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Recurrent Bubbles

Takashi KAMIHIGASHI

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Research Institute for Economics and Business Administration
Kobe University
2-1 Rokkodai, Nada, Kobe 657-8501 JAPAN

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Takashi Kamihigashi†

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Abstract

We study rational bubbles in a standard linear asset price model. We first consider a class of bubble processes driven by multiplicative iid shocks. We show that a bubble process in this class either diverges to infinity with probability one, converges to zero with probability one, or keeps fluctuating forever with probability one, depending on investors' "confidence" in expected bubble growth. We call a bubble process having the last property "recurrent." We develop sufficient conditions for a bubble process to be recurrent when it is driven by non-iid shocks, when the risk-free interest rate is not constant, and when the process is driven by non-iid shocks and the risk-free interest rate is not constant. In the last case we demonstrate via simulation that there can be a prolonged period in which both the bubble and the interest rate stay close to zero.

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†RIEB, Kobe University, Rokkodai, Nada, Kobe 657-8501 JAPAN. Email: tkamihig@rieb.kobe-u.ac.jp. Tel/Fax: +81-78-803-7015.

1 Introduction

Since the 17th century, there have been numerous episodes of dramatic rises in asset prices followed by sharp declines. Well-known historical examples include the Tulip mania (1637), the Mississippi and South See bubbles (1720); more recent examples are the Japanese asset bubble in the 1980s, the IT bubble in the 1990s, and the US housing bubble in the 2000s.[1] One possible explanation for this excessive "rise and fall" phenomenon is the presence of "rational asset bubbles."

An asset bubble is said to exist if the market price of an asset exceeds its fundamental value. The simplest model to analyze "rational" bubbles would be the following intertemporal no-arbitrage condition:

$$P_t = (1+r)^{-1} E_t(P_{t+1} + D_{t+1}), \tag{1.1}$$

where P_t is the asset price in period t, $D_t \geq 0$ is the dividend in period t, and $r > 0$ is the risk-free interest rate. Equation (1.1) says that the expected gross return on the asset, $E_t(P_{t+1} + D_{t+1})/P_t$, must be equal to the gross risk-free interest rate, $1 + r$. This simple model has been the basis for much of the literature on rational bubbles (e.g., Flood and Hodrick, 1990; Froot and Obstfeld, 1991; Evans, 1991; Gürkaynak, 2008).

In this model the fundamental value of the asset in period t is defined as the expected present discounted value of the dividend sequence from period $t + 1$ onward:

$$P_t^* = E_t \sum_{i=1}^{\infty} (1+r)^{-i} D_{t+i}. \tag{1.2}$$

It is easy to see that $\{P_t^*\}$ solves (1.1), but there are many other solutions to (1.1). Indeed, for any nonnegative process $\{B_t\}$ following

$$E_t B_{t+1} = (1+r) B_t, \tag{1.3}$$

the price process $\{P_t\}$ defined by $P_t = P_t^* + B_t$ also solves (1.1). Therefore, if (1.1) is the only requirement on the price process $\{P_t\}$, then one cannot rule out the possibility that the asset price P_t exceeds the fundamental value P_t^*.

Although the concept of rational bubbles as defined above has attracted considerable attention in the economic literature, showing the presence of

[1] See Garber (2000), Kindleberger and Aliber (2005), and Reinhart and Rogoff (2009) for accounts of these episodes.

bubbles has been a challenging problem, both theoretically and empirically. On the theoretical side, bubbles are often ruled out in a representative agent model using a transversality condition or other optimality-based arguments (e.g., Kamihigashi 1998, 2001, 2003). Bubbles are also ruled out in general overlapping generations models if the value of aggregate wealth is finite (Wilson, 1981; Santos and Woodford, 1997). Recently, however, it has been shown that there are representative agent models with a wealth effect in which bubbles are consistent with a transversality condition (Kamihigashi 2008a, 2008b).[2] In such cases the optimality requirement for bubbles reduces to an equation similar to (1.1).

On the empirical side, it is well-known that bubbles are difficult to detect, partly because sample paths of bubble processes do not necessarily appear explosive. This was first pointed out by Evans (1991) who demonstrated via simulation that "periodically collapsing bubbles" are not detectable by using standard tests. This point has further been reinforced by Charemza and Deadman (1995) and Gürkaynak (2008).[3]

The purpose of this paper is to stimulate further investigation of bubbles by characterizing the asymptotic behavior of their sample paths and by illustrating them with simulations. We start by formalizing Evan's (1991) point that bubbles may not appear explosive. For this purpose we consider three asymptotic properties: we say that a bubble process $\{B_t\}$ is "explosive" if it diverges to infinity with probability one, "implosive" if it converges to zero with probability one, and "recurrent" if with probability one, there are two levels $\underline{B}, \overline{B} > 0$ such that $B_t > \underline{B}$ infinitely often and $B_t < \overline{B}$ infinitely often. We are particularly interested in the last property. A recurrent bubble process (almost) always reappears even though it may temporarily become arbitrarily small, and always collapses to a certain level even though it may temporarily become arbitrarily large. This property may be consistent with the repeated phenomenon of excessive rises and falls observed in various asset price data.

We first consider a class of bubble processes—nonnegative, nontrivial stochastic processes satisfying the bubble equation (1.3)—with a multiplicative iid shock. We show that these bubble processes are explosive, implosive, or recurrent depending on investors' "confidence" in expected bubble growth

[2]As surveyed in Brunnermeier (2007) and Iraola and Santos (2007), bubbles are also possible in models with asymmetric information, heterogeneous beliefs, limited rationality, limited arbitrage, or agency problems.

[3]See Phillips, Wu, and Yu (2009) for recent developments.

$E_t B_{t+1}/B_t$. We define confidence as a measure of how sure or confident investors are about this expected value. We show that the bubble process $\{B_t\}$ is explosive if confidence is sufficiently high, and implosive if confidence is sufficiently low. There is also a knife-edge level of confidence such that the bubble process is recurrent.[4]

These results, which assume that confidence is constant, suggest that if confidence varies with the bubble, it may be possible that the bubble tends to shrink when it is large, and tends to grow when it is small. Such a bubble process would be recurrent in a robust way, and more stable than the knife-edge case mentioned above. Indeed, since the bubble equation (1.3) does not require the shock to be iid, the distribution of the shock, or confidence, is allowed to depend on the current bubble. Under this assumption we show that a bubble process is recurrent if confidence becomes sufficiently low when the bubble is extremely large, and becomes sufficiently high when the bubble is extremely small.

A similar stabilizing effect can be obtained even if confidence is constant, provided that the interest rate r varies with the bubble. We show that a bubble process is recurrent if the interest rate becomes sufficiently small when the bubble is extremely large, and becomes sufficiently large when the bubble is extremely small. Since one may find this negative relation counterfactual, we also consider a model in which the interest rate reacts positively to changes in the bubble, and confidence reacts negatively to changes in the interest rate. In this case the bubble process can once again be recurrent. We demonstrate via simulation that there can be a prolonged period in which both the bubble and the interest rate stay close to zero.

The rest of the paper is organized as follows. Section 2 introduces basic definitions. Section 3 models the bubble equation (1.3) as a stochastic difference equation with a multiplicative iid shock, and offers conditions under which the induced bubble process is explosive, implosive, or recurrent. Section 4 interpret these conditions in terms of investors' confidence in expected bubble growth. Section 4 endogenizes confidence as a function of the current bubble, and provides sufficient conditions for recurrence. Section 6 endogenizes the interest rate instead of confidence as a function of the current bubble, and provides sufficient conditions for recurrence. Section 7 considers a case in which both the interest rate and confidence are endogenous, and

[4]Section 3 discusses related results in the literature (e.g., Ikeda and Shibata, 1992, 1995; Salge, 1996; Kamihigashi, 2006).

offers sufficient conditions for recurrence. Section 8 concludes the paper. The Appendix contains technical proofs.

2 Definitions

In this section we introduce several definitions that we use throughout the paper. We say that a nonnegative stochastic process $\{B_t\}$ is a *bubble process* if it satisfies the bubble equation (1.3) and

$$B_0 > 0. \tag{2.1}$$

In other words we require a bubble process to be nonnegative and nontrivial in addition to satisfying the bubble equation (1.3). Nonnegativity often follows from the requirement that the asset price P_t cannot be smaller than the fundamental value P_t^*. Nontriviality requires (2.1), since if $B_0 = 0$, the bubble equation (1.3) and nonnegativity imply that $B_t = 0$ for all $t \in \mathbb{N}$.

Next we define some asymptotic properties that a given bubble process may or may not have. To simplify notation, let

$$\underline{\lim} = \liminf, \qquad \overline{\lim} = \limsup\,. \tag{2.2}$$

Let $P\{A\}$ denote the probability of the event A.[5] We say that a nonnegative stochastic process $\{X_t\}$ is

- *explosive* if
$$P\{\underline{\lim}_{t\uparrow\infty} X_t = \infty\} = 1. \tag{2.3}$$
- *implosive* if
$$P\{\overline{\lim}_{t\uparrow\infty} X_t = 0\} = 1. \tag{2.4}$$
- *downward recurrent* if
$$P\{\underline{\lim}_{t\uparrow\infty} X_t < \infty\} = 1. \tag{2.5}$$
- *upward recurrent* if
$$P\{\overline{\lim}_{t\uparrow\infty} X_t > 0\} = 1. \tag{2.6}$$

[5]There should be no confusion between $P\{\cdot\}$ and P_t; we never use them together, and P_t appears only in (6.1) in what follows.

- *recurrent* if it is both downward and upward recurrent.

In words, a stochastic process $\{X_t\}$ is explosive if it diverges to infinity with probability one, implosive if it converges to zero with probability one, and recurrent if with probability one, there exist two levels $\underline{X}, \overline{X} > 0$ such that $X_t < \overline{X}$ infinitely often and $X_t > \underline{X}$ infinitely often.[6]

3 Three Benchmarks with IID Shocks

A natural way to model the bubble equation (1.3) would be to introduce an additive iid shock:

$$B_{t+1} = (1+r)B_t + A_{t+1}, \qquad E_t A_{t+1} = 0, \tag{3.1}$$

where $\{A_t\}$ is a sequence of iid shocks on $\mathbb{R}$. In this formulation, however, nonnegativity is violated with positive probability when B_0 is sufficiently small, provided that A_t is a nontrivial random variable (i.e., $P\{A_t < 0\} > 0$).[7] Indeed, if the support of A_t is unbounded below, then we have $P\{B_1 < 0\} > 0$ for any $B_0 > 0$. Even if the support of A_t is bounded, we have

$$P\{B_1 < 0\} = P\{(1+r)B_0 + A_1 < 0\} = P\{A_1 < -(1+r)B_0\}, \tag{3.2}$$

which is strictly positive if B_0 is sufficiently close to zero. Therefore, under (3.1), it is not possible to ensure nonnegativity for all $B_0 > 0$.

An alternative way to model the bubble equation (1.3) is to introduce a multiplicative shock:

$$B_{t+1} = (1+r)B_t S_{t+1}, \tag{3.3}$$

where $\{S_t\}$ is a sequence of iid shocks[8] such that

$$E_t S_{t+1} = 1, \qquad S_{t+1} > 0. \tag{3.4}$$

[6] The recurrence property defined here is known as "non-evanescence" in the literature on Markov processes (Meyn and Tweedi, 2009, p. 206). Under regularity conditions, non-evanescence implies "recurrence" and "Harris recurrence" in Meyn and Tweedie's (2009, p. Theorem 9.2.2) terminology.

[7] Diba and Grossman (1988) recognize this point, and consider the stochastic difference equation (3.1) without assuming that $\{A_t\}$ is an iid process.

[8] Throughout the paper we follow the convention that if $\{\epsilon_t\}$ is a sequence of iid shocks, then ϵ_{t+1} is independent of information at time t; thus $Ef(\epsilon_{t+1}) = E_t f(\epsilon_{t+1})$ for any measurable function f (provided that both sides are well defined).

In this formulation it is guaranteed that the stochastic process $\{B_t\}$ obeying (3.3) with $B_0 > 0$ is always nonnegative and thus is a bubble process. A special case of (3.3) with a log-normal shock is considered by Charemza and Deadman (1995) and Salge (1997). In this section we study asymptotic properties of the bubble process generated by (3.3).

We should mention that we do not specify the source of uncertainty here: S_t can be a "fundamental" or "sunspot" shock. If S_t is a fundamental shock, or a shock that depends on D_t, then the process here can be viewed as an "intrinsic" bubble (Froot and Obstfeld, 1991). If S_t is a sunspot shock, then it can be interpreted that the fluctuations in the bubble process are driven by "animal spirits" (e.g., Farmer, 2008).

Taking logarithms, we can express (3.3) in additive form:

$$b_{t+1} = b_t + \rho + s_{t+1}, \tag{3.5}$$

where

$$b_t = \ln B_t, \tag{3.6}$$
$$\rho = \ln(1+r), \tag{3.7}$$
$$s_t = \ln S_t. \tag{3.8}$$

In this section we assume that

$$E|s_t| < \infty, \quad r \geq 0. \tag{3.9}$$

By Jensen's inequality and (3.4) we have

$$\mu \equiv E s_t = E \ln S_t \leq \ln E S_t = 0. \tag{3.10}$$

The above inequality holds strictly as long as S_t is a nontrivial random variable.

To see the importance of the inequality in (3.10), define

$$\zeta_t = s_t - \mu. \tag{3.11}$$

Then (3.5) can be rewritten as

$$b_{t+1} = b_t + \rho + \mu + \zeta_{t+1}. \tag{3.12}$$

Since $E_t(\zeta_{t+1} - \mu) = 0$, we see that $\{b_t\}$ is a random walk with drift. The drift parameter $\rho + \mu$ determines the direction in which the process tends to drift; it also determines the asymptotic behavior of the process, as we see below.

Note from (3.5) and (3.11) that for any $t \geq 1$, we have

$$b_t = b_0 + t(\rho + \mu) + \sum_{i=1}^{t} \zeta_i. \tag{3.13}$$

Dividing through by t, we obtain

$$\frac{b_t}{t} = \frac{b_0}{t} + \rho + \mu + \frac{1}{t} \sum_{i=1}^{t} \zeta_i. \tag{3.14}$$

The right-hand side converges to $\rho + \mu$ with probability one by the law of large numbers (White, 2001, p. 32):

$$\lim_{t \uparrow \infty} \frac{b_t}{t} = \rho + \mu \qquad \text{with probability one.} \tag{3.15}$$

The following result is based on this simple observation.

Proposition 3.1. *Consider the bubble process $\{B_t\}$ given by (3.3) and (3.4). Assume (3.9).*

(a) *Suppose that*

$$\rho + \mu > 0. \tag{3.16}$$

Then $\{B_t\}$ is explosive. In particular, for any $\delta \in (0, \rho + \mu)$, with probability one, there exists $\underline{t} \in \mathbb{N}$ such that

$$\forall t \geq \underline{t}, \quad B_t > e^{\delta t}. \tag{3.17}$$

(b) *Suppose that*

$$\rho + \mu < 0. \tag{3.18}$$

Then $\{B_t\}$ is implosive. In particular, for any $\delta \in (0, -\mu - \rho)$, with probability one, there exists $\underline{t} \in \mathbb{N}$ such that

$$\forall t \geq \underline{t}, \quad B_t < e^{-\delta t}. \tag{3.19}$$

Proof. To show part (a), let $\delta \in (0, \rho + \mu)$. From (3.15), with probability one, there exists $\underline{t} \in \mathbb{N}$ such that

$$\forall t \geq \underline{t}, \quad \frac{\ln B_t}{t} > \delta. \tag{3.20}$$

Thus (3.17) follows. The proof of part (b) is similar. □

Part (a) of Proposition 3.1 shows that under (3.16), the bubble process $\{B_t\}$ is explosive, asymptotically growing exponentially. This property is expected from the deterministic case, and is often considered to be a characteristic of bubbles.

Part (b), on the other hand, shows that under (3.18), the bubble process is implosive, asymptotically decaying exponentially. One might find this result rather counterintuitive since the bubble equation (1.3) implies that the deterministic sequence $\{EB_t\}$ of expected bubbles is explosive.

To understand this "exponential decay" property, consider the special case in which $\rho = 0$ (i.e., $r = 0$) and S_t takes only two values as follows:

$$S_t = \begin{cases} 0.5 & \text{with probability } 1/2, \\ 1.5 & \text{with probability } 1/2. \end{cases} \tag{3.21}$$

In this specification we have (3.4) and (3.9). Suppose that $B_0 = 100, S_1 = 0.5$, and $S_2 = 1.5$. Then $B_1 = 150$ and $B_2 = 75$. In other words, a 50% decrease followed by a 50% increase results in a 25% overall decrease. We have $B_2 = 75$ even if we interchange the values of S_1 and S_2: a 50% increase followed by a 50% decrease results in a 25% overall decrease.

More generally, if the number of good shocks is the same as that of bad shocks over a period of time, then the bubble B_t shrinks in the end. Since good and bad shocks are equally likely, the bubble is deemed to decay over time. Therefore the seemingly neutral assumption $E_t S_{t+1} = 1$ in fact places downward pressure on B_t.[9]

If $r > 0$, then this places upward pressure on B_t, and the overall effect on B_t is determined by the relative strength of upward pressure $\rho = \ln(1+r)$ to downward pressure $|\mu| = |E \ln S_t|$. Proposition 3.1 shows that the asymptotic behavior of the bubble process $\{B_t\}$ is indeed determined by the sign of $\rho + \mu$.

[9] See Froster and Hart (2009) and the references therein for similar arguments and results related to Proposition 3.1 in the context of measures of riskiness.

There are some results in the literature closely related to part (b) of Proposition 3.1. It is in fact a consequence of Kamihigashi's (2006, Theorem 3.1) result that almost every feasible path in a one-sector stochastic growth model with a multiplicative iid shock converges to zero exponentially fast if the marginal product of capital is finite at zero and if the shock is sufficiently volatile. Salge (1997, Proposition 4.2) shows a special case of Proposition 3.1(b) with a log-normal shock. The log-normal case can be viewed as a discrete-time version of the geometric Brownian motion, which is also known to be implosive (in our definition) under a continuous-time counterpart of (3.18) (Kushner, 1967, p. 57). Based on a similar argument, Ikeda and Shibata (1992, 1995) show that continuous-time intrinsic bubbles can be implosive.[10]

Given these results in the literature, Proposition 3.1(b) may not be surprising. Nonetheless it seems worth emphasizing that a bubble process can decay exponentially, since it is still widely believed that "in most models bubbles burst, while in reality bubbles seem to deflate over several weeks or even months" (Brunnermeier, 2008). Exponential decay captures this realistic feature of bubbles.

In the knife-edge case $\rho + \mu = 0$, which is not covered by Proposition 3.1, the logarithmic bubble process $\{b_t\}$ is a random walk without drift (recall (3.12)).[11] It turns out that in this case, the bubble process $\{B_t\}$ is recurrent under an additional condition. In the next result, we let $V(s)$ denote the variance of the random variable s.

Proposition 3.2. *Let $\{B_t\}$ be the bubble process given by (3.3) and (3.4). Assume (3.9). Suppose that*

$$\rho + \mu = 0, \tag{3.22}$$

$$V(s_{t+1}) < \infty. \tag{3.23}$$

Then $\{B_t\}$ is recurrent.

Proof. See the Appendix. □

[10] Turnovsky and Weintraub (1971), Kiernan and Madan (1989), and Kelly (1992) use similar arguments in different contexts.

[11] Recognizing both explosive and implosive cases, Lansing (2009) focuses on intrinsic bubbles that can be described as a geometric random walk without drift, and also considers "near-rational" bubbles. Branch and Evans (2010) study "recurrent" bubbles and crashes under learning dynamics.

Recurrence of $\{B_t\}$ means that with probability one, there are two levels $\underline{B}, \overline{B} > 0$ such that $B_t < \overline{B}$ infinitely often and $B_t > \underline{B}$ infinitely often. The proof of Proposition 3.2 in fact shows that under its hypotheses, $\underline{B}$ and $\overline{B}$ can be chosen in such a way that $\underline{B} < \overline{B}$ and $B_t \in [\underline{B}, \overline{B}]$ infinitely often. This property is similar to mean reversion, but notice that $\{B_t\}$ has no finite long run mean since $\{EB_t\}$ is explosive by the bubble equation (1.3).

Under (3.23), Propositions 3.1 and 3.2 allow us to classify all bubble processes obeying (3.3) and (3.4) into three classes depending on $\rho + \mu$:

Corollary 3.1. *Consider the bubble process $\{B_t\}$ given by (3.3) and (3.4). Assume (3.9) and (3.23). Then $\{B_t\}$ is either explosive, recurrent, or implosive, depending on $\rho + \mu \gtreqless 0$.*

Kolmogorov's zero-one law (e.g., Ash, 1972, p. 278) implies that

$$P\{\varliminf_{t \uparrow \infty} B_t = \infty\} = 0 \text{ or } 1, \tag{3.24}$$

$$P\{\varlimsup_{t \uparrow \infty} B_t = 0\} = 0 \text{ or } 1. \tag{3.25}$$

By definition, if the first probability is one, then the bubble process $\{B_t\}$ is explosive; if the second probability is one, then the process is implosive. If neither probability is one, then the process is recurrent. Therefore $\{B_t\}$ is either explosive, recurrent, or implosive even without (3.23). Corollary 3.1 specifies in terms of $\rho + \mu$ exactly when the process is explosive, recurrent, or implosive.

To illustrate the results so far by simulating the bubble process $\{B_t\}$ given by (3.3) and (3.4), suppose that there exists a sequence $\{\epsilon_t\}$ of iid shocks such that

$$s_t = \mu + \sigma \epsilon_t, \tag{3.26}$$

where

$$\mu \leq 0, \quad \sigma \geq 0, \quad \epsilon_t \sim N(0, 1). \tag{3.27}$$

Note that μ is the expected value of s_t as defined earlier, while σ is its standard deviation. These parameters are related as follows:

$$\mu = -\frac{\sigma^2}{2}. \tag{3.28}$$

This is because from (3.4) and normality of ϵ_t, we have

$$1 = E_t S_{t+1} = E_t e^{s_{t+1}} = e^{\mu + \frac{\sigma^2}{2}}. \tag{3.29}$$

Propositions 3.1 and 3.2 indicate that with $r > 0$ fixed, as μ gradually decreases from 0 to a value below $-\rho = -\ln(1+r)$, the bubble process $\{B_t\}$ is initially explosive, becomes recurrent when $\mu = -\rho$, and then becomes implosive. Figures 1 and 2 demonstrate how this transition takes place. In each figure, the bubble process $\{B_t\}$ is initially explosive with $\rho + \mu > 0$, becomes recurrent when $\rho+\mu = 0$, and then becomes implosive with $\rho+\mu < 0$.

All plots in each figure share a common sample path of $\{\epsilon_t\}$ (or a common sequence of pseudo-random variables); the only difference between a pair of plots in each figure is that they use different values of μ. In both Figures 1 and 2, μ is changed exactly in the same way; the only difference between the two figures is that they use different sample paths of $\{\epsilon_t\}$.

These figures suggest that plots of the same process can appear quite different depending on the sample path of $\{\epsilon_t\}$. This point is further illustrated in Figures 3, 4, and 5. Each of these figures shows eight sample paths of the same process with different sample paths of $\{\epsilon_t\}$, while these figures share a common set of sample paths of $\{\epsilon_t\}$. For example, panel (b) of Figure 3, panel (b) of Figure 4, and panel (b) of Figure 5 all use a common sample path of $\{\epsilon_t\}$.

Figure 3 shows eight sample paths of an explosive bubble process (with identical parameter and initial values). Although Proposition 3.1 ensures that each sample path eventually grows exponentially, and all the plots in the figure exhibit an overall upward trend, they appear to have rather different growth patterns. Notice that the maximum values of these plots are significantly different. Likewise the plots in Figure 4 exhibit an overall downward trend, but they appear to have rather different decay patterns.

Figure 5 shows eight sample paths of a recurrent bubble process. As expected, the plots in this figure exhibit neither an overall upward trend nor an overall downward trend, though the recurrence property is not always clear in each individual panel. This may not be surprising given that the bubble process here is a geometric random walk.

4 Confidence

In the previous section we studied the asymptotic properties of the bubble process given by (3.3) and (3.4) without offering any economic interpretation. To better understand these properties, we introduce a parameter that can be interpreted as a measure of investors' confidence. To be specific, in addition

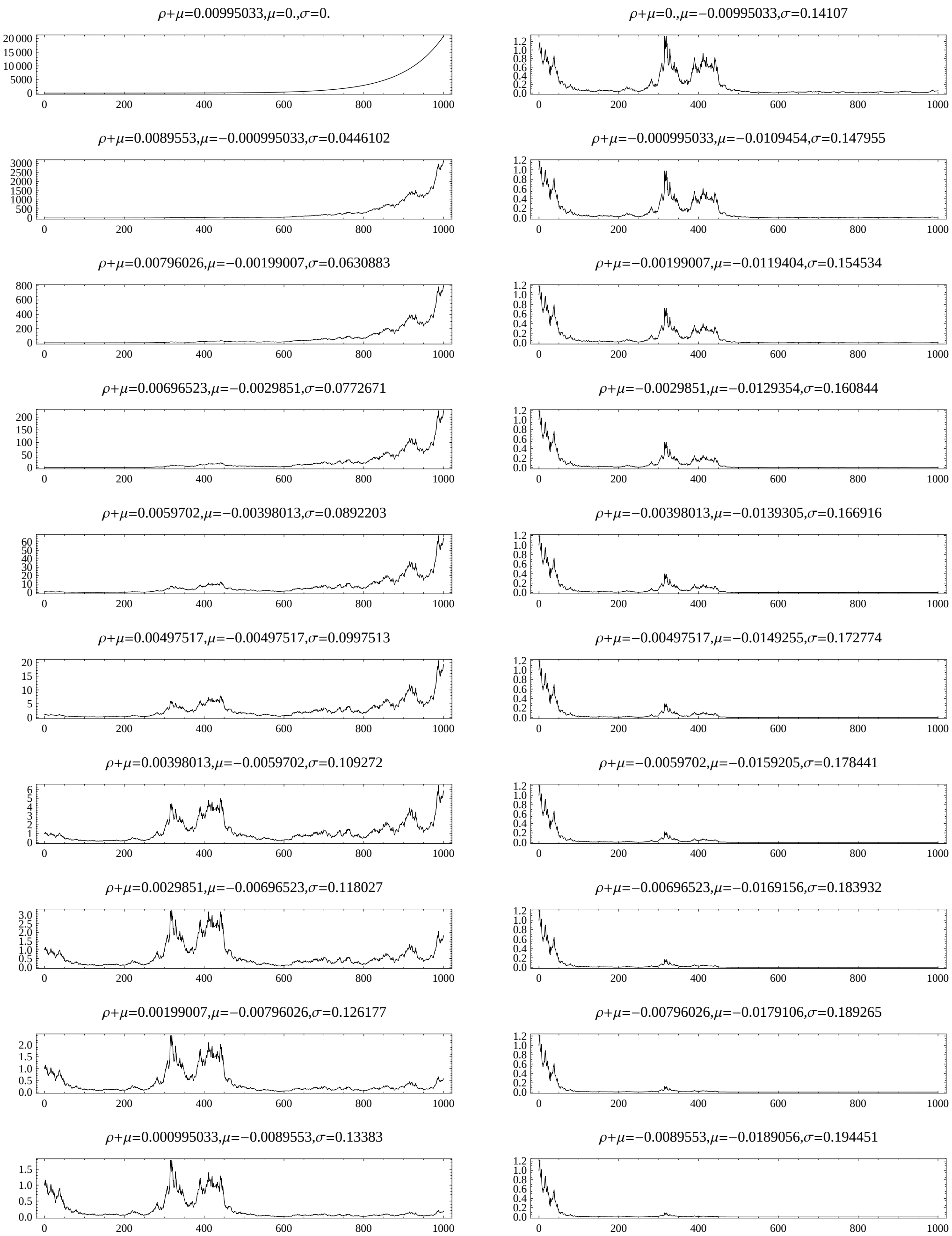


Figure 1: Explosive to implosive bubbles: plots of B_t under (3.3), (3.4), (3.26), and (3.27) with $r = 0.01, B_0 = 1$, and different values of μ

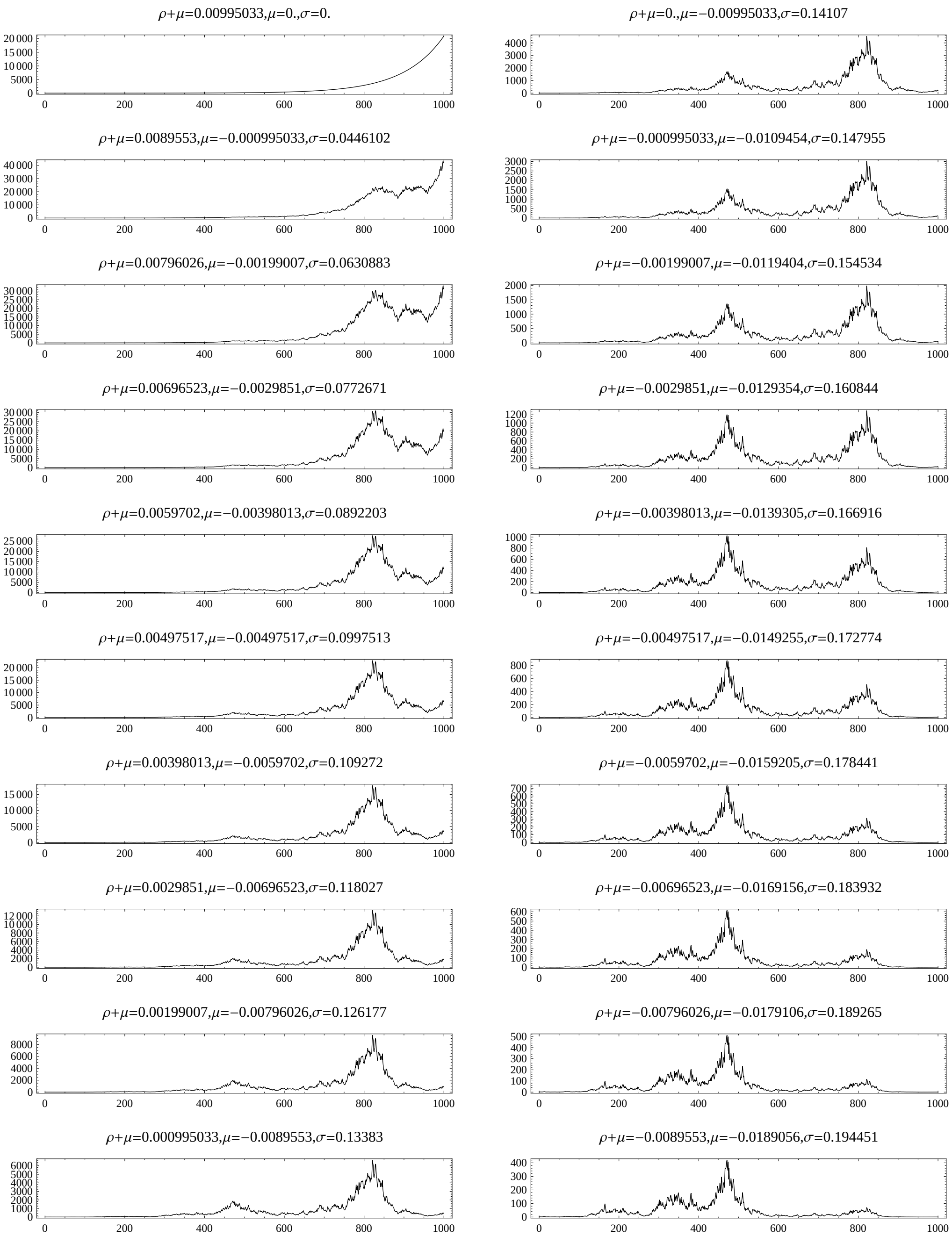


Figure 2: Explosive to implosive bubbles (another example): plots of B_t under (3.3), (3.4), (3.26), and (3.27) with $r = 0.01, B_0 = 1$, and different values of μ, and with different sample path of $\{\epsilon_t\}$ than in Figure 1

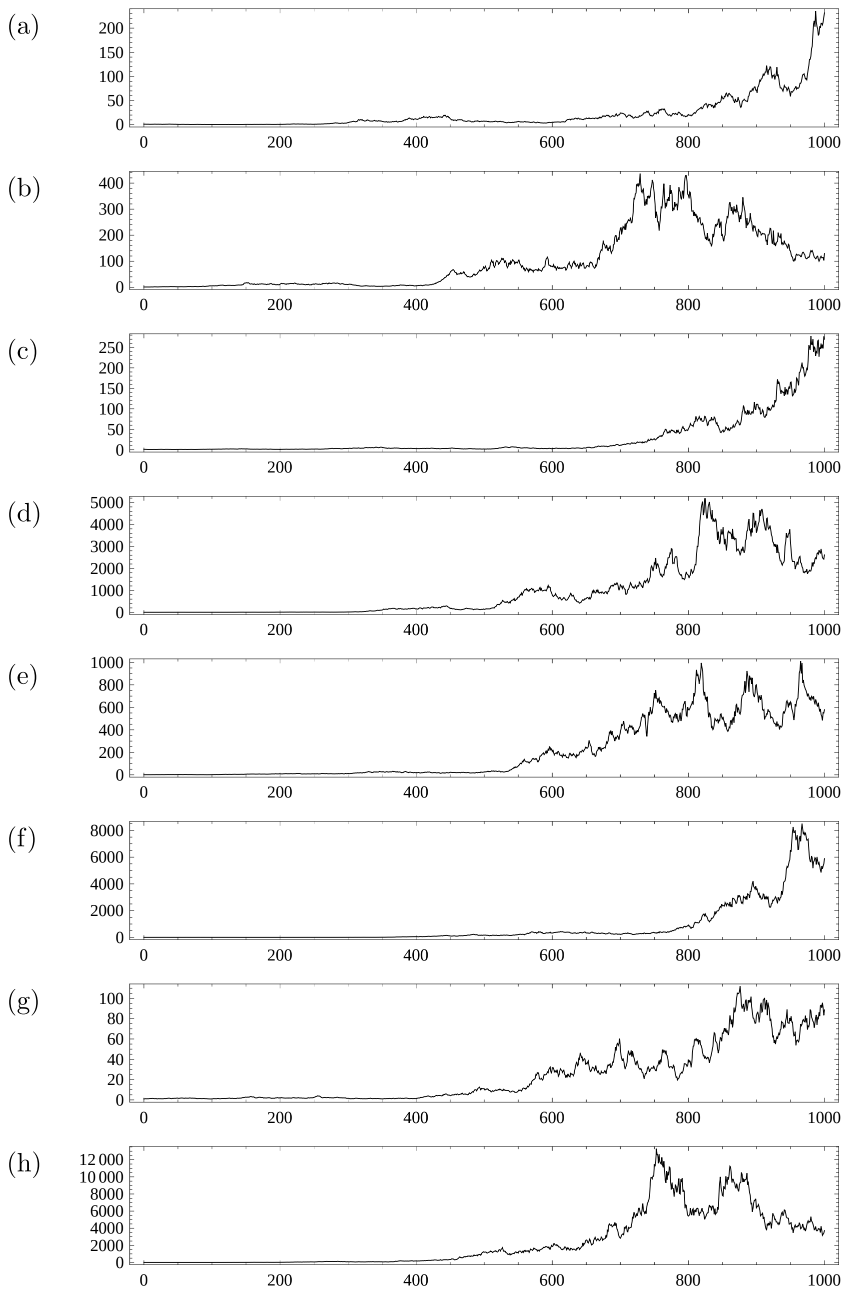


Figure 3: Explosive bubbles: plots of B_t under (3.3), (3.4), (3.26), and (3.27) with $r = 0.01$, $\rho + \mu = 0.007$, $B_0 = 1$, and different sample paths of $\{\epsilon_t\}$

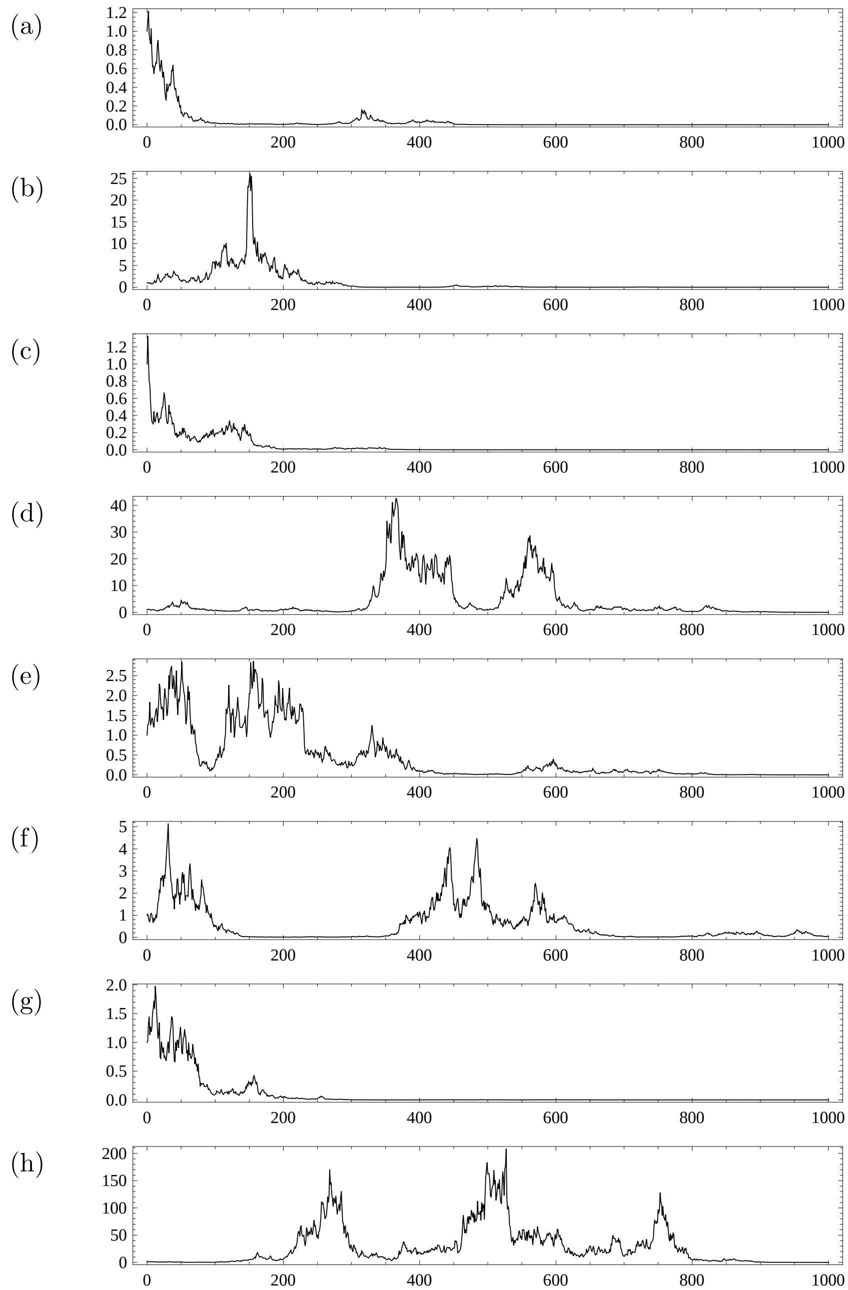


Figure 4: Implosive bubbles: plots of B_t under (3.3), (3.4), (3.26), and (3.27) with $r = 0.01$, $\rho + \mu = -0.007$, $B_0 = 1$, and different sample paths of $\{\epsilon_t\}$

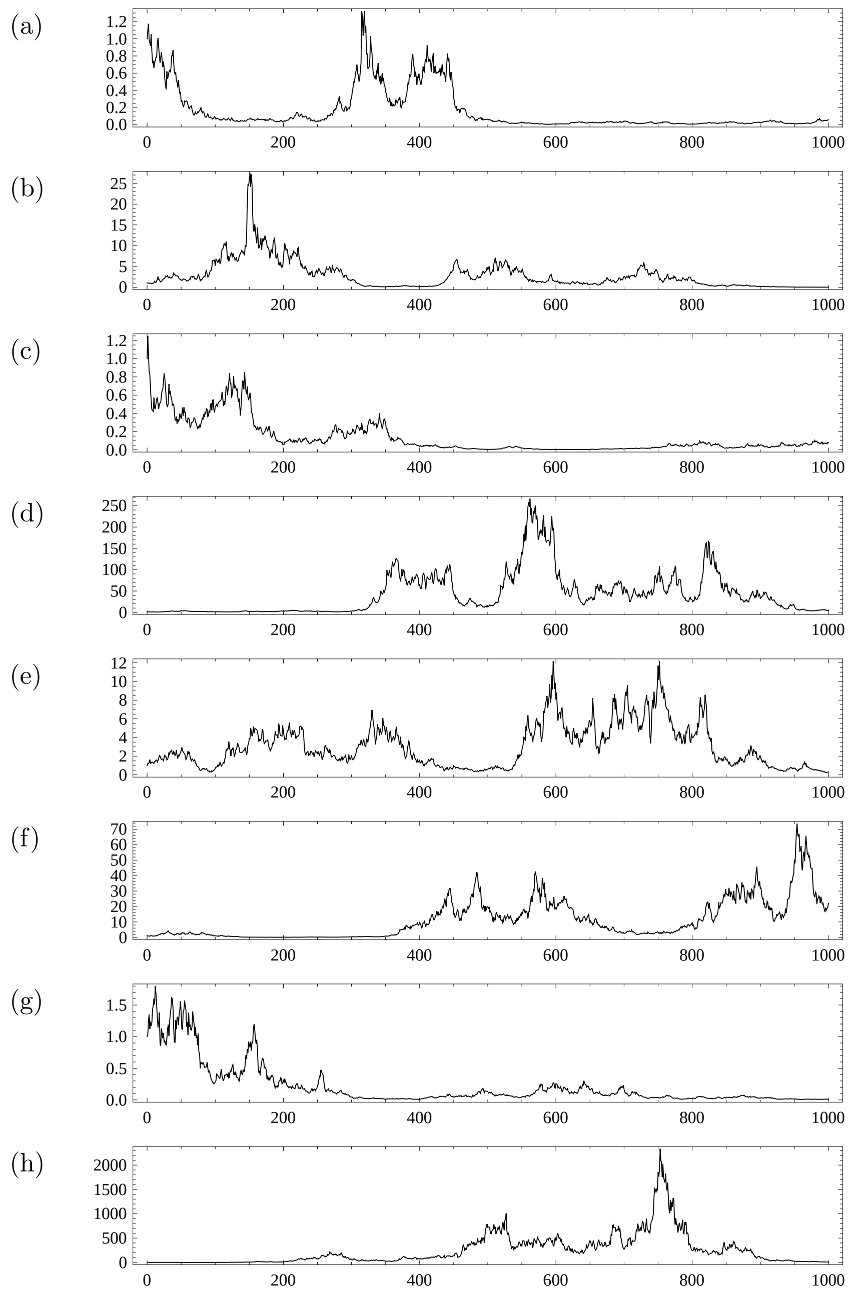


Figure 5: Recurrent bubbles: plots of B_t under (3.3), (3.4), (3.26), and (3.27) with $r = 0.01$, $\rho + \mu = 0$, $B_0 = 1$, and different sample paths of $\{\epsilon_t\}$

to (3.3) and (3.4), suppose that the logarithmic shock s_t takes the form of (3.26), and that $\{\epsilon_t\}$ is a sequence of of iid shocks such that

$$E_t\epsilon_{t+1} = 0, \quad V_t(\epsilon_{t+1}) = 1, \quad \forall h > 0, E_t e^{h\epsilon_{t+1}} < \infty. \tag{4.1}$$

Here V_t denotes the conditional variance at time t. Unlike in (3.26) we do not assume that ϵ_{t+1} is log-normal unless otherwise indicated.

Suppose now that there is a confidence index, denoted c, which measures investors' confidence in expected bubble growth $E_t B_{t+1}/B_t = 1/(1+r)$. Our use of the term "confidence" derives from its use in probability theory as in "confidence interval." It is slightly different from that of Akerlof and Shiller (2009), who consider "confidence" as one of the key aspects of animal spirits.

In our definition, if investors are almost perfectly confident, i.e., if $c \approx \infty$, then they believe that realized bubble growth B_{t+1}/B_t is likely to be close to its expected value $1 + r$. This can be a self-fulfilling belief provided that σ is endogenously determined in such a way that $c \approx \infty$ implies $\sigma \approx 0$, in which case there is little ex post volatility. On the other hand, if investors have little confidence in expected bubble growth, i.e., if $c \approx 0$, then they believe that realized bubble growth will be highly volatile. This can once again be a self-fulfilling belief provided that σ is endogenously determined in such a way that $c \approx 0$ implies $\sigma \approx \infty$.

To formalize this idea, we assume that σ is a function of c having the properties described above. This is the case if, for example, $\sigma = 1/c$. For simplicity we assume this relationship and replace σ in (3.26) with $1/c$:

$$s_t = \mu + \epsilon_t/c. \tag{4.2}$$

We assume that

$$\mu < 0, \qquad c > 0. \tag{4.3}$$

Note from (3.4) that

$$1 = E_t S_{t+1} = E_t e^{s_{t+1}} = E_t e^{\mu+\epsilon_{t+1}/c}. \tag{4.4}$$

Therefore

$$e^{-\mu} = E_t e^{\epsilon_{t+1}/c}. \tag{4.5}$$

Let μ_c be the value of μ satisfying this equation:

$$\mu_c = -\ln E_t e^{\epsilon_{t+1}/c}. \tag{4.6}$$

Define ν_c as the volatility of bubble growth B_{t+1}/B_t $(= (1+r)S_{t+1})$:

$$\nu_c = [V_t(B_{t+1}/B_t)]^{1/2} = [V_t((1+r)S_{t+1})]^{1/2}. \tag{4.7}$$

The following result shows that μ_c, and thus the drift parameter $\rho + \mu_c$, increases as confidence c increases, and that ν_c decreases as confidence c increases at least when S_{t+1} has a log-normal distribution.

Lemma 4.1. *Assume (3.3), (3.4), and (4.2)–(4.3).*

(a) *For any $c, c' > 0$ with $c < c'$, we have*

$$\mu_c < \mu_{c'}. \tag{4.8}$$

(b) *Suppose that ϵ_t has the standard normal distribution:*

$$\epsilon_t \sim N(0,1). \tag{4.9}$$

Then for any $c, c' > 0$ with $c < c'$, we have

$$\nu_c > \nu_{c'}. \tag{4.10}$$

Proof. See the Appendix. □

Part (a) of the above result can easily be verified under (4.9), in which case

$$\mu_c = -\frac{1}{2c^2} \tag{4.11}$$

(recall (3.28)). This confirms that the expected logarithmic growth rate of the bubble B_t (which is given by $\rho + \mu_c$) depends positively on confidence. In other words the bubble is likely to grow faster when confidence is higher.

Part (b) of Lemma 4.1 shows that the volatility of bubble growth B_{t+1}/B_t depends negatively on confidence under (4.9). It follows from the proof of Lemma 4.1 that

$$\nu_c = (1+r)[e^{1/c^2} - 1]^{1/2}. \tag{4.12}$$

Since ν_c can also be viewed as uncertainty about B_{t+1}/B_t, the conclusion of part (b) says that when confidence is low, ex ante uncertainty about bubble growth is high. This results in high ex post volatility, justifying low confidence and high uncertainty in a self-fulfilling way.

We can now restate Propositions 3.1 and 3.2 in terms of confidence: Given $r > 0$, the bubble process $\{B_t\}$ given by (3.3), (3.4), and (4.2)–(4.3) is explosive if confidence c is sufficiently high, and implosive if c is sufficiently low. There is also a knife-edge level of c such that the bubble process is recurrent.

5 Recurrent Bubbles with Endogenous Confidence

We have assumed so far that $\{S_t\}$ in (3.3) is an iid process. This is not an assumption required by the bubble equation (1.3). In fact the bubble equation only requires $\{S_t\}$ to satisfy (3.4) (with the strict inequality replaced by the weak one). In this section we continue to consider bubble processes obeying (3.3) and (3.4) while allowing $\{S_t\}$ to be non-iid.

More specifically we assume that $\{s_t\}$ takes the form

$$s_{t+1} = \mu(B_t) + \epsilon_{t+1}/c(B_t), \tag{5.1}$$

where $\mu : (0, \infty) \to (-\infty, 0)$ and $c : (0, \infty) \to (0, \infty)$ are measurable functions, and $\{\epsilon_t\}$ is a sequence of iid shocks satisfying (4.1). As in (4.6), $\mu(B_t)$ and $c(B_t)$ are related as follows:

$$\mu(B_t) = -\ln E_t e^{\epsilon_{t+1}/c(B_t)}. \tag{5.2}$$

In this setting the logarithmic bubble process $\{b_t\}$ follows

$$b_{t+1} = b_t + \rho + \mu(B_t) + \epsilon_{t+1}/c(B_t). \tag{5.3}$$

Proposition 3.1 suggests that the bubble B_t tends to grow when $\rho + \mu(B_t) > 0$, and tends to shrink when $\rho + \mu(B_t) < 0$. This is the basic idea of the next proposition, in which $[x]_+$ and $[x]_-$ denote the plus and minus parts of x:

$$[x]_+ = \max\{x, 0\}, \quad [x]_- = \max\{-x, 0\}. \tag{5.4}$$

Proposition 5.1. *Let $\{B_t\}$ be the bubble process given by (3.3), (3.4), (4.1), and (5.1). Let ϵ be a random variable having the same distribution as ϵ_t.*

(a) *Suppose that*

$$\rho + \varlimsup_{B\uparrow\infty} \mu(B) + \varlimsup_{B\uparrow\infty} \frac{E[\epsilon]_+}{c(B)} - \varliminf_{B\uparrow\infty} \frac{E[\epsilon]_-}{c(B)} < 0, \tag{5.5}$$

$$\varliminf_{B\uparrow\infty} c(B) > 0, \qquad \varlimsup_{B\uparrow\infty} c(B) < \infty. \tag{5.6}$$

Then $\{B_t\}$ is downward recurrent.

(b) *Suppose that*

$$\rho + \varliminf_{B\downarrow 0} \mu(B) + \varliminf_{B\downarrow 0} \frac{E[\epsilon]_+}{c(B)} - \varlimsup_{B\downarrow 0} \frac{E[\epsilon]_-}{c(B)} > 0, \tag{5.7}$$

$$\varliminf_{B\downarrow 0} c(B) > 0, \qquad \varlimsup_{B\downarrow 0} c(B) < \infty. \tag{5.8}$$

Then $\{B_t\}$ is upward recurrent.

(c) *If (5.5)–(5.8) hold, then $\{B_t\}$ is recurrent.*

Proof. See the Appendix. □

Inequality (5.5) essentially implies that the bubble B_t tends to shrink when it is extremely large, which in turn implies that it cannot stay at an extremely large level forever. Likewise, inequality (5.7) implies that the bubble cannot stay at an extremely small level forever. The inequalities in (5.6) and (5.8) are regularity conditions to ensure that these claims are indeed true. If all of these conditions hold, the bubble process can neither diverge to infinity nor converge to zero; thus it is recurrent.

The following result offers simpler sufficient conditions for recurrence.

Corollary 5.1. *Let $\{B_t\}$ be the bubble process given by (3.3), (3.4), (4.1), and (5.1). Let ϵ be a random variable having the same distribution as ϵ_t. Suppose that $c : (0, \infty) \to (0, \infty)$ is decreasing and satisfies*

$$0 < \lim_{B\uparrow\infty} c(B) \le \lim_{B\downarrow 0} c(B) < \infty. \tag{5.9}$$

Suppose further that

$$E[\epsilon]_+ = E[\epsilon]_-, \tag{5.10}$$

$$\rho + \lim_{B\uparrow\infty} \mu(B) < 0 < \rho + \lim_{B\downarrow 0} \mu(B). \tag{5.11}$$

Then $\{B_t\}$ is recurrent.

Proof. Since $c(\cdot)$ is decreasing, $\mu(\cdot)$ is decreasing by Lemma 4.1(a). Thus by (5.9) and (5.10), inequalities (5.5) and (5.7) reduce to (5.11). Since (5.6) and (5.8) follow from (5.9), the conclusion holds by Proposition 5.1(c). □

The inequalities in (5.11) together with (5.10) imply that when the bubble is extremely large, investors lose confidence and the bubble tends to shrink, and that when the bubble is extremely small, investors restores confidence and the bubble tends to grow.

Equation (5.10) holds if ϵ_t has a distribution symmetric around zero, such as the standard normal distribution. For the rest of this section, let us assume that ϵ_t has the standard normal distribution. To illustrate Corollary 5.1, we further assume that $c(\cdot)$ takes the following form:

$$c(B) = [2(\theta - e^{-\alpha B})]^{-1/2}, \tag{5.12}$$

where

$$\alpha > 0, \quad \theta > 1. \tag{5.13}$$

Note that $c(\cdot)$ is strictly decreasing and satisfies (5.9). From (4.11) we have

$$\mu(B) = e^{-\alpha B} - \theta. \tag{5.14}$$

This implies that

$$\lim_{B \downarrow 0} \mu(B) = 1 - \theta, \quad \lim_{B \uparrow \infty} \mu(B) = -\theta. \tag{5.15}$$

We also assume that

$$\theta - 1 < \rho < 1. \tag{5.16}$$

These inequalities together with (5.15) imply (5.11). Since $\mu(\cdot)$ is continuous, strictly decreasing, and satisfies (5.15), given any α, θ, and ρ as above, there exists a unique $B^* > 0$ such that $\mu(B^*) + \rho = 0$. We choose the values of α, θ, and r for which $B^* = 1$:

$$r = 0.01, \quad \alpha \approx 0.00498758, \quad \theta \approx 1.00498. \tag{5.17}$$

Figure 6 shows the graph of $-\mu(\cdot)$ along with a horizontal line representing $\rho = \ln(1 + r)$. When $B_t < 1$, we have $\rho + \mu(B_t) > 0$, so that we can expect B_t to grow; when $B_t > 1$, we have $\rho + \mu(B_t) > 0$, so that we can expect B_t to shrink. To put it in terms of confidence, when the bubble is small, confidence is high and the bubble is expected to grow, while when the bubble is large, confidence is low and the bubble is expected to shrink.

Figure 7 confirms this intuition, showing eight sample paths of the recurrent bubble process $\{B_t\}$ constructed above. This figure uses the same

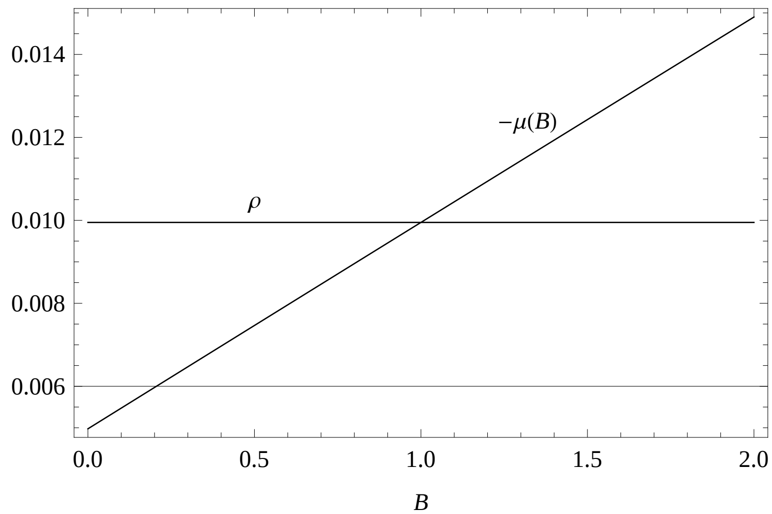


Figure 6: Graph of $-\mu(B)$ (given by (5.14) and (5.17)) and horizontal line $\rho = \ln(1.01)$

set of sample paths of $\{\epsilon_t\}$ as in Figures 3–5. One can see from the plots in Figure 7 that the bubble process tends to shrink quickly when it is large, but repeatedly becomes visibly large. Note also that the maximum values of the plots in this figure do not differ as much as those in Figure 5. In other words, the bubble process in Figure 7 appears more stable than that in Figure 5.

6 Recurrent Bubbles with an Endogenous Interest Rate

We have so far assumed that the interest rate r is constant over time. This assumption can easily be relaxed by assuming instead that the interest rate between periods t and $t+1$ is a stochastic process whose realization is known in period t. In this section we show that a stabilizing effect similar to that observed in Figure 7 is obtained even if confidence is constant, provided that the interest rate changes in such a way as to make the bubble shrink when it is extremely large, and to make it grow when extremely small.

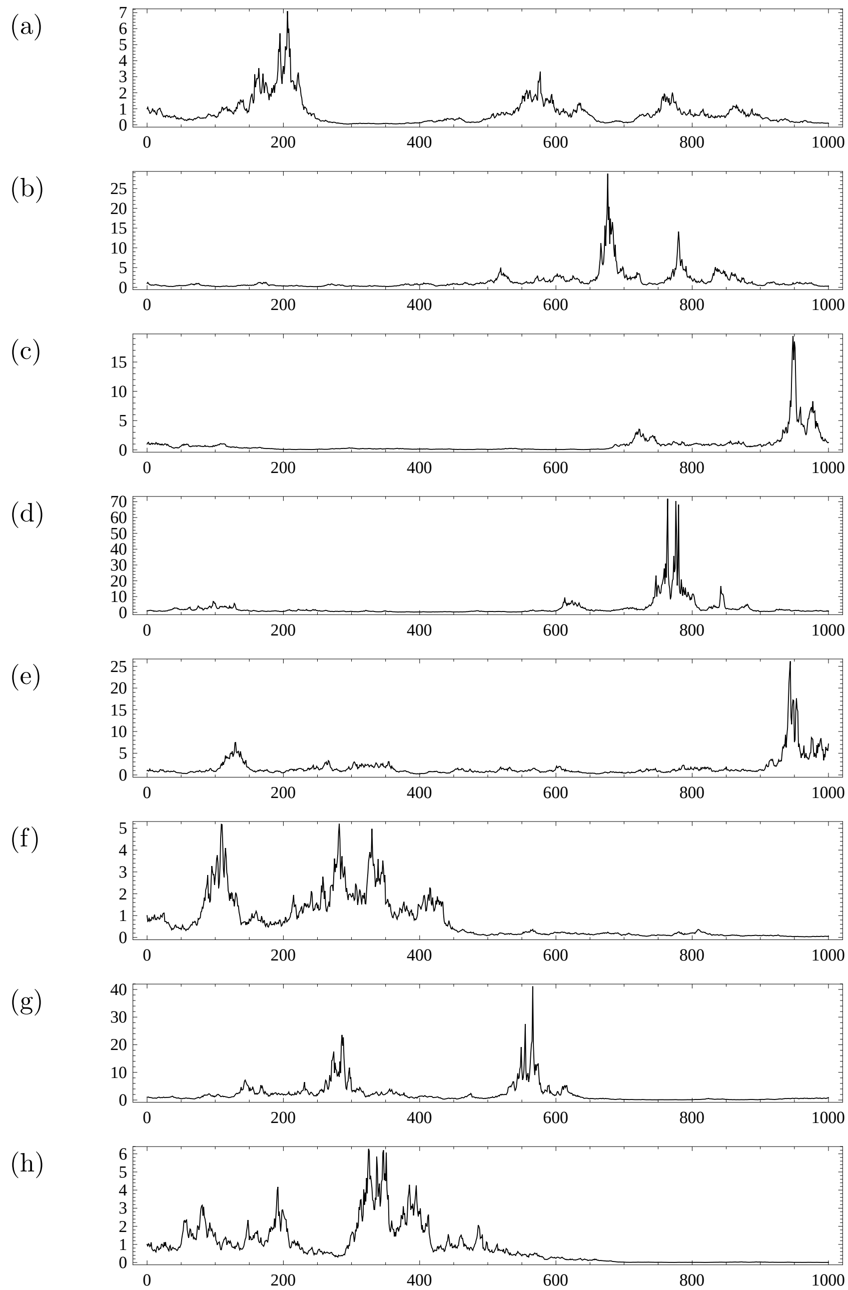


Figure 7: Recurrent bubbles with endogenous confidence: plots of B_t under (3.3), (3.4), (4.9), (5.1), (5.12), (5.14), and (5.17) with $B_0 = 1$ and different sample paths of $\{\epsilon_t\}$

To this end, let r_t be the risk-free interest rate between periods t and $t+1$, and replace r by r_t in the intertemporal no-arbitrage condition (1.1):

$$P_t = (1 + r_t)^{-1} E_t(P_{t+1} + D_{t+1}). \tag{6.1}$$

The bubble equation (1.3) then modifies to

$$E_t B_{t+1} = (1 + r_t) B_t. \tag{6.2}$$

Similarly (3.3) modifies to

$$B_{t+1} = (1 + r_t) B_t S_{t+1}, \tag{6.3}$$

where $\{S_t\}$ is a sequence of iid shocks satisfying (4.2)–(4.3). We further assume that the interest rate r_t is a function of the current bubble B_t:

$$r_t = r(B_t), \tag{6.4}$$

where $r : (0, \infty) \to (0, \infty)$ is a measurable function.

Equation (6.4) can be justified by assuming the presence of a central bank that attempts to control the bubble process by adjusting the interest rate. It can alternatively be justified as an implication of a theoretical model with a wealth effect (see Kamihigashi, 2008, Eq. (27)).

Under (6.4) the logarithmic bubble process $\{b_t\}$ follows

$$b_{t+1} = b_t + \rho(B_t) + \mu + \epsilon_{t+1}/c, \tag{6.5}$$

where

$$\rho(B_t) = \ln(1 + r(B_t)). \tag{6.6}$$

Since $r(B) > 0$ for all $B > 0$, we have

$$\forall B > 0, \quad \rho(B) > 0. \tag{6.7}$$

The following result shows that the endogenous interest rate formulated in (6.4) can results in a recurrent bubble process.

Proposition 6.1. *Let $\{B_t\}$ be the bubble process given by (6.3), (6.4), and (4.2)–(4.3).*

(a) *If*

$$\varlimsup_{B \uparrow \infty} \rho(B) + \mu < 0, \tag{6.8}$$

then $\{B_t\}$ is downward recurrent.

(b) *If*
$$\lim_{B\downarrow 0} \rho(B) + \mu > 0, \tag{6.9}$$
then $\{B_t\}$ *is upward recurrent.*

(c) *If both (6.8) and (6.9) hold, then* $\{B_t\}$ *is recurrent.*

Proof. See the Appendix. □

Inequality (6.8) implies that when the bubble B_t is extremely large, the interest rate r_t becomes low enough to make it shrink; inequality (6.9) implies that when the bubble is extremely small, the interest rate becomes high enough to make it grow.[12] Thus under these conditions, the bubble process is recurrent.

To construct a numerical example satisfying (6.8) and (6.9), let
$$\overline{r}, \lambda > 0, \quad \mu \in (-\ln(1+\overline{r}), 0). \tag{6.10}$$
We assume that
$$r(B) = \overline{r}e^{-\lambda B}. \tag{6.11}$$
Then $r(\cdot)$ is strictly decreasing, and
$$\lim_{B\downarrow 0} r(B) = \overline{r}, \quad \lim_{B\uparrow\infty} r(B) = 0. \tag{6.12}$$
From (6.10), (6.12), and (6.6), we have
$$\lim_{B\downarrow 0} \rho(B) = \ln(1+\overline{r}) > -\mu, \tag{6.13}$$
$$\lim_{B\uparrow\infty} \rho(B) = 0 < -\mu. \tag{6.14}$$
These inequalities imply (6.8) and (6.9) for any $\overline{r}$, λ, and μ satisfying (6.10). Since $\rho(\cdot)$ is strictly decreasing, continuous, and satisfies (6.8) and (6.9), there exists a unique $B^* > 0$ such that $\rho(B^*) + \mu = 0$. We choose the values of $\overline{r}$, λ, and μ so that $B^* = 1$:
$$\overline{r} = 0.1, \quad \lambda \approx 2.30259, \quad \mu = \ln(1.01). \tag{6.15}$$

[12] One may find this negative relation between B_t and r_t counterfactual. A model in which the relation is reversed is considered in the next section.

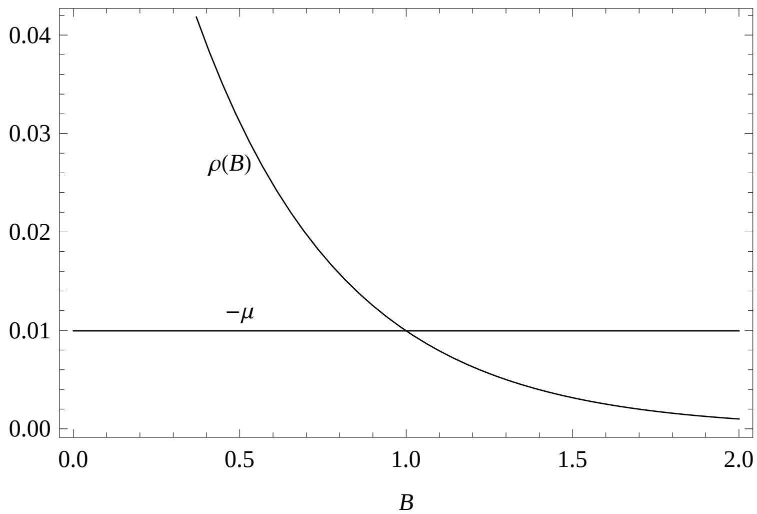


Figure 8: Graph of $\rho(B)$ (given by (6.6), (6.11), and (6.15)) and horizontal line $-\mu = \ln(1.01)$

Figure 8 shows the graph of $\rho(B)$ along with a horizontal line representing $-\mu = \ln(1.01)$. As in Figure 6, when $B_t < 1$, we have $\rho(B_t) + \mu > 0$, so that we can expect B_t to grow; when $B_t > 1$, we have $\rho(B_t) + \mu < 0$, so that we can expect B_t to shrink. In other words, the bubble tends to grow when it is small, and tends to shrink when it is large.

This behavior is confirmed in Figure 9, which shows eight sample paths of the bubble process $\{B_t\}$ constructed above (solid line scaled on the left) along with the corresponding paths of r_t (dashed line scaled on the right). The length and the sample paths of $\{\epsilon_t\}$ used in this figure are different from those in Figures 3–5 and 7. Since $r(\cdot)$ is decreasing, B_t and r_t move in the opposite directions. As depicted in the figure, when the bubble B_t is small, the interest rate r_t becomes high so that the bubble tends to grow via (6.3); when the bubble B_t is large, the interest rate r_t becomes low so that the bubble B_t tends to shrink.

(a) (b) (c) (d) (e) (f) (g) (h)

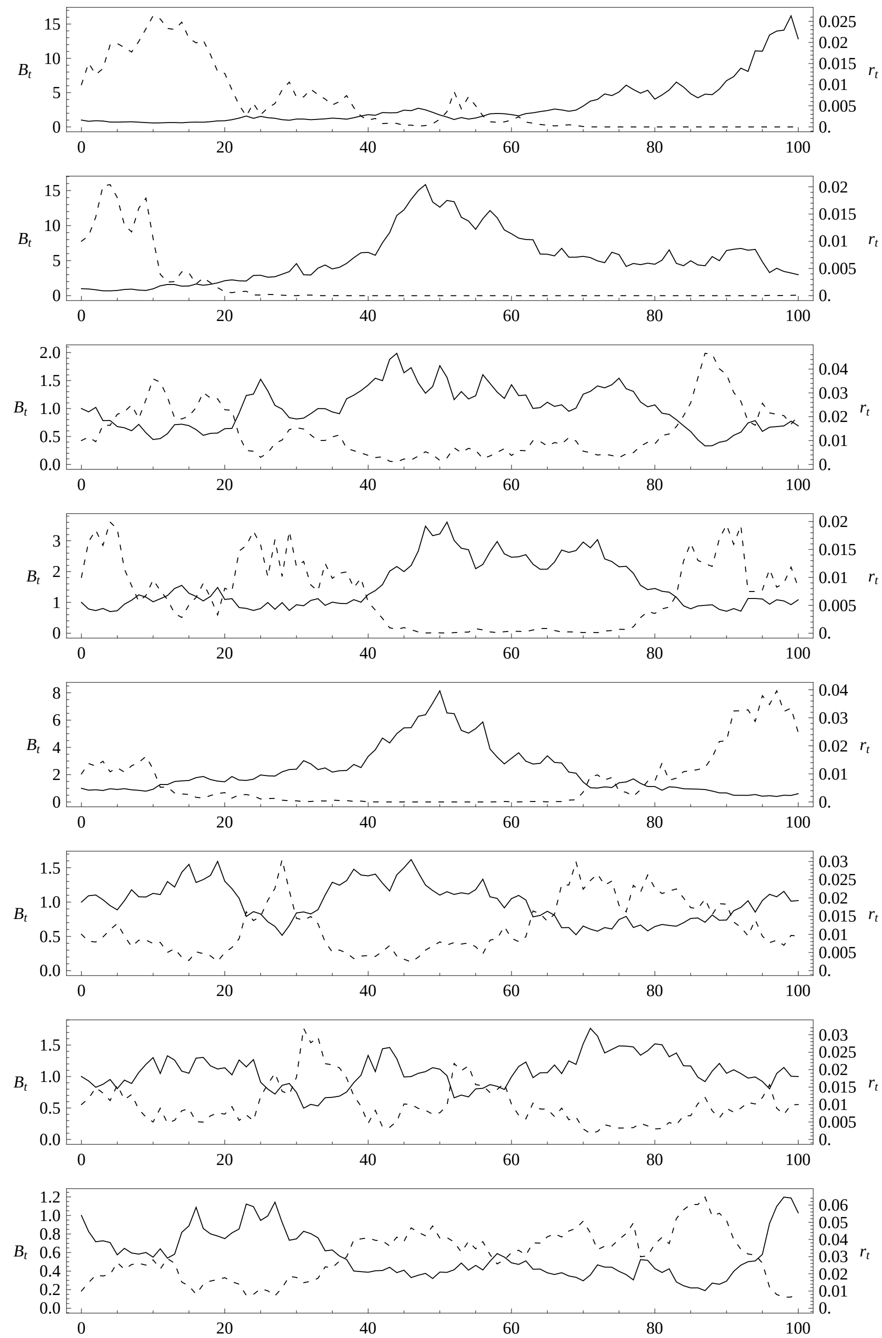


Figure 9: Plots of B_t (solid line scaled on right) and r_t (dashed line scaled on left) under (6.3), (6.4), (4.2)–(4.3), (6.10), (6.11), and (6.15).

7 Recurrent Bubbles with an Endogenous Interest Rate and Endogenous Confidence

For a bubble process to be recurrent, Proposition 6.1 essentially requires a negative relation between the interest rate and the bubble. This is inconsistent with the conventional wisdom that the central bank is expected to raise interest rates when the market is overheated, and to cut them when the market is performing poorly. Furthermore, the market is conventionally expected to react negatively when interest rates are raised, and positively when interest rates are cut.

To capture these ideas, we allow in this section the interest rate r_t to depend positively on the current bubble, while we allow confidence to depend negatively on r_t. The bubble equation (6.2) remains the same as in the previous section, and we continue to assume (6.3) and (6.4). However we assume that

$$s_{t+1} = \mu(r(B_t)) + \epsilon_{t+1}/c(r(B_t)), \tag{7.1}$$

where $r : (0, \infty) \to (0, \infty), c : (0, \infty) \to (0, \infty)$, and $\mu : (0, \infty) \to (0, \infty)$ are measurable functions. The last two functions are related as in (5.2).

Equation (7.1) means that the shock process $\{S_t\}$ depends on confidence, which in turn depends on the risk-free interest rate $r_t = r(B_t)$. In this case the logarithmic bubble process $\{b_t\}$ follows

$$b_{t+1} = b_t + \rho(B_t) + \mu(r(B_t)) + \epsilon_{t+1}/c(r(B_t)), \tag{7.2}$$

where $\rho(\cdot)$ is given by (6.6).

The next result shows that a bubble process can be recurrent even if the interest rate depends positively on the bubble, provided that changes in confidence counteract the destabilizing effect of changes in the interest rate. The result is similar to Proposition 5.1, but the condition for upward recurrence in part (b) is somewhat different from that of Proposition 5.1. While the latter essentially requires that the constant interest rate r be strictly positive, the condition here, (7.5), allows $r(0) = 0$; we do not rule out the zero interest-rate bound.

Proposition 7.1. *Let $\{B_t\}$ be the bubble process given by (6.3), (6.4), (7.1), and (4.1). Let ϵ be a random variable having the same distribution as ϵ_t. Suppose that both $r : (0, \infty) \to (0, \infty)$ and $\mu : (0, \infty) \to (-\infty, 0)$ are bounded.*

(a) *Suppose that*

$$\varlimsup_{B\uparrow\infty}[\rho(B)+\mu(r(B))]+\varlimsup_{B\uparrow\infty}\frac{E[\epsilon]_+}{c(r(B))}-\varliminf_{B\uparrow\infty}\frac{E[\epsilon]_-}{c(r(B))}<0, \tag{7.3}$$

$$\varliminf_{B\uparrow\infty}c(r(B))>0, \qquad \varlimsup_{B\uparrow\infty}c(r(B))<\infty. \tag{7.4}$$

Then $\{B_t\}$ *is downward recurrent.*

(b) *Suppose that*

$$\exists\overline{B}>0, \forall B\in(0,\overline{B}], \quad \rho(B)+\mu(r(B))\geq\frac{E[\epsilon]_-}{c(r(B))}. \tag{7.5}$$

Then $\{B_t\}$ *is upward recurrent.*

(c) *If (7.3)–(7.5) hold, then* $\{B_t\}$ *is recurrent.*

Proof. See the Appendix. □

Corollary 7.1. *Let* $\{B_t\}$ *be the bubble process given by (6.3), (6.4), (7.1), and (4.1). Let* ϵ *be a random variable having the same distribution as* ϵ_t*. Assume (5.10). Suppose that* $r:(0,\infty)\to(0,\infty)$ *is bounded and increasing, and that* $c:(0,\infty)\to(-\infty,0)$ *is decreasing and satisfies*

$$0<\lim_{B\uparrow\infty}c(r(B))<\infty. \tag{7.6}$$

Suppose further that

$$\lim_{B\uparrow\infty}\rho(B)+\lim_{B\uparrow\infty}\mu(r(B))<0, \tag{7.7}$$

$$\varliminf_{B\downarrow 0}c(r(B))[\rho(B)+\mu(r(B))]=\infty. \tag{7.8}$$

Then $\{B_t\}$ *is recurrent.*

Proof. Since $c(\cdot)$ is decreasing and satisfies (7.6), $\mu(\cdot)$ is decreasing and bounded by Lemma 4.1(a) and (4.6). Thus both limits in (7.7) exist. By (5.10) and monotonicity of $c(\cdot)$ and $r(\cdot)$, the last two terms on the right-hand side of (7.3) cancel out each other. Hence (7.3) reduces to (7.7). We obtain (7.4) from (7.6). It is easy to see that (7.8) implies (7.5). Therefore the conclusion holds by Proposition 7.1. □

If $\lim_{B\downarrow 0} r(B) > 0$, then it is easy to construct a recurrent bubble process satisfying the conditions of Corollary 7.1; in fact we can easily modify the parametric example constructed in Section 5 in this case. In what follows we construct a more complicated bubble process to allow for the zero interest-rate bound: $\lim_{B\downarrow 0} r(B) = 0$.[13] The difficulty here is to ensure (7.8) when both $\rho(B)$ and $\mu(r(B))$ converge to zero as B goes to zero.

We start by specifying the interest rate function

$$r(B) = \overline{r}(1 - e^{-B}), \tag{7.9}$$

where

$$\overline{r} \in (0, 1). \tag{7.10}$$

We have

$$\lim_{B\downarrow 0} r(B) = 0, \quad \lim_{B\uparrow\infty} r(B) = \overline{r}, \quad \lim_{B\downarrow 0} r'(B) = \overline{r}. \tag{7.11}$$

The interest rate is lowered to zero when there is no bubble, and is raised toward $\overline{r}$ when the bubble grows without bound. It follows that

$$\lim_{B\downarrow 0} \rho(B) = 0, \quad \lim_{B\uparrow\infty} \rho(B) = \ln(1 + \overline{r}) < 1, \quad \lim_{B\downarrow 0} \rho'(B) = \overline{r}. \tag{7.12}$$

We assume that ϵ_t has the standard normal distribution. Let us specify the confidence function

$$c(z) = 2^{-1/2}(1 - e^{-\alpha r^{-1}(z)})^{-2}, \tag{7.13}$$

where α is a strictly positive parameter and $r^{-1}(\cdot)$ is the inverse of $r(\cdot)$:

$$\forall z \in (0, \overline{r}), \quad r^{-1}(z) = -\ln\left(1 - \frac{z}{\overline{r}}\right). \tag{7.14}$$

Note from (4.11) that

$$\mu(r(B)) = -\frac{1}{2c(r(B))^2} = -(1 - e^{-\alpha B})^4. \tag{7.15}$$

We define

$$\sigma(B) = 1/c(r(B)) = 2^{1/2}(1 - e^{-\alpha B})^2. \tag{7.16}$$

[13] See Robinson and Stone (2006) for an analysis of the effect of this bound in a simple linear model.

We have

$$\lim_{B\downarrow 0}\mu(r(B))=0,\quad \lim_{B\uparrow\infty}\mu(r(B))=-1,\quad \lim_{B\downarrow 0}d\mu(r(B))/dB=0, \tag{7.17}$$

$$\lim_{B\downarrow 0}\sigma(B)=0,\quad \lim_{B\uparrow\infty}\sigma(B)=2^{1/2},\quad \lim_{B\downarrow 0}\sigma'(B)=0. \tag{7.18}$$

Furthermore, since $1-e^{-\alpha B}<1$ for all $B>0$, we have

$$\forall B>0,\quad -\mu(r(B))<\sigma(B). \tag{7.19}$$

From (7.10), (7.12), and (7.17), we obtain (7.7). From (7.12) and (7.19) we have

$$c(r(B_t))[\rho(B)+\mu(r(B))]=\frac{\rho(B)}{\sigma(B)}+\frac{\mu(r(B))}{\sigma(B)}>\frac{\rho(B)}{\sigma(B)}-1. \tag{7.20}$$

The rightmost side tends to ∞ as $B\downarrow 0$ by the derivative conditions in (7.12) and (7.18). Thus we obtain (7.8). It follows by Corollary 7.1 that $\{B_t\}$ is recurrent.

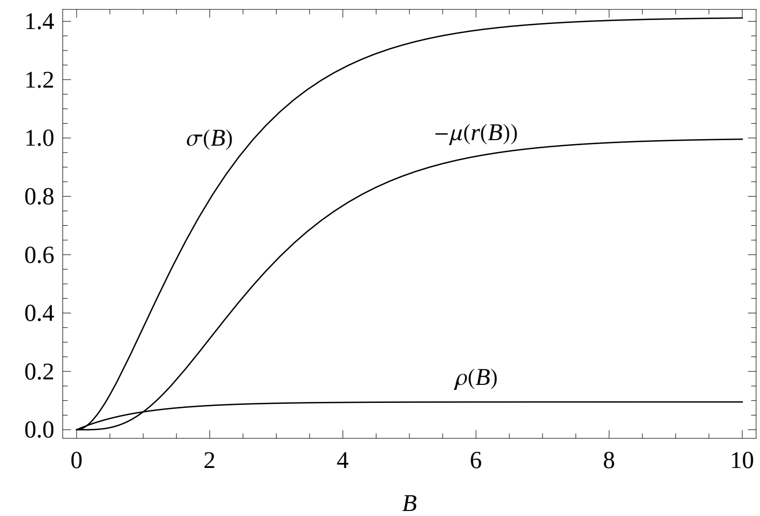
It follows from (7.7), (7.12), and (7.17) that there exists $B^*>0$ such that $\rho(B^*)+\mu(r(B^*))=0$. We specify the values of $\bar{r}$ and α in such a way that $B^*=1$:

$$\bar{r}=0.1,\quad \alpha\approx 0.688302. \tag{7.21}$$

Figure 10 shows the graphs of $\rho(B),-\mu(r(B))$, and $\sigma(B)$ in this specification on two different domains. Panel (a) confirms (7.7) and shows that all these functions are increasing and bounded. Panel (b) illustrates the derivative conditions in (7.12), (7.17), and (7.18). Panel (a) also shows that $\rho(B)+\mu(r(B))>0$ for $B<1$ and $\rho(B)+\mu(r(B))<0$ for $B>1$. This suggests that the bubble B_t tends to grow when $B_t<1$, and tends to shrink when $B_t>1$.

Figure 11 plots eight sample paths of the bubble process $\{B_t\}$ constructed above. This figure uses the same set of sample paths of $\{\epsilon_t\}$ as in Figure 9. By construction the bubble B_t and the interest rate $r(B_t)$ move in the same direction. Some of the plots, especially those in panels (b) and (e), suggest that there can be a prolonged period in which both the bubble and the interest rate stay close to zero. This is because when the bubble is close to zero, the interest rate is also close to zero, which implies that the bubble grows very slowly even though confidence is extremely high. Nevertheless the bubble process here is recurrent, so that it is only a matter of time before it starts to reappear.

(a)



(b)

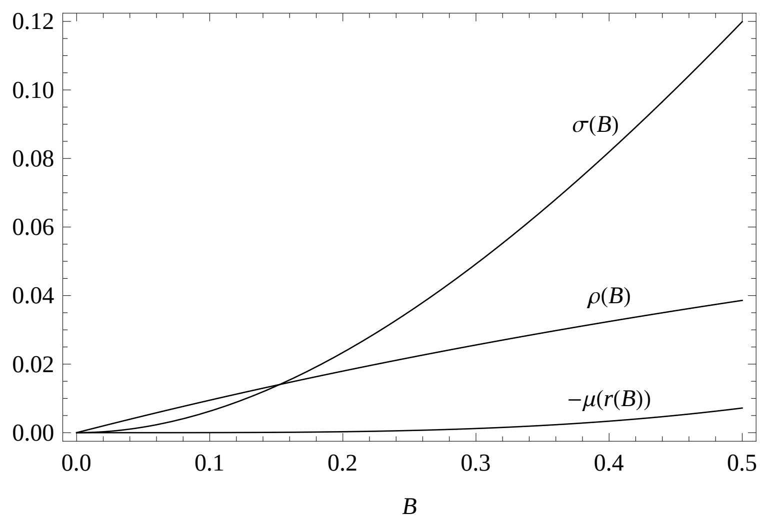


Figure 10: Graphs of $\rho(B), -\mu(r(B))$, and $\sigma(B) = 1/C(r(b))$ (given by (7.9), (6.6), (7.15), (7.16), and (7.21))

(a)
(b)
(c)
(d)
(e)
(f)
(g)
(h)

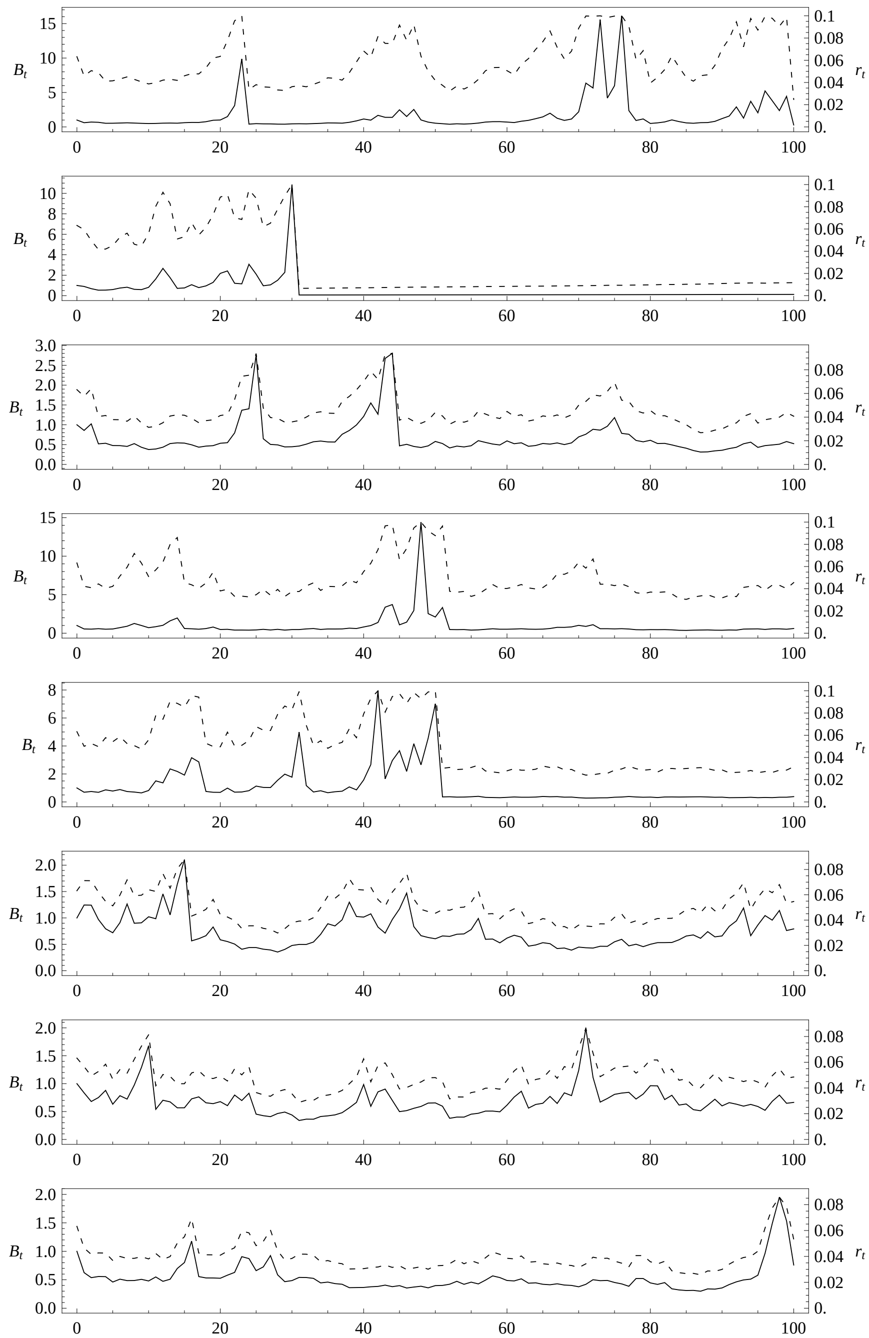


Figure 11: Plots of B_t (solid line scaled on left) and r_t (dashed line scaled on right) under (6.3), (6.4), (7.1), (4.9), (7.9), (7.13), (7.15), and (7.21).

8 Concluding Comments

In this paper we have studied rational bubbles in a simple linear asset price model. We have characterized the asymptotic behavior of bubble processes driven by multiplicative iid shocks in terms of investors' confidence in expected bubble growth. We have shown that a bubble process in this class is explosive if confidence is sufficiently high, and implosive if confidence is sufficiently low. There is also a knife-edge level of confidence such that the bubble process is recurrent. We have also developed sufficient conditions for recurrence when confidence depends on the current bubble, when the interest rate depends on the current bubble, and when the interest rate depends on the current bubble and confidence depends on the current interest rate. Most of our conditions for recurrence are based on the simple idea that a bubble process should be recurrent if confidence becomes sufficiently low when the bubble is extremely large, and becomes sufficiently high when the bubble is extremely small. We have also illustrated our results with simulated sample paths of bubble processes.

Of particular interest for future research is to examine if any of the models studied here performs well on real data. In doing so, it would be useful to note that our results in the last two sections do not require that the interest rate function be continuous, although our numerical examples use continuous interest rate functions for simplicity. For example, one can use an interest rate function that takes discrete values so that the interest rate is changed much less frequently than in our simulations.

This paper, especially Section 7, is partly motivated by Japan's post-bubble experience. Since the collapse of the asset bubble in the early 1990s, until this writing, Japan has been suffering from low asset prices and low interest rates. The simulation results in Section 7 suggest that the combination of low asset prices and low interest rates could be an endogenous phenomenon generated by a simple mechanism like (7.2), even though those results are based on some ad hoc assumptions and may not offer a plausible explanation.

In concluding this paper we wish to emphasize that equations such as (1.3) and (6.2) can be derived from various economic models of serious interest, and our analysis can be extended to such models. We hope that the results in this paper will facilitate and stimulate further investigation of economic problems involving bubbles.

A Proofs

A.1 Proof of Proposition 3.2

Under (3.22), $\{b_t\}$ is a random walk. Meyn and Tweedie (2009, Proposition 9.4.5) show that a random walk satisfying (3.23) is "non-evanescent" (Meyn and Tweedie, 2009, p. 206).[14] This means that with probability one, there exist $\underline{b}, \bar{b} \in \mathbb{R}$ such that $b_t \in [\underline{b}, \bar{b}]$ infinitely often, i.e., $B_t \in [\underline{B}, \overline{B}]$ infinitely often with $\underline{B} = e^{\underline{b}}$ and $\overline{B} = e^{\bar{b}}$. Therefore $\{B_t\}$ is recurrent.

A.2 Proof of Lemma 4.1

(a) We show equivalently that

$$c > c' \quad \Rightarrow \quad \mu_c > \mu_{c'}. \tag{A.1}$$

To this end, let $c, c' > 0$ with

$$c > c'. \tag{A.2}$$

Define $\sigma = 1/c$ and $\sigma' = 1/c'$. Let ϵ be a random variable having the same distribution as ϵ_{t+1}. We have

$$e^{\sigma' \epsilon} = e^{(\sigma' - \sigma)\epsilon + \sigma\epsilon} \geq e^{\sigma\epsilon} + \epsilon e^{\sigma\epsilon}(\sigma' - \sigma), \tag{A.3}$$

where the inequality holds by convexity of the exponential function. Note that

$$\begin{aligned} \epsilon e^{\sigma\epsilon} &= 1\{\epsilon \geq 0\}\epsilon e^{\sigma\epsilon} + 1\{\epsilon < 0\}\epsilon e^{\sigma\epsilon} && \text{(A.4)} \\ &\geq 1\{\epsilon \geq 0\}\epsilon + 1\{\epsilon < 0\}\epsilon = \epsilon, && \text{(A.5)} \end{aligned}$$

where $1\{\cdot\}$ is the indicator function. The above inequality holds strictly as long as $\epsilon \neq 0$. From this, (A.3)–(A.5), and (4.1), we have

$$Ee^{\sigma'\epsilon} > Ee^{\sigma\epsilon} + (\sigma' - \sigma)E\epsilon = Ee^{\sigma\epsilon}. \tag{A.6}$$

Thus $\ln Ee^{\sigma'\epsilon} > \ln Ee^{\sigma\epsilon}$, i.e.,

$$\mu_{c'} = -\ln Ee^{\sigma'\epsilon} < -\ln Ee^{\sigma\epsilon} = \mu_c. \tag{A.7}$$

[14] There is a typo in the statement of their Proposition 9.4.5: $\mathbb{R}_+$ should be replaced with $\mathbb{R}$.

This establishes (A.1).

(b) Define

$$\hat{\nu}_c = [V_t(B_{t+1}/((1+r)B_t))]^{1/2} = [V_t(S_{t+1})]^{1/2}. \tag{A.8}$$

Since r is constant, it suffices to show that $\hat{\nu}_c$, instead of ν_c, is decreasing in c. To this end, let $c > 0$ and $\sigma = 1/c$. Assume (4.9). Let ϵ be a standard normal random variable, and let $S = e^{\mu+\sigma\epsilon}$. Note that

$$\hat{\nu}_c^2 = E(S-1)^2 = E(S^2 - 2S + 1) = ES^2 - 1. \tag{A.9}$$

Since $S^2 = (e^{\mu+\sigma\epsilon})^2 = e^{2\mu+2\sigma\epsilon}$ with $\mu = -\sigma^2/2$ (by (3.28)), we have

$$\hat{\nu}_c^2 + 1 = ES^2 = Ee^{2\mu+2\sigma\epsilon} = e^{2\mu+2\sigma^2} = e^{\sigma^2}. \tag{A.10}$$

Hence $\hat{\nu}_c$ is strictly increasing in $\sigma = 1/c$, i.e., strictly decreasing in c.

A.3 Proof of Proposition 5.1

To show part (a), assume (5.5) and (5.6). Define $\sigma(\cdot) = 1/c(\cdot)$. Recalling (5.2) we see that there exit $\overline{\mu}, \overline{\sigma}, \underline{\sigma} \in \mathbb{R}$ such that

$$\varlimsup_{B\uparrow\infty} \mu(B) < \overline{\mu} < 0, \tag{A.11}$$

$$\varlimsup_{B\uparrow\infty} \sigma(B) < \overline{\sigma} < \infty, \tag{A.12}$$

$$\varliminf_{B\uparrow\infty} \sigma(B) > \underline{\sigma} > 0, \tag{A.13}$$

$$\rho + \overline{\mu} + \overline{\sigma} E[\epsilon_t]_+ - \underline{\sigma} E[\epsilon_t]_- < 0. \tag{A.14}$$

Define

$$\tilde{\Omega} = \{\omega \in \Omega : \varliminf_{t\uparrow\infty} B_t(\omega) = \infty\}. \tag{A.15}$$

Suppose that $\{B_t\}$ is not downward recurrent. Then $P(\tilde{\Omega}) > 0$. Define

$$\Omega_+ = \left\{\omega \in \Omega : \lim_{t\uparrow\infty} \frac{1}{t} \sum_{i=1}^{t} [\epsilon_i(\omega)]_+ = E[\epsilon]_+ \right\}, \tag{A.16}$$

$$\Omega_- = \left\{\omega \in \Omega : \lim_{t\uparrow\infty} \frac{1}{t} \sum_{i=1}^{t} [\epsilon_i(\omega)]_- = E[\epsilon]_- \right\}, \tag{A.17}$$

$$\underline{\Omega} = \tilde{\Omega} \cap \Omega_+ \cap \Omega_-. \tag{A.18}$$

Since $P(\Omega_+ \cap \Omega_-) = 1$ by the law of large numbers, we have

$$P(\underline{\Omega}) = P(\tilde{\Omega}) > 0. \tag{A.19}$$

Let

$$\omega \in \underline{\Omega}. \tag{A.20}$$

By (5.3) and (A.11)–(A.13), there exists $\underline{t} \in \mathbb{Z}_+$ such that for all $t \geq \underline{t}$, we have

$$b_{t+1}(\omega) \leq b_t(\omega) + \rho + \overline{\mu} + \overline{\sigma}[\epsilon_t(\omega)]_+ - \underline{\sigma}[\epsilon_t(\omega)]_-. \tag{A.21}$$

Without loss of generality, we assume that $\underline{t} = 0$. Then we have

$$b_t(\omega) \leq b_0 + t(\rho + \overline{\mu}) + \overline{\sigma} \sum_{i=1}^{t} [\epsilon_i(\omega)]_+ - \underline{\sigma} \sum_{i=1}^{t} [\epsilon_i(\omega)]_-. \tag{A.22}$$

Dividing through by t, we obtain

$$\frac{b_t(\omega)}{t} \leq \frac{b_0}{t} + \rho + \overline{\mu} + \overline{\sigma}\frac{1}{t} \sum_{i=1}^{t} [\epsilon_i(\omega)]_+ - \underline{\sigma}\frac{1}{t} \sum_{i=1}^{t} [\epsilon_i(\omega)]_-. \tag{A.23}$$

Recalling (A.16) and (A.17), we see that the right-hand side converges to

$$\rho + \overline{\mu} + \overline{\sigma}E[\epsilon]_+ - \underline{\sigma}E[\epsilon]_- < 0, \tag{A.24}$$

where ϵ is a random variable with the same distribution as ϵ_t, and the inequality holds by (A.14). Let $\delta > 0$ be such that

$$\rho + \overline{\mu} + \overline{\sigma}E[\epsilon]_+ - \underline{\sigma}E[\epsilon]_- < -\delta. \tag{A.25}$$

Then for t sufficiently large, we have

$$\frac{b_t(\omega)}{t} < -\delta, \tag{A.26}$$

i.e., $b_t(\omega) < -\delta t$, or $B_t(\omega) < e^{-\delta t}$, which contradicts (A.20). This completes the proof of part (a).

The proof of part (b) is similar. Part (c) follows from parts (a) and (b).

A.4 Proof of Proposition 6.1

To show part (a), assume (6.8). Let $\sigma = 1/c$. It follows from (6.5) that for $t \geq 1$, we have

$$\frac{b_t}{t} = \frac{b_0}{t} + \frac{1}{t}\sum_{i=1}^{t}[\rho(B_i) + \mu + \sigma\epsilon_i]. \tag{A.27}$$

Define

$$\tilde{\Omega} = \{\omega \in \Omega : \varliminf_{t\uparrow\infty} B_t(\omega) = \infty\}. \tag{A.28}$$

Suppose that $\{B_t\}$ is not downward recurrent. Then $P(\tilde{\Omega}) > 0$. Define

$$\hat{\Omega} = \left\{\omega \in \Omega : \lim_{t\uparrow\infty} \frac{1}{t}\sum_{i=1}^{t}\epsilon_i(\omega) = 0\right\}, \tag{A.29}$$

$$\underline{\Omega} = \tilde{\Omega} \cap \hat{\Omega}. \tag{A.30}$$

Since $P(\hat{\Omega}) = 1$ by the law of large numbers, we have

$$P(\underline{\Omega}) = P(\tilde{\Omega}) > 0. \tag{A.31}$$

Let

$$\omega \in \underline{\Omega}. \tag{A.32}$$

Let $\overline{\rho} > 0$ be such that

$$\varlimsup_{B\uparrow\infty} \rho(B) < \overline{\rho} < -\mu. \tag{A.33}$$

It follows from (A.27) and (A.33) that for $t \geq 1$ sufficiently large, we have

$$\frac{b_t(\omega)}{t} < \overline{\rho} + \mu + \sigma\frac{1}{t}\sum_{i=1}^{t}\epsilon_i(\omega). \tag{A.34}$$

Let $\delta \in (0, -\overline{\rho} - \mu)$. Recalling (A.29), we see that for $t \geq 1$ sufficiently large, we have $b_t(\omega) < -\delta t$, or $B_t(\omega) < e^{-\delta t}$, contradicting (A.28).

The proof of part (b) is similar. Part (c) follows from parts (a) and (b).

A.5 Proof of Proposition 7.1

Part (a) can be shown by slightly modifying the proof of part (a) of Proposition 5.1. Part (b) requires a different argument because we wish to allow $\lim_{B\downarrow 0} r(B) = 0$. To show part (b), note first that it suffices to show that

$$\exists \underline{b} \in \mathbb{R}, \quad b_t \geq \underline{b} \text{ infinitely often.} \tag{A.35}$$

This means that $B_t \geq \underline{B}$ infinitely often with $\underline{B} = e^{\underline{b}}$, which implies that $\{B_t\}$ is upward recurrent.

To show (A.35), we express (7.2) as

$$b_{t+1} = f(b_t, \epsilon_{t+1}), \tag{A.36}$$

where

$$f(b, \epsilon) = b + \rho(e^b) + \mu(r(e^b)) + \epsilon/c(r(e^b)). \tag{A.37}$$

We extend this function to $b = \infty$ by defining $f(\infty, \epsilon) = \infty$. Note that the stochastic process $\{b_t\}$ given by (A.36) is a Markov process on $(-\infty, \infty]$. It follows from Meyn and Tweedie (2009, Theorem 9.4.1) that if there exists a function $w : (-\infty, \infty] \to \mathbb{R}_+$ such that

$$\exists \bar{b} \in \mathbb{R}, \forall b \leq \bar{b}, \quad Ew(f(b, \epsilon)) \leq w(b), \tag{A.38}$$

$$\lim_{b\downarrow -\infty} w(b) = \infty, \tag{A.39}$$

then (A.35) holds.

We verify that the function w defined below satisfies (A.38) and (A.39):

$$w(b) = [b]_- = |b| 1\{b \leq 0\}. \tag{A.40}$$

Since $w(b) = |b|$ for $b \leq 0$, we have (A.39). Regarding (A.38), since both $\rho(\cdot)$ and $\mu(\cdot)$ are bounded by hypothesis, there exists $\bar{b} \leq \ln \overline{B}$ (with $\overline{B}$ given by (7.5)) such that

$$\forall b \leq \bar{b}, \quad b + \rho(e^b) + m(e^b) \leq 0, \tag{A.41}$$

where

$$m(e^b) = \mu(r(e^b)). \tag{A.42}$$

Fix $b \leq \bar{b}$. To simplify notation, define

$$b' = b + \rho(B) + m(B) + \sigma(B)\epsilon, \tag{A.43}$$

where

$$B = e^b, \qquad \sigma(B) = 1/c(r(B)). \tag{A.44}$$

Note that $b' = f(b, \epsilon)$. We have

$$\begin{aligned} Ew(b') &= E[-b'1\{b' \leq 0\}] && \text{(A.45)} \\ &= -[b + \rho(B) + m(B)]1\{b' \leq 0\} - \sigma(B)E\epsilon 1\{b' \leq 0\} && \text{(A.46)} \\ &\leq -[b + \rho(B) + m(B)] + \sigma(B)E[\epsilon]_- && \text{(A.47)} \\ &\leq -b = w(b), && \text{(A.48)} \end{aligned}$$

where (A.47) uses (A.41), and (A.48) uses (7.5). Since $b \leq \bar{b}$ was arbitrary, we have verified (A.38). This completes the proof of part (b).

Finally part (c) follows from parts (a) and (b).

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