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Abstract

We study a one-dimensional voting game in which voters choose a policy from a one-dimensional policy set over which voters have single-peaked preferences. The purpose of this paper is to analyze coalitional behaviors under any given voting mechanism. We employ the notion of strong Nash equilibrium and identify a necessary and sufficient condition for a voting mechanism to possess a strong Nash equilibrium by using the minimax theorem. We moreover show that any strong Nash outcome, if it exists, results in an outcome recommended by a particular augmented median voter rule.

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Key words: Single-peakedness, Augmented median voter rule, Strong Nash equilibrium, Coalition-proof Nash Equilibrium, Minimax theorem, Manipulation.

1 Introduction

We study a one-dimensional voting game in which voters choose a policy from a one-dimensional policy set according to a given voting mechanism over which voters single-peaked preferences (Black [9]). In this environment, it is well known that there exist strategy-proof, efficient and anonymous social choice functions, contrary to the Gibbard-Satterthwaite theorem (Gibbard [17]; Satterthwaite [33]). A typical example of such a function is the median voter rule, which chooses the median of voters' peaks. Moreover, Moulin [27] provided a characterization of strategy-proof social choice functions. He showed that a social choice function is strategy-proof, efficient, anonymous and peak-only if and only if it is a "generalized median voter rule," which chooses the median of n voters' peaks and $n-1$ exogenous parameters. Because of the existence of "reasonable"

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and strategy-proof social choice functions, many studies on this environment have focused on strategy-proof voting rules[1].

On the other hand, we have not paid much attention to what happens under any given voting mechanism that does not possess a dominant strategy[2]. The purpose of this paper is to analyze the consequence of strategic votes under any given voting mechanisms, including indirect mechanisms, satisfying a mild condition that we call "own range continuity."

In voting situations, voters with similar thoughts often form a coalition to influence the outcome of the vote. For example, in a congress, most of its members belong to political parties, because an independent member usually has little power to influence political decisions. When there is an election, some voters with a common interest form an interest group to influence political decisions. The importance of analyzing coalitional behaviors in voting situations is explained by Sertel and Sanver [35], that studied coalitional manipulations in general voting games and implied by the fact that coalitional notions, such as the core, strong Nash implementation, and coalitional strategy-proofness, are often applied to the studies of political situations. This is why we pay attention to coalitional behaviors theroughout this paper. To do so, we employ the notions of *strong Nash equilibrium* (Aumann [2]) and *coalition-proof Nash equilibrium* (Bernheim, Peleg and Whinston [8]) and obtain the following results.

First, we identify a necessary and sufficient condition for a voting mechanism to possess a non-empty strong Nash equilibrium (Theorem 1). So far as the author knows, there have been no environments where we can provide a necessary and sufficient condition for the existence of strong Nash equilibria, though, several studies, such as Ichiishi [19], Konishi, Le Breton, and Weber [21] and Bochet, Sakai and Thomson [11] have found sufficient conditions. In this sense, a one-dimensional voting game is an interesting environment in the literature of game theory, as well as in that of social choice theory.

Second, we analyze what happens in strong Nash equilibria and show that if a voting mechanism possess a non-empty strong Nash equilibrium, the set of strong Nash outcomes must be single-valued and equivalent to an outcome recommended by a particular augmented median voter rule (Theorem 2). Augmented median voter rules are known as the only social choice functions satisfying strategy-proofness and continuity (Ching [13].) Theorem 2 says that even if we do not use a direct revelation mechanism of an augmented median voter rule, voters' strategic votes must result in a particular augmented median

[1] See for example, Barbera, Gul, and Stachetti [5], Ching [13], and so on. Sprumont [39], Barbera [3] and Jackson [20] offer surveys in this literature.

[2] Recently, some papers have analyzed properties of direct reveration mechanisms of some manipulable rules. See, for example, Renault and Trannoy [30] [31] and Yamamura and Kawasaki [43].

voter rule, as long as coalitional behaviors are permitted.

In the proof of Theorem 1 and Theorem 2, we make use of the well-known minimax theorem (von Neumann [42]). In a two-person zerosum game $G = \{\{1,2\}, \{A_1, A_2\}, \{u_1, u_2\}\}$, thanks to the minimax theorem, we can judge whether G has a Nash equilibrium or not and specify what happens in a Nash equilibrium. That is,

(1) G has a Nash equilibrium if and only if $\max_{a_1 \in A_1} \inf_{a_2 \in A_2} u_1(a_1, a_2) = \min_{a_2 \in A_2} \sup_{a_1 \in A_1} u_1(a_1, a_2)$; and

(2) if (a_1^*, a_2^*) is a Nash equilibrium of G, then $u_1(a_1^*, a_2^*) = \max_{a_1 \in A_1} \inf_{a_2 \in A_2} u_1(a_1, a_2) = \min_{a_2 \in A_2} \sup_{a_1 \in A_1} u_1(a_1, a_2)$.

Observe that a two-person zerosum game G is a special case of a one-dimensional voting game, because in a two-person zerosum game the outcome space can be identified with $range(u_1)$, that is a one-dimensional set, and two players have single-peaked preferences over the outcome space $range(u_1)$, whose peaks are $\sup range(u_1)$ and $\inf range(u_1)$, respectively. Such similarity between two-person zerosum games and one-dimensional voting games enables us to expand the results of the minimax theorem into one-dimensional voting games.

Third, we study what happens in coalition-proof Nash equilibria. We show that though the set of coalition-proof Nash equilibria is not always equivalent to that of strong Nash equilibria (Example 1), these two coincide under a large class of voting mechanisms (Theorem 3). As long as the equivalence between strong Nash equilibria and coalition-proof Nash equilibria holds, we can also state that by Theorem 2 any coalition-proof Nash outcome is equivalent to a particular augmented median voter rule.

Our theorems help us analyze strtategic manipulations of a direct revelation mechanism of a given social choice function[3]. Since we consider a large class of indirect voting mechanisms containing direct mechanisms, the analysis of direct mechanisms can be obtained as corollaries. Recently, strategic manipulations of some manipulable rules, such as average voting rules, have been studied by Renault and Trannoy [30] [31] and Yamamura and Kawasaki [43] based on dynamically stable Nash equilibrium. They show that there exists a class of direct revelation mechanisms in which voters learn to play a unique Nash equilibrium that implements an augmented median voter rule. Our theorems reinforce their results from the standpoint of coalitional stability.

[3]There have been some studies that analyzes the consequence of strategic manipulations in several economic models. See, for example, Hurwicz [18], Otani and Sicilian [28] and Kranich [23] in divisible goods economies, Tadenuma and Thomson [40] and Fujinaka and Sakai [15] [16] in the economies of one indivisible good and money, Bochet, Sakai and Thomson [11] in the division problem with single-peaked preferences, Ma [24] , Alcade [1], Shin and Suh [36], Sonmez [34] and Takamiya [41] in matching problems and Sertel and Sanver [35] in general voting situations.

Our theorems are also useful when we characterize the class of of strategy-proof social choice functions; and when we consider what happens in a two-person division problem with single-peaked preferences. We will show how to apply our theorems to these situations.

The rest of this paper is organized as follows: in section 2, we introduce the model and definitions; in section 3, we indicate our main result; in section 4, we propose applications for our theorems; and in section 5, we conclude this paper.

2 Notation

2.1 Basic Definitions

Let $N \equiv \{1, 2, \cdots, n\}$ be the set of *voters*. A denotes the *policy set*. Through this paper, we suppose that A is a non-empty and closed subset of the extended real number line $\mathbb{R} \cup \{-\infty, \infty\}$[4]. For each voter $i \in N$, i has a complete, transitive and single-peaked *preference* R_i over A. The symmetric and asymmetric parts of R_i are denoted by I_i and P_i respectively. A preference relation R_i over A is said to be *single-peaked* if there exists a *peak* $p(R_i) \in A$ such that for each $c, d \in A$, $c < d \leq p(R_i)$ implies dP_ic and $p(R_i) \leq c < d$ implies cP_id. Let $\mathcal{R}_i$ be the set of i's single-peaked preferences.

$S \subseteq A$ is said to be an *interval of* A if for any $a, b, c \in A$, $a \leq b \leq c$, if $a, c \in S$, then $b \in A$. For each $a, b \in A$, $a \leq b$, $[a, b]_A$ denotes the set $\{c \in A |\ a \leq c \leq b\}$ and is called an *closed interval of* A.

2.2 Voting Mechanism

A *voting mechanism* is defined by a pair $\Gamma = (\{M_i\}_{i \in N}, g)$ where M_i denotes voter i's *message space*, and $g : M \to A$ the *outcome function* that associates with each message profile $m \equiv \{m_i\}_{i \in N} \in M \equiv \{M_i\}_{i \in N}$ a policy $g(m) \in A$.

We moreover introduce some properties of voting mechanisms.

Own Range Continuity: *For any* $i \in N$ *and any* $m_{-i} \in M_{-i} \equiv \{M_j\}_{j \in N \setminus \{i\}}$, $g(M_i, m_{-i})$ is an interval of A.

Own range continuity requires that for any $a, b, c \in A$, $a \leq b \leq c$, whenever a voter can change a policy from a to c by changing his message, he is also able to change a policy from a to b. The policy change from a to c is more radical than the policy change from a to b. In this sense, own range continuity is a condition that makes it possible for

[4] We allow A to be either finite or a closed interval.

a voter to change a policy moderately. Throughout this paper, we assume that a voting mechanism satisfies own range continuity.

Optimizability: *For any $i \in N$, there exist $\widetilde{m}_i, \overline{m}_i \in M_i$ such that for any $m_{-i} \in M_{-i}$, $g(M_i, m_{-i}) \subseteq [g(\widetilde{m}_i, m_{-i}), g(\overline{m}_i, m_{-i})]_A$.*

Optimizability requires that whenever a voter wants to make a policy the lowest (or the highest), he is able to do so by reporting $\widetilde{m}_i$ (or $\overline{m}_i$). That is, under an optimizable voting mechanism, a voter can minimize (or maximize) a policy without expecting other voters' messages.

2.3 Equilibrium Notions

Under a voting mechanism Γ, each voter reports his own messsage and a policy is decided according to the reported messages. Then, each voter will vote strategically to his advantage. We expect the consequences of strategic votes based on strong Nash equilibrium (Aumann [2]) and coalition-proof Nash equilibrium (Bernheim, Peleg and Whinston [8]), which take coalitional deviations into consideration.

First, let us define a strong Nash equilibrium. Given a voting mechanism $\Gamma = (\{M_i\}_{i\in N}, g)$ and a preference profile $R \in \mathcal{R}$, we say a coalition $S \subseteq N$ has a *deviation* $m'_S \in M_S \equiv \{M_i\}_{i\in S}$ at a voting profile $m \in M$, if $g(m'_S, m_{-S})\, P_i g(m)$, $\forall i \in S$. A message profile m is a *strong Nash equilibrium* if there is no coalition that has a deviation at m. Given a voting mechanism Γ and a preference profile R, let

$$SN_v(\Gamma, R) \equiv \{m \in M \mid \neg\, (\exists S \subseteq N \, such\, that \, S \, has \, a \, deviation \, at \, m.)\}.$$

be the *set of Nash equilibrium voting profiles* and

$$SN(\Gamma, R) \equiv g(SN_v(\Gamma, N))$$

be the *set of strong Nash outcomes* of a voting mechanism Γ under a preference profile R.

Next, we introduce the notion of coalition-proof Nash equilibrium. Given Γ and R, we say a coalition $S \subseteq N$ has a *credible deviation* $m'_S \in M_S$ at a voting profile m, if $g(m'_S, m_{-S})\, P_i g(m)$, $\forall i \in S$ and there is no $T \subseteq S$, $T \neq S$ such that T has a credible deviation at (m'_S, m_{-S}). A message profile m is a *coalition-proof Nash equilibrium* if there is no coalition that has a credible deviation at m. The *set of coalitional-proof Nash* *voting profiles* $CN_v(\Gamma, R)$ is defined as follows.

$$CN_v(\Gamma, R) \equiv \{m \in M \mid \neg\,(\exists S \subseteq N \, such\, that\, S\, has\, a\, credible\, deviation\, at\, m.)\}\,.$$

Let

$$CN(\Gamma, R) \equiv g(CN_v(\Gamma, N))$$

be the *set of coalitional-proof Nash outcomes.*

By the definitions of these two equilibrium notions, we can easily check that

$$SN(\Gamma, R) \subseteq CN(\Gamma, R), \text{ for any } \Gamma \text{ and any } R.$$

2.4 Social Choice Function

A *social choice function* is defined as a mapping $f : \mathcal{R} \to A$ that associates with each preference profile $R \in \mathcal{R}$ a policy $f(R) \in A$ that is considered to be socially desirable. Here are examples of social choice functions.

Augmented median voter rule: *For each* $S \subseteq N$*, there exists* $a_S \in A$ *such that* $S \subseteq T$ *implies* $a_S \geq a_T$ *and for each* $R \in R$*, whenever* $p(R_{i(1)}) \leq p(R_{i(2)}) \leq \cdots \leq p(R_{i(n)})$*,*

$$f(R) = m(\{p(R_{i(1)}), \cdots, p(R_{i(n)}), a_\phi, a_{\{i_1\}}, a_{\{i_1, i_2\}}, \cdots, a_N\}),$$

where $m(\{x_1, \cdots, x_n\})$ *denotes the median of* $\{x_1, \cdots, x_n\}$*.*

Given a list of parameters $a \equiv \{a_S\}_{S \subseteq N}$ such that $S \subseteq T \Longrightarrow a_S \geq a_T$, m_a denotes an augmented median voter rule such that for each $R \in R$, if $p(R_{i(1)}) \leq p(R_{i(2)}) \leq \cdots \leq p(R_{i(n)})$, then

$$m_a(R) = m(\{p(R_{i(1)}), \cdots, p(R_{i(n)}), a_\phi, a_{\{i_1\}}, a_{\{i_1, i_2\}}, \cdots, a_N\}).$$

Augmented median voter rules are known as the only rules satisfying strategy-proofness and an additional condition (Moulin [27], Ching [13]), such as continuity, peak-only, peak-monotonicity, uncompromisingness or range continuity.

Generalized median voter rule: *An augmented median voter rule* m_a *such that* $a_\phi = \max A$*,* $a_N = \min A$ *and* $|S| = |T| \Longrightarrow a_S = a_T$*, where* $|X|$ *denotes the cardinality of* X*.*

It is well known that a social choice satisfies efficiency, anonymity, strategy-proofness and an additional condition, if and only if it is a generalized median voter rules (Moulin

[27], Ching [13]).

Generalized average voting rule: *Suppose that* A is a closed interval $[a, b]$. *There exists a continuous and strictly increasing function* $g : [na.nb] \to [a, b]$ *such that* $g(na) = a$, $g(nb) = b$, *and for each* $R \in \prod_{i \in N} \mathcal{R}_i$, $f(R) = g(\sum_{i \in N} p(R_i))$.

Though generalized average voting rules are not strategy-proof, some studies, such as Renault and Trannoy [30] [31] and Yamamura and Kawasaki [43] pay attention to the behaviors of generalized average voting rules, because a direct revelation mechanism of a generalized average voting rule implements a generalized median voter rule in a dynamically stable and unique Nash equilibrium. This is a merit any direct revelation mechanism of an augmented median voter rule does not possess; as pointed out by Saijo, Sjostrom and Yamato [32], there is no augmented median voter rule that can exclude inefficient Nash outcomes.

Next, We introduce some properties of social choice functions.

Own Range Continuity: *For any* $i \in N$ *and any* $R_{-i} \in \mathcal{R}_{-i}$, $f(\mathcal{R}_i, R_{-i})$ is an interval of A.

Optimizability: *For any* $i \in N$, *there exist* $\widetilde{R}_i, \overline{R}_i \in \mathcal{R}_i$ *such that for any* $R_{-i} \in \mathcal{R}_{-i}$, $f(R_i, R_{-i}) \subseteq \left[f(\widetilde{R}_i, R_{-i}), f(\overline{R}_i, R_{-i})\right]_A$.

It can be easily checked that both median voter rules and average voting rules satisfy all of these properties.

These properties are somewhat similar to those Bochet, Sakai and Thomson [11] imposed on division rules in the context of the division problem with single-peaked preferences. However, our properties are different from theirs, because we replace efficiency[5], anonymity, own peak continuity and own peak monotonicity, which Bochet, Sakai and Thomson [11] imposed on division rules, with own range continuity and optimizability. We can obtain similar results to Bochet, Sakai and Thomson [11] even if peak continuity and own peak monotonicity are weakened to own range continuity and optimizability, respectively.

Given a social choice funtion f, a natural (but not always rational) way to achieve

[5] In the problem of one-dimensional voting, we can easily be checked that a rule f satisfies *efficiency* if and only if for any $R \in \mathcal{R}$, there exists $i, j \in N$, such that $p(R_i) \leq f(R) \leq p(R_j)$.

f is to ask each voter their preference directly and a policy is chosen according to a social choice function f. Such a voting mechanism is called a *direct revelation mechanism associated with* f and denoted by $\Gamma_f \equiv (\{\mathcal{R}_i\}_{i\in N}, f)$.

A social choice funtion f is said to be *strategy-proof* if for any $R \in \mathcal{R}$, R is a Nash equilibrium of Γ_f and *coalitionally strategy-proof* if for any $R \in \mathcal{R}$, R is a strong Nash equilibrium of Γ_f.

3 Results

3.1 Strong Nash Equilibrium

The purpose of this paper is to analyze the consequence of coalitional behaviors under any range continuous voting mechanism. First, we provide a necessary and sufficient condition for a voting rule to possess a non-empty strong Nash equilibrium.

Throughout this chapter, we assume without a loss of generality that $p(R_1) \leq \cdots \leq p(R_n)$.

Theorem 1. *Let* Γ *be any own range continuous voting mechanism. Then for any* $R \in \mathcal{R}$, $SN_v(\Gamma, N) \neq \phi$ *if and only if for any* $S \subseteq N$, *the policies*

$$\max_{m_{N\backslash S}} \inf_{m_S} g(m_S, m_{N\backslash S}), \min_{m_S} \sup_{m_{N\backslash S}} g(m_S, m_{N\backslash S}),$$

exist and satisfy the following two conditions;

Condition 1: $\max_{m_{N\backslash S}} \inf_{m_S} g(m_S, m_{N\backslash S}) = \min_{m_S} \sup_{m_{N\backslash S}} g(m_S, m_{N\backslash S})$;

Condition 2: *for any* $i \in N$, $S \subseteq N\backslash\{i\}$, *and any* $c \in \left[\max_{m_{N\backslash S\cup\{i\}}} \inf_{m_{S\cup\{i\}}} g(m_S, m_{N\backslash S}), \max_{m_{N\backslash S}} \inf_{m_S} g(m_S, m_{N\backslash S})\right]_A$, *there exists* $m_i^* \in M_i$ *such that the policies*

$$\max_{m_{N\backslash S\cup\{i\}}} \inf_{m_S} g(m_S, m_{N\backslash S\cup\{i\}}, m_i^*), \min_{m_S} \sup_{m_{N\backslash S\cup\{i\}}} g(m_S, m_{N\backslash S\cup\{i\}}, m_i^*)$$

exist and satisfy

$$\max_{m_{N\backslash S\cup\{i\}}} \inf_{m_S} g(m_S, m_{N\backslash S\cup\{i\}}, m_i^*), \min_{m_S} \sup_{m_{N\backslash S\cup\{i\}}} g(m_S, m_{N\backslash S\cup\{i\}}, m_i^*) = c.$$

Proof of Theorem 1.

In order to prove theorem 1, we make use of the following "minimax theorem."

Theorem (Minimax Theorem[6].) *Let* $f : X \times Y \to \mathbb{R} \cup \{-\infty, \infty\}$ *be an extended real-valued mapping. There exists* $(x^*, y^*) \in X \times Y$ *such that for any* $x \in X$ *and* $y \in Y$,

$$f(x^*, y) \leq f(x^*, y^*) \leq f(x, y^*),$$

if and only if the two quantities

$$\max_y \inf_x f(x, y), \min_x \sup_y f(x, y)$$

exist and the following equality holds:

$$\max_y \inf_x f(x, y) = \min_x \sup_y f(x, y).$$

First, we shall prove the "only if" part. For each $S \subseteq N$, consider a preference profile $R \in \mathcal{R}$ such that $p(R_j) = \min A$, $\forall j \in S$ and $p(R_j) = \max A$, $\forall j \in N \backslash S$[7]. Under this preference profile R, we can easily check that $m^* \in SN(\Gamma, R)$ if and only if for any $m_S \in M_S$ and any $m_{N\backslash S} \in M_{N\backslash S}$,

$$g(m^*_S, m_{N\backslash S}) \leq g(m^*) \leq g(m_S, m^*_{N\backslash S}).$$

Then, by applying the minimax theorem, there exists a strong Nash message profile $m^* \in SN(\Gamma, R)$, only if the policies

$$\max_{m_{N\backslash S}} \inf_{m_S} g(m_S, m_{N\backslash S}), \min_{m_S} \sup_{m_{N\backslash S}} g(m_S, m_{N\backslash S}),$$

exist and satisfy the following property;

$$(1) \max_{m_{N\backslash S}} \inf_{m_S} g(m_S, m_{N\backslash S}) = \min_{m_S} \sup_{m_{N\backslash S}} g(m_S, m_{N\backslash S}).$$

The rest of the proof of the "only if" part is to show that the condition (2) is also required.

[6] For example, Petrosjan and Zenkevich [29] provide a proof of this statement.

[7] Since A is a closed subset of $\mathbb{R} \cup \{-\infty, \infty\}$, $\min A$ and $\max A$ are well-defined.

For any $i \in N$, $S \subseteq N \setminus \{i\}$, and any $c \in \left[\max_{m_{N\setminus(S\cup\{i\})}} \inf_{m_{(S\cup\{i\})}} g(m_S, m_{N\setminus S}), \max_{m_{N\setminus S}} \inf_{m_S} g(m_S, m_{N\setminus S})\right]_A$, consider a preference profile $R' \in \mathcal{R}$ such that $p(R'_j) = \min A$, $\forall j \in S$ and $p(R'_j) = \max A$, $\forall j \in N \setminus (S \cup \{i\})$ and $p(R'_i) = c$. In this case, $m^* \in SN(\Gamma, R')$ only if for any $m_S \in M_S$ and any $m_{N\setminus(S\cup\{i\})} \in M_{N\setminus S}$,

$$g(m^*_S, m_{N\setminus(S\cup\{i\})}, m^*_i) \leq g(m^*) \leq g(m_S, m^*_{N\setminus(S\cup\{i\})}, m^*_i).$$

Then, by applying the minimax theorem again, there exists a strong Nash message profile $m^* \in SN(\Gamma, R')$, only if the policies

$$\max_{m_{N\setminus S\cup\{i\}}} \inf_{m_S} g(m_S, m_{N\setminus S\cup\{i\}}, m^*_i), \min_{m_S} \sup_{m_{N\setminus S\cup\{i\}}} g(m_S, m_{N\setminus S\cup\{i\}}, m^*_i)$$

exist and satisfy

$$\max_{m_{N\setminus S\cup\{i\}}} \inf_{m_S} g(m_S, m_{N\setminus S\cup\{i\}}, m^*_i) = \min_{m_S} \sup_{m_{N\setminus S\cup\{i\}}} g(m_S, m_{N\setminus S\cup\{i\}}, m^*_i).(a)$$

Now suppose that there exists $m^* \in SN(\Gamma, R')$ such that $p(R_i) > g(m^*)$. Then, since $\min_{m_S} \sup_{m_{N\setminus S\cup\{i\}}} g(m_S, m_{N\setminus S\cup\{i\}}, m_i) \leq \min_{m_S} \sup_{m_{N\setminus S}} g(m_S, m_{N\setminus S}, m_i)$, we can take $m'_{N\setminus S} \in M_{N\setminus S}$ such that

$$g(m^*) < g(m^*_S, m'_{N\setminus S}) \leq p(R_i),$$

by own range continuity of Γ. So, $g(m^*_S, m'_{N\setminus S}) P_i$, $\forall j \in N \setminus S$, that contradicts $m^* \in SN(\Gamma, R')$. Hence, $m^* \in SN(\Gamma, R')$ only if $p(R_i) \leq g(m^*)$.

We can similarly show that whenever $m^* \in SN(\Gamma, R')$, $p(R_i) \geq g(m^*)$, so we have $m^* \in SN(\Gamma, R')$ only if $p(R_i) = g(m^*)$. (b)

Combining (a) with (b), we have that $m^* \in SN(\Gamma, R')$, only if the policies

$$\max_{m_{N\setminus S\cup\{i\}}} \inf_{m_S} g(m_S, m_{N\setminus S\cup\{i\}}, m^*_i), \min_{m_S} \sup_{m_{N\setminus S\cup\{i\}}} g(m_S, m_{N\setminus S\cup\{i\}}, m^*_i)$$

exist and satisfy

$$\max_{m_{N\setminus S\cup\{i\}}} \inf_{m_S} g(m_S, m_{N\setminus S\cup\{i\}}, m_i) = \min_{m_S} \sup_{m_{N\setminus S\cup\{i\}}} g(m_S, m_{N\setminus S\cup\{i\}}, m_i) = c,$$

that shows the necessity of the condition (2). The proof of the "only if" part is completed.■

Next, we shall show the "if" part. Before we proceed to the proof of the "if" part, we introduce 2^n policies that are useful for our analysis. Let Γ be any own range continuous voting mechanism that satisfies the conditions of Theorem 1. For each $S \subseteq N$, define $\Gamma_S \in A$ as

$$\Gamma_S = \max_{m_{N\setminus S}} \inf_{m_S} g(m_S, m_{N\setminus S}) = \min_{m_S} \sup_{m_{N\setminus S}} g(m_S, m_{N\setminus S}).$$

This Γ_S indicates an outcome when members of S minimize the policy and the rest of the members maximize it. These 2^n policies $\{\Gamma_S\}_{S\subseteq N}$ helps us analyze the consequence of strategic votes, because they actually play the roles of 2^n exogenous parameters of an augmented median voter rule.

Case 1. $\big\{i \in N \mid p(R_i) = m(\big\{p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N\big\})\big\} = \phi$.
In this case, there exists $j \in \{0, 1, \cdots, n\}$ such that

$$\Gamma_{\{1,\cdots,j\}} = m(\big\{p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N\big\}).$$

Because the value of $\Gamma_{\{1,\cdots,j\}}$ is the median of $2n+1$ policies, we can choose $j \in \{0, 1, \cdots, n\}$ to satisfy the following equality,

$$\big|\big\{i \in N | p(R_i) < \Gamma_{\{1,\cdots,j\}}\big\}\big| + j = \big|\big\{i \in N | p(R_i) > \Gamma_{\{1,\cdots,j\}}\big\}\big| + (n - j).$$

Then, since both sides of the above equality must equal n, we obtain

$$\big|\big\{i \in N | p(R_i) < \Gamma_{\{1,\cdots,j\}}\big\}\big| = n - j \text{ and } \big|\big\{i \in N | p(R_i) > \Gamma_{\{1,\cdots,j\}}\big\}\big| = j.$$

Let us consider the following message profile m^* such that

$$\begin{aligned} m^*_{\{1,\cdots,n-j\}} &\in Arg \min_{m_{\{1,\cdots,n-j\}}} \sup_{m_{\{n-j+1,\cdots,n\}}} g(m_{\{1,\cdots,n-j\}}, m_{\{n-j+1,\cdots,n\}}), \text{ and} \\ m^*_{\{n-j+1,\cdots,n\}} &\in Arg \max_{m_{\{n-j+1,\cdots,n\}}} \inf_{m_{\{1,\cdots,n-j\}}} g(m_{\{1,\cdots,n-j\}}, m_{\{n-j+1,\cdots,n\}}). \end{aligned}$$

By assumption, we can take such a message profile m^* and

$$\begin{aligned} g(m^*) &= \min_{m_{\{1,\cdots,n-j\}}} \sup_{m_{\{n-j+1,\cdots,n\}}} g(m_{\{1,\cdots,n-j\}}, m_{\{n-j+1,\cdots,n\}}) \\ &= \max_{m_{\{n-j+1,\cdots,n\}}} \inf_{m_{\{1,\cdots,n-j\}}} g(m_{\{1,\cdots,n-j\}}, m_{\{n-j+1,\cdots,n\}}) \\ &= \Gamma_{\{1,\cdots,j\}}. \end{aligned}$$

We will show that this message profile m^* is a strong Nash equilibrium. Suppose that there exist $S \subseteq N$ and $m'_S \in M_S$ such that $g(m'_S, m^*_{N\setminus S}) P_i g(m^*)$, $\forall i \in S$. We moreover suppose without a loss of generality that $g(m'_S, m^*_{N\setminus S}) < g(m^*)$. Then,

$$S \subseteq \{1, \cdots, n-j\},$$

because $g(m^*)P_i g(m'_S, m^*_{N\setminus S})$, $\forall i \in \{n-j+1, \cdots, n\}$. However, since $m^*_{\{1,\cdots,n-j\}} \in Arg \min_{m_{\{1,\cdots,n-j\}}} g(m_{\{1,\cdots,n-j\}}, m^*_{\{n-j+1,\cdots,n\}})$

$$g(m'_S, m^*_{N\setminus S}) \geq g(m^*), \ \forall m'_S \in M_S,$$

which contradicts $g(m'_S, m^*_{N\setminus S}) < g(m^*)$. Hence, m^* must be a strong Nash equilibrium.■

Case 2. $\{i \in N \mid p(R_i) = m(\{p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N\})\} \neq \phi$. In this case, by definition of the median, we can take $i \in N$, and $j \in \{0, 1, \cdots, n\}$ with $\Gamma_{\{1,2,\cdots,j\}} \leq p(R_i)$ such that

$$p(R_i) = m(\{p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N\})$$

and $j + (i-1) = n$ or $j = n$.

By definition of the median again, we must also have

$$\Gamma_{\{1,2,\cdots,n-i\}} \leq p(R_i) \leq \Gamma_{\{1,2,\cdots,n-i+1\}}.$$

Let us consider the following message profile m^* such that

$$\begin{aligned} m^*_{\{1,\cdots,i-1\}} &\in Arg \min_{m_{\{1,\cdots,i-1\}}} \sup_{m_{\{i+1,\cdots,n\}}} g(m_{\{1,\cdots,i-1\}}, m^*_i, m_{\{i+1,\cdots,n\}}), \text{ and} \\ m^*_{\{i+1,\cdots,n\}} &\in Arg \max_{m_{\{i+1,\cdots,n\}}} \inf_{m_{\{1,\cdots,i-1\}}} g(m_{\{1,\cdots,i-1\}}, m^*_i, m_{\{i+1,\cdots,n\}}), \end{aligned}$$

and $g(m^*) = p(R_i)$. By assumption, we can take such a message profile m^* and

$$\begin{aligned} g(m^*) &= \min_{m_{\{1,\cdots,i-1\}}} \sup_{m_{\{i+1,\cdots,n\}}} g(m_{\{1,\cdots,i-1\}}, m^*_i, m_{\{i+1,\cdots,n\}}) \\ &= \max_{m_{\{i+1,\cdots,n\}}} \inf_{m_{\{1,\cdots,i-1\}}} g(m_{\{1,\cdots,i-1\}}, m^*_i, m_{\{i+1,\cdots,n\}}) \\ &= p(R_i). \end{aligned}$$

We will show that this message profile m^* is a strong Nash equilibrium. Suppose that there exist $S \subseteq N$ and $m'_S \in M_S$ such that $g(m'_S, m^*_{N\setminus S})P_i g(m^*)$, $\forall i \in S$. We moreover suppose without a loss of generality that $g(m'_S, m^*_{N\backslash S}) < g(m^*)$. Then,

$$S \subseteq \{1, 2, \cdots, i-1\},$$

because $g(m^*)P_j g(m'_S, m^*_{N\backslash S})$, $\forall j \in \{i, i+1, \cdots, n\}$.

However, since $m^*_{\{1,\cdots,i-1\}} \in Arg \min_{m_{\{1,\cdots,i-1\}}} g(m_{\{1,\cdots,i-1\}}, m^*_i, m^*_{\{i+1,\cdots,n\}})$

$$g(m'_S, m^*_{N\backslash S}) \geq g(m^*),\ \forall m'_S \in M_S,$$

which contradicts $g(m'_S, m^*_{N\backslash S}) < g(m^*)$. Hence, m^* must be a strong Nash equilibrium.■

Theorem 1 establishes the necessary and sufficient condition for a voting mechanism to possess a non-empty strong Nash equilibrium. The proof of sufficiency part of Theorem 1 is constructive. First, focus on the policy $a^* \in A$ that is the median of $\left\{p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N\right\}$. Next, construct a message profile $m^* \in M$ such that

$$\begin{cases} m^*_{N\backslash\{S\cup T\}} \in \left\{ m'_{N\backslash\{S\cup T\}} | \min_{m_S} \sup_{m_T} g(m_S, m'_{N\backslash\{S\cup T\}}, m_T) = a^* \right\} & \\ m^*_S \in Arg \min_{m_S} \sup_{m_T} g(m_S, m^*_{N\backslash\{S\cup T\}}, m_T), & S = \{i \in N | p(R_i) < a^*\} \\ m^*_T \in Arg \max_{m_T} \inf_{m_S} g(m_S, m^*_{N\backslash\{S\cup T\}}, m_T), & T = \{i \in N | p(R_i) > a^*\} \end{cases}$$

and show that this message profile m^* is a strong Nash equilibrium

As a next step, we specify strong Nash outcomes.

Theorem 2. *Let Γ be any own range continuous voting mechanism that satisfies all conditions in Theorem 1. Then, for any $R \in \mathcal{R}$,*

$$SN(\Gamma, R) = \{m_{a^\Gamma}(R)\},$$

where $a^\Gamma_S = \Gamma_S$, for each $S \subseteq N$.

Proof of Theorem 2.

In Theorem 1, we have already shown that for any $R \in \mathcal{R}$, $m_{a^\Gamma}(R) \in SN(\Gamma, R)$. So, it is sufficient to show that $SN(\Gamma, R)$ does not contain any other policy.

Suppose that there exists $m \in SN(\Gamma, R)$, such that

$$g(m) > m(\left\{p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N\right\}).$$

Let

$S \equiv \left\{ i \in N \mid p(R_i) \leq m(\left\{ p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N \right\}) \right\} = \{1, \cdots, i\}$ and

$T \equiv \left\{ j \in \{0, \cdots, n\} \mid \Gamma_{\{1,2,\cdots,j\}} \leq m(p(R_1), \cdots, p(R_n), f_1, \cdots, f_{n-1}) \right\} = \{j, j+1, \cdots, n\}.$

Then, since $|S| + |T| \geq n + 1$,

$$\Gamma_S = \min_{m_S} \sup_{m_{N \setminus S}} g(m_S, m_{N \setminus S}) \in T.$$

By own range continuity, we can take $m'_S \in M_S$ such that

$$\Gamma_S \leq m\left(\left\{ p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N \right\}\right) \leq g(m'_S, m_{N \setminus S}) < g(m).$$

In this case, since for any $i \in S$, $p(R_i) \leq m\left(\left\{ p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N \right\}\right)$, we obtain

$$g(m'_S, m_{N \setminus S}) P_i g(m), \ \forall i \in S.$$

Hence, S has a deviation $m'_S \in M_S$, which contradicts $m \in SN(\Gamma, R)$. That is, whenever $m \in SN(\Gamma, R)$,

$$g(m) \leq m(\left\{ p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N \right\}).$$

We can similarly show that whenever $m \in SN(\Gamma, R)$,

$$g(m) \geq m(\left\{ p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{1\}}, \Gamma_{\{1,2\}}, \cdots, \Gamma_N \right\}).\blacksquare$$

Theorem 1 and Theorem 2 together characterize strong Nash outcomes. As a direct consequence of Theorem 1 and Theorem 2, we can state that if a voting mechanism satisfies range continuity, and all conditions of Theorem1, then whenever $p(R_1) \leq \cdots \leq p(R_n)$,

$$SN(\Gamma, R) = \left\{ m(p(R_1), \cdots, p(R_n), \Gamma_\phi, \Gamma_{\{i_i\}}, \cdots, \Gamma_N) \right\},$$

which implies that strategic votes must result in an augmented median voter rule when communication among voters is allowed.

In the context of implementation theory, Theorem 1 and Theorem 2 "almost" characterize the class of social choice rules that can be implemented in strong Nash equilibrium. These theorems tell us that a social choice rule that can be implemented in strong Nash equilibrium by an own range continuous mechnism must be single-valued and an augmented median voter rule. The way to establish this result is quite different from the "standard" approach to implementation theory, such as Maskin [25] [26], Dutta and Sen [14], and Suh [37], in which a particular mechanism is constructed to show the implemenbility of a social choice rule.

In the last of this section, we illustrate how to use Theorem 1 and Theorem 2. The following Proposition 1 exhibits how many voting mechanisms possess a non-empty strong Nash equilibrium.

Proposition 1. *Let Γ be any voting mechanism that satisfies own range continuity and optimizability. Then, for any $R \in R$,*

$$SN(\Gamma, R) = \{m_{a^{\Gamma}}(R)\} \neq \phi,$$

where $a_S^{\Gamma} = g(\widetilde{m}_S, \overline{m}_{N \backslash S})$, in which for any $i \in N$, any $m_i \in M_i$, and any $m_{-i} \in M_{-i}$, $g(\widetilde{m}_i, m_{-i}) \leq g(m_i, m_{-i}) \leq g(\overline{m}_i, m_{-i})$.

Proof of Proposition 1.
By Theorem 1 and Theorem 2, it is sufficient to show that an optimizable voting mechanism Γ satisfies all conditions in Theorem 1. If, Γ satisfies optimizability, then for any $S \subseteq N$,

$$g(\widetilde{m}_S, \overline{m}_S) = \max_{m_{N \backslash S}} \inf_{m_S} g(m_S, m_{N \backslash S}) = \min_{m_S} \sup_{m_{N \backslash S}} g(m_S, m_{N \backslash S}).$$

Hence, Condition (1) of Theorem 1 is satisfied.

For any $i \in N$, $S \subseteq N \backslash \{i\}$, and any $c \in \left[\max_{m_{N \backslash S \cup \{i\}}} \inf_{m_{S \cup \{i\}}} g(m_S, m_{N \backslash S}), \max_{m_{N \backslash S}} \inf_{m_S} g(m_S, m_{N \backslash S}) \right]_A$, by range continuity, we can take $m_i^* \in M_i$ such that

$$\begin{aligned} g(\widetilde{m}_S, \overline{m}_{N \backslash (S \cup \{i\})}, m_i^*) &= \max_{m_{N \backslash (S \cup \{i\})}} \inf_{m_S} g(m_S, m_{N \backslash (S \cup \{i\})}, m_i^*) \\ &= \min_{m_S} \sup_{m_{N \backslash S \cup \{i\}}} g(m_S, m_{N \backslash (S \cup \{i\})}, m_i^*) \\ &= c. \end{aligned}$$

Hence, Condition (2) of Theorem 1 is also satisfied.■

3.2 Coalition-proof Nash Equilibrium

Next, we study what happens in a coalition-proof Nash equilibrium. Though the set of coalition-proof Nash outcomes is not always equivalent to that of strong Nash outcomes, we can show the equivalence between $CN(\Gamma, R)$ and $SN(\Gamma, R)$ in a large class of voting mechanisms. The following Example 1 exhibits the inequivalence between $CN(\Gamma, R)$ and $SN(\Gamma, R)$.

Example 1. Let $N = \{1, 2\}$, $A = \{1, 2, 3\}$. Consider the following voting mechanism $\Gamma = (\{M_1, M_2\}, g)$ such that $M_1 = M_2 = \{x, y, z\}$ and g is expressed by the following Table 1.

Table 1.

$1\backslash 2$	x	y	z
x	1	1	1
y	1	3	2
z	1	2	3

Then, under a preference profile $R = (R_1, R_2)$ such that $p(R_1) = 2$ and $p(R_2) = 3$, since $SN_v(\Gamma, R) = \phi \neq \{(x, x)\} = CN_v(\Gamma, R)$,

$$SN(\Gamma, R) = \phi \neq \{1\} = CN(\Gamma, R).$$

Thus, the equivalence between $SN_v(\Gamma, R)$ and $CN_v(\Gamma, R)$ does not always hold.

On the other hand, as long as Γ satisfies optimizability, the equivalence between coalition-proof Nash equilibrium and strong Nash equilibrium is assured.

Theorem 3. *Let Γ be any voting mechanism that satisfies own range continuity and optimizability. Then, for any $R \in \mathcal{R}$,*

$$SN_v(\Gamma, R) = CN_v(\Gamma, R) \neq \phi.$$

Hence, for any $R \in \mathcal{R}$,

$$SN(\Gamma, R) = CN(\Gamma, R) = \{m_{a^\Gamma}(R)\},$$

where $a_S^\Gamma = g(\widetilde{m}_S, \overline{m}_{N\backslash S})$, in which for any $i \in N$, any $m_i \in M_i$, and any $m_{-i} \in M_{-i}$, $g(\widetilde{m}_i, m_{-i}) \leq g(m_i, m_{-i}) \leq g(\overline{m}_i, m_{-i})$.

Proof of Theorem 3.
It is sufficient to show that there exists $S' \subseteq N$ which has a credible deviation at

$m \in M$, whenever there exists $S \subseteq N$ which has a deviation at $m \in M$. Suppose that $S \subseteq N$ has a deviation $m'_S \in M_S$ at $m \in M$. We moreover assume without a loss of generality $g(m'_S, m_{-S}) < g(m)$. Then, for any $i \in S$, $p(R_i) < g(m)$. Let $S \equiv \{i_1, \cdots, i_k\}$ be such that $p(R_{i_1)} \leq \cdots \leq p(R_{i_k})$. Consider $\widetilde{m}_S \in M_S$, such that $g(\widetilde{m}_i, m_{-i}) \leq g(m_i, m_{-i})$ $\forall i \in S$, $\forall m_i \in M_i$.and $\forall m_{-i} \in M_{-i}$ Then, there exists $h \in \{1, \cdots, k\}$such that

$$g\left(\widetilde{m}_{\{i_1,\cdots,i_{h-1}\}}, m_{i \in N/\{i_1,\cdots,i_{h-1}\}}\right) = g(m), \text{ and}$$

$$g\left(\widetilde{m}_{\{i_1,\cdots,i_h\}}, m_{N/\{i_1,\cdots,i_h\}}\right) < g(m).$$

Let $S' \equiv \{i_1, \cdots, i_h\}$ and take $m''_{S'} \in M_{S'}$ such that $m''_i = \widetilde{m}_i$, $\forall i \in \{i_1, \cdots, i_{h-1}\}$, and

$$m''_{i_h} = \begin{cases} \widetilde{m}_{i_h} & \text{if } g\left(\widetilde{m}_{S'}, m_{N/S'}\right) > p(R_{i_h}) \\ \widehat{m}_{i_h},\ \textit{such that}\, g\left(\widetilde{m}_{\{i_1,\cdots,i_{h-1}\}}, \widehat{m}_{i_h}, m_{N/S'}\right) = p(R_{i_h}) & \text{if } g\left(\widetilde{m}_{S'}, m_{N/S'}\right) \leq p(R_{i_h}). \end{cases}$$

We shall show that $m''_{S'}$ is a credible deviation of S' at m.

If $g\left(\widetilde{m}_{S'}, m_{N/S'}\right) > p(R_{i_h})$, then

$$g(m''_{S'}, m_{-S'}) \leq g(m'''_{S'}, m_{-S'}),\ \forall m'''_{S'} \in M_{S'}$$

by optimizability, so we have

$$g(m''_{S'}, m_{-S'}) R_i g(m'''_{s''}, m''_{S' \setminus S''}, m_{-S'}), \forall S'' \subseteq S',\ \forall m_{S''} \in M_{S''},\ \forall i \in S''.$$

Hence, there is no $S'' \subset S'$ that has a credible deviation at $(m''_{S'}, m_{-S'})$. So, $m''_{S'}$ is a credible deviation of S' at m.

If $g\left(\widetilde{m}_{S'}, m_{N/S'}\right) \leq p(R_{i_h})$, then

$$g(m''_{S'}, m_{-S'}) R_{i_h} g(m'''_{S'}, m_{-S'}),\ \forall m'''_{S'} \in M_{S'},$$

because $p(R_{i_h}) = g(m''_{S'}, m_{-S'})$. Hence, if there exists $S'' \subset S'$ that has a credible deviation at $(m''_{S'}, m_{-S'})$, then $i_h \notin S''$. However, by optimizability,

$$g\left(\widetilde{m}_{S' \setminus \{i_h\}}, m''_{i_h}, m_{N \setminus S'}\right) \leq g\left(m'''_{S' \setminus \{i_h\}}, m''_{i_h}, m_{N \setminus S'}\right), \forall m'''_{S' \setminus \{i_h\}} \in M_{S' \setminus \{i_h\}},$$

that implies

$$g(m''_{S'}, m_{-S'}) R_i g(m'''_{S''}, m''_{S' \setminus S''}, m_{-S'}), \forall S'' \subseteq S' \setminus \{i_h\}, \forall m_{S''} \in M_{S''}, \forall i \in S''.$$

Therefore, there is no $S'' \subset S'$ that has a credible deviation at $(m''_{S'}, m_{-S'})$. So, $m''_{S'}$ is a credible deviation of S' at m.■

Note that by Theorem 3, we find a new sufficient condition for the equivalence between strong Nash and coalition-proof Nash equilibria. Konishi, Breton, and Weber [22] showed sufficient conditions for the equivalence between these two solutions, but we cannot apply them to our case.

4 Applications

4.1 Direct Revelation Games

The original motivation of this study is to analyze what happens as a consequence of coalitional manipulations under a given direct revelation mechanisms as in other environments, such as Shin and Suh [36], Selter and Sanver [35] and Bochet, Sakai and Thomson [11]. Since our theorems cover direct revelation mechanisms as special cases, we can propose the following proposition as corollaries of our Theorem 1, 2 and 3.

Proposition 2. *Let f be any own range continuous and optimizable social choice function. Then for any $R \in \mathcal{R}$,*

$$SN_v(\Gamma_f, R) = CN_v(\Gamma_f, R) \neq \phi \text{ and } SN(\Gamma_f, R) = CN(\Gamma_f, R) = \{m_{a^\Gamma}(R)\},$$

where $a^\Gamma_S = f(\widetilde{R}_S, \overline{R}_{N \setminus S})$, in which for any $i \in N$, any $R_i \in \mathcal{R}_i$, and any $R_{-i} \in \mathcal{R}_{-i}$, $f(\widetilde{R}_i, R_{-i}) \leq f(R_i, R_{-i}) \leq f(\overline{R}_i, R_{-i})$.

This Proposition 2 exhibits no matter what kind of direct revelation mechanism we use, coalitional manipulations result in an outcome that a particular augmented median voter rule suggests, as long as an associated social choice function satisfies own range continuity and optimizability. Notice that optimizability is a natural property for a social choice function to be satisfied. It is implied by the fact that the following condition called "own peak monotonicity" implies optimizability.

Own Peak Monotonicity: *For any* $i \in N$, $R_i, R_i' \in \mathcal{R}_i$ *any and any* $R_{-i} \in \mathcal{R}_{-i}$, *if* $p(R_i) \leq p(R_i')$, then $f(R_i, R_{-i}) \leq f(R_i, R_{-i})$.

4.2 Characterization of Strategy-proof Social Choice Functions: A Simple Proof

Our Theorem 2 is, in fact, useful for characterizing strategy-proof social choice functions. Though Moulin [27] and Ching [13] have already provided the proofs of the characterization theorems, Theorem 2 can make them much simpler. The following Theorem 4 is an alternative characterization theorem of strategy-proof social choice functions with a simple proof.

Theorem 4. *A Social function* f *satisfies (coalitional) startegy-proofness and own range continuity if and only if it is an augmented median voter rule.*

Proof of Theorem 4.
Barbera, Berga and Moreno [4] have shown that in one-dimensional voting situations, strategy-proofness is equivalent to coalitional strategy-proofness. So, it is sufficient to characterize the class of coalitional strategy-proof social choice functions. In this paper we provide the proof of the "only if" part only.

Let f be any coalitionally strategy-proof and own range continuous social choice function. Then, since for any $R \in \mathcal{R}$, $R \in SN_v(\Gamma_f, R)$, we have $f(R) \in SN(\Gamma_f, R)$. By Theorem 2, we obtain $SN(\Gamma_f, R) = \{f(R)\}$ and $f(R)$ must be a particular augmented median voter rule. Thus, if an own range continuous social choice function satisfies coalitional strategy-proofness, it must be an augmented median voter rule.■

Ching [13] has shown that a social choice function f is an augmented median voter rule if and only if f satisfies strategy-proofness and one of the following five conditions;

Continuity: *For any* $i \in N$, *any* $R_{-i} \in \mathcal{R}_{-i}$, *any* $R_i \in \mathcal{R}_i$ *and any* $\{R_i^k\}_{k=1}^{\infty} \subset \mathcal{R}_i$, if $\lim_{k \to \infty} R_i^k = R_i$, then $\lim_{k \to \infty} f(R_i^k, R_{-i}) = f(R_i, R_{-i})$;

Peak Only: *For any* $R, R' \in \mathcal{R}$, *if for any* $i \in N$, $p(R_i) = p(R_i')$, *then* $f(R) = f(R')$;

Own Peak Monotonicity: *For any* $i \in N$, $R_i, R_i' \in \mathcal{R}_i$ *any and any* $R_{-i} \in \mathcal{R}_{-i}$, *if* $p(R_i) \leq p(R_i')$, then $f(R_i, R_{-i}) \leq f(R_i, R_{-i})$;

Uncompromisingness: *For any* $R \in \mathcal{R}$, $i \in N$, and any $R'_i \in \mathcal{R}_i$, if $[f(R) < p(R_i)$ and $f(R) \leq p(R'_i)]$ or $[f(R) > p(R_i)$ and $f(R) \geq p(R'_i)]$, then $f(R) = f(R'_i, R_{-i})$;

Range Continuity: *There exists* $\widetilde{R}, \overline{R} \in \mathcal{R}$ *such that* $f(\mathcal{R}) = \left[f(\widetilde{R}), f(\overline{R})\right]_A$.

Notice that our Theorem 4 is independent of Ching [13]'s characterization theorem, because own range continuity implies none of these five conditions. The following Example 2 indicates that own range continuity does not imply range continuity.

Example 2. Let $A = [0, 1]$ and consider the following social choice funciton f such that

$$f(R) = \begin{cases} \max\limits_{i \in N} p(R_i) & \text{if } p(R_i) > 0, \ \forall i \in N, \\ 1 & \text{otherwise.} \end{cases}$$

Since for any $i \in N$, and any $R_{-i} \in \mathcal{R}_{-i}$,

$$f(\mathcal{R}_i, R_{-i}) = \begin{cases} \left[\max\limits_{j \neq i} p(R_j), 1\right] & \text{if } p(R_j) > 0, \ \forall j \neq i, \\ \{1\} & \text{otherwise,} \end{cases}$$

f satisfies own range continuity. However, f fails to satisfy range continuity, because $f(\mathcal{R}) = (0, 1]$.

4.3 Division Problem in Single-peaked Preferences: Two-person Case

A *division problem with single peaked preferences* is a situation where a fixed amount of a resource should be shared among agents who have single-peaked preferences over the amount of a resource. In this environment, it is known that there exists a strategy-proof, efficient and anonymous division rule; the uniform rule (Benassy [7]; Sprumout [38]).

The division problem with single-peaked preferences can be described as follows. Let $N = \{1, \cdots, n\}$ be the set of *agents* and there is a fixed amount of a *resource* $\Omega > 0$ to be shared among N. The set of feasible allocations is denoted by $A = \left\{\{x_i\}_{i \in N} \in \mathbb{R}^n_+ | \sum_{i \in N} x_i = \Omega\right\}$. Each agent $i \in N$ has a single-peaked preference over the amount of a resource $[0, \Omega]$ allocated to i. $p(R_i) \in [0, \Omega]$ denotes agent i's *peak*.

An *allocation mechanism* is defined by the pair $\Gamma = (\{M_i\}_{i \in N}, g)$ where M_i denotes voter i's *message space*, and $g : M \to A$ the *outcome function* that associates with each message profile $m \in M$ an allocation $g(m) \in A$. An allocation mechanism Γ is said to be *own range continuous* if for any $i \in N$ and any $m_{-i} \in M_{-i}$, $g_i(M_i, m_{-i})$ is an interval of $[0, \Omega]$, and to be *optimizable* if for any $i \in N$, there exist $\widetilde{m}_i, \overline{m}_i \in M_i$ such that for any $m_{-i} \in M_{-i}$, $g_i(M_i, m_{-i}) \subseteq [g_i(\widetilde{m}_i, m_{-i}), g_i(\overline{m}_i, m_{-i})]$.

Observe that A can be regarded as an $n-1$ dimensional simplex. Hence, when $n = 2$, as Barbera, Jackson and Neme [6] have pointed out, we can identify A with the set of agent 1's allotments, that is a one-dimensional interval, and two agents have single-peaked preference over 1's allotments; agent 1's peak and agent 2's peak are $p(R_1)$, $\Omega - p(R_2)$, respectively. That is, when $n = 2$, we can apply our results to the analysis of division problems in sigle-peaked preferences.

When $n = 2$, our Theorem 1 tells us whether any given own range continuous division mechanism has a non-empty strong Nash equilibrium or not; our Theorem 2 says that the set of strong Nash outcomes must be single-valued and specified as a particular sequential allotment rule (Barbera, Jackson and Neme [6]); and our Theorem 3 shows us that as long as a division mechanism satisfies optimizability, coalition-proof Nash equilibriua must be equivalent to strong Nash equilibria.

Our results above are independent of Bochet and Sakai [10] and Bochet Sakai and Thomson [11], that analyze stragtegic manipulations in direct revelation mechanisms, in the sense that when $n = 2$, our results cover a larger class of division mechanisms, including indirect mechanisms, than their studies. On the other hand, our results cannot apply to the cases $n > 2$, because the minimax theorem is useful only for the cases where the outcome space is one-dimensional. Hence, in order to analyze more than two-person cases, a different approach is required.

5 Concluding Remarks

Through the analysis of strategic votes, we reveal the strong position that augmented median voter rules possess. They are not only strategy-proof but also always expected as a result of coalitional votes under any given voting mechanism satisfying own range continuity. This result has a somewhat negative message in the context of implementation theory. No matter what kind of own range continuous voting mechanism we use, we cannot escape from augmented median voter rules, as long as coalitional behaviors are permitted. The class of social choice rules that an own range continuous voting mechanism can implemented in strong Nash equilibrium is quite restricted.

In one-dimensional voting games, most voters intend either to minimize the policy or to maximize it. Then, the interest of those who want to minimize the policy perfectly conflicts with that of those who want to maximize it. We can regard the situation as a kind of two-person zerosum game between minimizers and maximizers. This is why we can apply the minimax theorem to the analysis of one-dimensional voting games.

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