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Existence, Uniqueness and Stability of Stationary Distributions: An Extension of the Hopenhayn-Prescott Theorem*

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Abstract

This paper strengthens the Hopenhayn-Prescott stability theorem for monotone economies by extending it to a significantly larger class of models. We provide general conditions for existence, uniqueness and stability of stationary distributions. The conditions in our main result are both necessary and sufficient for global stability of monotone economies that satisfy a weak mixing condition introduced in the paper. Through our analysis we develop new insights on the nature and causes of stability and instability.

JEL Classification: C62, C63
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1 Introduction

Hopenhayn and Prescott's (1992) stability theorem is a standard tool for analysis of dynamics and stationary equilibria. For example, Huggett (1993) used the the-

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orem to study asset distributions in incomplete-market economies with infinitely-lived agents.[1] The same theorem was applied to variants of Huggett's model with features such as habit formation, endogenous labor supply, capital accumulation and international trade (Díaz *et al.*, 2003; Joseph and Weitzenblum, 2003; Pijoan-Mas, 2006; Marcet *et al.*, 2007; Portes, 2009). It was used to study the classical one-sector optimal growth model by Hopenhayn and Prescott (1992), a stochastic endogenous growth model by de Hek (1999), and a small open economy by Chatterjee and Shukayev (2010). It has been used in a wide range of OLG models with features such as credit rationing (Aghion and Bolton, 1997; Piketty, 1997), human capital (Owen and Weil, 1998; Lloyd-Ellis, 2000; Cardak, 2004; Couch and Morand, 2005; Cabrillana, 2009), international trade (Ranjan, 2001; Das, 2006), nonconcave production (Morand and Reffett, 2007), and occupational choice (Lloyed-Ellis and Bernhardt, 2000; Antunes and Cavalcanti, 2007; Antunes *et al.*, 2008). Other well-known applications include variants of Hopenhayn and Rogerson's (1993) model of job turnover (Cabrales and Hopenhayn, 1997; Samaniego, 2008) as well as variants of Hopenhayn's (1993) model of entry and exit (Cooley and Quadrini, 2001; Samaniego, 2006).

Although Hopenhayn and Prescott's theorem has already proved to be important, there are important economic models to which it does not apply. A typical problem is that the theorem requires that the state space be compact and order bounded. This condition is not satisfied if, for example, we are working with a macroeconomic model where exogenous productivity follows an AR(1) process with normal (or lognormal) shocks, or with a model of wealth or firm size distributions where the interest is in whether the right-hand tail follows a power law.

In this paper we show that it is possible to significantly weaken the conditions of Hopenhayn and Prescott's theorem. We begin by introducing a mixing condition called "order reversing" that is weaker than the mixing condition used by Hopenhayn and Prescott. We also relax the restriction that the state space be compact and order bounded. In this setting, we obtain general conditions for monotone, order reversing processes to attain global stability. The conditions are also necessary, and hence we are able to fully characterize global stability for monotone economies that satisfy this very weak mixing condition.

To date, one major difficulty in extending the Hopenhayn-Prescott theorem has been due to the fact that the proof of the existence of a stochastic steady state uses

[1] See Kam and Lee (2011) for a recent extension of Huggett's (1993) analysis.

the Knaster-Tarski fixed point theorem in the space of distributions on the state. For non-compact state spaces the Knaster-Tarski theorem cannot be applied, since a chain in the space of distributions need not have a supremum or an infimum. Our fixed point argument is new, combining order-theoretic and topological results to obtain existence of the stochastic steady state. Freeing Hopenhayn and Prescott's existence result from the Knaster-Tarski theorem has the obvious benefit of permitting more general state spaces. On a deeper level, it also relaxes a tension that is present in the original formulation of the stability theorem: On one hand, a compact state space is needed to apply the Knaster-Tarski theorem, which yields existence of a stationary distribution. On the other hand, the restriction to compact state spaces requires that shocks have relatively small supports, which in turn implies less mixing. Since mixing is associated with uniqueness and stability of stationary distributions, reduced mixing means that these properties are less likely to hold.[2]

To put these ideas in a different light, suppose that we have a model with normal shocks, and hence the state space is unbounded. It is potentially possible to truncate these shocks, thereby creating a version of the model with a compact state space. One immediate problem is that we are approximating in an ad hoc manner. A second problem, alluded to above, is that the stability problem may now be significantly *harder*, because we have reduced the amount of mixing in the model. A third problem is that certain questions become more difficult to address, such as whether large shocks are stabilizing or destabilizing, or whether the tails are well modeled by a Pareto distribution. For all of these reasons it may be preferable to work with the original model. As we show below, this can be done in a natural and convenient way.

Our results are illustrated in two applications: a model of renewable resource exploitation and an overlapping generations model with borrowing constraints. In both applications, we illustrate situations where the conditions of our extended Hopenhayn-Prescott theorem are satisfied, while those of the theorem in its original formulation do not hold. In fact no current theory from the literature on Markov processes can be used to obtain existence of a stationary distribution in these cases. Our applications also shed some light on the extent to which large shocks are destabilizing. Our results suggest that, provided that the fundamentals of the model act against divergence, large shocks are not destabilizing. On the contrary, large shocks generate mixing, and mixing promotes stability.

[2]For a discussion of the relationship between mixing and stability see Stokey *et al.* (1989, p. 380).

Concerning related literature, monotone economic models with the Markov property have been studied by Razin and Yahav (1979), Bhattacharya and Lee (1988), Stokey, Lucas and Prescott (1989), Hopenhayn and Prescott (1992), Bhattacharya and Majumdar (2001), Bhattacharya *et al*. (2010) and Szeidl (2012). Hopenhayn and Prescott (1992) is an extension of the work by Stokey, Lucas and Prescott (1989), which in turn extends Razin and Yahav (1979). The papers by Bhattacharya and Lee (1988), Bhattacharya and Majumdar (2001) and Bhattacharya *et al*. (2010) studied stability in the monotone setting via a mixing condition called "splitting." Our order reversing condition is weaker than splitting. At the same time, the literature on splitting contains many important results not treated in this paper.

The paper by Szeidl (2012) is also a direct extension of the Hopenhayn-Prescott stability result for monotone economies. It studies processes that satisfy a certain "weak mixing" condition. Our order reversing condition is weaker than this weak mixing condition, and the main stability results in Szeidl's paper are special cases of theorems 3.1 and 3.2 below. Nonetheless, Szeidl's paper contains many interesting and thoughtful arguments, and his weak mixing condition is a useful way to check order reversing.

The rest of the paper is structured as follows: Section 2 reviews some basic definitions and introduces the concept of order reversing. Section 3 states the main results and compares them to the original formulation of the Hopenhayn-Prescott stability theorem. Section 4 gives applications and section 5 concludes.

2 Preliminaries

At each time $t = 0, 1, \dots$, the state of the economy is described by a point X_t in topological space S. The space S is equipped with its Borel sets $\mathscr{B}_S$ and a closed partial order $\leq$. An order interval of S is a set of the form $[a, b] := \{x \in S : a \leq x \leq b\}$. A function $f \colon S \to \mathbb{R}$ is called *increasing* if $f(x) \leq f(y)$ whenever $x \leq y$. A subset B of S is called *order bounded* if there exists an order interval $[a, b] \subset S$ with $B \subset [a, b]$. In addition, B is called *increasing* if its indicator function $\mathbb{1}_B$ is increasing, and *decreasing* if $\mathbb{1}_B$ is decreasing.

To simplify terminology, we often use the word "distribution" to mean "probability measure on $(S, \mathscr{B}_S)$". The set of all probability measures on $(S, \mathscr{B}_S)$ will be denote by $\mathscr{P}_S$. We let cbS denote the continuous bounded functions from S to $\mathbb{R}$, and ibS denote the set of increasing bounded measurable functions from S to $\mathbb{R}$. We adopt the standard definitions of convergence in distribution and stochastic domination: Given sequence $\{\mu_n\}_{n=0}^{\infty}$ in $\mathscr{P}_S$, we say that μ_n *converges to* μ and write $\mu_n \to \mu_0$ if $\int h \, d\mu_n \to \int h \, d\mu_0$ for all $h \in cbS$. We say that μ_2 *stochastically dominates* μ_1 and write $\mu_1 \preceq \mu_2$ if $\int h \, d\mu_1 \leq \int h \, d\mu_2$ for all $h \in ibS$.

Following Hopenhayn and Prescott (1992), we assume that S is a normally ordered Polish space.[3] Hopenhayn and Prescott assume in addition that S is compact, with least element a and greatest element b.[4] Since we wish to include more general state spaces such as $\mathbb{R}^n$, we make the weaker assumption that a subset of S is compact if and only if it is closed and order bounded. This is obviously the case in Hopenhayn and Prescott's setting, where all subsets of S are order bounded, and any closed subset is compact. It also holds for $S = \mathbb{R}^n$ with its standard partial order, since order boundedness is then equivalent to boundedness. In addition, it holds in common state spaces such as $\mathbb{R}^n_+$ or $\mathbb{R}^n_{++}$, or in any set of the form $I_1 \times \cdots \times I_n \subset \mathbb{R}^n$, where each I_i is an open, closed, half-open or half-closed interval in $\mathbb{R}$.[5]

2.1 Markov Properties

Throughout the paper, we suppose that the model under consideration is time-homogeneous and Markovian. The dynamics of such a model can be summarized by a stochastic kernel Q, where $Q(x, B)$ represents the probability that the state moves from $x \in S$ to $B \in \mathscr{B}_S$ in one unit of time. As usual, we require that $Q(x, \cdot) \in \mathscr{P}_S$ for each $x \in S$, and that $Q(\cdot, B)$ is measurable for each $B \in \mathscr{B}_S$. For each $t \in \mathbb{N}$, let Q^t be the t-th order kernel, defined by

$$Q^1 := Q, \quad Q^t(x, B) := \int Q^{t-1}(y, B) Q(x, dy) \qquad (x \in S, \, B \in \mathscr{B}_S).$$

The value $Q^t(x, B)$ represents the probability of transitioning from x to B in t steps.

Here and below, $(\Omega, \mathscr{F}, \mathbb{P})$ denotes a fixed probability space on which all random variables are defined, and $\mathbb{E}$ is the corresponding expectations operator. Given $\mu \in \mathscr{P}_S$ and stochastic kernel Q, an S-valued stochastic process $\{X_t\}_{t \in \mathbb{Z}_+}$ is called

[3] A Polish space is a separable and completely metrizable topological space. The space $(S, \leq)$ is normally ordered if, given any closed increasing set I and closed decreasing set D with $I \cap D = \emptyset$, there exists an f in $ibS \cap cbS$ such that $f(x) = 0$ for all $x \in D$ and $f(x) = 1$ for all $x \in I$.

[4] A point a is called a *least element* of S if $a \in S$ and $a \leq x$ for all $x \in S$. A point b is called a *greatest element* of S if $b \in S$ and $x \leq b$ for all $x \in S$.

[5] A simple example that does not satisfy our assumptions is $S = (0, 1) \cup (2, 3)$. In this case the order interval $[0.5, 2.5]$ is closed and order bounded but not compact.

(Q, μ)-*Markov* if X_0 has distribution μ and $Q(x, \cdot)$ is the conditional distribution of X_{t+1} given $X_t = x$.[6] If μ is the distribution $\delta_x \in \mathscr{P}_S$ concentrated on $x \in S$, we call $\{X_t\}$ (Q, x)-*Markov*. We call $\{X_t\}$ Q-*Markov* if $\{X_t\}$ is (Q, μ)-Markov for some $\mu \in \mathscr{P}_S$.

Example 2.1. Many economic models result in processes for the state variables represented by nonlinear, vector-valued stochastic difference equations. As a generic example, consider the S-valued process

$$X_{t+1} = F(X_t, \xi_{t+1}), \qquad \{\xi_t\} \overset{\text{IID}}{\sim} \phi, \tag{1}$$

where $\{\xi_t\}$ takes values in $Z \subset \mathbb{R}^m$, the function $F \colon S \times Z \to S$ is measurable, and ϕ is a probability measure on the Borel sets of Z. Let Q_F be the kernel

$$Q_F(x, B) := \mathbb{P}\{F(x, \xi_t) \in B\} = \phi\{z \in Z : F(x, z) \in B\}. \tag{2}$$

Then $\{X_t\}$ in (1) is Q_F-Markov.[7]

For each Q we define two operators, sometimes called the left and right Markov operators. The left Markov operator maps $\mu \in \mathscr{P}_S$ into $\mu Q \in \mathscr{P}_S$, where

$$(\mu Q)(B) := \int Q(x, B)\mu(dx) \qquad (B \in \mathscr{B}_S). \tag{3}$$

The right Markov operator maps bounded measurable function $h \colon S \to \mathbb{R}$ into bounded measurable function Qh, where

$$(Qh)(x) := \int h(y)Q(x, dy) \qquad (x \in S).$$

The interpretation of the left Markov operator $\mu \mapsto \mu Q$ is that it shifts the distribution for the state forward by one time period. In particular, if $\{X_t\}$ is (Q, μ)-Markov, then μQ^t is the distribution of X_t. The interpretation of the right Markov operator $h \mapsto Qh$ is that $(Q^t h)(x)$ is the expectation of $h(X_t)$ given $X_0 = x$. If Q_F is the kernel in (2), then $(Q_F h)(x) = \int h[F(x, z)]\phi(dz)$. Also, given any $x \in S$, $B \in \mathscr{B}_S$ and $t \in \mathbb{N}$, the t-th order kernel and the left and right Markov operators are related by $Q^t(x, B) = (\delta_x Q^t)(B) = (Q^t \mathbb{1}_B)(x)$. Here $\mathbb{1}_B$ is the indicator function of B.

[6] More formally, $\mathbb{P}[X_{t+1} \in B \,|\, \mathscr{F}_t] = Q(X_t, B)$ almost surely for all $B \in \mathscr{B}_S$, where $\mathscr{F}_t$ is the σ-algebra generated by the history $X_0, \ldots, X_t$.

[7] Although the process (1) is only first order, models including higher order lags of the state and shock process can be rewritten in the form of (1) by redefining the state variables.

A stochastic kernel Q is called *bounded in probability* if the sequence $\{Q^t(x, \cdot)\}_{t \geq 0}$ is tight for all $x \in S$.[8] If $\mu^* \in \mathscr{P}_S$ and $\mu^* Q = \mu^*$, then μ^* is called *stationary* (or *invariant*) for Q. If Q has a unique stationary distribution μ^* in $\mathscr{P}_S$, and, in addition, $\mu Q^t \to \mu^*$ as $t \to \infty$ for all $\mu \in \mathscr{P}_S$, then Q is called *globally stable*. In this case, μ^* is naturally interpreted as the long-run equilibrium of the economic system. If μ^* is stationary, then any (Q, μ^*)-Markov process $\{X_t\}$ is strict-sense stationary with $X_t \sim \mu^*$ for all t.

If $\mu \in \mathscr{P}_S$ and $\mu Q \preceq \mu$, then μ is called *excessive*. If $\mu \preceq \mu Q$, then μ is called *deficient*. If Q satisfies $\mu Q \preceq \mu' Q$ whenever $\mu \preceq \mu'$, then Q is called *increasing*.[9] It is in fact sufficient to check that $Q(x, \cdot) \preceq Q(x', \cdot)$ whenever $x \leq x'$. A third equivalent condition is that $Qh \in ibS$ whenever $h \in ibS$. If, on the other hand, $Qh \in cbS$ whenever $h \in cbS$, then Q is called *Feller*.

Remark 2.1. Let Q be an increasing stochastic kernel. If A is an increasing set, then $x \mapsto Q(x, A)$ is increasing. If A is a decreasing set, then $x \mapsto Q(x, A)$ is decreasing.

Remark 2.2. If S has a least element a, then δ_a is deficient for any kernel Q, because $\delta_a \preceq \mu$ for every $\mu \in \mathscr{P}_S$, and hence $\delta_a \preceq \delta_a Q$. Similarly, if S has a greatest element b, then δ_b is excessive for Q.

Remark 2.3. Let F and Q_F be as in example 2.1. If $x \mapsto F(x, z)$ is increasing, then Q_F is increasing. If $x \mapsto F(x, z)$ is continuous, then Q_F is Feller.

2.2 Order Reversing

Our next step is to introduce a new order-theoretic mixing condition. We will say that a stochastic kernel Q is *order reversing* if, given any x and x' in S with $x' \leq x$, and independent Q-Markov processes $\{X_t\}$ and $\{X'_t\}$ starting at x and x' respectively, there exists a $t \in \mathbb{N}$ with $\mathbb{P}\{X_t \leq X'_t\} > 0$. In other words, the initial ordering is reversed at some point in time with positive probability.

[8] Sequence $\{\mu_n\} \subset \mathscr{P}_S$ is called *tight* if, for all $\epsilon > 0$, there exists a compact $K \subset S$ such that $\mu_n(K \setminus S) \leq \epsilon$ for all n.

[9] Many examples of models with increasing kernels were given in the introduction. Other examples not discussed there include various infinite horizon optimal growth models with features such as irreversible investment, renewable resources, distortions, and capital-dependent utility. Increasing kernels are also found in stochastic OLG models besides those mentioned previously, such as models with limited commitment, and in a variety of stochastic games. See, for example, Amir (2002, 2005), Gong et al. (2010), Balbus et al. (2010), Olson (1989), Olson and Roy (2000), Datta et al. (2002) and Mirman et al. (2008). For an empirical test of the increasing property, see Lee *et al.* (2009).

Two remarks on the definition are as follows: First, in verifying order reversing, it is clearly sufficient to check the existence of a t with $\mathbb{P}\{X_t \leq X'_t\} > 0$ for *arbitrary* pair $x, x' \in S$. Often this is just as easy, and much of the following discussion proceeds accordingly. Second, it is not entirely clear from the definition given here that order reversing is a property of Q alone. This fact is clarified in the technical appendix, where we give an alternative (more formal) definition.

Below we give some examples that illustrate order reversing, and show that for any increasing kernel Q, order reversing is weaker than the monotone mixing condition (MMC) used in Hopenhayn and Prescott (1992). For increasing kernels, order reversing is also weaker than the splitting condition used by Bhattacharya and Majumdar (2001), the "weak mixing" condition used by Szeidl (2012), and the "order mixing" condition used by Kamihigashi and Stachurski (2011a). The proofs are quite straightforward, and details are available from the authors.

Example 2.2. Suppose we are studying a dynamic model of household wealth. Informally, the model is order reversing if, for two households receiving idiosyncratic shocks from the same distribution, the wealth of the first household is less than that of the second at some point in time with non-zero probability, regardless of initial wealth for each of the two households.

Example 2.3. Let S be a compact metric space with least element a and greatest element b, and let Q be an increasing kernel on S. In this setting, Q is said to satisfy the MMC whenever

$$\exists\, \bar{x} \in S \text{ and } k \in \mathbb{N} \text{ such that } Q^k(a, [\bar{x}, b]) > 0 \text{ and } Q^k(b, [a, \bar{x}]) > 0. \tag{4}$$

Under these conditions, Q is order reversing: If we start independent Q-Markov processes $\{X_t^a\}$ and $\{X_t^b\}$ at a and b respectively, then (4) implies the order reversal $X_k^b \leq X_k^a$ occurs at time k with positive probability. Since Q is increasing, closer initial conditions only make this event more likely.[10]

Example 2.4. Consider the stochastic kernel $Q(x, B) = \mathbb{P}\{\rho x + \xi_t \in B\}$ on $S = \mathbb{R}$ associated with the linear Gaussian model

$$X_{t+1} = \rho X_t + \xi_{t+1}, \qquad \{\xi_t\} \overset{\text{IID}}{\sim} N(0,1). \tag{5}$$

[10] To be precise, let $\bar{x}$ and k be as in (4). Fix $x, x' \in S$ and let $\{X_t\}$ and $\{X'_t\}$ be independent, (Q, x)-Markov and (Q, x')-Markov respectively. By independence and $\{X_k \leq \bar{x} \leq X'_k\} \subset \{X_k \leq X'_k\}$, we have $\mathbb{P}\{X_k \leq \bar{x}\}\mathbb{P}\{\bar{x} \leq X'_k\} = \mathbb{P}\{X_k \leq \bar{x} \leq X'_k\} \leq \mathbb{P}\{X_k \leq X'_k\}$. But $\mathbb{P}\{\bar{x} \leq X'_k\} = Q^k(x, [a, \bar{x}])$ and $\mathbb{P}\{X_k \leq \bar{x}\} = Q^k(x, [\bar{x}, b])$ are strictly positive by (4) and remark 2.1. Hence Q is order reversing.

This kernel does not satisfy the MMC because $S = \mathbb{R}$. On the other hand, it is order reversing. To see this, fix $(x, x') \in \mathbb{R}^2$, and take a second, independent Q-Markov process $X'_{t+1} = \rho X'_t + \xi'_{t+1}$ with $X'_0 = x'$, where $\{\xi_t\}$ and $\{\xi'_t\}$ are IID, standard normal, and independent of each other. We can see that, for any given pair (x, x') of initial conditions, $\mathbb{P}\{X_t \leq X'_t\} > 0$ is satisfied with $t = 1$, because

$$\mathbb{P}\{X_1 \leq X'_1\} = \mathbb{P}\{\rho x + \xi_1 \leq \rho x' + \xi'_1\} = \mathbb{P}\{\xi_1 - \xi'_1 \leq \rho(x' - x)\}.$$

Since $\xi_1 - \xi'_1$ is Gaussian, this probability is strictly positive.

3 Results

We can now state our main results, which concern stability of increasing, order reversing stochastic kernels.

3.1 Global Stability

Our first result extends Hopenhayn and Prescott's stability theorem to a broader class of models. It also characterizes the set of increasing order reversing kernels that are globally stable. The proof is in section 6.

Theorem 3.1. *Let Q be a stochastic kernel that is both increasing and order reversing. Then Q is globally stable if and only if*

1. *Q is bounded in probability, and*
2. *Q has either a deficient or an excessive distribution.*

Remark 3.1. In terms of sufficient conditions for global stability, the order reversing assumption cannot be omitted, even for *existence* of a stationary distribution. In particular, there exist increasing kernels that are bounded in probability and possess an excessive or deficient distribution, but have no stationary distribution.[11]

[11] An example is the kernel Q associated with the deterministic process on $S = \mathbb{R}_+$ defined by $X_{t+1} = 1/2 + \sum_{n=0}^{\infty} \mathbb{1}\{n \leq X_t < n+1\}(n + (X_t - n)/2)$. It is easy to check that $X_{t+1} > X_t$ with probability one, and hence X_{t+1} and X_t can never have the same distribution. On the other hand, Q is increasing, bounded in probability (because each interval $[n, n+1)$ is absorbing) and has the deficient distribution δ_0 (cf., remark 2.2).

To see that the conditions of theorem 3.1 are weaker than those of the original Hopenhayn-Prescott stability theorem, suppose as they do that S is a compact metric space with least element a and greatest element b, and Q is an increasing kernel satisfying the MMC. The conditions of theorem 3.1 then hold. First, Q is increasing by assumption. Second, Q is order reversing, as shown in example 2.3. Third, Q is bounded in probability, since S is compact and hence $\{Q^t(x, \cdot)\}$ is always tight. Fourth, Q has a deficient distribution because S has a least element (see remark 2.2).

To see that the conditions of theorem 3.1 are strictly weaker than those of Hopenhayn and Prescott, consider the linear Gaussian model (5) with $\rho \in [0, 1)$. Here the Gaussian shocks force us to choose the state space $S = \mathbb{R}$, which is not compact, and the Hopenhayn-Prescott theorem in its original formulation cannot be applied. On the other hand, all the conditions of theorem 3.1 are satisfied.[12] (Of course this is an extremely simple example. Nontrivial applications are presented in section 4.)

Regarding the proof of theorem 3.1, boundedness in probability and existence of an excessive or deficient distribution generalize Hopenhayn and Prescott's assumption that S is compact and has a least and greatest element. As Hopenhayn and Prescott show, if S is compact and has a least and greatest element, then the Knaster-Tarski fixed point theorem implies that every increasing stochastic kernel has a stationary distribution. Adding the MMC then yields uniqueness and global stability. In our setting, the same fixed point argument cannot be applied. As remark 3.1 shows, our mixing condition plays an essential role in the proof of existence, and the proof is fundamentally different to the Knaster-Tarski fixed point argument.

We make two final remarks. First, one of the most attractive features of the MMC is that it is straightforward to check in applications when it holds. In section 3.2.3, we provide conditions for order reversing that are also straightforward to verify when they hold. Second, there is no continuity requirement in theorem 3.1. However, in many applications the kernel Q will have the Feller property (see remark 2.3). If Q is Feller, then condition 2 can be omitted. Since this result is likely to be useful, we state it as a second theorem.

Theorem 3.2. *Let Q be increasing, order reversing, and Feller. Then Q is globally stable if and only if Q is bounded in probability.*

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[12] That the model is order reversing was shown in example 2.3. Monotonicity follows from remark 2.3. Boundedness in probability is shown below. For existence of a μ with $\mu \preceq \mu Q$, we can take $\mu = N(0, (1-\rho^2)^{-1})$.

3.2 Verifying the Conditions

Theorem 3.1 requires that Q is increasing, order reversing, bounded in probability, and possesses an excessive or deficient distribution. A sufficient condition for Q to be increasing was given in remark 2.3. In this section, we present a number of sufficient conditions for the remaining properties.

3.2.1 Checking Boundedness in Probability

Boundedness in probability is a standard condition in the Markov process literature. As is well known, if Q is a stochastic kernel on either $S = \mathbb{R}^n$ or $S = \mathbb{R}^n_+$, then Q is bounded in probability whenever $\sup_t \mathbb{E} \|X_t\| < \infty$ for any (Q, x)-Markov process $\{X_t\}$. (The norm $\| \cdot \|$ can be any norm on $\mathbb{R}^n$.) For example, it is easy to show by this method that the process (5) is bounded in probability whenever $|\rho| < 1$. More systematic approaches to establishing boundedness in probability can be found in Meyn and Tweedie (2009, chapter 12).

3.2.2 Finding Excessive and Deficient Distributions

Condition 2 of theorem 3.1 requires existence of either an excessive or a deficient distribution. If S has a least element or a greatest element then the condition always holds (see remark 2.2). However, there are many settings where S has neither ($S = \mathbb{R}^n$ and $S = \mathbb{R}^n_{++}$ are obvious examples), and the existence is harder to verify. In this case, one can work more carefully with the definition of the model to construct excessive and deficient distributions. One example is Zhang (1997), who constructs such distributions for the stochastic optimal growth model. However, it is useful to have a more systematic method that is relatively straightforward to check in different applications. To this end we provide the following result:

Proposition 3.1. *Let Q be a stochastic kernel on S. If there exists another kernel Q' such that Q' is Feller, bounded in probability and $Q \preceq Q'$ (resp., $Q' \preceq Q$), then Q has an excessive (resp., deficient) distribution.*[13]

3.2.3 Checking the Order Reversing Property

In this section we give sufficient conditions for order reversing. To state them, we introduce two new definitions: We call kernel Q on S *upward reaching* if, given any

[13]The statement $Q \preceq Q'$ means that $\mu Q \preceq \mu Q'$ for all $\mu \in \mathscr{P}_S$.

(Q, x)-Markov process $\{X_t\}$ and c in S, there exists a $t \in \mathbb{N}$ such that $\mathbb{P}\{X_t \geq c\} > 0$. We call Q *downward reaching* if, given any (Q, x)-Markov process $\{X_t\}$ and c in S, there exists a $t \in \mathbb{N}$ such that $\mathbb{P}\{X_t \leq c\} > 0$. For example, the linear Gaussian process in (5) is both upward and downward reaching: If we fix x, c in $S = \mathbb{R}$ and take $t = 1$, then $\mathbb{P}\{X_1 \leq c\} = \mathbb{P}\{\rho x + \xi_1 \leq c\} = \mathbb{P}\{\xi_1 \leq c - \rho x\}$. This term is always strictly positive because ξ_t is Gaussian. Hence Q is downward reaching. The proof of upward reaching is similar.

Proposition 3.2. *Suppose that Q is bounded in probability. If Q is either upward or downward reaching, then Q is order reversing.*

It follows that the statements in theorem 3.1 and theorem 3.2 remain valid if order reversing is replaced by either upward or downward reaching.

4 Applications

We now turn to more substantial applications of the results described above.

4.1 Optimal Exploitation of a Renewable Resource

Consider an elementary model of renewable resource exploitation, where a single planner maximizes $\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to $y_{t+1} = \xi_t f(y_t - c_t)$. Here y_t is the stock of the resource, c_t is consumption, all variables are nonnegative and $\{\xi_t\} \stackrel{\text{IID}}{\sim} \phi$. For simplicity, we assume that u is bounded with $u' > 0$, $u'' < 0$, and $u'(0) = \infty$. The growth function f for the resource is assumed to satisfy $f(0) = 0$, $f' > 0$, $f'(0) = \infty$ and $f'(\infty) = 0$. Since f is biologically determined, we do not assume it is concave. To study dynamics, we take y_t as the state variable, and consider the optimal process $y_{t+1} = \xi_t f(y_t - \sigma(y_t))$, where $\sigma(\cdot)$ is an optimal consumption policy. Let Q be the corresponding stochastic kernel. For the state space we take $S = (0, \infty)$. Zero is deliberately excluded from S so that any stationary distribution on S is automatically non-trivial. Regarding the shock process $\{\xi_t\}$, we assume that arbitrarily bad shocks are possible. In particular, we assume that $\mathbb{P}\{\xi_t \leq z\} > 0$ for all $z \in S$. We also replace the common assumption that shocks are bounded with the small tail conditions $\mathbb{E}\, \xi_t < \infty$ and $\mathbb{E}\,(1/\xi_t) < \infty$.

The dynamics of models similar to the one described above have been studied by various authors, including Nishimura and Stachurski (2005), Olson and Roy (2006) and Mitra and Roy (2006). Here one difficulty is that f is not concave, implying that the optimal policy is not continuous. As a result, the stochatic kernel Q is not Feller. Moreover, without additional assumptions, the MMC does not apply, Q is not irreducible, the splitting condition fails, the model is not an expected contraction, the state space is unbounded and the standard Harris recurrence conditions are not satisfied.[14] On the other hand, theorem 3.1 can easily be applied. Q is still increasing and bounded in probability (see, e.g., Nishimura and Stachurski, 2005). Existence of an excessive distribution is not difficult to establish.[15] Moreover, the process is downward reaching (and hence order reversing, cf., proposition 3.2) because if y_0 and $\bar{y}$ in S are given, then

$$\mathbb{P}\{y_1 \leq \bar{y}\} = \mathbb{P}\{\xi_1 f(y_0 - \sigma(y_0)) \leq \bar{y}\} = \mathbb{P}\{\xi_1 \leq \bar{y}/f(y_0 - \sigma(y_0))\} > 0. \quad (6)$$

Hence theorem 3.1 applies, and Q is globally stable.

Regarding this argument, it is interesting to note that in order to prove stability we used order reversing, and to prove order reversing we relied on nonzero probability of arbitrarily bad shocks. These shocks are stabilizing rather than destabilizing because the Inada conditions prevent divergence, and the large shocks generate mixing.

Figure 1 shows a collection of stationary distributions for $\log y_t$, each one corresponding to a different value of the discount factor β.[16] For this model, a sudden shift in the optimal harvest policy occurs around $\beta = 0.965$. As a result, a very small difference in the patience of the agent can lead to a large difference in the steady state population of the stock.

[14] For a discussion of irreducibility and Harris recurrence, see Meyn and Tweedie (2009). On the splitting condition, see, e.g., Bhattacharya and Lee (1988), or Bhattacharya and Majumdar (2001).

[15] Since $f' > 0$ and $f'(\infty) = 0$, we can choose positive constants α, β with $\alpha \mathbb{E}\, \xi_t < 1$ and $f(x) \leq \alpha x + \beta$. Now take $G(x,z) := z(\alpha x + \beta)$, so that $F(x,z) := zf(x - \sigma(x)) \leq zf(x) \leq G(x,z)$. Letting Q_F and Q_G be the corresponding kernels, the last inequality implies $Q_F \preceq Q_G$. It can be shown that Q_G is both bounded in probability and Feller (for details see the working paper version, Kamihigashi and Stachurski, 2011b), so proposition 3.1 applies.

[16] The utility function is $u(x) = 1 - \exp(-\theta x^\gamma)$ and production is $f(x) = x^\alpha \ell(x)$, where ℓ is the logistic function $\ell(x) = a + (b-a)/(1 + \exp(-c(x-d)))$. The parameters are $a = 1$, $b = 2$, $c = 20$, $d = 1$, $\theta = 0.5$, $\gamma = 0.9$ and $\alpha = 0.5$. The discount factor β ranges from 0.945 to 0.99. The shock is lognormal $(-0.1, 0.2)$. For details on the calculations including full justification of consistency, see the working paper version (Kamihigashi and Stachurski, 2011b).

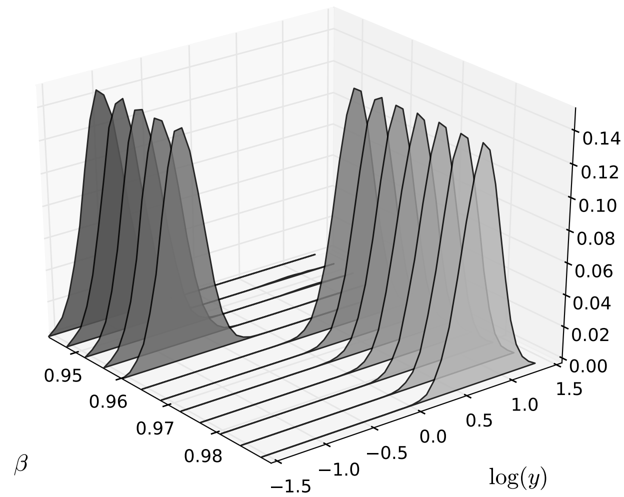

Figure 1: Stationary distributions as a function of β

4.2 Wealth Distribution Dynamics: A Two-Dimensional Example

Next we consider an OLG model of wealth distribution. The model can be viewed as a stochastic version of the small open economy of Matsuyama (2004), but we introduce persistence in inequality by assuming that an old agent provides financial support to her child.[17] Agents live for two periods, consuming only when old. Households consist of one old agent and one child. There is a unit mass of such households indexed by $i \in [0, 1]$. In each period t, the old agent of household i provides financial support b_t^i to her child. The child has the option to become an entrepreneur, investing one unit of the consumption good in a "project," and receiving stochastic output $\theta + \eta_{t+1}^i$ in period $t+1$. Let $k_{t+1}^i \in \{0, 1\}$ be young agent i's investment in the project. If the remainder $b_t^i - k_{t+1}^i$ is positive, then she invests this quantity at the world risk-free rate R. If it is negative then she borrows $k_{t+1}^i - b_t^i$ at the same risk-free rate. Independent of her investment choice, she receives an endowment of e_{t+1}^i units of the consumption good when old. Suppressing the i superscript to simplify notation, her wealth at the beginning of period $t+1$ is therefore

$$w_{t+1} = (\theta + \eta_{t+1})k_{t+1} - R(k_{t+1} - b_t) + e_{t+1}. \tag{7}$$

[17]This is a common assumption in the literature on wealth distribution (see, e.g., Antunes and Cavalcanti, 2007; Antunes et al., 2008; Cardak, 2004; Couch and Morand, 2005; Lloyd-Ellis, 2000; Lloyd-Ellis and Bernhardt, 2000; Owen and Weil, 1998; Piketty, 1997; and Ranjan, 2001).

We assume that

$$e_{t+1} = \rho e_t + \epsilon_{t+1}, \quad 0 < \rho < 1. \tag{8}$$

The idiosyncratic shocks $\{\eta_t\}$ and $\{\epsilon_t\}$ are taken to be IID and nonnegative, and ϵ_t satisfies $\mathbb{P}\{\epsilon_t > z\} > 0$ for any $z \geq 0$. (For example, ϵ_t might be lognormal.) We also assume that $R < \theta$, which implies that becoming an entrepreneur is always profitable, even *ex-post*, and every agent would choose to do so absent additional constraint. Due to a credit market imperfection, however, each agent may borrow only up to a fraction $\lambda \in (0,1)$ of $\theta + \rho e_t$, the minimum possible value of her old-age income. That is,

$$R(k_{t+1} - b_t) \leq \lambda(\theta + \rho e_t). \tag{9}$$

As becoming an entrepreneur is always profitable, young agents do so whenever feasible, implying

$$k_{t+1} = \kappa(b_t, e_t) := \mathbb{1}\{R(1 - b_t) \leq \lambda(\theta + \rho e_t)\}. \tag{10}$$

(Here $\mathbb{1}\{\cdot\}$ is an indicator function.) Let c_{t+1} denote consumption at $t+1$. It is common in the literature on wealth distribution to assume that each agent derives utility from her own consumption and financial support to her child. Following this approach, we assume that young agents maximize $\mathbb{E}_t[c_{t+1}^{1-\gamma} b_{t+1}^{\gamma}]$ subject to (7), (9), and the budget constraint $c_{t+1} + b_{t+1} = w_{t+1}$. Regarding the parameter γ we assume that $\gamma R < 1$. Maximization of $c_{t+1}^{1-\gamma} b_{t+1}^{\gamma}$ subject to the budget constraint implies that $b_{t+1} = \gamma w_{t+1}$. Combining this equality, (7) and (8), we obtain

$$b_{t+1} = \gamma[(\theta + \eta_{t+1} - R)\kappa(b_t, e_t) + R b_t + \rho e_t + \epsilon_{t+1}]. \tag{11}$$

Together, (8) and (11) define a Markov process with state vector $X_t := (b_t, e_t)$ taking values in state space $S := \mathbb{R}_+^2$. Let Q denote the corresponding stochastic kernel.[18]

Recalling that $R < \theta$, $\rho \in (0,1)$ and $\eta_{t+1} \geq 0$, and observing that $\kappa(b_t, e_t)$ is increasing in (b_t, e_t), we can see from (8) and (11) that (b_{t+1}, e_{t+1}) is increasing in (b_t, e_t) when the values of the shocks are held fixed. Hence Q is increasing (cf., remark 2.3). On the other hand, (11) is discontinuous in (b_t, e_t), so Q is not Feller.

As far as we are aware, no existing Markov process theory can be used to show that Q is globally stable unless additional conditions are imposed. In contrast, global stability can be obtained in a straightforward way from theorem 3.1. To begin, let

[18]We do not exclude $(0,0)$ from the state space since it is not an absorbing state.

$m_\eta := \mathbb{E}\,\eta_t$ and $m_\epsilon := \mathbb{E}\,\epsilon_t$. To see that Q is bounded in probability, we can take expectations of (8) and iterate backwards to obtain

$$\mathbb{E}\,e_t \le m_\epsilon/(1-\rho) + \rho^t e_0 \le m_\epsilon/(1-\rho) + e_0 =: \bar{e} \tag{12}$$

for all t. In addition, it follows from (11) and (12) that

$$\mathbb{E}\,b_{t+1} \le \gamma[\theta + m_\eta - R + R\mathbb{E}\,b_t + \bar{e}].$$

Using $\gamma R < 1$ and iterating backwards, we obtain the bound

$$\mathbb{E}\,b_t \le \gamma[\theta + m_\eta - R + \bar{e}]/(1-\gamma R) + b_0 \tag{13}$$

for all t. Together, (12) and (13) imply that Q is bounded in probability.[19] Since $\mathbb{P}\{\epsilon_t > z\} > 0$ for any $z \ge 0$, and since both b_t and e_t can be made arbitrarily large by choosing ϵ_t sufficiently large (see (8) and (11)), it follows that Q is upward reaching, and thus order reversing by proposition 3.2. In view of these results and theorem 3.1, Q will be globally stable whenever it has a deficient or excessive distribution. Since $(0,0)$ is a least element for S, remark 2.2 implies that Q has a deficient distribution, and we conclude that Q is globally stable.

Figure 2 shows smoothed histograms representing the marginal stationary distribution of wealth at two different values of λ, computed by simulation.[20] The shift in the densities shows how the distribution of wealth in the stationary equilibrium can be highly sensitive to the value of the borrowing constraint parameter λ.

5 Conclusion

The Hopenhayn-Prescott stability theorem has become an important tool for assessing the dynamics of stochastic economic models. This paper significantly extends their theorem. Our results are necessary and sufficient for global stability of monotone models satisfying a very weak mixing condition. Two applications were discussed.

[19]The function $V(b,e) = |b| + |e|$ is a norm on $\mathbb{R}^2$. Equations (12) and (13) yield $\sup_t \mathbb{E}\,[V(b_t, e_t)] \le \sup_t \mathbb{E}\,[b_t] + \sup_t \mathbb{E}\,[e_t] < \infty$, implying boundedness in probability. See section 3.2.1.

[20]The values of λ are 0.57 and 0.58. The other parameters are $\gamma = 0.2$, $R = 1.05$, $\theta = 1.1$ and $\rho = 0.9$. The shock ϵ is lognormal with parameters $\mu = -3$ and $\sigma = 0.1$. The shock η is beta with shape parameters 3,10. For full details on the calculations, see the working paper version (Kamihigashi and Stachurski, 2011b).

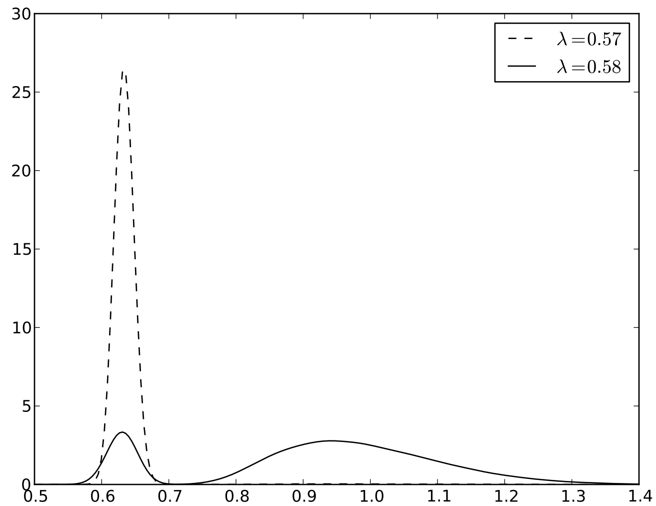

Figure 2: Stationary distribution of wealth

6 Technical Appendix

Before proving theorem 3.1, we need some additional results and notation. To begin, let Q be any stochastic kernel on S, let $x \in S$ and let S-valued stochastic process $\{X_t\}$ be (Q, x)-Markov. The joint distribution of $\{X_t\}$ over the sequence space S^∞ will be denoted by $\mathbf{P}_x^Q$. For example, $\mathbf{P}_x^Q\{X_t \in B\} = Q^t(x, B)$ for any $B \subset S$, and $\mathbf{P}_x^Q \cup_{t=0}^\infty \{X_t \in B\}$ is the probability that the process ever enters B. The symbol $\mathbf{E}_x^Q$ represents the expectations operator corresponding to $\mathbf{P}_x^Q$. For given kernel Q, we say that Borel set $B \subset S$ is

- *strongly accessible* if $\mathbf{P}_x^Q \cup_{t=0}^\infty \{X_t \in B\} = 1$ for all $x \in S$, and
- *C-accessible* if, for all compact $K \subset S$, there exists an $n \in \mathbb{N}$ with $\inf_{x \in K} Q^n(x, B) > 0$.

The following lemma is fundamental to our results, although the proofs are delayed to maintain continuity.

Lemma 6.1. *Let B be a Borel subset of S. If Q is bounded in probability and B is C-accessible, then B is strongly accessible.*

It is helpful to provide a second definition of order reversing. To do so, let

$$\mathbb{G} := \text{graph}(\leq) := \{(y, y') \in S \times S : y \leq y'\},$$

so that $y \leq y'$ iff $(y, y') \in \mathbb{G}$. Also, let Q be a stochastic kernel on S, and consider the product kernel $Q \times Q$ on $S \times S$ defined by

$$(Q \times Q)((x, x'), A \times B) = Q(x, A)Q(x', B) \tag{14}$$

for $(x, x') \in S \times S$ and $A, B \in \mathscr{B}_S$.[21] The product kernel represents the stochastic kernel of the joint process $\{(X_t, X_t')\}$ when $\{X_t\}$ and $\{X_t'\}$ are independent Q-Markov processes. Using this notation, Q is order reversing if and only if

$$\forall\, x, x' \in S \text{ with } x' \leq x, \;\; \exists\, t \in \mathbb{N} \;\text{ such that }\; (Q \times Q)^t((x, x'), \mathbb{G}) > 0. \tag{15}$$

This second definition emphasizes the fact that order reversing is a property of the kernel Q alone (taking S and $\leq$ as given). Condition (15) can alternatively be written as

$$\forall\, x, x' \in S \text{ with } x' \leq x, \;\; \exists\, t \geq 0 \text{ such that } \mathbf{P}_{x,x'}^{Q \times Q}\{X_t \leq X_t'\} > 0, \tag{16}$$

where $\{X_t\}$ and $\{X_t'\}$ are independent of each other and (Q, x)-Markov and (Q, x')-Markov respectively. Following Kamihigashi and Stachurski (2011a), Q is called *order mixing* if $\mathbf{P}_{x,x'}^{Q \times Q} \cup_{t=0}^{\infty} \{X_t \leq X_t'\} = 1$ for all $x, x' \in S$. Put differently, Q is order mixing if $\mathbb{G}$ is strongly accessible for the product kernel $Q \times Q$.

Lemma 6.2. *If Q is bounded in probability on S, then so $Q \times Q$ on $S \times S$.*

Lemma 6.3. *If Q is increasing and bounded in probability, then $\{\mu Q^t\}$ is tight for all $\mu \in \mathscr{P}_S$.*

Lemma 6.4. *If Q is increasing and order reversing, then $\mathbb{G}$ is C-accessible for $Q \times Q$.*

Proofs are given at the end of this section.

Let us now turn to the proof of theorem 3.1. The proof proceeds as follows: First we show that under the conditions of the theorem, Q is order mixing. Using order mixing, we then go on to prove existence of a stationary distribution, and global stability.

Lemma 6.5. *If Q is increasing, bounded in probability and order reversing, then Q is order mixing.*

Proof. To show that Q is order mixing we need to prove that $\mathbb{G}$ is strongly accessible for $Q \times Q$ under the conditions of theorem 3.1. Since Q is bounded in probability, $Q \times Q$ is also bounded in probability (lemma 6.2), and hence, by lemma 6.1, it suffices to show that $\mathbb{G}$ is C-accessible for $Q \times Q$. This follows from lemma 6.4. □

[21] Sets of the form $A \times B$ with $A, B \in \mathscr{B}_S$ provide a semi-ring in the product σ-algebra $\mathscr{B}_S \otimes \mathscr{B}_S$ that also generates $\mathscr{B}_S \otimes \mathscr{B}_S$. Defining the probability measure $Q((x, x'), \cdot)$ on this semi-ring uniquely defines $Q((x, x'), \cdot)$ on all of $\mathscr{B}_S \otimes \mathscr{B}_S$. See, e.g., Dudley (2002, theorem 3.2.7).

We now prove global stability, making use of order mixing. In the sequel, we define $icbS$ to be the bounded, increasing and continuous functions from S to $\mathbb{R}$ (i.e., $icbS = ibS \cap cbS$). To simplify notation, we will also use inner product notation to represent integration, so that

$$\langle \mu, h \rangle := \int h(x)\mu(dx) \quad \text{for } \mu \in \mathscr{P}_S \text{ and } h \in ibS \cup cbS.$$

It is well known (see, e.g., Stokey *et al.* 1989, p. 219) that the left and right Markov operators are adjoint, in the sense that, for any such h and any $\mu \in \mathscr{P}_S$, we have $\langle \mu, Qh \rangle = \langle \mu Q, h \rangle$.

We will make use of the following results, which are proved at the end of this section.

Lemma 6.6. *Let $\mu, \mu', \mu_n \in \mathscr{P}_S$.*

1. *$\mu \preceq \mu'$ iff $\langle \mu, h \rangle \leq \langle \mu', h \rangle$ for all $h \in icbS$,*
2. *$\mu = \mu'$ iff $\langle \mu, h \rangle = \langle \mu', h \rangle$ for all $h \in icbS$, and*
3. *$\mu_n \to \mu$ iff $\{\mu_n\}$ is tight and $\langle \mu_n, h \rangle \to \langle \mu, h \rangle$ for all $h \in icbS$.*

Proof of theorem 3.1. We begin by showing that if Q is globally stable, then conditions 1–2 of the theorem hold. Regarding condition 1, fix $x \in S$. Global stability implies that $\{\mu Q^t\}$ is convergent for each $\mu \in \mathscr{P}_S$, and hence $\{Q^t(x, \cdot)\} = \{\delta_x Q^t\}$ is convergent. Since convergent sequences are tight (Dudley, 2002, proposition 9.3.4) and $x \in S$ was arbitrary, we conclude that Q is bounded in probability, and condition 1 is satisfied. Condition 2 is trivial, because global stability implies existence of a stationary distribution, and every stationary distribution is both deficient and excessive.

Next we show that if Q is increasing, order reversing and conditions 1–2 of theorem 3.1 hold, then Q has at least one stationary distribution. By lemma 6.5, Q is order mixing, and hence, by Kamihigashi and Stachurski (2011a, theorem 3.1), for any ν and ν' in $\mathscr{P}_S$ we have

$$\lim_{t \to \infty} |\langle \nu Q^t, h \rangle - \langle \nu' Q^t, h \rangle| = 0, \qquad \forall\, h \in ibS. \tag{17}$$

By condition 2 of theorem 3.1 there exists a $\mu \in \mathscr{P}_S$ that is either excessive or deficient. In what follows we will assume it is deficient, since the excessive case only changes the direction of inequalities. Since μ is deficient we have $\mu \preceq \mu Q$. Since Q is increasing, we can iterate on this inequality to establish that the sequence $\{\mu Q^t\}$ is monotone increasing in $\preceq$. By condition 1 of theorem 3.1 and lemma 6.3, the sequence $\{\mu Q^t\}$ is also tight.

By Prohorov's theorem (Dudley, 2002, theorem 11.5.4), tightness implies existence of a subsequence of $\{\mu Q^t\}$ converging to some $\psi^* \in \mathscr{P}_S$. Since $\{\mu Q^t\}$ is $\preceq$-increasing, it follows that, for any given $h \in icbS$, the entire sequence $\langle \mu Q^t, h \rangle$ converges up to $\langle \psi^*, h \rangle$. Because $\{\mu Q^t\}$ is tight, part 3 of lemma 6.6 implies that $\mu Q^t \to \psi^*$.

In addition to $\mu Q^t \to \psi^*$, we also have $\mu Q^t \preceq \psi^*$ for all $t \geq 0$, because for any $h \in icbS$ and $t \geq 0$ we have

$$\langle \mu Q^t, h \rangle \leq \sup_{t \geq 0} \langle \mu Q^t, h \rangle = \lim_{t \to \infty} \langle \mu Q^t, h \rangle = \langle \psi^*, h \rangle.$$

The inequality $\mu Q^t \preceq \psi^*$ now follows from part 1 of lemma 6.6.

Next, we claim that $\psi^* \preceq \psi^* Q$. To see this, pick any $h \in icbS$. Since $\mu Q^t \preceq \psi^*$ for all t, and since $Qh \in ibS$,

$$\langle \mu Q^t, Qh \rangle \leq \langle \psi^*, Qh \rangle = \langle \psi^* Q, h \rangle.$$

Using this inequality and the fact that $h \in cbS$, we obtain

$$\langle \psi^*, h \rangle = \lim_{t \to \infty} \langle \mu Q^{t+1}, h \rangle = \lim_{t \to \infty} \langle \mu Q^t, Qh \rangle \leq \langle \psi^* Q, h \rangle.$$

Hence $\langle \psi^*, h \rangle \leq \langle \psi^* Q, h \rangle$ for all $h \in icbS$, and $\psi^* \preceq \psi^* Q$ as claimed. Iterating on this inequality we obtain $\psi^* \preceq \psi^* Q^t$ for all t.

To summarize our results so far, we have $\mu Q^t \preceq \psi^* \preceq \psi^* Q \preceq \psi^* Q^t$ for all $t \geq 0$, and hence

$$\langle \mu Q^t, h \rangle \leq \langle \psi^*, h \rangle \leq \langle \psi^* Q, h \rangle \leq \langle \psi^* Q^t, h \rangle \quad \text{for all} \quad h \in icbS.$$

Applying (17), we obtain $\langle \psi^*, h \rangle = \langle \psi^* Q, h \rangle$ for all $h \in icbS$. By lemma 6.6, this implies that $\psi^* = \psi^* Q$. In other words, ψ^* is stationary for Q.

It remains to show that Q is globally stable. Fixing $\nu \in \mathscr{P}_S$ and applying (17) again, we have

$$\langle \nu Q^t, h \rangle \to \langle \psi^*, h \rangle, \quad \forall\, h \in ibS. \tag{18}$$

Since $icbS \subset ibS$ and $\{\nu Q^t\}$ is tight (cf., lemma 6.3), this implies that $\nu Q^t \to \psi^*$ (lemma 6.6, part 3). Finally, uniqueness is also immediate, because if ν is also stationary, then by (18) we have $\langle \nu, h \rangle = \langle \psi^*, h \rangle$ for all $h \in icbS$. By lemma 6.6, we then have $\nu = \psi^*$. □

Proof of theorem 3.2. Under the conditions of the theorem, Q is order mixing, as proved in lemma 6.5. In addition, boundedness in probability and the Feller property guarantee the existence of a stationary distribution by the Krylov-Bogolubov theorem (Meyn and Tweedie, 2009, proposition 12.1.3 and lemma D.5.3). Given existence of a stationary distribution ψ^*, the proof that Q is globally stable is now identical to the proof of the same claim given for theorem 3.1 (see the preceding paragraph). □

Proof of proposition 3.1. Suppose that Q' is Feller and bounded in probability with $Q' \preceq Q$. By the Krylov-Bogolubov theorem (Meyn and Tweedie, 2009, proposition 12.1.3 and lemma D.5.3), Q' has at least one stationary distribution μ. For this μ we have $\mu = \mu Q' \preceq \mu Q$. In other words, μ is deficient for Q. A similar argument shows that if Q' is Feller and bounded in probability with $Q \preceq Q'$ then Q has an excessive distribution. □

Proof of proposition 3.2. Let Q be bounded in probability. Suppose first that Q is upward reaching. Pick any $(x, x') \in S \times S$. Let $\{X_t\}$ and $\{X'_t\}$ be independent, (Q, x)-Markov and (Q, x')-Markov respectively. We need to prove existence of a $k \in \mathbb{N}$ such that $\mathbb{P}\{X_k \leq X'_k\} > 0$. Since Q is bounded in probability, there exists a compact $C \subset S$ with $\mathbb{P}\{X_t \in C\} > 0$ for all $t \geq 0$. Since compact sets are assumed to be order bounded, we can take an order interval $[a, b]$ of S with $C \subset [a, b]$. For this a, b we have $\mathbb{P}\{a \leq X_t \leq b\} > 0$ for all $t \geq 0$. As Q is upward reaching, there is a $k \in \mathbb{N}$ such that $\mathbb{P}\{b \leq X'_k\} > 0$. Using independence, we now have

$$\mathbb{P}\{X_k \leq X'_k\} \geq \mathbb{P}\{X_k \leq b \leq X'_k\} = \mathbb{P}\{X_k \leq b\}\mathbb{P}\{b \leq X'_k\} > 0,$$

as was to be shown. The proof for the downward reaching case is similar. □

Finally, we complete the proof of all remaining lemmas stated in this section.

Proof of lemma 6.1. Let B be a C-accessible subset of S. To prove the lemma, it suffices to show that $\mathbf{P}_x^Q \cup_t \{X_t \in B\} = 1$ whenever $\{Q^t(x, \cdot)\}$ is tight. To this end, fix $x \in S$, and assume that $\{Q^t(x, \cdot)\}$ is tight. Let $\tau := \inf\{t \geq 0 : X_t \in B\}$. Evidently we have $\cup_{t=0}^{\infty}\{X_t \in B\} = \{\tau < \infty\}$. Thus, we need to show that $\mathbf{P}_x^Q\{\tau < \infty\} = 1$.

Fix $\epsilon > 0$. Since $\{Q^t(x, \cdot)\}$ is tight, there exists a compact set C such that

$$\inf_t \mathbf{P}_x^Q\{X_t \in C\} = \inf_t Q^t(x, C) \geq 1 - \epsilon.$$

Since B is C-accessible, there exists an $n \in \mathbb{N}$ and $\delta > 0$ such that $\inf_{y \in C} Q^n(y, B) \geq \delta$. For $t \in \mathbb{N}$, define $p_t := \mathbf{P}_x^Q\{\tau \leq tn\}$. We wish to obtain a relationship between p_t and p_{t+1}. To this end, note that

$$\begin{aligned}
\mathbb{1}\{\tau \leq (t+1)n\} &= \mathbb{1}\{\tau \leq tn\} + \mathbb{1}\{\tau > tn\}\mathbb{1}\{\tau \leq (t+1)n\} \\
&\geq \mathbb{1}\{\tau \leq tn\} + \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{(t+1)n} \in B\} \\
&\geq \mathbb{1}\{\tau \leq tn\} + \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbb{1}\{X_{(t+1)n} \in B\}.
\end{aligned}$$

Taking expectations yields

$$p_{t+1} \geq p_t + \mathbf{E}_x^Q \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbb{1}\{X_{(t+1)n} \in B\}.$$

We estimate the last expectation as follows:

$$\begin{aligned}
&\mathbf{E}_x^Q \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbb{1}\{X_{(t+1)n} \in B\} \\
&\qquad = \mathbf{E}_x^Q[\mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbf{E}_x^Q[\mathbb{1}\{X_{(t+1)n} \in B\}|\mathscr{F}_{tn}]] \\
&\qquad = \mathbf{E}_x^Q[\mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}Q^n(X_{tn}, B)] \\
&\qquad \geq \mathbf{E}_x^Q \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\delta \\
&\qquad = \mathbf{E}_x^Q(1 - \mathbb{1}\{\tau \leq tn\})\mathbb{1}\{X_{tn} \in C\}\delta \\
&\qquad = \mathbf{E}_x^Q \mathbb{1}\{X_{tn} \in C\}\delta - \mathbf{E}_x^Q \mathbb{1}\{\tau \leq tn\}\mathbb{1}\{X_{tn} \in C\}\delta \\
&\qquad \geq (1 - \epsilon)\delta - \mathbf{E}_x^Q \mathbb{1}\{\tau \leq tn\}\delta \\
&\qquad = (1 - \epsilon)\delta - p_t\delta.
\end{aligned}$$

$$\therefore \quad p_{t+1} \geq p_t + (1-\epsilon)\delta - p_t\delta = (1-\delta)p_t + (1-\epsilon)\delta.$$

The unique, globally stable fixed point of $q_{t+1} = (1-\delta)q_t + (1-\epsilon)\delta$ is $1-\epsilon$, so $1-\epsilon \leq \lim_{t\to\infty} p_t = \mathbf{P}_x^Q\{\tau < \infty\} \leq 1$. Since ϵ was arbitrary, we obtain $\mathbf{P}_x^Q\{\tau < \infty\} = 1$. □

Proof of lemma 6.2. Fix $x, x' \in S$ and $\epsilon > 0$. Since Q is bounded in probability, we can choose compact sets C and C' such that

$$Q^t(x, C) \geq (1-\epsilon)^{1/2} \quad \text{and} \quad Q^t(x', C') \geq (1-\epsilon)^{1/2} \quad \text{for all } t.$$

$$\therefore \quad (Q \times Q)^t((x, x'), C \times C') = Q^t(x, C)Q^t(x', C') \geq 1-\epsilon \quad \text{for all } t.$$

Since $C \times C'$ is compact in the product space, $Q \times Q$ is bounded in probability. □

Proof of lemma 6.3. Fix $\mu \in \mathscr{P}_S$ and $\epsilon > 0$. Since individual elements of $\mathscr{P}_S$ are tight (Dudley, 2002, theorem 11.5.1), we can choose a compact set $C_\mu \subset S$ with $\mu(C_\mu) \geq 1-\epsilon$. By assumption, we can take an order interval $[a, b]$ of S with $C_\mu \subset [a, b]$. For this a, b, we have

$$\mu([a, b]^c) = \mu(S \setminus [a, b]) \leq \epsilon. \tag{19}$$

By hypothesis, $\{Q^t(x, \cdot)\}$ is tight for all $x \in S$, so we choose compact subsets C_a and C_b of S with $Q^t(a, C_a) \geq 1-\epsilon$ and $Q^t(b, C_b) \geq 1-\epsilon$ for all t. Since $C_a \cup C_b$ is also compact, we can take an order interval $[\alpha, \beta]$ of S with $C_a \cup C_b \subset [\alpha, \beta] \subset S$. We then have $Q^t(a, [\alpha, \beta]) \geq 1-\epsilon$ and $Q^t(b, [\alpha, \beta]) \geq 1-\epsilon$ for all t. Letting $I_\alpha := \{x \in S : x \geq \alpha\}$ and $D_\beta := \{x \in S : x \leq \beta\}$, this leads to

$$Q^t(a, I_\alpha) \geq 1-\epsilon \quad \text{and} \quad Q^t(b, D_\beta) \geq 1-\epsilon \quad \text{for all } t. \tag{20}$$

In view of remark 2.1 and (20), we have

$$a \leq x \implies Q^t(x, I_\alpha) \geq Q^t(a, I_\alpha) \geq 1-\epsilon,$$

and, by a similar argument,

$$x \leq b \implies Q^t(x, D_\beta) \geq Q^t(b, D_\beta) \geq 1-\epsilon.$$

Since $[\alpha, \beta] := \{x \in S : \alpha \leq x \leq \beta\} = I_\alpha \cap D_\beta$, we have

$$Q^t(x, [\alpha, \beta]^c) = Q^t(x, D_\beta^c \cup I_\alpha^c) \leq 2 - Q^t(x, D_\beta) - Q^t(x, I_\alpha).$$

This leads to the estimate

$$a \leq x \leq b \implies Q^t(x, [\alpha, \beta]^c) \leq 2\epsilon. \tag{21}$$

Combining (19) and (21), we now have

$$\begin{aligned}\mu Q^t([\alpha,\beta]^c) &= \int Q^t(x,[\alpha,\beta]^c)\mu(dx)\\ &= \int_{[a,b]} Q^t(x,[\alpha,\beta]^c)\mu(dx) + \int_{[a,b]^c} Q^t(x,[\alpha,\beta]^c)\mu(dx)\\ &\leq \int_{[a,b]} 2\epsilon\,\mu(dx) + \mu([\alpha,\beta]^c) \;\leq\; 3\epsilon.\end{aligned}$$

Since $[\alpha,\beta]$ is compact and t is arbitrary, we conclude that $\{\mu Q^t\}$ is tight. □

Proof of lemma 6.4. Let C be any compact subset of $S \times S$. We need to prove existence of an $n \in \mathbb{N}$ and $\delta > 0$ such that $(Q \times Q)^n((x,x'),\mathbb{G}) \geq \delta$ whenever $(x,x') \in C$. To do so, we introduce the function

$$\psi_n(x,x') := (Q \times Q)^n((x,x'),\mathbb{G}) = \mathbf{P}^{Q\times Q}_{x,x'}\{X_n \leq X'_n\},$$

where (X_n, X'_n) is $(Q \times Q, (x,x'))$-Markov. Intuitively, since Q is increasing, the event $\{X_n \leq X'_n\}$ becomes less likely as x rises and x' falls, and hence $\psi_n(x,x')$ is decreasing in x and increasing in x' for each n. A routine argument confirms this is the case.

Since $C \subset S \times S$ is compact, we can take an order interval $[a,b]$ of S with $C \subset [a,b] \times [a,b]$.[22] Moreover, since Q is order reversing, we can take an $n \in \mathbb{N}$ such that $\delta := \psi_n(b,a) > 0$. Observe that

$$(x,x') \in C \implies (x,x') \in [a,b] \times [a,b] \implies x \leq b \text{ and } x' \geq a.$$

$$\therefore \quad (x,x') \in C \implies (Q \times Q)^n((x,x'),\mathbb{G}) = \psi_n(x,x') \geq \psi_n(b,a) = \delta.$$

In other words, $\mathbb{G}$ is C-accessible for $Q \times Q$. □

Proof of lemma 6.6. The statement $\mu \preceq \mu'$ iff $\langle \mu, h\rangle \leq \langle \mu', h\rangle$ for all $h \in icbS$ holds for every normally ordered space, as shown by Whitt (1980, theorem 2.6). Moreover, since $\preceq$ is a partial order on $\mathscr{P}_S$ (Kamae and Krengel, 1978, theorem 2), and hence antisymmetric, it follows that $\mu = \mu'$ iff $\langle \mu, h\rangle = \langle \mu', h\rangle$ for all $h \in icbS$. Regarding the third assertion of the lemma, observe first that if $\mu_n \to \mu$, then since S is Polish the sequence $\{\mu_n\}$ is tight (Dudley, 2002, theorem 11.5.3). The statement $\langle \mu_n, h\rangle \to \langle \mu, h\rangle$ whenever $h \in icbS$ is obvious. To prove the converse, suppose that $\{\mu_n\}$ is tight and $\langle \mu_n, h\rangle \to \langle \mu, h\rangle$ for all $h \in icbS$. Take

[22] To see this, let K be a compact subset of S with $C \subset K \times K$. (Such a K can be obtained by projecting C onto the first and second axis, and defining K as the union of these projections.) Since K is order bounded in S by assumption, we just choose $a, b \in S$ with $K \subset [a,b]$.

any subsequence $\{\mu_n\}_{n\in\mathbb{N}_1}$ of $\{\mu_n\}$. By tightness and Prohorov's theorem (Dudley, 2002, theorem 11.5.4), this subsequence has a subsubsequence converging to some $\nu \in \mathscr{P}_S$:

$$\exists\, \mathbb{N}_2 \subset \mathbb{N}_1 \quad \text{such that} \quad \lim_{n\in\mathbb{N}_2} \langle \mu_n, h\rangle = \langle \nu, h\rangle \quad \text{for all} \quad h \in cbS.$$

Since $\langle \mu_n, h\rangle \to \langle \mu, h\rangle$ for all $h \in icbS$, we now have $\lim_{n\in\mathbb{N}_2}\langle \mu_n, h\rangle = \langle \nu, h\rangle = \langle \mu, h\rangle$ for all $h \in icbS$, and hence $\nu = \mu$. We have now shown that every subsequence of $\{\mu_n\}$ has a subsubsequence converging to μ, and hence the entire sequence also converges to μ. □

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