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A Note on Monotone Markov Processes*[†]

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Abstract

This note contains some technical results developed for Kamihigashi and Stachurski (2010). We first consider a stochastic kernel on an arbitrary measurable space and establish some general results. We then introduce a preorder and consider an increasing stochastic kernel. None of our results requires any topological assumption. To make this note self-contained, we include some of the definitions reviewed or discussed in Kamihigashi and Stachurski (2010).

*This note refines and generalizes the technical results initially developed in an earlier manuscript (Kamihigashi and Stachurski, 2009) that has never been considered for publication anywhere, and that is no longer intended for publication. The textbook by Stachurski (2009) includes a discussion of some aspects of that manuscript. This note was originally intended to be an appendix to Kamihigashi and Stachurski (2010), but we have separated the note from the latter to focus on economic analysis in the latter. This note is not being considered for publication anywhere.

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1 Basic Definitions

We begin with basic definitions concerning discrete-time Markov processes on an arbitrary measurable space $(\mathcal{X}, \mathscr{X})$. Let $\mathcal{P}_{\mathscr{X}}$ be the probability measures on $\mathscr{X}$, let $\mathcal{X}^\infty = \times_{t=0}^\infty \mathcal{X}$ be the set of all $\mathcal{X}$-valued sequences, and let $\mathscr{X}^\infty = \otimes_{t=0}^\infty \mathscr{X}$ be the product σ-algebra. A *stochastic kernel* on $\mathcal{X}$ is a function $Q\colon \mathcal{X} \times \mathscr{X} \to [0,1]$ such that (a) $Q(x, \cdot) \in \mathcal{P}_{\mathscr{X}}$ for each $x \in \mathcal{X}$, and (b) $Q(\cdot, B)$ is $\mathscr{X}$-measurable for each $B \in \mathscr{X}$.

Given a stochastic kernel Q on $\mathcal{X}$, a discrete-time, $\mathcal{X}$-valued stochastic process $\{X_t\}_{t=0}^\infty$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be *Markov-*(Q, μ) if X_0 has distribution $\mu \in \mathcal{P}_{\mathscr{X}}$, and $Q(x, \cdot)$ is the conditional distribution of X_{t+1} given $X_t = x$:

$$\mathbb{P}[X_{t+1} \in B \mid \mathscr{F}_t^X] = Q(X_t, B) \qquad \forall B \in \mathscr{X}. \tag{1.1}$$

Here $\mathscr{F}_t^X$ is the σ-algebra generated by the history $X_0, \ldots, X_t$. If the initial distribution μ is the probability measure $\delta_x \in \mathcal{P}_{\mathscr{X}}$ concentrated on $x \in \mathcal{X}$, we call $\{X_t\}$ Markov-(Q, x) rather than Markov-(Q, δ_x). We say that $\{X_t\}$ is *Markov-*Q if $\{X_t\}$ is Markov-(Q, μ) for some $\mu \in \mathcal{P}_{\mathscr{X}}$. Whenever we introduce a Markov process, we implicitly take the underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ as given.

It is well-known (e.g., Pollard, 2002, p. 101) that for each $\mu \in \mathcal{P}_{\mathscr{X}}$ and stochastic kernel Q on $(\mathcal{X}, \mathscr{X})$, there exists a unique probability measure $\mathbf{P}_Q^\mu$ on $(\mathcal{X}^\infty, \mathscr{X}^\infty)$ with the property that $\mathbf{P}_Q^\mu$ is the joint distribution of any Markov-(Q, μ) process. That is, if $\{X_t\}$ is Markov-(Q, μ), then $\mathbb{P}\{\{X_t\} \in C\} = \mathbf{P}_Q^\mu(C)$ for all $C \in \mathscr{X}^\infty$. If $\mu = \delta_x$, we write $\mathbf{P}_Q^x$ rather than $\mathbf{P}_Q^{\delta_x}$.

For each $n \in \mathbb{N}$, let Q^n be the n-th order kernel, defined by

$$Q^1 := Q, \qquad Q^n(x, B) := \int Q^{n-1}(y, B) Q(x, dy) \qquad (x \in \mathcal{X},\ B \in \mathscr{X}).$$

Each Q^n is a stochastic kernel in its own right, and $Q^n(x, B)$ represents the probability of transitioning from x to B in n steps. We extend the definition of Q^n to the case $n = 0$ by letting

$$Q^0(x, B) := \mathbb{1}_B(x) \qquad (x \in \mathcal{X},\ B \in \mathscr{X}).$$

Let $h \colon \mathcal{X} \to \mathbb{R}$ be measurable and bounded, and let $\mu \in \mathcal{P}_{\mathscr{X}}$. We define the *right Markov operator* $h \mapsto Qh$ by

$$(Qh)(x) := \int h(y)Q(x, dy) \qquad (x \in \mathcal{X}), \tag{1.2}$$

and the *left Markov operator* $\mu \mapsto \mu Q$ by

$$(\mu Q)(B) := \int Q(x, B)\mu(dx) \qquad (B \in \mathscr{X}). \tag{1.3}$$

The t-th iterates of these operators can be interpreted as follows:

$$(Q^t h)(x) = \mathbb{E}[h(X_t) \,|\, X_0 = x],$$
$$(\mu Q^t)(B) = \mathbb{P}[X_t \in B \,|\, X_0 \sim \mu],$$

where $\{X_t\}$ is any Markov-Q process. We also define

$$\mu h := \int h(x)\mu(dx). \tag{1.4}$$

Given two independent Markov-Q processes $\{X_t\}_{t=0}^{\infty}$ and $\{X'_t\}_{t=0}^{\infty}$, the $\mathcal{X} \times \mathcal{X}$-valued process $\{(X_t, X'_t)\}_{t=0}^{\infty}$ is also a Markov process. Indeed, if we define the *product kernel* $Q \times Q$ on $\mathcal{X} \times \mathcal{X}$ by

$$(Q \times Q)((x, x'), A) = \int \int \mathbb{1}_A(y, y')Q(x, dy)Q(x', dy') \tag{1.5}$$

for $(x, x') \in \mathcal{X} \times \mathcal{X}$ and $A \in \mathscr{X} \otimes \mathscr{X}$, then $Q \times Q$ is a stochastic kernel on $\mathcal{X} \times \mathcal{X}$, and $\{(X_t, X'_t)\}_{t=0}^{\infty}$ is Markov-$Q \times Q$. When $A = B \times B'$ for some $B, B' \in \mathscr{X}$, then (1.5) reduces to

$$(Q \times Q)((x, x'), B \times B') = Q(x, B)Q(x', B). \tag{1.6}$$

2 Some General Results

Fix a stochastic kernel Q on a measurable space $(\mathcal{X}, \mathscr{X})$. Given a Markov-$Q$ process $\{X_t\}$, a random variable $\xi : \Omega \to \mathbb{Z}_+ \cup \{\infty\}$ is called a *stopping time* if the event $\{\xi = n\} \in \mathscr{F}_n^X$ for all $n \in \mathbb{Z}_+$. For any $x \in \mathcal{X}, C \in \mathscr{X}$, and Markov-$Q$ process $\{X_t\}$, let

$$\tau_C^x := \inf\{t \geq 0 : X_t \in C\}$$

be the *first hitting time* of C. In this definition, we adhere to the usual convention that $\inf \emptyset = \infty$. Note also that the distribution of τ_C^x is determined by Q and x alone, as all Markov-(Q, x) processes have the same distribution $\mathbf{P}_Q^x$. Let $\eta(x, C)$ denote the probability that a Markov-(Q, x) process never visits C:

$$\eta(x, C) := \mathbb{P}\{\tau_C^x = \infty\} = \lim_{t \to \infty} \mathbb{P}\{\tau_C^x \geq t\}. \tag{2.1}$$

We establish two useful properties of this function below.

Lemma 2.1. *Let ξ be a stopping time for a Markov-Q process $\{X_t\}_{t=0}^{\infty}$. For any $x \in \mathcal{X}$ and $C \in \mathscr{X}$ we have*

$$\begin{aligned} \eta(x, C) &\leq \mathbb{E}\, \mathbb{1}\{\xi < \infty\} \mathbb{1}\{\tau_C^x \geq \xi\} \eta(X_\xi, C) + \mathbb{P}\{\xi = \infty\} && (2.2) \\ &\leq \mathbb{E}\, \mathbb{1}\{\xi < \infty\} \eta(X_\xi, C) + \mathbb{P}\{\xi = \infty\}. && (2.3) \end{aligned}$$

Proof. To simplify notation, let $\tau := \tau_C^x$. Note that

$$\begin{aligned} \mathbb{1}\{\tau = \infty\} &= \prod_{t=0}^{\infty} \mathbb{1}\{X_t \notin C\} \\ &= \mathbb{1}\{\tau \geq \xi\} \prod_{t=\xi}^{\infty} \mathbb{1}\{X_t \notin C\} \\ &\leq \mathbb{1}\{\xi < \infty\} \mathbb{1}\{\tau \geq \xi\} \prod_{t=\xi}^{\infty} \mathbb{1}\{X_t \notin C\} + \mathbb{1}\{\xi = \infty\}. \end{aligned}$$

Taking expectations we have

$$\begin{aligned} \eta(x, C) &\leq \mathbb{E}\left[\mathbb{1}\{\xi < \infty\} \mathbb{1}\{\tau \geq \xi\} \prod_{t=\xi}^{\infty} \mathbb{1}\{X_t \notin C\}\right] + \mathbb{P}\{\xi = \infty\} \\ &= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}\{\xi < \infty\} \mathbb{1}\{\tau \geq \xi\} \prod_{t=\xi}^{\infty} \mathbb{1}\{X_t \notin C\} \,\Big|\, \mathscr{F}_\xi^X\right]\right] + \mathbb{P}\{\xi = \infty\} \\ &= \mathbb{E}\left[\mathbb{1}\{\xi < \infty\} \mathbb{1}\{\tau \geq \xi\} \mathbb{E}\left[\prod_{t=\xi}^{\infty} \mathbb{1}\{X_t \notin C\} \,\Big|\, \mathscr{F}_\xi^X\right]\right] + \mathbb{P}\{\xi = \infty\}, \end{aligned} \tag{2.4}$$

where $\mathscr{F}_\xi^X$ is the σ-algebra associated with the stopping time ξ:

$$\mathscr{F}_\xi^X := \{A \in \mathscr{F} : \forall n \in \mathbb{Z}_+, \{\xi = n\} \cap A \in \mathscr{F}_n^X\} \tag{2.5}$$

(see Meyn and Tweedie, 2009, p. 66). We have

$$\mathbb{E}\left[\prod_{t=\xi}^{\infty} \mathbb{1}\{X_t \notin C\} \,\middle|\, \mathscr{F}_{\xi}^{X}\right] = \eta(X_{\xi}, C) \tag{2.6}$$

on $\{\xi < \infty\}$ by the strong Markov property (Meyn and Tweedie, 2009, Proposition 3.4.6). Substituting (2.6) into (2.4) yields the inequality in (2.2). The inequality in (2.3) holds since $\mathbb{1}\{\tau \geq \xi\} \leq 1$. $\square$

Lemma 2.2. *Let $C \in \mathscr{X}$. If there exists a measurable function $w\colon \mathcal{X} \to [1, \infty)$ and a $\lambda \in [0, 1)$ such that $(Qw)(x) \leq \lambda w(x)$ whenever $x \notin C$, then $\eta(x, C) = 0$ for all $x \in \mathcal{X}$.*

Proof. Pick any $x \in \mathcal{X}$, and let $\{X_t\}_{t=0}^{\infty}$ be Markov-(Q, x). Let $\tau := \tau_C^x$, and let $M_t := w(X_t)\mathbb{1}\{\tau \geq t\}$ for $t \in \mathbb{Z}_+$. For any $t \in \mathbb{Z}_+$, we have

$$\begin{aligned}
\mathbb{E}[M_{t+1} \mid \mathscr{F}_t^X] &= \mathbb{E}[w(X_{t+1})\mathbb{1}\{\tau \geq t+1\} \mid \mathscr{F}_t^X] \\
&= \mathbb{E}[w(X_{t+1}) \mid \mathscr{F}_t^X]\mathbb{1}\{\tau \geq t+1\} \\
&= (Qw)(X_t)\mathbb{1}\{\tau \geq t+1\} \\
&\leq \lambda w(X_t)\mathbb{1}\{\tau \geq t+1\} \\
&\leq \lambda w(X_t)\mathbb{1}\{\tau \geq t\} \\
&= \lambda M_t.
\end{aligned}$$

Taking expectations we obtain $\mathbb{E}M_{t+1} \leq \lambda \mathbb{E}M_t$. Iterating backwards to M_0, we have

$$\mathbb{E}M_t \leq \lambda^t M_0 \qquad \forall t \in \mathbb{N}.$$

We have

$$\begin{aligned}
\eta(x, C) &\leq \mathbb{P}\{\tau \geq t\} \\
&= \mathbb{E}\mathbb{1}\{\tau \geq t\} \\
&\leq \mathbb{E}w(X_t)\mathbb{1}\{\tau \geq t\} \\
&= \mathbb{E}M_t \\
&\leq \lambda^t M_0.
\end{aligned}$$

Since this is true for any $t \in \mathbb{N}$, we obtain $\eta(x, C) = 0$. $\square$

3 Anticipation

For $C \subset \mathcal{X}$ and $n \in \mathbb{Z}_+$, let $\mathcal{V}_C^n \subset \mathcal{X}$ be the set of $\mathcal{X}$-valued sequences which visit C at time n, and let $\mathcal{V}_C \subset \mathcal{X}^\infty$ be the set of $\mathcal{X}$-valued sequences which visit C at least once over an infinite horizon:

$$\mathcal{V}_C^n := \{\{x_t\}_{t=0}^\infty \in \mathcal{X}^\infty : x_n \in C\}, \quad \mathcal{V}_C := \cup_{n \geq 0} \mathcal{V}_C^n. \tag{3.1}$$

Given any Markov-(Q, x) process $\{X_t\}_{t=0}^\infty$, we have

$$\mathbb{P}\{X_n \in C\} = \mathbf{P}_Q^x(\mathcal{V}_C^n) = Q^n(x, C),$$
$$\mathbb{P} \cup_{t=0}^\infty \{X_t \in C\} = \mathbf{P}_Q^x(\mathcal{V}_C).$$

Given any $B, C \in \mathscr{X}$, we say that

- *B weakly anticipates C with respect to Q* (written $B \xrightarrow[Q]{w.a.} C$) if[1]

  $$\mathbf{P}_Q^x(\mathcal{V}_C) > 0 \qquad \forall x \in B.$$

- *B totally anticipates C with respect to Q* (written $B \xrightarrow[Q]{t.a.} C$) if

  $$\mathbf{P}_Q^x(\mathcal{V}_C) = 1 \qquad \forall x \in B.$$

- *B simultaneously anticipates C with respect to Q* (written $B \xrightarrow[Q]{s.a.} C$) if there exists an $n \in \mathbb{Z}_+$ and an $\epsilon > 0$ such that

  $$\mathbf{P}_Q^x(\mathcal{V}_C^n) = Q^n(x, C) \geq \epsilon \quad \forall x \in B.$$

Observe that

$$B \xrightarrow[Q]{w.a.} C \qquad \iff \qquad \forall x \in B, \quad \eta(x, C) < 1, \tag{3.2}$$

$$B \xrightarrow[Q]{t.a.} C \qquad \iff \qquad \forall x \in C, \quad \eta(x, C) = 0. \tag{3.3}$$

[1] The following property is known as *accessibility* in the literature on Markov processes (Meyn and Tweedie, 2009). We use our nonstandard terminology to make the binary relations defined here "readable."

Lemma 3.1. *The binary relations* $\xrightarrow[Q]{w.a.}$, $\xrightarrow[Q]{s.a.}$, *and* $\xrightarrow[Q]{t.a.}$ *are all preorders on* $\mathscr{X}$.[2]

Proof. Reflexivity of all of the binary relations $\xrightarrow[Q]{w.a.}$, $\xrightarrow[Q]{t.a.}$, and $\xrightarrow[Q]{s.a.}$ follows from the fact that any sequence starting from $x \in B$ belongs to $\mathcal{V}_B^0$. For the rest of the proof, let $\{X_t\}_{t=0}^{\infty}$ be Markov-(Q, x).

For transitivity of $\xrightarrow[Q]{s.a.}$, suppose that $A \xrightarrow[Q]{s.a.} B \xrightarrow[Q]{s.a.} C$. This means that there exists an $\epsilon_B > 0$ and an $n_B \in \mathbb{N}$ such that $Q^{n_B}(x, B) \geq \epsilon_B$ for all $x \in A$, and there exists an $\epsilon_C > 0$ and an $n_C \in \mathbb{N}$ such that $Q^{n_C}(x, C) \geq \epsilon_C$ for all $x \in B$. To conclude that $A \xrightarrow[Q]{s.a.} C$, it suffices to show that $Q^n(x, C) \geq \epsilon$ for all $x \in A$ with $n := n_B + n_C$ and $\epsilon := \epsilon_B \epsilon_C$. To this end, note that

$$\mathbb{1}\{X_n \in C\} \geq \mathbb{1}\{X_{n_B} \in B\}\mathbb{1}\{X_{n_B+n_C} \in C\}.$$

Taking expectations we have

$$\begin{aligned}
Q^n(x, C) &\geq \mathbb{E}\mathbb{1}\{X_{n_B} \in B\}\mathbb{1}\{X_{n_B+n_C} \in C\} \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbb{1}\{X_{n_B} \in B\}\mathbb{1}\{X_{n_B+n_C} \in C\} \;\middle|\; \mathscr{F}_{n_B}^X\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}\{X_{n_B} \in B\}\mathbb{E}\left[\mathbb{1}\{X_{n_B+n_C} \in C\} \;\middle|\; \mathscr{F}_{n_B}^X\right]\right] \\
&= \mathbb{E}\left[\mathbb{1}\{X_{n_B} \in B\}Q(X_{n_B}, C)\right] \\
&\geq \mathbb{E}\mathbb{1}\{X_{n_B} \in B\}\epsilon_C \\
&= Q(x, B)\epsilon_C \\
&\geq \epsilon_B \epsilon_C.
\end{aligned}$$

It follows that $A \xrightarrow[Q]{s.a.} C$.

For transitivity of $\xrightarrow[Q]{t.a.}$, suppose that $A \xrightarrow[Q]{t.a.} B \xrightarrow[Q]{t.a.} C$. To establish $A \xrightarrow[Q]{t.a.} C$, it suffices to show that $\eta(x, C) = 0$ for all $x \in A$. To this end, fix $x \in A$, and note that $\tau_B^x < \infty$ almost surely. Hence by (2.3) with $\xi = \tau_B^x$, we have

$$\eta(x, C) \leq \mathbb{E}\eta(X_{\tau_B^x}, C).$$

[2] A binary relation is called a *preorder* if it is reflexive and transitive. Although the fact that these binary relations are preorders is not used in Kamihigashi and Stachurski (2010), it is natural to ask whether they are preorders, and we answer this question here.

But since $X_{\tau_B^x} \in B$ and $\eta(y, C) = 0$ for any $y \in B$, we obtain $\eta(x, C) = 0$. Since x was arbitrary, we conclude that $\mathcal{X} \xrightarrow[Q]{t.a.} C$.

For transitivity of $\xrightarrow[Q]{w.a.}$, suppose that $A \xrightarrow[Q]{w.a.} B \xrightarrow[Q]{w.a.} C$. Fix $x \in A$. Applying (2.3) with $\xi = \tau_B^x$, we obtain

$$\begin{aligned}\eta(x, C) &\leq \mathbb{E}\mathbb{1}\{\tau_B^x < \infty\}\eta(X_{\tau_B^x}, C) + \mathbb{P}\{\tau_B^x = \infty\} \\ &< \mathbb{E}\mathbb{1}\{\tau_B^x < \infty\} + \mathbb{E}\mathbb{1}\{\tau_B^x = \infty\} \\ &= 1,\end{aligned}$$

where the strict inequality holds since $\mathbb{P}\{\tau_B^x < \infty\} > 0$ and $\eta(y, C) < 1$ for any $y \in B$. Since x was arbitrary, we conclude that $\mathcal{X} \xrightarrow[Q]{w.a.} C$. □

Lemma 3.2. *For any $B, C \in \mathscr{X}$, if $\mathcal{X} \xrightarrow[Q]{t.a.} B$ and $B \xrightarrow[Q]{s.a.} C$, then $\mathcal{X} \xrightarrow[Q]{t.a.} C$.*

Proof. Suppose that $\mathcal{X} \xrightarrow[Q]{t.a.} B \xrightarrow[Q]{s.a.} C$. Since $B \xrightarrow[Q]{s.a.} C$, there exists an $n \in \mathbb{N}$ and an $\epsilon > 0$ such that $Q^n(x, C) \geq \epsilon$ for all $x \in B$. Define

$$\overline{\eta} := \sup_{x \in \mathcal{X}} \eta(x, C).$$

We show that $\overline{\eta} = 0$. By (2.2) with $\xi = n + 1$, for any $x \in B$, we have

$$\begin{aligned}\eta(x, C) &\leq \mathbb{E}\mathbb{1}\{\tau_C^x \geq n + 1\}\eta(X_{n+1}, C) \\ &\leq \mathbb{E}\mathbb{1}\{\tau_C^x \geq n + 1\}\overline{\eta} \\ &\leq \mathbb{P}\{X_n \notin C\}\overline{\eta} \\ &= [1 - Q^n(x, C)]\overline{\eta} \\ &\leq (1 - \epsilon)\overline{\eta}.\end{aligned}$$

For any $x \in \mathcal{X}$, we have $\tau_B^x < \infty$ almost surely, so that by (2.3) with $\xi = \tau_B^x$,

$$\eta(x, C) \leq \mathbb{E}\eta(X_{\tau_B^x}, C) \leq (1 - \epsilon)\overline{\eta}.$$

Taking the supremum of the leftmost side over all $x \in \mathcal{X}$, we obtain $\overline{\eta} \leq (1 - \epsilon)\overline{\eta}$, which implies that $\overline{\eta} = 0$. □

4 Markov Processes on Preordered Spaces

Let $(\mathcal{X}, \mathscr{X})$ be a measurable space, and suppose that $\mathcal{X}$ is endowed with a preorder $\leq$. Given $a, b \in \mathcal{X}$, we define

$$\begin{aligned}(-\infty, b] &:= \{x \in \mathcal{X} : x \leq b\}, \\ [a, \infty) &:= \{x \in \mathcal{X} : a \leq x\}, \\ [a, b] &:= \{x \in \mathcal{X} : a \leq x \leq b\}.\end{aligned}$$

The *graph* of the preorder $\leq$ is the set

$$\mathbb{G} := \{(x, x') \in \mathcal{X} \times \mathcal{X} : x \leq x'\},$$

i.e., $(x, x') \in \mathbb{G}$ iff $x \leq x'$.

In this section, we assume the following:

Assumption 4.1. The state space $\mathcal{X}$ is endowed with a preorder $\leq$, the σ-algebra $\mathscr{X}$ is generated by $\{(-\infty, b] : b \in \mathcal{X}\} \cup \{[a, \infty) : a \in \mathcal{X}\}$,[3] and the graph $\mathbb{G}$ of $\leq$ is measurable in the product space $\mathcal{X} \times \mathcal{X}$ (i.e., $\mathbb{G} \in \mathscr{X} \otimes \mathscr{X}$).

A set $C \in \mathscr{X}$ is called *increasing* if $[a, \infty) \subset C$ whenever $a \in C$, *decreasing* if $(-\infty, b] \subset C$ whenever $b \in C$, and *order bounded* if there exists a pair $a, b \in \mathcal{X}$ such that $C \subset [a, b]$. Let $\mathscr{X}^i$ denote the set of increasing measurable subsets of $\mathcal{X}$. We say that a function $h : \mathcal{X} \to \mathbb{R}$ is *increasing* if $h(x) \leq h(y)$ whenever $x \leq y$, and *decreasing* if $-h$ is increasing. Let $ib\mathcal{X}$ denote the set of increasing bounded measurable functions from $\mathcal{X}$ to $\mathbb{R}$.

Let Q be a stochastic kernel on $\mathcal{X}$. We say that Q is *increasing* if $Qh \in ib\mathcal{X}$ for any $h \in ib\mathcal{X}$. It is immediate from (1.2) that if Q is increasing, then Q^n is increasing for any $n \in \mathbb{N}$.

A distribution $\mu^* \in \mathcal{P}_{\mathscr{X}}$ is called *stationary* if $\mu^* Q = \mu^*$. Given any $\mu^* \in \mathcal{P}_{\mathscr{X}}$ and sequence $\{\mu_t\} \subset \mathcal{P}_{\mathscr{X}}$, we write $\mu_t \to \mu^*$ if

$$\lim_{t \to \infty} \mu_t h = \mu^* h \qquad \forall h \in ib\mathcal{X} \tag{4.1}$$

[3] If $\mathcal{X} \subset \mathbb{R}^n$, this generates the Borel subsets of $\mathcal{X}$ in the usual topology of $\mathbb{R}^n$; see Folland (1999, p. 22, 23) and Aliprantis and Border (1999, p. 135, 146).

(recall (1.4)). Note that, in $\mathbb{R}^n$ with the usual partial order, this convergence criterion implies weak convergence.[4] We say that μ^* is *globally stable* if μ^* is a unique stationary distribution and $\mu Q^t \to \mu^*$ for all $\mu \in \mathcal{P}_{\mathscr{X}}$. We say that Q is *globally stable* if Q has a globally stable distribution.

Lemma 4.1. *Suppose that*

$$\forall \mu, \nu \in \mathcal{P}_{\mathscr{X}}, \forall h \in ib\mathcal{X}, \quad \limsup_{t\to\infty}[(\mu Q^t)h - (\nu Q^t)h] \leq 0. \tag{4.2}$$

Then

(a) *Q has at most one stationary distribution.*

(b) *If Q has a stationary distribution, then Q is globally stable.*

Proof. Suppose that Q satisfies (4.2). Then, reversing the roles of μ and ν, we also have

$$\forall \mu, \nu \in \mathcal{P}_{\mathscr{X}}, \forall h \in ib\mathcal{X}, \quad \limsup_{t\to\infty}[(\nu Q^t)h - (\mu Q^t)h] \leq 0.$$

This inequality is equivalent to

$$\liminf_{t\to\infty}[(\mu Q^t)h - (\nu Q^t)h] \geq 0.$$

This combined with (4.2) yields

$$\forall \mu, \nu \in \mathcal{P}_{\mathscr{X}}, \forall h \in ib\mathcal{X}, \quad \lim_{t\to\infty}[(\mu Q^t)h - (\nu Q^t)h] = 0. \tag{4.3}$$

Suppose that $\mu^*, \nu^* \in \mathcal{P}_{\mathscr{X}}$ are both stationary. Then, for any $B \in \mathscr{X}^i$, by (4.3) with $h = \mathbb{1}_B$, we have $\mu^* \mathbb{1}_B = \nu^* \mathbb{1}_B$, i.e., $\mu^*(B) = \nu^*(B)$. Let $\sigma(\mathscr{X}^i)$ be the σ-algebra generated by $\mathscr{X}^i$. We have $\sigma(\mathscr{X}^i) \subset \mathscr{X}$ since $\mathscr{X}^i \subset \mathscr{X}$ and $\mathscr{X}$ is a σ-algebra. To see that $\mathscr{X} \subset \sigma(\mathscr{X}^i)$, note that

$$\mathscr{X}^i \supset \{[a, \infty) : a \in \mathcal{X}\} \cup \{\mathcal{X} \setminus (-\infty, b] : b \in \mathcal{X}\},$$

[4] Recall that weak convergence in $\mathbb{R}^n$ means that $\lim_{t\to\infty} \mu_t((-\infty, b]) = \mu^*((-\infty, b])$ for all continuity points b of the distribution function $F(x) := \mu^*(-\infty, x]$. Since $\mathbb{1}_{(-\infty,b]}$ is decreasing and thus $1 - \mathbb{1}_{(-\infty,b]} = \mathbb{1}_{\mathcal{X}\setminus(-\infty,b]}$ is increasing, (4.1) with $h = \mathbb{1}_{\mathcal{X}\setminus(-\infty,b]}$ implies that $\lim_{t\to\infty} \mu_t((-\infty, b]) = \mu^*((-\infty, b])$ for *all* $b \in \mathcal{X}$.

which implies that

$$\sigma(\mathscr{X}^i) \supset \{(-\infty, b] : b \in \mathcal{X}\} \cup \{[a, \infty) : a \in \mathcal{X}\}.$$

Since the right-hand side generates $\mathscr{X}$, it follows that $\sigma(\mathscr{X}^i) \supset \mathscr{X}$. Therefore, $\sigma(\mathscr{X}^i) = \mathscr{X}$. Since, in addition, $\mathscr{X}^i$ is a π-system (i.e., are closed under finite intersections), we have $\mu^*(B) = \nu^*(B)$ for all $B \in \mathscr{X}$, i.e., $\mu^* = \nu^*$ (see Billingley, 1995, p. 42). We have verified part (a).

To see part (b), suppose that Q has a stationary distribution μ^*. By part (a), μ^* is the unique stationary distribution. Let $\mu \in \mathcal{P}_{\mathscr{X}}$, and let $h \in ib\mathcal{X}$. By (4.3) with $\nu = \mu^*$ and stationarity, we have

$$\lim_{t\to\infty} [(\mu Q^t)h - \mu^* h] = 0,$$

i.e., $\lim_{t\to\infty}(\mu Q^t)h = \mu^* h$. Since μ and h were arbitrary, it follows that Q is globally stable. □

Lemma 4.2. *If Q is increasing and $\mathcal{X} \times \mathcal{X} \xrightarrow[Q\times Q]{t.a.} \mathbb{G}$, then Q satisfies (4.2).*

Proof. Suppose that Q is increasing and $\mathcal{X} \times \mathcal{X} \xrightarrow[Q\times Q]{t.a.} \mathbb{G}$. To verify (4.2), let $\{X_t\}_{t=0}^\infty$ and $\{Y_t\}_{t=0}^\infty$ be independent Markov-(Q, μ) and Markov-(Q, ν) processes, respectively. Then the bivariate process $\{(X_t, Y_t)\}$ is Markov-$(Q \times Q, \mu \times \nu)$. Let

$$\tau := \inf\{t \in \mathbb{Z}_+ : X_t \le Y_t\}.$$

Since $\mathcal{X} \times \mathcal{X} \xrightarrow[Q\times Q]{t.a.} \mathbb{G}$, we have

$$\lim_{t\to\infty} \mathbb{P}\{\tau \ge t\} = 0. \tag{4.4}$$

Let $h \in ib\mathcal{X}$ and $t \in \mathbb{N}$. We have

$$
\begin{aligned}
(\nu Q^t)h &= \mathbb{E} h(Y_t) \\
&\geq \mathbb{E} \mathbb{1}\{\tau \leq t\} h(Y_t) \\
&= \mathbb{E}[\mathbb{E}[\mathbb{1}\{\tau \leq t\} h(Y_t) | \mathscr{F}_\tau^{(X,Y)}]] \\
&= \mathbb{E}[\mathbb{1}\{\tau \leq t\} \mathbb{E}[h(Y_t) | \mathscr{F}_\tau^{(X,Y)}]] \\
&= \mathbb{E}[\mathbb{1}\{\tau \leq t\} (Q^{t-\tau} h)(Y_\tau)] \qquad (4.5) \\
&\geq \mathbb{E}[\mathbb{1}\{\tau \leq t\} (Q^{t-\tau} h)(X_\tau)] \qquad (4.6) \\
&= \mathbb{E}[\mathbb{1}\{\tau \leq t\} \mathbb{E}[h(X_t) | \mathscr{F}_\tau^{(X,Y)}]] \qquad (4.7) \\
&= \mathbb{E}[\mathbb{E}[\mathbb{1}\{\tau \leq t\} h(X_t) | \mathscr{F}_\tau^{(X,Y)}]] \\
&= \mathbb{E} \mathbb{1}\{\tau \leq t\} h(X_t) \\
&= \mathbb{E} h(X_t) - \mathbb{E} \mathbb{1}\{\tau \geq t+1\} h(X_t) \\
&\geq (\mu Q^t)h - \mathbb{P}\{\tau \geq t+1\} M,
\end{aligned}
$$

where the definition of $\mathscr{F}_\tau^{(X,Y)}$ is analogous to (2.5) and $M := \sup_{x \in \mathcal{X}} |h(x)|$. The equality in (4.5) uses the strong Markov property and independence, (4.6) holds because $Q^{t-\tau}h$ is increasing, and (4.7) uses the strong Markov property and independence again. It follows that

$$(\mu Q^t)h - (\nu Q^t)h \leq \mathbb{P}\{\tau \geq t+1\} M.$$

Since the right-hand side converges to zero by (4.4), we obtain the inequality in (4.2). Since μ, ν, and h were arbitrary, we have (4.2). □

The following result is immediate from the preceding two lemmas.

Proposition 4.1. *Q is globally stable if the following three conditions hold:*

(i) *Q is increasing.*

(ii) *Q has a stationary distribution.*

(iii) $\mathcal{X} \times \mathcal{X} \xrightarrow[Q \times Q]{t.a.} \mathbb{G}$.

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