline{\epsilon}_{1cG}$$

On the contrary, when $\sigma < \tilde{\sigma}$, $D(0) < 0$ for any $\epsilon_{1cG} > 0$. Let us now introduce a second bound

$$\hat{\sigma} \equiv \frac{(1-s)\theta\left(1 - \frac{1}{H}\right)}{1 + \beta_1} < \tilde{\sigma}$$

such that

$$1 + \beta_1 - \frac{(1-s)\theta}{\sigma}\left(1 - \frac{1}{H}\right) \gtreqless 0 \quad \Leftrightarrow \quad \sigma \gtreqless \hat{\sigma}$$

It follows that when $\sigma > \hat{\sigma}$, $P(-1) \gtreqless 0$ if and only if

$$\epsilon_{1cG} \lesseqgtr \frac{1 + \beta_1 - \frac{(1-s)\theta}{\sigma}\left[1 - \frac{1}{H} - \frac{1}{2\epsilon_{1cc}}\left(1 - \beta_1 + \frac{(1-s)\theta}{sH}\right)\right]}{\frac{s}{\epsilon_{1cc}}\left(1 - \beta_1 + \frac{(1-s)\theta}{sH}\right)} \equiv \bar{\epsilon}_{1cG}$$

with $\bar{\epsilon}_{1cG} > \underline{\epsilon}_{1cG}$. Finally, when $\sigma \in (0, \hat{\sigma})$, we can define the following bound

$$\bar{\epsilon}_{1cc} \equiv \frac{(1-s)\theta\left(1 - \beta_1 + \frac{(1-s)\theta}{sH}\right)}{2(1+\beta_1)(\hat{\sigma} - \sigma)} > 0$$

and we conclude that when $\epsilon_{1cc} < \bar{\epsilon}_{1cc}$, $P(-1) \gtreqless 0$ if and only if $\epsilon_{1cG} \lesseqgtr \bar{\epsilon}_{1cG}$. Recall then that $P(1) > 0$ and $\lim_{\lambda\to\pm\infty} P(\lambda) = +\infty$ for any $\alpha \in [0, \hat{\alpha})$. On this basis, we conclude from Lemma 6.1 that the following results hold:

i) if $\sigma > \tilde{\sigma}$, then

- the second root of the characteristic polynomial satisfies $\lambda_2 \in (0,1)$ when $\epsilon_{1cG} \in [0, \underline{\epsilon}_{1cG})$ and the steady state is saddle-point stable with monotone convergence;
- the second root of the characteristic polynomial satisfies $\lambda_2 \in (-1,0)$ when $\epsilon_{1cG} \in (\underline{\epsilon}_{1cG}, \bar{\epsilon}_{1cG})$, and the steady state is saddle-point stable with damped oscillations;
- the second root of the characteristic polynomial satisfies $\lambda_2 < -1$ when $\epsilon_{1cG} > \bar{\epsilon}_{1cG}$ and the steady state is totally unstable. But when $\epsilon_{1cG} = \bar{\epsilon}_{1cG}$, $P(-1) = 0$ and a flip bifurcation occurs so that saddle-point stable (locally unstable) period-two cycles occur in a right (left) neighborhood of $\bar{\epsilon}_{1cG}$.

ii) if $\sigma \in (\hat{\sigma}, \tilde{\sigma})$, the same results as in i) are satisfied with $\underline{\epsilon}_{1cG} = 0$.

iii) if $\sigma \in (0, \hat{\sigma})$, the same results as in ii) are satisfied provided $\epsilon_{1cc} < \bar{\epsilon}_{1cc}$. On the contrary, when $\epsilon_{1cc} > \bar{\epsilon}_{1cc}$, both characteristic roots are outside the unit circle and the steady state is totally unstable for any $\epsilon_{1cG} \geq 0$. Note also that if $\epsilon_{1cG} = 0$, $\bar{\epsilon}_{1cc}$ becomes a flip bifurcation value. □

6.4 Proof of Proposition 3

If $\epsilon_{1cG} = 0$, we easily derive from (13) that when α is varied over the interval $[0, \hat{\alpha})$, $\mathcal{D}$ and $\mathcal{T}$ evolve along the following line

$$\begin{aligned} \mathcal{D} &= \frac{1-s\left(1-\frac{1}{H}\right)}{1-s\left(1-\frac{1}{H}\right)-\frac{1-s}{\epsilon_{1cc}}\left(1-\beta_1-\frac{\theta}{H}\right)}\mathcal{T} \\ &- \frac{\sigma\beta_1\left[1-s\left(1-\frac{1}{H}\right)\right]+\frac{\sigma(1-s)}{\epsilon_{1cc}}\left(1-\beta_1-\frac{\theta}{H}\right)+\frac{\theta(1-s)}{H}\left(1+\frac{\theta(1-s)}{\epsilon_{1cc}s}\right)}{1-s\left(1-\frac{1}{H}\right)-\frac{1-s}{\epsilon_{1cc}}\left(1-\beta_1-\frac{\theta}{H}\right)} \equiv \mathcal{S}\mathcal{T} - \mathcal{C} \end{aligned} \tag{20}$$

We then need to compute the starting and end points of the line. We get from Lemma 1:

$$\begin{aligned} \mathcal{D}(0) &= \frac{\sigma-\theta(1-s)\left(1-\frac{1}{H}\right)}{\sigma\beta_1} \\ \mathcal{T}(0) &= \frac{\sigma(1+\beta_1)-\theta(1-s)\left(1-\frac{1}{H}\right)}{\sigma\beta_1} + \frac{\theta(1-s)\left[1-\beta_1+\frac{\theta(1-s)}{sH}\right]}{\sigma\beta_1\epsilon_{1cc}} \\ \mathcal{D}(\hat{\alpha}) &= \frac{\sigma\left(1-\tau+\frac{\tau\theta}{1-\beta_1}\right)-\theta(1-s)(1-\tau)\left(1-\frac{1}{H}+\frac{\tau\theta}{(1-\beta_1)s(1-\tau)}\right)}{\sigma\beta_1\left(1-\tau+\frac{\tau\theta}{1-\beta_1}\right)} \\ \mathcal{T}(\hat{\alpha}) &= \frac{\sigma(1+\beta_1)\left(1-\tau+\frac{\tau\theta}{1-\beta_1}\right)-\theta(1-s)(1-\tau)\left(1-\frac{1}{H}+\frac{\tau\theta}{(1-\beta_1)s(1-\tau)}\right)}{\sigma\beta_1\left(1-\tau+\frac{\tau\theta}{1-\beta_1}\right)} \\ &+ \frac{\theta(1-s)(1-\tau)\left[(1-\beta_1)\left(1+\frac{\tau\theta}{(1-\beta_1 s(1-\tau}\right)+\frac{\theta(1-s)}{sH}\right]}{\sigma\beta_1\epsilon_{1cc}\left(1-\tau+\frac{\tau\theta}{1-\beta_1}\right)} \end{aligned}$$

As shown previously, we have $\mathcal{D}(0) \gtreqless 0$ if and only if $\sigma \gtreqless \tilde{\sigma}$ and $P(-1)|_{\alpha=0} > 0$ if $\sigma \geq \hat{\sigma}$ or $\sigma < \hat{\sigma}$ and $\epsilon_{1cc} < \bar{\epsilon}_{1cc}$, while $P(-1)|_{\alpha=0} < 0$ if $\sigma < \hat{\sigma}$ and $\epsilon_{1cc} > \bar{\epsilon}_{1cc}$. We also derive that $\mathcal{D}(\hat{\alpha}) \gtreqless 0$ if and only if

$$\sigma \gtreqqless \frac{\theta(1-s)(1-\tau)\left(1-\frac{1}{H}+\frac{\tau\theta}{(1-\beta_1)s(1-\tau)}\right)}{1-\tau+\frac{\tau\theta}{1-\beta_1}} \equiv \bar{\sigma} \;)$$

with $\bar{\sigma} > \tilde{\sigma}$, and $P(-1)|_{\alpha=\hat{\alpha}} > 0$ if

$$\sigma \geq \frac{\theta(1-s)(1-\tau)\left(1-\frac{1}{H}+\frac{\tau\theta}{(1-\beta_1)s(1-\tau)}\right)}{(1+\beta_1)\left(1-\tau+\frac{\tau\theta}{1-\beta_1}\right)} \equiv \underline{\sigma}$$

or $\sigma < \underline{\sigma}$ and

$$\epsilon_{1cc} < \frac{\theta(1-s)(1-\tau)\left[(1-\beta_1)\left(1+\frac{\tau\theta}{(1-\beta_1)s(1-\tau)}\right)+\frac{\theta(1-s)}{sH}\right]}{2(1+\beta_1)\left(1-\tau+\frac{\tau\theta}{1-\beta_1}\right)(\underline{\sigma}-\sigma)} \equiv \tilde{\epsilon}_{1cc}$$

while $P(-1)|_{\alpha=\hat{\alpha}} < 0$ if $\sigma < \underline{\sigma}$ and $\epsilon_{1cc} > \tilde{\epsilon}_{1cc}$. We also easily derive that there exists $\underline{\beta}_1 \in (0,1)$ such that when $\beta_1 \in (\underline{\beta}_1, 1)$, $\underline{\sigma} > \tilde{\sigma}$, so that we have the following ranking: $\bar{\sigma} > \underline{\sigma} > \tilde{\sigma} > \hat{\sigma}$. Moreover, when $\sigma < \hat{\sigma}$ we have $\bar{\epsilon}_{1cc} > \tilde{\epsilon}_{1cc}$. Recall finally from Lemma 6.1 that $P(1) < 0$ and $\lim_{\lambda\to\pm\infty} P(\lambda) = +\infty$ for any $\alpha \in [0, \hat{\alpha})$. We then conclude from all this:

- If $\sigma > \bar{\sigma}$ then $\mathcal{D}(0) > 0$, $\mathcal{D}(\hat{\alpha}) > 0$, $P(-1)|_{\alpha=0} > 0$ and $P(-1)|_{\alpha=\hat{\alpha}} > 0$. It follows that the steady state is saddle-point stable with monotone convergence for any $\alpha \in [0, \hat{\alpha})$.

- If $\sigma \in (\underline{\sigma}, \bar{\sigma})$ then $\mathcal{D}(0) > 0$, $\mathcal{D}(\hat{\alpha}) < 0$, $P(-1)|_{\alpha=0} > 0$ and $P(-1)|_{\alpha=\hat{\alpha}} > 0$. It follows that there exists $\underline{\alpha} \in (0,1)$ such that the steady state is saddle-point stable with monotone convergence when $\alpha \in [0, \underline{\alpha})$ and saddle-point stable with damped fluctuations when $\alpha \in (\underline{\alpha}, \hat{\alpha})$.

- If $\sigma \in (\tilde{\sigma}, \underline{\sigma})$ then $\mathcal{D}(0) > 0$, $\mathcal{D}(\hat{\alpha}) < 0$, $P(-1)|_{\alpha=0} > 0$ and $P(-1)|_{\alpha=\hat{\alpha}} \gtreqqless 0$ if and only if $\epsilon_{1cc} \lesseqqgtr \tilde{\epsilon}_{1cc}$. Therefore, when $\epsilon_{1cc} \leq \tilde{\epsilon}_{1cc}$ we get the same conclusion as in the previous case, but when $\epsilon_{1cc} > \tilde{\epsilon}_{1cc}$, there exist $\hat{\alpha} > \bar{\alpha} > \underline{\alpha} > 0$ such that the steady state is saddle-point stable with monotone convergence when $\alpha \in [0, \underline{\alpha})$, saddle-point stable with damped fluctuations when $\alpha \in (\underline{\alpha}, \bar{\alpha})$, undergoes a flip bifurcation when $\alpha = \bar{\alpha}$ and becomes locally unstable with oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$. Moreover, there exist saddle-point stable (locally unstable) period-two cycles in a right (left) neighborhood of $\bar{\alpha}$.

- If $\sigma \in (\hat{\sigma}, \tilde{\sigma})$ then $\mathcal{D}(0) < 0$, $\mathcal{D}(\hat{\alpha}) < 0$, $P(-1)|_{\alpha=0} > 0$ and $P(-1)|_{\alpha=\hat{\alpha}} \gtreqqless 0$ if and only if $\epsilon_{1cc} \lesseqqgtr \tilde{\epsilon}_{1cc}$. Therefore, when $\epsilon_{1cc} \leq \tilde{\epsilon}_{1cc}$ the steady state is saddle-point stable with damped fluctuations for any $\alpha \in [0, \hat{\alpha})$, but when $\epsilon_{1cc} > \tilde{\epsilon}_{1cc}$, there exists $\bar{\alpha} \in (0, \hat{\alpha})$ such that the steady state is saddle-point stable with damped fluctuations when $\alpha \in [0, \bar{\alpha})$, undergoes a flip bifurcation when $\alpha = \bar{\alpha}$ and becomes locally unstable with oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$. Moreover, there exist saddle-point stable (locally unstable) period-two cycles in a right (left) neighborhood of $\bar{\alpha}$.

- If $\sigma \in (0, \hat{\sigma})$ then $\mathcal{D}(0) < 0$, $\mathcal{D}(\hat{\alpha}) < 0$, $P(-1)|_{\alpha=0} \gtreqless 0$ if and only if $\epsilon_{1cc} \lesseqgtr \bar{\epsilon}_{1cc}$ and $P(-1)|_{\alpha=\hat{\alpha}} \gtreqless 0$ if and only if $\epsilon_{1cc} \lesseqgtr \tilde{\epsilon}_{1cc}$, with $\bar{\epsilon}_{1cc} > \tilde{\epsilon}_{1cc}$. Therefore, when $\epsilon_{1cc} \leq \tilde{\epsilon}_{1cc}$ the steady state is saddle-point stable with damped fluctuations for any $\alpha \in [0, \hat{\alpha})$ and when $\epsilon_{1cc} > \bar{\epsilon}_{1cc}$ the steady state is locally unstable with fluctuations for any $\alpha \in [0, \hat{\alpha})$. On the contrary, when $\epsilon_{1cc} \in (\tilde{\epsilon}_{1cc}, \bar{\epsilon}_{1cc})$, there exists $\bar{\alpha} \in (0, \hat{\alpha})$ such that the steady state is saddle-point stable with damped fluctuations when $\alpha \in [0, \bar{\alpha})$, undergoes a flip bifurcation when $\alpha = \bar{\alpha}$ and becomes locally unstable with oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$. Moreover, there exist saddle-point stable (locally unstable) period-two cycles in a right (left) neighborhood of $\bar{\alpha}$.

The results follow.

□

6.5 Proof of Proposition 4

Let $\alpha \in (0, \hat{\alpha})$. In the limit case $\alpha = \hat{\alpha}$, we get $\Delta = 0$ and thus from Lemma 1:

$$D(\hat{\alpha}) = 1 - \frac{(1-s)\theta}{\sigma\beta_1 s} \quad \text{and} \quad P(-1) = 2\left(2 - \frac{(1-s)\theta}{\sigma\beta_1 s}\right)$$

It follows that $D(\hat{\alpha}) < 0$ if and only if $\sigma < (1-s)\theta/(\beta_1 s) \equiv \bar{\sigma}$, and $P(-1) < 0$ if and only if $\sigma < (1-s)\theta/(2\beta_1 s) \equiv \underline{\sigma}$.

Assume that $\delta < s < 1/2$ and $\beta_1 > 1/(1+s-\delta)$. It follows that $\sigma_{CIM} > \bar{\sigma} > \underline{\sigma}$. Obvious computations also show that $\bar{\sigma}, \underline{\sigma} > \tilde{\sigma}$. We then conclude from Lemma 6.1 and Proposition 2 that there exists $\bar{\alpha} \in [0, \hat{\alpha})$ such that the following results hold:

i) if $\sigma > \bar{\sigma}$, for any $\epsilon_{1cG} > 0$ the steady state is saddle-point stable with monotone convergence when $\alpha \in (\bar{\alpha}, \hat{\alpha})$.

ii) if $\sigma \in (\underline{\sigma}, \bar{\sigma})$, for any $\epsilon_{1cG} > 0$ the steady state is saddle-point stable with damped oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$.

iii) if $\sigma \in (0, \underline{\sigma})$, for any $\epsilon_{1cG} > 0$ the steady state is totally unstable with oscillations when $\alpha \in (\bar{\alpha}, \hat{\alpha})$.

The results follow considering Proposition 2 and the fact that $0 < \hat{\sigma} < \tilde{\sigma} < \underline{\sigma} < \bar{\sigma} < \sigma_{CIM}$.

□

6.6 Proof of Lemma 2

Let us consider the expression of the bound $\sigma_{\mathcal{I}}$ as given by (16). We get $\sigma_{\mathcal{I}} > 1$ if and only if

$$g(\alpha) \equiv 1 - \theta - \frac{\alpha\theta^2}{\beta_1(1-\tau)s} > 0$$

We have $g(0) > 0$ and $g'(\alpha) < 0$. We then need to evaluate this function at the upper bound $\hat{\alpha} = \tau\beta_1/(1-\beta_1)$. Using the expression of θ we derive

$$g(\hat{\alpha}) = \frac{\beta_1(1-\beta_1)(1-\delta)(1-\tau)s-\tau\theta^2}{(1-\beta_1)(1-\tau)s}$$

Assuming $\delta = 0$ we get

$$g(\hat{\alpha}) = \frac{\beta_1(1-\delta)(1-\tau)s-\tau(1-\beta_1)}{(1-\tau)s}$$

and this expression is strictly positive when $\beta_1 = 1$. Then there exist $\underline{\beta}_1 \in (0,1)$ and $\bar{\delta} \in (0,1)$ such that when $\delta \in (0,\bar{\delta})$ and $\beta_1 \in (\underline{\beta}_1, 1)$, then $\sigma_{\mathcal{I}} > 1$ for any $\alpha \in [0,\hat{\alpha})$.
□

References

[1] Becker, R. (1980): "On the Long-Run Steady State in a Simple Dynamic Model of Equilibrium with Heterogeneous Households," *Quarterly Journal of Economics* **95**, 375-382.

[2] Becker, R., Bosi, S., Le Van, C. and T. Seegmuller (2013): "On Existence and Bubbles of Ramsey Equilibrium with Borrowing Constraints," *AMSE Working Paper* 2012-31.

[3] Becker, R., J. Boyd III, and C. Foias (1991): "The Existence of Ramsey Equilibrium," *Econometrica* **59**, 441-460.

[4] Becker, R. and C. Foias (1987): "A Characterization of Ramsey Equilibrium," *Journal of Economic Theory* **41**, 173-184.

[5] Becker, R. and C. Foias (1994): "The Local Bifurcation of Ramsey Equilibrium," *Economic Theory* **4**, 719-744.

[6] Bosi, S. and T. Seegmuller (2010): "On the Ramsey Equilibrium with Heterogeneous Consumers and Endogenous Labor Supply," *Journal of Mathematical Economics* **46**, 475-492.

[7] Chirinko, R., (2008): "The CES Production Function in the Theory and Empirics of Economic Growth," *Journal of Macroeconomics* **30**, 671-686.

[8] de la Croix, D. and P. Michel (2002): *A Theory of Economic Growth, Dynamics and Policy in Overlapping Generations.* Cambridge University Press.

[9] Duffy, J., and C. Papageorgiou (2000): "A Cross-Country Empirical Investigation of the Aggregate Production Function Specification," *Journal of Economic Growth* **5**, 87-120.

[10] Guo, J.-T. and S. Harrison (2008): "Useful Government Spending and Macroeconomic (In)Stability under Balanced-Budget Rules," *Journal of Public Economic Theory* **10**, 383-397.

[11] Herndon, T., M. Ash and R. Pollin (2013): "Does High Public Debt Consistently Stifle Economic Growth? A Critique of Reinhart and Rogoff," *Working Paper 322*, University of Massachusetts Amherst.

[12] Ilzetzki, E., E. Mendoza and C. Vègh (2013): "How Big (Small?) are Fiscal Multipliers?," *Journal of Monetary Economics* **60(2)**, 239-254.

[13] Klump, R., P. McAdam and A. Willman, (2007): "Factor Substitution and Factor-Augmenting Technical Progress in the United States: a Normalized Supply-Side System Approach," *Review of Economics and Statistics* **89**, 183-192.

[14] León-Ledesma, M., P. McAdam and A. Willman (2010): "Identifying the Elasticity of Substitution with Biased Technical Change," *American Economic Review* **100**, 1330-1357.

[15] Nishimura, K., C. Nourry, T. Seegmuller and A. Venditti (2013): "Public Spending as a Source of Endogenous Business Cycles in a Ramsey Model with Many Agents," forthcoming in *Macroeconomic Dynamics*.

[16] Ramsey, F. (1928): "A Mathematical Theory of Saving," *Economic Journal* **38**, 543-559.

[17] Reinhart, C. and K. Rogoff (2010): "Growth in a Time of Debt," *American Economic Review* **100**, 573-578.

[18] Reinhart, C., V. Reinhart and K. Rogoff (2012): "Public Debt Overhangs: Advanced-Economy Episodes since 1800," *Journal of Economic Perspectives* **26**, 69-86.

[19] Sorger, G. (1994): "On the Structure of Ramsey Equilibrium: Cycles, Indeterminacy, and Sunspots," *Economic Theory* **4**, 745-764.

[20] Sorger, G. (2002): "On the Long-Run Distribution of Capital in the Ramsey Model," *Journal of Economic Theory* **105**, 226-243.

[21] Sutherland, D., P. Hoeller, R. Merola and V. Zieman (2012): "Debt and Macroeconomic Stability," *OECD Economic Department Working Paper* **1003**.