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Abstract

This paper presents new order-theoretic conditions for global stability of monotone Markov processes with possibly non-compact state spaces. Our main result shows that a Markov process induced by a continuous and increasing transition law is globally stable if it admits a Lyapunov-like function, and becomes larger than any given element of the state space with positive probability, or smaller than any given element of the state space with positive probability. This result applies to a wide range of stochastic economic models.

Keywords: Monotonicity, stability, Markov process, stochastic dynamics

*This paper evolved out of an earlier manuscript (Kamihigashi and Stachurski, 2009) that has never been considered for publication anywhere, and that is no longer intended for publication. The textbook by Stachurski (2009) includes a discussion of some aspects of that manuscript. However, our results (Theorems 3.1 and 3.2) in the present paper are new and unpublished. Section 4.4 and footnotes 4, 12, 34, and 35 provide further details on the related results in Kamihigashi and Stachurski (2009) and Stachurski (2009). Incidentally, the technical results developed in the earlier manuscript (Kamihigashi and Stachurski, 2009) have been refined and generalized in a technical note (Kamihigashi and Stachurski, 2010). The latter was originally intended to be an appendix to this paper, but we have separated the note from this paper to keep it to a reasonable length and focus on the results most relevant to economic analysis. The note is not being considered for publication anywhere.

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1 Introduction

Assessing stability of stochastic economic models is necessary for both theoretical and quantitative research. Stability of stochastic dynamics is connected to such phenomena as existence and uniqueness of stationary equilibria, convergence, and stationarity and ergodicity in time series. In the economic literature, a popular approach to stability is to utilize monotonicity, which is a natural property of various economic models.

A seminal contribution in this regard was made by Razin and Yahav (1979), who derived a stability condition for monotone Markov models that satisfy what was to become known as a "monotone mixing condition." Their ideas were extended to more general settings by Stokey, Lucas, and Prescott (1989) and Hopenhayn and Prescott (1992). Further contributions were provided by Bhattacharya and Lee (1988) and Bhattacharya and Majumdar (2001), who studied stability in the monotone setting via a "splitting condition," defined in terms of an ordering on the state space. These results were significant advances both to economic theory and to the theory of stability for Markov processes. They have proved useful for treating a variety of economic problems (see, e.g., Huggett, 1993; Aghion and Bolton, 1997; Zhang, 2007; or Bhattacharya and Majumdar, 2007).

There are, however, some rather standard cases in which these conditions are not satisfied, especially when the underlying state space is not compact. For example, the elementary AR(1) process $X_{t+1} = \rho X_t + Z_{t+1}$ with $0 < \rho < 1$ and IID shocks with finite mean is globally stable in the sense that the distribution of X_t converges to a unique stationary distribution as $t \to \infty$, but neither monotone mixing nor splitting is satisfied.[1]

In the recent mathematical literature on Markov processes, non-compact state spaces are usually treated by establishing "drift conditions" with respect to "Lyapunov-like" functions. These conditions prevent the process in question from escaping to the boundary of the state space. If a continuous (or, more precisely, Feller) Markov process admits such a function, then it has a station-

[1]This is trivial for the monotone mixing condition, which requires a compact state space. However, we show in Section 4 that the AR(1) process violates a common implication of both monotone mixing and splitting that applies even to non-compact state spaces.

ary distribution. If, in addition, the process is irreducible and aperiodic, then it is globally stable.[2] Unfortunately, for many economic models, irreducibility either fails, is difficult to establish, or requires implausible assumptions.

In this paper we propose alternative, order-theoretic conditions that can be used to establish global stability for these models. In particular, the main result of this paper shows that a Markov process $\{X_t\}_{t=0}^{\infty}$ on a well-behaved subset of $\mathbb{R}^n$ with the usual partial order $\leq$ is globally stable if it is induced by a continuous and increasing transition law, admits a Lyapunov-like function, and satisfies one of the following three conditions:[3]

(a) Given a second process $\{Y_t\}_{t=0}^{\infty}$ independent of $\{X_t\}_{t=0}^{\infty}$ but obeying the same transition law, and given any initial conditions X_0 and Y_0 with $X_0 \geq Y_0$, there exists a $t \in \mathbb{N}$ such that $X_t \leq Y_t$ with positive probability.

(b) Given any initial condition X_0 and element c in the state space, there exists a $t \in \mathbb{N}$ such that $X_t \leq c$ with positive probability.

(c) Given any initial condition X_0 and element c in the state space, there exists a $t \in \mathbb{N}$ such that $X_t \geq c$ with positive probability.

Condition (a) plays a role similar to those of irreducibility, monotone mixing, and splitting in establishing global stability. In fact, it is a generalization of the latter two conditions, and easily applies to standard models such as the AR(1) process discussed above.[4] When combined with the existence of a Lyapunov-like function, conditions (b) and (c) each imply condition (a).

Each of the three conditions has a simple economic interpretation. For example, suppose that X_t represents household wealth in a model of savings

[2]For recent treatments of these topics within the mathematical literature, see, e.g., Meyn and Tweedie (2009, Theorems 12.1.3, 15.0.1, and 16.1.2) or Hernández-Lerma and Lasserre (2003). For applications of these ideas to economic problems, see, e.g., Nishimura and Stachurski (2005), Kamihigashi (2007), or Kristensen (2008). For more general discussions of Markov processes with economic applications, see Bhattacharya and Majumdar (2007) or Stachurski (2009). We refer the reader to those textbooks for more comprehensive discussions of Markov processes, and to Olson and Roy (2006) for a recent survey on stochastic growth.

[3]This result is in fact proved for a more general topological space equipped with a preorder. See Assumptions A.1 and A.2.

[4]See Section 4 for details. Condition (a) is also a considerable generalization of the "order mixing" condition discussed in Kamihigashi and Stachurski (2009) and Stachurski (2009).

and investment. Condition (a) means that, given two households X and Y that obey the same model but face independent, idiosyncratic shocks, even if household X is far richer than household Y initially, there is a small probability that household Y will be richer than household X at some point in the future. Condition (b) means that any household can be arbitrarily close to bankruptcy after an extremely unlucky sequence of shocks. Condition (c) means that any household can be arbitrarily rich after an extremely lucky sequence of shocks. These are natural properties of many economic models, and can easily be satisfied whether the state space is compact or not.

We emphasize that our result requires only one of these conditions to be satisfied, so that it can be used in various settings. To illustrate this point, we analyze a range of models including AR(1) processes, the benchmark Brock-Mirman (1972) model, the Brock-Mirman model with irreversible investment (Olson, 1989), and a stochastic version of the small open economy of Matsuyama (2004) with correlated shocks. The last two applications offer new results; in particular, the former may be of independent interest.

The rest of the paper is structured as follows. Section 2 reviews basic definitions concerning general and monotone Markov processes. Section 3 states the main result and then specializes to the case where the process is generated by a stochastic difference equation. Section 4 discusses the relations between our results and those based on monotone mixing and splitting. Section 5 provides economic applications. Remaining proofs are given in Appendices A and B.

2 Preliminaries

In this section we review basic definitions concerning both general and monotone discrete-time Markov processes.

2.1 Discrete-Time Markov Process

We begin with basic definitions concerning discrete-time Markov processes on an arbitrary measurable space $(\mathcal{X}, \mathscr{X})$. Let $\mathcal{P}_{\mathscr{X}}$ be the probability measures on $\mathscr{X}$. A *stochastic kernel* on $\mathcal{X}$ is a function $Q\colon \mathcal{X} \times \mathscr{X} \to [0, 1]$ such that

(i) $Q(x,\cdot) \in \mathcal{P}_{\mathscr{X}}$ for each $x \in \mathcal{X}$, and (ii) $Q(\cdot, B)$ is $\mathscr{X}$-measurable for each $B \in \mathscr{X}$.

Given a stochastic kernel Q on $\mathcal{X}$, a discrete-time, $\mathcal{X}$-valued stochastic process $\{X_t\}_{t=0}^{\infty}$ on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ is said to be *Markov-(Q, μ)* if (i) X_0 has distribution $\mu \in \mathcal{P}_{\mathscr{X}}$, and (ii) $Q(x,\cdot)$ is the conditional distribution of X_{t+1} given $X_t = x$:

$$\mathbb{P}[X_{t+1} \in B \mid \mathscr{F}_t^X] = Q(X_t, B) \qquad \forall B \in \mathscr{X}.$$

Here $\mathscr{F}_t^X$ is the σ-algebra generated by the history $X_0, \dots, X_t$. If the initial distribution μ is the probability measure $\delta_x \in \mathcal{P}_{\mathscr{X}}$ concentrated on $x \in \mathcal{X}$, we call $\{X_t\}$ Markov-(Q, x) rather than Markov-(Q, δ_x). We say that $\{X_t\}$ is *Markov-Q* if $\{X_t\}$ is Markov-(Q, μ) for some $\mu \in \mathcal{P}_{\mathscr{X}}$. Whenever we introduce a stochastic process, we implicitly take the underlying probability space $(\Omega, \mathscr{F}, \mathbb{P})$ as given.

For each $n \in \mathbb{N}$, let Q^n be the n-th order kernel, defined by

$$Q^1 := Q, \qquad Q^n(x, B) := \int Q^{n-1}(y, B) Q(x, dy) \qquad (x \in \mathcal{X},\ B \in \mathscr{X}). \tag{2.1}$$

Each Q^n is a stochastic kernel in its own right, and $Q^n(x, B)$ represents the probability of transitioning from x to B in n steps.

Let $h\colon \mathcal{X} \to \mathbb{R}$ be measurable and bounded, and let $\mu \in \mathcal{P}_{\mathscr{X}}$. We define the *right Markov operator $h \mapsto Qh$* by

$$(Qh)(x) := \int h(y) Q(x, dy) \qquad (x \in \mathcal{X}), \tag{2.2}$$

and the *left Markov operator $\mu \mapsto \mu Q$* by

$$(\mu Q)(B) := \int Q(x, B) \mu(dx) \qquad (B \in \mathscr{X}). \tag{2.3}$$

The t-th iterates of these operators can be interpreted as follows:

$$(Q^t h)(x) = \mathbb{E}[h(X_t) \mid X_0 = x] \quad \text{and} \quad (\mu Q^t)(B) = \mathbb{P}[X_t \in B \mid X_0 \sim \mu],$$

where $\{X_t\}_{t=0}^{\infty}$ is any Markov-Q process. Given any $B \in \mathscr{X}$, letting $\mathbb{1}_B$ denote the indicator function of B,[5] we have

$$Q(x, B) = (Q \mathbb{1}_B)(x) = \int \mathbb{1}_B(y) Q(x, dy) \qquad \forall x \in \mathcal{X}.$$

[5] That is, $\mathbb{1}_B(y) = 1$ if $y \in B$, and $= 0$ otherwise.

A distribution $\mu^* \in \mathcal{P}_{\mathscr{X}}$ is called *stationary* if $\mu^* Q = \mu^*$. If $\mathcal{X}$ has a topology and $\mathscr{X}$ is the Borel sets, then Q is called *Feller* if Qh is continuous on $\mathcal{X}$ whenever $h \colon \mathcal{X} \to \mathbb{R}$ is continuous and bounded.

Let $V \colon \mathcal{X} \to \mathbb{R}_+$ be a measurable function. The following condition is often called a "drift" condition:

$$\exists\, \alpha \in [0,1),\ \exists\, \beta \in \mathbb{R}_+,\ \forall\, x \in \mathcal{X}, \qquad (QV)(x) \le \alpha V(x) + \beta. \tag{2.4}$$

This condition implies that the process drifts towards areas of the state space where V is small.

Given independent Markov-Q processes $\{X_t\}_{t=0}^{\infty}$ and $\{X'_t\}_{t=0}^{\infty}$, the $\mathcal{X}\times\mathcal{X}$-valued process $\{(X_t, X'_t)\}_{t=0}^{\infty}$ is also a Markov process. Indeed, if we define the *product kernel* $Q\times Q$ on $\mathcal{X}\times\mathcal{X}$ by

$$(Q\times Q)((x,x'),A) = \int\int \mathbb{1}_A(y,y')Q(x,dy)Q(x',dy') \tag{2.5}$$

for $(x,x') \in \mathcal{X}\times\mathcal{X}$ and $A \in \mathscr{X}\otimes\mathscr{X}$, then $Q\times Q$ is a stochastic kernel on $\mathcal{X}\times\mathcal{X}$, and $\{(X_t, X'_t)\}_{t=0}^{\infty}$ is Markov-$Q\times Q$. When $A = B\times B'$ for some $B, B' \in \mathscr{X}$, then (2.5) reduces to

$$(Q\times Q)((x,x'),B\times B') = Q(x,B)Q(x',B). \tag{2.6}$$

2.2 Monotone Markov Processes

For the rest of the paper (except for Appendix A), let $\mathcal{X}$ be a Borel subset of $\mathbb{R}^m$ with $m \in \mathbb{N}$, and let $\mathscr{X}$ be the Borel subsets of $\mathcal{X}$.[6] To introduce monotone Markov processes, let $\le$ be the usual partial order on $\mathbb{R}^m$.[7] Given $a, b \in \mathcal{X}$, we define

$$\begin{aligned}
(-\infty, b] &:= \{x \in \mathcal{X} : x \le b\}, \\
[a, \infty) &:= \{x \in \mathcal{X} : a \le x\}, \text{ and} \\
[a, b] &:= \{x \in \mathcal{X} : a \le x \le b\}.
\end{aligned}$$

[6] Our general results, Theorems 3.1 and 3.2, are in fact established for a more general state space. See Assumptions A.1 and A.2 for details.

[7] That is, $(x_1, \dots, x_m) \le (y_1, \dots, y_m)$ whenever $x_i \le y_i$ for all $i = 1, \dots, m$. We define $\ge$ similarly.

These subsets of $\mathcal{X}$ are called *order intervals*. The *graph* of $\leq$ is the set

$$\mathbb{G} := \{(x, x') \in \mathcal{X} \times \mathcal{X} : x \leq x'\},$$

so that $(x, x') \in \mathbb{G}$ iff $x \leq x'$.

A set $C \in \mathscr{X}$ is called *increasing* if $[a, \infty) \subset C$ whenever $a \in C$, *decreasing* if $(-\infty, b] \subset C$ whenever $b \in C$, and *order bounded* if there exists a pair $a, b \in \mathcal{X}$ such that $C \subset [a, b]$. We say that a function $h : \mathcal{X} \to \mathbb{R}$ (or $\mathcal{X}$) is *increasing* if $h(x) \leq h(y)$ whenever $x \leq y$, and *decreasing* if $h(x) \geq h(y)$ whenever $x \leq y$. Let $ib\mathcal{X}$ ($db\mathcal{X}$) denote the set of increasing (decreasing) bounded measurable functions from $\mathcal{X}$ to $\mathbb{R}$.

Let Q be a stochastic kernel on $\mathcal{X}$. We say that Q is *increasing* if $Qh \in ib\mathcal{X}$ for any $h \in ib\mathcal{X}$. It is immediate from (2.2) and (2.1) that if Q is increasing, then Q^n is increasing for any $n \in \mathbb{N}$, and $Qg \in db\mathcal{X}$ for any $g \in db\mathcal{X}$.[8]

Remark 2.1. If Q is increasing, then for any $c \in \mathcal{X}$, we have $Q(\cdot, [c, \infty)) = Q\mathbb{1}_{[c,\infty)} \in ib\mathcal{X}$, and $Q(\cdot, (-\infty, c]) = Q\mathbb{1}_{(-\infty,c]} \in db\mathcal{X}$, since $\mathbb{1}_{[c,\infty)} \in ib\mathcal{X}$ and $\mathbb{1}_{(-\infty,c]} \in db\mathcal{X}$.

Given any $\mu^* \in \mathcal{P}_{\mathscr{X}}$ and sequence $\{\mu_t\} \subset \mathcal{P}_{\mathscr{X}}$, we write $\mu_t \to \mu^*$ if

$$\lim_{t\to\infty} \int h(x)\mu_t(dx) = \int h(x)\mu^*(dx) \qquad \forall h \in ib\mathcal{X}. \tag{2.7}$$

This convergence criterion is stronger than the standard notion of "weak" convergence.[9] We say that μ^* is *globally stable* if μ^* is a unique stationary distribution and $\mu Q^t \to \mu^*$ for all $\mu \in \mathcal{P}_{\mathscr{X}}$. We say that Q is *globally stable* if Q has a globally stable distribution.

3 The Main Result

The main result of this paper provides sufficient conditions for a stochastic kernel Q on $\mathcal{X}$ to be globally stable. Those sufficient conditions are based on

[8]To see that $Qg \in db\mathcal{X}$, note that $-Qg = Q[-g]$ by (2.2), and that $-g \in ib\mathcal{X}$. Therefore, $-Qg \in ib\mathcal{X}$, i.e., $Qg \in db\mathcal{X}$.

[9]Recall that weak convergence in $\mathbb{R}^m$ means that $\lim_{t\to\infty} \mu_t((-\infty, c]) = \mu^*((-\infty, c])$ for all continuity points c of the distribution function $F(x) := \mu^*(-\infty, x]$ (see Billingsley, 1995, p. 378). Since $\mathbb{1}_{\mathcal{X}\setminus(-\infty,c]} = 1 - \mathbb{1}_{(-\infty,c]} \in ib\mathcal{X}$, (2.7) with $h = \mathbb{1}_{\mathcal{X}\setminus(-\infty,c]}$ implies that $\lim_{t\to\infty} \mu_t((-\infty, c]) = \mu^*((-\infty, c])$ for *all* $c \in \mathcal{X}$.

the order-theoretic concepts that we introduce below.

We say that a function $V : \mathcal{X} \to \mathbb{R}_+$ *has order bounded sublevel sets* if the sublevel set $\{x \in \mathcal{X} : V(x) \leq r\}$ is order bounded for each $r > 0$. We say that Q is *order constricting* if there exists a measurable function $V : \mathcal{X} \to \mathbb{R}_+$ satisfying the drift condition (2.4) and having order bounded sublevel sets. We assume the following for the rest of the paper (except for Appendix A).

Assumption 3.1. For all $a, b \in \mathcal{X}$ with $a \leq b$, the order interval $[a, b]$ is compact.[10]

We say that Q is

(a) *order reversing* if, given any $x, x' \in \mathcal{X}$ with $x \geq x'$, there exists an $n \in \mathbb{N}$ such that $(Q \times Q)^n((x, x'), \mathbb{G}) > 0$.

(b) *downward reaching* if, given any $x, c \in \mathcal{X}$, there exists an $n \in \mathbb{N}$ such that $Q^n(x, (-\infty, c]) > 0$.

(c) *upward reaching* if, given any $x, c \in \mathcal{X}$, there exists an $n \in \mathbb{N}$ such that $Q^n(x, [c, \infty)) > 0$.

The statement that Q is order reversing means that, given any independent Markov-(Q, x) and Markov-(Q, x') processes $\{X_t\}_{t=0}^\infty$ and $\{X'_t\}_{t=0}^\infty$ with $x \geq x'$, there exists an $n \in \mathbb{N}$ such that $X_n \leq X'_n$ with positive probability. In other words, if a pair of initial conditions are ordered, the order can be reversed with positive probability in finite time. The definition above clarifies the fact that order reversal is a property of Q, and does not depend on the particular Markov-Q processes $\{X_t\}$ and $\{X'_t\}$. The following is a simple sufficient condition for order reversal that does not involve the product kernel $Q \times Q$.

Remark 3.1. If, given any $x, x' \in \mathcal{X}$ with $x \geq x'$, there exists a $c \in \mathcal{X}$ and an $n \in \mathbb{N}$ such that $Q^n(x, (-\infty, c]) > 0$ and $Q^n(x', [c, \infty)) > 0$, then Q is order reversing.[11]

[10] This assumption is satisfied in all common state spaces. For example it is satisfied if $\mathcal{X}$ is an increasing subset of $\mathbb{R}^m$, if $\mathcal{X}$ can be expressed as the cartesian product of intervals in $\mathbb{R}$ (each open, closed, or half open), or if $\mathcal{X}$ is any closed subset of $\mathbb{R}^m$.

[11] This follows from (2.6) with $B = (-\infty, c]$ and $B' = [c, \infty)$, and the inclusion $\mathbb{G} \supset (-\infty, c] \times [c, \infty)$.

The statement that Q is downward reaching means that any Markov-Q process $\{X_t\}_{t=0}^{\infty}$ becomes smaller than any element c in $\mathcal{X}$ with positive probability at some point in the future. The case in which Q is upward reaching is similar.

We are now ready to state the main result of this paper.

Theorem 3.1. *Let Assumption 3.1 hold. Suppose that Q is Feller, increasing, and order constricting. Suppose further that Q is either* (a) *order reversing,* (b) *downward reaching, or* (c) *upward reaching. Then Q is globally stable.*

Proof. See Appendix A. □

In the proof, the only role of the Feller property is to ensure existence of a stationary distribution. More specifically, it is shown that Q has a stationary distribution if it is Feller and order constricting. Given the existence of a stationary distribution, we need only assume that Q is increasing, order constricting, and order reversing to prove that Q is globally stable. The proof of Theorem 3.1 is then completed by verifying that Q is order reversing if it is order constricting and either downward reaching or upward reaching.[12]

As suggested by the outline of the proof in the preceding paragraph, among conditions (a)–(c), order reversal is the more fundamental concept, while (b) and (c) are more directed towards applications. As discussed in the Introduction, order reversal is a generalization of several existing stability conditions. These issues are elaborated on in Section 4. Before doing so, we first develop a second version of Theorem 3.1 that is convenient for studying many economic models.

3.1 Stochastic Difference Equations

In economic modelling, many Markov processes arise as stochastic difference equations of the form

$$X_{t+1} = F(X_t, Z_{t+1}) \qquad (t \geq 0), \tag{3.1}$$

[12] As long as compact sets are order bounded, part (a) of Theorem 3.1 generalizes the main result of Kamihigashi and Stachurski (2009, Theorem 5.2), which assumes that $\mathcal{X} \subset \mathbb{R}^n$ and Q is given by (3.1) and (3.2). By contrast, Theorem 3.1 is proved in Appendix A under only Assumptions A.1, A.2, and 3.1. Parts (b) and (c) of the theorem are entirely new.

where $\{Z_t\}_{t=1}^{\infty}$ is an IID shock process taking values in metric space $\mathcal{Z}$ with Borel sets $\mathscr{Z}$, and $F\colon \mathcal{X} \times \mathcal{Z} \to \mathcal{X}$ is measurable.[13] In what follows, we develop a version of Theorem 3.1 that is directly applicable to (3.1).

Let $\phi : \mathscr{Z} \to [0,1]$ be the common distribution of Z_t. For $x \in \mathcal{X}$ and $B \in \mathscr{X}$, define

$$Q(x, B) := \mathbb{P}\{F(x, Z_1) \in B\} = \int \mathbb{1}_B(F(x,z))\phi(dz). \tag{3.2}$$

Then Q is a stochastic kernel on $\mathcal{X}$, the stochastic process $\{X_t\}_{t=0}^{\infty}$ in (3.1) is Markov-Q, and the right Markov operator (2.2) takes the form

$$(Qh)(x) = \int h(F(x,z))\phi(dz) \qquad (x \in \mathcal{X}). \tag{3.3}$$

Each finite path of shock realizations $\{z_t\}_{t=1}^{n} \subset \mathcal{Z}$ and initial condition $X_0 = x \in \mathcal{X}$ determines a path $\{x_t\}_{t=0}^{n}$ for the state variable up until time n via (3.1). Let $F^n(x, \{z_t\}_{t=1}^{n})$ denote the value of x_n determined this way.[14] We then have the relation

$$Q^n(x, B) = \mathbb{P}\{F^n(x, \{Z_t\}_{t=1}^{n}) \in B\} \qquad (x \in \mathcal{X},\ B \in \mathscr{X},\ n \in \mathbb{N}).$$

We assume the following whenever we consider a model of the form (3.1).

Assumption 3.2. The state space $\mathcal{X}$ satisfies Assumption 3.1, $\mathcal{Z}$ is a separable metric space with Borel sets $\mathscr{Z}$, and $\mathcal{Z}$ is equal to the support of ϕ.[15] The function F is continuous on $\mathcal{X} \times \mathcal{Z}$ with respect to the product topology, and $F(\cdot, z)$ is increasing for each $z \in \mathcal{Z}$.

It easily follows from this assumption and (3.3) that Q is Feller and increasing. Further, the drift condition (2.4) can now be written as

$$\exists\, \alpha \in [0,1),\ \exists\, \beta \in \mathbb{R}_+,\ \forall\, x \in \mathcal{X}, \qquad \int V(F(x,z))\phi(dz) \leq \alpha V(x) + \beta. \tag{3.4}$$

[13]This formulation is relatively general. Many models with additional lags and non-IID shocks can be expressed in the form (3.1) by readjusting the definition of the state variables.

[14]Formally, $F^1 := F$ and $F^{i+1}(x, \{z_t\}_{t=1}^{i+1}) := F(F^i(x, \{z_t\}_{t=1}^{i}), z_{i+1})$ for all $i \in \mathbb{N}$.

[15]That is, $\phi(\mathcal{Z}) = 1$, and $\phi(G) > 0$ for any nonempty open $G \subset \mathcal{Z}$. This can be assumed without loss of generality as long as ϕ is a distribution on a separable metric space; for then ϕ has a unique support (see Aliprantis and Border, 1999, p. 374, 73), and we can simply let $\mathcal{Z}$ denote this support.

If a measurable function $V : \mathcal{X} \to \mathbb{R}_+$ satisfying (3.4) and having order bounded sublevel sets can be found, then, in view of Theorem 3.1, global stability will be established whenever Q is either order reversing, downward reaching, or upward reaching. In the present setting, we can obtain sufficient conditions for these properties using the extra structure provided by F:

Theorem 3.2. *Let Assumption 3.2 hold. Suppose that there exists a measurable function $V : \mathcal{X} \to \mathbb{R}_+$ satisfying (3.4) and having order bounded sublevel sets. Then Q is globally stable if one of the following conditions holds:*

(a) *For any pair $x, x' \in \mathcal{X}$ with $x \geq x'$, there exist finite $\mathcal{Z}$-valued sequences $\{z_t\}_{t=1}^n$ and $\{z'_t\}_{t=1}^n$ such that $F^n(x, \{z_t\}_{t=1}^n) \ll F^n(x', \{z'_t\}_{t=1}^n)$.*[16]

(b) *For any pair $x, c \in \mathcal{X}$, there exists a finite $\mathcal{Z}$-valued sequence $\{z_t\}_{t=1}^n$ such that $F^n(x, \{z_t\}_{t=1}^n) \ll c$.*

(c) *For any pair $x, c \in \mathcal{X}$, there exists a finite $\mathcal{Z}$-valued sequence $\{z_t\}_{t=1}^n$ such that $c \ll F^n(x, \{z_t\}_{t=1}^n)$.*

Proof. See Appendix A. □

Remark 3.2. As suggested above, under the hypotheses of Theorem 3.2, conditions (a), (b), and (c) imply that the corresponding stochastic kernel Q is order reversing, downward reaching, and upward reaching, respectively.

4 Discussions

As we remarked after Theorem 3.1, among conditions (a)–(c) in the theorem, order reversal is the more fundamental concept, while (b) and (c) are more directed towards applications. In this section we discuss some well-known stability conditions for monotone Markov processes, and clarify the relations between order reversal and those conditions. We also illustrate our results in a simple setting by applying them to an AR(1) process. Furthermore, we discuss

[16]The notation $(x_1, \ldots, x_m) \ll (x'_1, \ldots, x'_m)$ means that $x_i < x'_i$ for all $i = 1, \ldots, m$. When $\mathcal{X}$ is a more general state space (as in Appendix A), $x \ll x'$ means that (x, x') is interior to $\mathbb{G}$ with respect to the product topology.

a stability condition introduced in Kamihigashi and Stachurski (2009) and its relation to our results.

To facilitate the comparison, we introduce an additional concept: We say that Q is *simultaneously order inducing* if there exists an $\epsilon > 0$ and an $n \in \mathbb{N}$ such that, given any $x, x' \in \mathcal{X}$, we have $(Q \times Q)^n((x, x'), \mathbb{G}) \geq \epsilon$. If Q is simultaneously order inducing, then there exists a fixed $n \in \mathbb{N}$ and $\epsilon > 0$ such that the event $\{X_n \leq X'_n\}$ occurs with probability at least ϵ, regardless of initial conditions. Evidently, simultaneous order inducement is considerably stricter than order reversal. The following is a sufficient condition for simultaneous order inducement that does not involve the product kernel $Q \times Q$.

Remark 4.1. If there exists a $c \in \mathcal{X}$, an $\epsilon > 0$, and an $n \in \mathbb{N}$ such that $Q^n(x, (-\infty, c]) \geq \epsilon$ and $Q^n(x, [c, \infty)) \geq \epsilon$ for all $x \in \mathcal{X}$, then Q is simultaneously order inducing.[17]

4.1 Monotone Mixing

Consider the well-known Monotone Mixing Condition (MMC) of Razin and Yahav (1979), Stokey, Lucas, and Prescott (1989), and Hopenhayn and Prescott (1992). To state the MMC, assume that $\mathcal{X}$ is compact and takes the form $[a, b]$ for some $a, b \in \mathcal{X}$. In this setting, a stochastic kernel Q on $\mathcal{X}$ satisfies the MMC whenever there exists a $c \in \mathcal{X}$ and an $n \in \mathbb{N}$ such that $Q^n(a, [c, b]) > 0$ and $Q^n(b, [a, c]) > 0$.

The MMC implies simultaneous order inducement. To see this, let ϵ be the smaller of $Q^n(a, [c, b])$ and $Q^n(b, [a, c])$. Since Q is increasing, Q^n is also increasing, so that $Q^n(x, [c, \infty))$ is increasing in x by Remark 2.1. Hence, for all $x \in \mathcal{X}$,

$$Q^n(x, [c, \infty)) = Q^n(x, [c, b]) \geq Q^n(a, [c, b]) \geq \epsilon.$$

The same bound holds for $Q^n(x, (-\infty, c])$ over all $x \in \mathcal{X}$. Thus Q is simultaneously order inducing by Remark 4.1.

Stokey, Lucas, and Prescott (1989, Theorem 12.12) state that if Q is increasing, Feller, and satisfies the MMC, then Q is globally stable. This is a special case of Theorem 3.1. To see this, recall that the MMC implies simultaneous

[17] See the discussion in the footnote to Remark 3.1.

order inducement, and hence order reversal. To verify that Q is order constricting, let V be identically zero. Then V trivially satisfies the drift condition (2.4) and has order bounded sublevel sets. Indeed, (2.4) holds with $\alpha = \beta = 0$, and the sublevel set $\{x \in \mathcal{X} : V(x) \leq r\} = [a, b]$ is order bounded for any $r > 0$. The conditions of the theorem are now verified.

The stability result of Hopenhayn and Prescott (1992, Theorem 2) is stronger than that of Stokey, Lucas, and Prescott in that Q is not required to be Feller. This is possible because the Knaster-Tarski fixed point theorem can be used to ensure existence of a stationary distribution when $\mathcal{X}$ is a compact order interval and Q is increasing. In our case, since we do not even assume that $\mathcal{X}$ is order bounded, we use a continuity-based approach to ensure existence. Therefore, our result and that of Hopenhayn and Prescott are not directly comparable due to the difference in approach to existence. Once existence is established, however, order reversal considerably weakens the MMC, as we further illustrate below.

4.2 Splitting

Dubin and Friedman (1966), Bhattacharya and Lee (1988), and Bhattacharya and Majumdar (2001) consider a "splitting" condition and its relationship to stability for monotone random systems. To define the condition, consider the setting of Section 3.1. The splitting condition (Bhattacharya and Lee, 1988) requires existence of a $c \in \mathcal{X}$ and an $n \in \mathbb{N}$ satisfying

$$\mathbb{P}\{\forall x \in \mathcal{X}, F^n(x, \{Z_t\}_{t=1}^n) \leq c\} > 0, \quad \text{and} \tag{4.1}$$

$$\mathbb{P}\{\forall x \in \mathcal{X}, F^n(x, \{Z_t\}_{t=1}^n) \geq c\} > 0. \tag{4.2}$$

We show that the splitting condition implies simultaneous order inducement. Let ϵ be the smaller of the two probabilities in (4.1) and (4.2). Then, given any $x \in \mathcal{X}$, we have

$$\begin{aligned} Q^n(x, (-\infty, c]) &= \mathbb{P}\{F^n(x, \{Z_t\}_{t=1}^n) \leq c\} \\ &\geq \mathbb{P}\{\forall y \in \mathcal{X}, F^n(y, \{Z_t\}_{t=1}^n) \leq c\} \geq \epsilon. \end{aligned}$$

The same bound holds for $Q^n(x, [c, \infty))$ over all $x \in \mathcal{X}$. Thus Q is simultaneously order inducing by Remark 4.1

Dubin and Friedman (1966) and Bhattacharya and Majumdar (2001) establish global stability of Q under the splitting condition and additional assumptions when $\mathcal{X}$ is an interval in $\mathbb{R}$; Bhattacharya and Lee (1988) and Bhattacharya, Majumdar, and Hashimzade (2009) consider the case $\mathcal{X} \subset \mathbb{R}^n$ with $n \in \mathbb{N}$. Since they allow $F(x, z)$ to be increasing or decreasing in x depending on z, their results are not directly comparable to ours.[18] However, we argue below that the splitting condition (as well as the MMC) is not satisfied even in some standard cases where our results easily apply.

4.3 AR(1) Processes

As discussed above, the MMC and the splitting condition are both stricter than simultaneous order inducement. However, simultaneous order inducement is itself too strong for many standard models. The problem can be illustrated by considering an elementary scalar AR(1) process of the form

$$X_{t+1} = \rho X_t + Z_{t+1}, \qquad 0 < \rho < 1, \tag{4.3}$$

where $\{Z_t\}_{t=1}^{\infty}$ is IID with finite mean. Let ϕ be the distribution of Z_t, and let $\mathcal{Z} \subset \mathbb{R}$ be the support of ϕ. Define $F(x, z) := \rho x + z$, and let Q be the stochastic kernel on $\mathcal{X} := \mathbb{R}$ associated with (4.3) via (3.2).

To see that Q is not simultaneously order inducing, let $\{Z_t\}_{t=1}^{\infty}$ and $\{Z'_t\}_{t=1}^{\infty}$ be independent IID processes with common distribution ϕ. Simultaneous order inducement requires existence of an $\epsilon > 0$ and an $n \in \mathbb{N}$ such that, for any $x, x' \in \mathcal{X}$,

$$\mathbb{P}\{F^n(x, \{Z_t\}_{t=1}^n) \leq F^n(x', \{Z'_t\}_{t=1}^n)\} \geq \epsilon.$$

However, if n is fixed, then the left-hand side goes to zero as $x \to \infty$ and $x' \to -\infty$. Hence Q is not simultaneously order inducing.

On the other hand, Q is order reversing whenever $\mathcal{Z}$ contains at least two distinct elements. To see this, let $z, z' \in \mathcal{Z}$ with $z < z'$. Let $\{z_t\}_{t=1}^{\infty}$ and $\{z'_t\}_{t=1}^{\infty}$ be constant sequences equal to z and z' respectively. Then, for any $x, x' \in \mathbb{R}$ with $x \geq x'$, $F^n(x, \{z_t\}_{t=1}^n) \to z/(1-\rho)$ and $F^n(x', \{z'_t\}_{t=1}^n) \to z'/(1-\rho)$ as

[18] Even if $F(x, z)$ is not monotone in x, the splitting condition implies simultaneous order inducement since Remark 4.1 holds true for any stochastic kernel Q.

$n \to \infty$. Since $z < z'$, condition (a) of Theorem 3.2 is established, and Q is order reversing.

To further illustrate our results, let us now show that Q is globally stable using Theorem 3.2. It remains only to show that there exists a measurable function $V : \mathcal{X} \to \mathbb{R}_+$ satisfying the drift condition (3.4) and having order bounded sublevel sets. For this purpose, note that $V(x) := |x|$ has order bounded sublevel sets because, for any $r > 0$,

$$\{x \in \mathcal{X} : V(x) \leq r\} = \{x \in \mathcal{X} : |x| \leq r\} = [-r, r].$$

Moreover, V satisfies the drift condition (3.4) since

$$\int V(F(x,z))\phi(dz) = \int |\rho x + z|\phi(dz) \leq \rho V(x) + \int |z|\phi(dz),$$

which implies (3.4) with $\alpha = \rho$ and $\beta = \int |z|\phi(dz)$. Thus, Q is globally stable by Theorem 3.2.

If ϕ has a well-behaved density component, then global stability can be established via irreducibility. However, our techniques do not rely on such an additional requirement. Indeed, Theorem 3.2 applies even when $\mathcal{Z}$ is entirely discrete, as long as it contains at least two elements. If $\mathcal{Z}$ is unbounded, it is even easier to establish global stability since it is trivial to verify condition (b) or (c) of Theorem 3.2.[19]

It should be easy to see that the above analysis can be extended to multidimensional state spaces in a straightforward way. It can also be extended to nonlinear processes of the form (3.1) under appropriate assumptions. For example, the drift condition (3.4) can be verified as above if there exist constants $\rho \in (0,1)$ and $\theta \geq 0$ such that $|F(x,z)| \leq \rho|x| + \theta|z|$.

4.4 Order Mixing

Kamihigashi and Stachurski (2009) consider a "order mixing" condition and its implications for stability. Specifically, a stochastic kernel Q on $\mathcal{X}$ is called *order*

[19] Of course the AR(1) process is so simple that only elementary methods are needed to check its stability. We make these points to illustrate the nature and scope of our results.

mixing if, given any independent Markov-(Q, x) and Markov-(Q, x') processes $\{X_t\}_{t=0}^{\infty}$ and $\{X'_t\}_{t=0}^{\infty}$ with arbitrary initial conditions $x, x' \in \mathcal{X}$, we have

$$\mathbb{P} \cup_{t=0}^{\infty} \{X_t \leq X'_t\} = \mathbb{P} \cup_{t=0}^{\infty} \{X_t \geq X'_t\} = 1. \tag{4.4}$$

Evidently this condition is far stronger than order reversal, and is rather difficult to verify directly. However, the condition can be a useful intermediate step toward stability. Indeed, it is shown in Kamihigashi and Stachurski (2009, Theorem 5.1, Corollary 5.1) that, in the setting of of Section 3.1, Q is globally stable if it is increasing, has a stationary distribution, and is order mixing.[20] Although the constructive proof of this result in Kamihigashi and Stachurski (2009) does not apply to our general setting, a generalized version of the result is proved in Kamihigashi and Stachurski (2010, Proposition 4.1) and used in the proof of Theorem 3.1 in Appendix A.

5 Economic Applications

Let us now consider some economic applications, beginning with the benchmark stochastic growth model of Brock and Mirman (1972).

5.1 The Brock-Mirman Model

Consider the following maximization problem:

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \mathbb{E} \sum_{t=0}^{\infty} \delta^t u(c_t) \tag{5.1}$$

subject to $c_t + k_{t+1} = Z_t f(k_t)$ and $c_t, k_{t+1} \geq 0$ with $k_0 > 0$ given, where $\{Z_t\}_{t=0}^{\infty}$ is IID with finite mean and distribution ϕ supported on $\mathcal{Z} \subset \mathbb{R}_+$. We assume that u is C^1 on $\mathbb{R}_{++}$, bounded, strictly increasing, strictly concave, and $\lim_{c \to 0} u'(c) = \infty$, while f is C^1 on $\mathbb{R}_{++}$, continuous, concave, and increasing, with $f(0) = 0$ and $\lim_{k \to \infty} f'(k) = 0$. In addition, we require that δ, f, and ϕ jointly satisfy

$$\delta \lim_{k \to 0} f'(k) > \int \frac{1}{z} \phi(dz). \tag{5.2}$$

[20] Stachurski (2009, Theorem 11.3.14) discusses a special case of this result along with an informal proof based on Kamihigashi and Stachurski (2009).

All of these conditions are relatively standard, apart from (5.2), which is weaker than the usual assumption $f'(0) = \infty$. Our assumptions are chosen to simplify the exposition, and can be weakened significantly by employing instead Assumptions 2.1–2.5 and 3.1–3.5 in Kamihigashi (2007).

Let $y_t := Z_t f(k_t)$ be the state variable, and let $c(\cdot)$ and $k(\cdot)$ be the optimal policies for consumption and investment, both of which are continuous and strictly increasing (c.f., e.g., Kamihigashi, 2007, Theorem 2.1). For any $y > 0$, these functions satisfy the Euler equation

$$u'(c(y)) = \delta \int u'(c(zf(k(y))))zf'(k(y))\phi(dz). \tag{5.3}$$

Define $F(y,z) := zf(k(y))$, and the corresponding stochastic kernel Q by (3.2).

Brock and Mirman (1972) and Hopenhayn and Prescott (1992) study stability of Q when

$$0 < \inf \mathcal{Z} < \sup \mathcal{Z} < \infty, \tag{5.4}$$

and $(\inf \mathcal{Z})f(k(y)) > y$ for $y > 0$ sufficiently small.[21] In this case, the state space $\mathcal{X}$ can be chosen as a compact interval in $\mathbb{R}_{++}$, and global stability can be established using various techniques.

On the other hand, Theorem 3.2 can be used without such additional assumptions. To be precise, let $\mathcal{X} := \mathbb{R}_{++}$. Using the Euler equation, Kamihigashi (2007, pp. 495–496) shows that $V(y) := u'(c(y))^{1/2} + y$ has compact sublevel sets and satisfies the drift condition (3.4). Since compact sets are order bounded, Theorem 3.2 implies global stability if one of conditions (a), (b), or (c) is satisfied.

Under (5.4), condition (a) in Theorem 3.2 easily follows from the arguments of Brock and Mirman (1972, Figure 5) or Hopenhayn and Prescott (1992, p. 1402). Global stability can be established even more easily if $\inf \mathcal{Z} = 0$ or $\sup \mathcal{Z} = \infty$.[22] Suppose, for example, that $\inf \mathcal{Z} = 0$. Then, for any $y, r \in \mathcal{X}$,

[21] See Olson and Roy (2006) and Mitra and Roy (2010) for discussions on the last property. Brock and Mirman (1972) derive this property by assuming that $\mathbb{P}\{Z_t = \inf \mathcal{Z}\} > 0$. See Chatterjee and Shukayev (2008) for an extension of Hopenhayn and Prescott's (1992) analysis.

[22] Such cases have been treated under conditions guaranteeing irreducibility by Stachurski (2002) and Kamihigashi (2007). In addition, Zhang (2007) studies a Brock-Mirman model with unbounded shocks, and provides a direct proof of global stability. His results can be established via Theorem 3.2.

there exists a $z \in \mathcal{Z}$ such that $zf(k(y)) < r$. This verifies condition (b) in Theorem 3.2, and Q is globally stable.

5.2 Irreversible Investment

The analysis of the previous section can be extended to a model with irreversible investment. Specifically, consider the problem of maximizing (5.1) subject to

$$c_t + k_{t+1} = Z_t f(k_t) + \rho k_t, \tag{5.5}$$

$$k_{t+1} \geq \rho k_t, \tag{5.6}$$

and $c_t, k_{t+1} \geq 0$, where $\rho \in [0, 1)$ is one minus the depreciation rate. We maintain all the other assumptions, including the conditions on u, f, and ϕ.

If we take $x_t := k_t$ and z_t as states, the Bellman equation is given by

$$v(x,z) = \max_{x' \in \Gamma(x,z)} \left\{ u(zf(x) + \rho x - x') + \delta \int v(x', z')\phi(dz') \right\}, \tag{5.7}$$

where v is the value function and $\Gamma(x,z) = [\rho x, zf(x) + \rho x]$. Let $c(x,z)$ and $k(x,z)$ be the optimal policies for c_t and k_{t+1}. It is easy to verify that $v(x,z)$, $c(x,z)$, and $k(x,z)$ are all increasing in x and continuous.[23]

Define $F(x,z) := k(x,z)$, and the corresponding stochastic kernel Q by (3.2). Due to the irreversibility constraint (5.6), the Euler equation corresponding to (5.3) is not guaranteed here. Instead, we obtain the following inequality whenever $x, z > 0$ with $z \in \mathcal{Z}$, as verified in Appendix B:

$$u'(c(x,z)) \geq \delta \int u'(c(k(x,z), z'))z' f'(k(x,z))\phi(dz'). \tag{5.8}$$

[23]Under additional assumptions, Olson (1989) establishes these properties. They can be shown more generally as follows. Continuity of v as well as concavity of $v(x,z)$ in x follows from a standard argument. That $v(x,z)$ is increasing in x follows from the fact that, given any $k_0, k_0' > 0$ with $k_0 < k_0'$ and optimal path $\{k_t\}$ from k_0, the path $\{k_t'\}$ defined by $k_{t+1}' = k_{t+1} + \rho(k_t' - k_t)$ is feasible from k_0' and offers strictly greater consumption in each period. Continuity of c and k follows from strict concavity of the maximand in (5.7) in x'. Since the maximand has increasing differences in (x, x'), $k(x,z)$ is increasing in x (see Topkis, 1998, p. 76). That $c(x,z)$ is increasing in x holds since the right-hand side of (5.7) can be written as $\max_{0 \leq c \leq f(x,z)} \{u(c) + \delta \int v(zf(x) + \rho x - c, z')\phi(dz')\}$, and this maximand has increasing differences in (x, c) by concavity of $v(x,z)$ in x.

This inequality suffices for our purpose since it implies that

$$V(x) := \left[\int u'(c(x,z))^{1/2} \phi(dz) \right]^{1/2} + x \tag{5.9}$$

satisfies the drift condition (3.4) and has order bounded sublevel sets; see Appendix B for details.

To establish global stability, Olson (1989, A.13) assumes, in addition to (5.4), that $k(x,z) \geq x$ for all $z \in \mathcal{Z}$ if x is sufficiently small. The last assumption is particularly difficult to verify under standard conditions, but it is unnecessary in our approach. To illustrate this point, let us focus on a case in which the assumption is obviously violated: $0 \in \mathcal{Z}$,[24] which implies that $k(x,0) = \rho x < x$. In this case, for any $x, r > 0$, $F^n(x, \{z_t\}_{t=1}^n) = \rho^n x < r$ for large enough $n \in \mathbb{N}$ if $z_t = 0$ for all $t = 1, \dots, n$. This verifies condition (b) of Theorem 3.2, and Q is globally stable.

5.3 An Open Economy with Borrowing Constraints

Finally, we illustrate an application of Theorem 3.1 (rather than Theorem 3.2) in an example in which the state space is not open and is two-dimensional. Specifically, consider a stochastic version of the small open economy of Matsuyama (2004) with correlated productivity shocks.[25] In this economy, agents live for two periods, with a unit mass of young born at the start of each period. At time t, the old own capital stock k_t. Combined with the labor of the young, this produces $f(k_t)\xi_t$ units of the consumption good. Here $\{\xi_t\}_{t=0}^\infty$ is a sequence of correlated productivity shocks satisfying

$$\xi_{t+1} = \rho \xi_t + Z_{t+1}, \tag{5.10}$$

where $0 < \rho < 1$ and $\{Z_t\}_{t=1}^\infty \overset{\text{IID}}{\sim} \phi$ with support $\mathcal{Z} \subset \mathbb{R}_+$. We assume that $\mathcal{Z}$ is unbounded, and that $\mu := \mathbb{E} Z_t = \int z \phi(dz) \in (0, \infty)$.

[24] However, from (5.2) we have $\mathbb{P}\{Z_t = 0\} = 0$. For our purpose, it actually suffices to assume that $\inf \mathcal{Z} = 0$.

[25] The case of IID productivity shocks can be treated using existing techniques; see Stachurski (2009). Here we focus on the case in which the shocks are correlated and the Markov process induced by the model is two-dimensional.

The production function f is as in Section 5.1 except for (5.2). We assume further that f is strictly concave with $\lim_{k \to 0} f'(k) = \infty$. Factor markets are competitive, with return on capital given by $f'(k_t)\xi_t$ and wages by

$$w_t = w(k_t, \xi_t) := [f(k_t) - k_t f'(k_t)]\xi_t.$$

Here we define $w(0, \xi) = 0$ for any $\xi \in \mathbb{R}_+$. Note that $w(\cdot, \cdot)$ is increasing and continuous on $\mathbb{R}_+ \times \mathbb{R}_+$.

The young invest their wages, either in international credit markets at the risk free world interest rate R, or in a domestic project which converts one unit of the consumption good into one unit of the capital good next period. Agents can start at most one such project, so $0 \leq k_{t+1} \leq 1$. Provided that $k_{t+1} > 0$, risk neutral profit maximization requires

$$R \leq f'(k_{t+1})\mathbb{E}\left[\xi_{t+1} | \xi_t\right] = f'(k_{t+1})(\rho\xi_t + \mu). \tag{5.11}$$

The excess cost of the project above wages (i.e., $1 - w_t$) is financed by borrowing abroad (which is possible even when $k_t = 0$). Due to imperfect credit markets, the liabilities $R(1 - w_t)$ of those agents borrowing abroad to finance the project cannot exceed a fraction $\lambda \in (0, 1)$ of expected net worth at $t + 1$, which is $f'(k_{t+1})\mathbb{E}\left[\xi_{t+1} | \xi_t\right]$. Thus we have the additional restriction $R(1 - w_t) \leq \lambda f'(k_{t+1})\mathbb{E}\left[\xi_{t+1} | \xi_t\right]$. This inequality and (5.11) are simultaneously satisfied if and only if

$$R \leq \Theta(w_t) f'(k_{t+1})(\rho\xi_t + \mu), \tag{5.12}$$

where $\Theta(w) := \min\{\lambda/(1 - w), 1\}$.

Since $f'(0) = \infty$ and $\mu > 0$, the right-hand side of (5.12) exceeds the left-hand side whenever k_{t+1} is sufficiently small. This guarantees that $k_{t+1} > 0$ given any $k_t \in [0, 1]$. Furthermore, if $k_{t+1} \in (0, 1)$, agents must be indifferent as to whether to invest internationally or domestically. Therefore, k_{t+1} satisfies (5.12) with equality if $k_{t+1} < 1$. In other words, letting g be the inverse of f', we have

$$k_{t+1} = h(k_t, \xi_t) := \min\left\{ g\left[\frac{R}{\Theta(w(k_t, \xi_t))(\rho\xi_t + \mu)}\right], 1 \right\}. \tag{5.13}$$

Equations (5.10) and (5.13) together with initial conditions k_0 and ξ_0 define a Markov process $\{(k_t, \xi_t)\}_{t=0}^{\infty}$ on $\mathcal{X} := [0,1] \times \mathbb{R}_+$. Define

$$F((k, \xi), z) := (h(k, \xi), \rho\xi + z),$$

and define the corresponding stochastic kernel Q by (3.2).

We prove global stability of Q using Theorem 3.1. First observe that for $V \colon \mathcal{X} \to \mathbb{R}_+$ defined by $V(k, \xi) := \xi$ we have

$$\begin{aligned}(QV)(k, \xi) &= \mathbb{E}\,[\,V(k_1, \xi_1) \,|\, (k_0, \xi_0) = (k, \xi)] \\ &= \mathbb{E}\,[\,\xi_1 \,|\, (k_0, \xi_0) = (k, \xi)] \\ &= \rho\xi + \mu = \rho V(k, \xi) + \mu.\end{aligned}$$

Since $\rho \in (0,1)$ and $\mu > 0$, we have verified the drift condition (2.4). Moreover, V has order bounded sublevel sets because, for any $r > 0$, the sublevel set

$$\{(k, \xi) \in \mathcal{X} : V(k, \xi) \leq r\} = \{(k, \xi) \in \mathcal{X} : \xi \leq r\} = [0,1] \times [0, r]$$

is order bounded. Therefore, Q is order constricting.

Since $F((k, \xi), z)$ is continuous in (k, ξ, z), and increasing in (k, ξ) when z is held fixed, Q is both Feller and increasing. Hence it remains to prove one of conditions (a), (b) or (c) in Theorem 3.1. We will prove (c), which requires that Q is upward reaching. For this purpose, we need to show that, given any $x, c \in \mathcal{X}$, there exists an $n \in \mathbb{N}$ such that $Q^n(x, [c, \infty)) > 0$.

To see this, fix $x = (k_0, \xi_0) \in \mathcal{X}$ and $c = (\bar{k}, \bar{\xi}) \in \mathcal{X}$. Let $\{(k_t, \xi_t)\}_{t=0}^{\infty}$ be the process generated by (5.10) and (5.13) with initial condition (k_0, ξ_0). It suffices to show that $(Q^2(x, [c, \infty)) =)\ \mathbb{P}\{(k_2, \xi_2) \geq (\bar{k}, \bar{\xi})\} > 0$. To this end, note that $\xi_2 = \rho^2 \xi_0 + \rho Z_1 + Z_2 \geq \rho Z_1$, and hence $\xi_2 \geq \bar{\xi}$ whenever $Z_1 \geq \bar{\xi}/\rho$. Regarding k_2, note that g is decreasing, that $\Theta(w) \geq \lambda$ for all $w \geq 0$, and that $\xi_1 \geq Z_1$. Thus

$$\begin{aligned}k_2 = h(k_1, \xi_1) &\geq \min\left\{ g\left(\frac{R}{\lambda(\rho\xi_1 + \mu)} \right),\ 1 \right\} \\ &\geq \min\left\{ g\left(\frac{R}{\lambda\rho Z_1} \right),\ 1 \right\}.\end{aligned}$$

In particular, $k_2 = 1 \geq \bar{k}$ whenever $g\left(R/(\lambda\rho Z_1)\right) \geq 1$ or, equivalently, $Z_1 \geq R/(\lambda\rho f'(1))$. In summary,

$$(k_2, \xi_2) \geq (\bar{k}, \bar{\xi}) \text{ whenever } Z_1 \geq \max\left\{\frac{\bar{\xi}}{\rho}, \frac{R}{\lambda\rho f'(1)}\right\}.$$

Since $\mathcal{Z}$ is unbounded, the event on the right-hand side has positive probability, and $Q^2(x, [c, \infty)) > 0$. We have shown that Q is upward reaching, and hence globally stable.

5.4 Concluding Comments

This paper has shown that a Markov process $\{X_t\}$ on $\mathbb{R}^n$ (or, indeed, on a preordered metric space) is globally stable if it has a continuous and increasing stochastic kernel, is order constricting (i.e., admits a Lyapunov-like function), and satisfies one of the following three conditions: (a) given any Markov process $\{Y_t\}$ independent of $\{X_t\}$ but having the same stochastic kernel Q, and given any initial conditions X_0 and Y_0 with $X_0 \geq Y_0$, there exists a $t \in \mathbb{N}$ such that $X_t \leq Y_t$ with positive probability (Q is order reversing); (b) given any element c of the state space, there exists a $t \in \mathbb{N}$ such that $X_t \leq c$ with positive probability (Q is downward reaching); and (c) given any element c of the state space, there exists a $t \in \mathbb{N}$ such that $X_t \geq c$ with positive probability (Q is upward reaching). We have also provided a version of this result that applies directly to stochastic difference equations. We have illustrated the usefulness of these results by applying them to a range of stochastic economic models.

Although many of the applications discussed here are one-dimensional, it is easy to see that our results readily apply to multidimensional models as well, as we noted in Section 4.3. We further illustrated this point by analyzing a two-dimensional economic model in Section 5.3. There are many other cases in which a multidimensional Markov process characterizing the dynamics of an economic model is increasing, continuous, and order constricting. The last property often follows from economic forces that make it undesirable or infeasible to exhaust or accumulate resources excessively fast.

The Markov process in question can also be downward or upward reaching for economic reasons. This is true, for example, if the economy under study converges to extinction when it keeps receiving bad shocks, or grows without bound when it keeps receiving good shocks. These economic mechanisms make for instability by themselves and, naturally, have often been avoided in the economic literature. However, when combined with the counterbalancing economic forces discussed above, these mechanisms make the process "well mixed" (or order reversing) and result in stochastic stability, as our results demonstrate. We believe that our results widen the range of Markov processes that economists can comfortably utilize in conducting stability analysis.

A Proofs of Theorems 3.1 and 3.2

This appendix proves Theorems 3.1 and 3.2. Before proving these results, we provide some preliminary results on existence of a stationary distribution, and introduce two binary relations based on stochastic kernels. In this appendix, we take $(\mathcal{X}, \mathscr{X})$ to be an arbitrary measurable space, and clarify the exact assumptions needed for our results.

A.1 Preliminaries

A.1.1 Existence of a Stationary Distribution

The purpose of this section is to clarify the topological assumptions on $(\mathcal{X}, \mathscr{X})$ that we use to ensure existence of a stationary distribution. The following is our basic topological requirement.

Assumption A.1. The state space $\mathcal{X}$ is a metrizable topological space, and the σ-algebra $\mathscr{X}$ is the Borel sets.

We say that a function $V : \mathcal{X} \to \mathbb{R}_+$ *has precompact sublevels sets* if the sublevel set $\{x \in \mathcal{X} : V(x) \leq r\}$ is precompact for each $r > 0$.[26] The following result gives a sufficient set of conditions for existence of a stationary distribution.[27]

[26] A set is called *precompact* if its closure is compact. Functions having precompact sublevel sets are often called *coercive* in the literature on Markov processes (Meyn and Tweedie, 2009).

[27] Lemma A.1 is equivalent to Meyn and Tweedie (2009, Proposition 12.1.3) except for the

Lemma A.1. *Let $(\mathcal{X}, \mathscr{X})$ be a measurable space satisfying Assumption A.1. Let Q be a Feller stochastic kernel on $\mathcal{X}$. Suppose that there exists a measurable function $V : \mathcal{X} \to \mathbb{R}_+$ having precompact sublevel sets and satisfying*

$$\exists x \in \mathcal{X}, \quad \limsup_{t \to \infty} (Q^t V)(x) < \infty. \tag{A.1}$$

Then Q has a stationary distribution.

Proof. Define $\mu_0 := \delta_x$, and $\mu_t := \mu_0 Q^t = Q^t(x, \cdot)$ for $t \in \mathbb{N}$. Let

$$s > \limsup_{t \to \infty} \int V(y) \mu_t(dy) = \limsup_{t \to \infty} (Q^t V)(x). \tag{A.2}$$

Discarding a finite number of elements if necessary, we assume that

$$\int V(y) \mu_t(dy) \leq s \qquad \forall t \geq 0.$$

Since V has precompact sublevel sets, this implies that $\{\mu_t\}$ is tight in the sense that, given any $\epsilon > 0$, there exists a compact subset K of $\mathcal{X}$ such that $\mu_t(K) \geq 1 - \epsilon$.[28] For $n \in \mathbb{N}$, define $\nu_n := \frac{1}{n} \sum_{t=0}^{n-1} \mu_t$. Since $\{\mu_t\}$ is tight, $\{\nu_n\}$ is also tight. By Pollard (2002, p. 185),[29] $\{\nu_n\}$ has a subsequence that weakly converges to a distribution in $\mathcal{P}_{\mathscr{X}}$. This distribution is stationary by Bhattacharya and Majumdar (2007, Theorem 11.1, p. 192). □

As is well-known, a simple sufficient condition for (A.1) is the drift condition (2.4). We state this observation here for later reference.[30]

Lemma A.2. *Let $(\mathcal{X}, \mathscr{X})$ be an arbitrary measurable space, and let Q be a stochastic kernel on $\mathcal{X}$. If there exists a measurable function $V : \mathcal{X} \to \mathbb{R}_+$ satisfying the drift condition (2.4), then*

$$\forall x \in \mathcal{X}, \quad \sup_{t \geq 0} (Q^t V)(x) < \infty. \tag{A.3}$$

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assumptions on $(\mathcal{X}, \mathscr{X})$. Since they assume that $\mathcal{X}$ is a locally compact, complete, separable, metrizable topological space (and this set of properties is quite common in the literature), we give an alternative proof that requires only Assumption A.1.

[28] To see this, let $\epsilon > 0$, and let $r > 0$ be such that $s/r \leq \epsilon$. Let K be the closure of the sublevel set $\{x \in \mathcal{X} : V(x) \leq r\}$. Since V has precompact sublevel sets, K is compact. For any $t \geq 0$, we have $s \geq \int V(y)\mu_t(dy) \geq \int V(y) \mathbb{1}_{\mathcal{X} \setminus K}(y) \mu_t(dy) \geq r\mu_t(\mathcal{X} \setminus K)$. Thus $\mu_t(\mathcal{X} \setminus K) \leq s/r \leq \epsilon$, i.e., $\mu_t(K) \geq 1 - \epsilon$.

[29] His definition of uniform tightness is weaker than the definition of tightness here, so that his result applies.

[30] In particular, Lemma A.2 plays an important role in the proof of Lemma A.6.

Proof. Let $x \in \mathcal{X}$. Iterating on the drift condition (2.4) gives the bound

$$\forall t \geq 0, \quad (Q^t V)(x) \leq \alpha^t V(x) + \frac{\beta}{1-\alpha} \leq V(x) + \frac{\beta}{1-\alpha}. \tag{A.4}$$

Since $V(x)$ is finite, we obtain (A.3). $\square$

The following result is immediate from the preceding two lemmas.

Corollary A.1. *Let $(\mathcal{X}, \mathscr{X})$ be a measurable space satisfying Assumption A.1. Let Q be a Feller stochastic kernel on $\mathcal{X}$. Suppose that there exists a measurable function $V : \mathcal{X} \to \mathbb{R}_+$ satisfying the drift condition (2.4) and having precompact sublevel sets. Then Q has a stationary distribution.*

A.1.2 Anticipation

Let $(\mathcal{X}, \mathscr{X})$ be an arbitrary measurable space, and let Q be a stochastic kernel on $\mathcal{X}$. We introduce two binary relations based on Q: Given any $B, C \in \mathscr{X}$, we say that

- *B simultaneously anticipates C with respect to Q* (written $B \xrightarrow[Q]{s.a.} C$) if there exists an $n \geq 0$ and an $\epsilon > 0$ such that $Q^n(x, C) \geq \epsilon$ for all $x \in B$.[31]
- *B totally anticipates C with respect to Q* (written $B \xrightarrow[Q]{t.a.} C$) if, for any $x \in B$ and Markov-(Q, x) process $\{X_t\}_{t=0}^{\infty}$, we have $\mathbb{P} \cup_{t \geq 0} \{X_t \in C\} = 1$.

It is shown in Kamihigashi and Stachurski (2010, Section 3) that the binary relations $\xrightarrow[Q]{s.a.}$ and $\xrightarrow[Q]{t.a.}$ are preorders;[32] furthermore, for any $C, G \in \mathscr{X}$, we have

$$\mathcal{X} \xrightarrow[Q]{t.a.} C \xrightarrow[Q]{s.a.} G \quad \Longrightarrow \quad \mathcal{X} \xrightarrow[Q]{t.a.} G. \tag{A.5}$$

Both binary relations are defined for an arbitrary stochastic kernel Q, which may itself be a product kernel on a product space. In what follows, we use these binary relations for a product space.

[31] For $n = 0$, we define $Q^n(x, C) := \mathbb{1}_C(x)$.

[32] A reflexive and transitive binary relation is called a *preorder*.

A.2 Proof of Theorem 3.1

Let us now prove Theorem 3.1. To clarify the exact assumptions needed for the result, we start with an arbitrary measurable space $(\mathcal{X}, \mathscr{X})$. Throughout the proof, we assume the following.

Assumption A.2. The state space $\mathcal{X}$ is endowed with a preorder $\leq$, the σ-algebra $\mathscr{X}$ is generated by $\{(-\infty, b] : b \in \mathcal{X}\} \cup \{[a, \infty) : a \in \mathcal{X}\}$,[33] and the graph $\mathbb{G}$ of $\leq$ is measurable in the product space $\mathcal{X} \times \mathcal{X}$ (i.e., $\mathbb{G} \in \mathscr{X} \otimes \mathscr{X}$).

Let Q be a stochastic kernel on $\mathcal{X}$. Under Assumption A.2, it is shown in Kamihigashi and Stachurski (2010, Proposition 4.1) that if (i) Q is increasing, (ii) Q has a stationary distribution, and (iii) Q satisfies

$$\mathcal{X} \times \mathcal{X} \xrightarrow[Q \times Q]{t.a.} \mathbb{G}, \tag{A.6}$$

then Q is globally stable.[34]

Since the statement of Theorem 3.1 already assumes that Q is increasing, we need not consider condition (i). Regarding condition (ii), let Assumptions A.1 and 3.1 hold. Suppose that Q is Feller and order constricting. Then there exists a measurable function $V : \mathcal{X} \to \mathbb{R}_+$ satisfying the drift condition (2.4) and having order bounded sublevel sets. Since order bounded sets are compact by Assumption 3.1, it follows that V has precompact sublevel sets. Thus Q has a stationary distribution by Corollary A.1.

To establish global stability, it remains to verify condition (iii), or (A.6). In particular, to prove part (a) of Theorem 3.1, it suffices to show the following.

Lemma A.3. *If Q is increasing, order constricting, and order reversing, then (A.6) holds.*[35]

[33] If $\mathcal{X} \subset \mathbb{R}^n$, this generates the Borel subsets of $\mathcal{X}$ in the usual topology of $\mathbb{R}^n$; see Folland (1999, p. 22, 23) and Aliprantis and Border (1999, p. 135, 146).

[34] This result generalizes Kamihigashi and Stachurski (2009, Theorem 5.1) (where Q is given by (3.1) and (3.2)) and Stachurski (2009, Theorem 11.3.14) (where $\mathcal{X} \subset \mathbb{R}^n$ in addition).

[35] A variant of this result based on a different "order constricting" property is shown in Kamihigashi and Stachurski (2009, Theorem 4.2) using a considerably more complicated argument. A special case of this variant is stated without proof in Stachurski (2009, Theorem 11.3.19). The latter result is not applicable here even in the setting of (3.1), because the definition of "order inducing sets" (Stachurski, 2009, Definition 11.3.16) is too restrictive for our purposes.

For the rest of this section (including the above lemma), we assume only Assumption A.2 unless otherwise indicated (i.e., we do not assume Assumptions A.1 and 3.1).

Using (A.5) we prove Lemma A.3 in two steps. More specifically, we first show that there exists an order bounded set $C \in \mathscr{X}$ such that $\mathcal{X}\times\mathcal{X} \xrightarrow[Q\times Q]{t.a.} C\times C$, and then show that any order bounded set $C \in \mathscr{X}$ satisfies $C\times C \xrightarrow[Q\times Q]{s.a.} \mathbb{G}$. This together with (A.5) establishes (A.6).

Lemma A.4. *If Q is order constricting, then there exists an order bounded set $C \in \mathscr{X}$ such that*

$$\mathcal{X}\times\mathcal{X} \xrightarrow[Q\times Q]{t.a.} C\times C. \tag{A.7}$$

Proof. It follows from Kamihigashi and Stachurski (2010, Lemma 2.2) that, given any $C \in \mathscr{X}$, if there exists a measurable function $w\colon \mathcal{X}\times\mathcal{X} \to [1,\infty)$ and a constant $\lambda \in [0,1)$ such that

$$((Q\times Q)w)(x,x') \le \lambda w(x,x') \qquad \forall (x,x') \notin C\times C, \tag{A.8}$$

then (A.7) holds. Thus it suffices to construct an order bounded set $C \in \mathscr{X}$, a measurable function w, and a constant $\lambda \in [0,1)$ jointly satisfying (A.8).

Since Q is order constricting, there exists a measurable function $V : \mathcal{X} \to \mathbb{R}_+$ having order bounded sublevel sets and satisfying the drift condition (2.4). Without loss of generality, we may assume that $V \ge 1$.[36] Let α and β be as in (2.4). Choose any $r > 0$ such that $\lambda := \alpha + \beta/r < 1$. Let $C := \{x \in \mathcal{X} : V(x) \le 2r\}$, which is order bounded since V has order bounded sublevel sets. Define $w : \mathcal{X}\times\mathcal{X} \to [1,\infty)$ by

$$w(x,x') := \frac{V(x)+V(x')}{2}.$$

By the drift condition (2.4), we have

$$((Q\times Q)w)(x,x') \le \alpha w(x,x') + \beta \qquad \forall (x,x') \in \mathcal{X}\times\mathcal{X}. \tag{A.9}$$

[36]Some algebra shows that if V satisfies the drift condition (2.4), then $V' := V + 1$ also satisfies the condition.

Now pick any $(x,x') \notin C \times C$. Then $V(x) + V(x') > 2r$, and hence $w(x,x') > r$. In view of (A.9), we have

$$\frac{((Q \times Q)w)(x,x')}{w(x,x')} \leq \alpha + \frac{\beta}{w(x,x')} < \alpha + \frac{\beta}{r} = \lambda.$$

This proves (A.8), and hence the lemma. □

Lemma A.5. *If Q is increasing and order reversing, then for any order bounded set $C \in \mathscr{X}$, we have*

$$C \times C \xrightarrow[Q \times Q]{s.a.} \mathbb{G}. \tag{A.10}$$

Proof. Let $C \in \mathscr{X}$ be order bounded. Then there exists a pair $a, b \in \mathcal{X}$ such that $C \subset [a,b]$. Since Q is order reversing, there exists an $n \in \mathbb{N}$ such that $\epsilon := (Q \times Q)^n((b,a), \mathbb{G}) > 0$. For $(x,x') \in \mathcal{X}$, define

$$\psi(x,x') := (Q \times Q)^n((x,x'), \mathbb{G}).$$

We claim that $\psi(x,x')$ is decreasing in x and increasing in x'. To see this, note from (2.5) that

$$\psi(x,x') = \int \int \mathbb{1}_{\mathbb{G}}(y,y') Q^n(x,dy) Q^n(x',dy'). \tag{A.11}$$

Calculating the inner integral, we have

$$\psi(x,x') = \int Q^n(x,(-\infty,y']) Q^n(x',dy').$$

Since Q^n is increasing, $Q^n(x,(-\infty,y])$ is decreasing in x by Remark 2.1, so that $\psi(x,x')$ is decreasing in x. Interchanging the order of integration in (A.11) and calculating the inner integral, we have

$$\psi(x,x') = \int Q^n(x',[y,\infty)) Q^n(x,dy).$$

Since $Q^n(x',[y,\infty))$ is increasing in x' by Remark 2.1, it follows that $\psi(x,x')$ is increasing in x'.

Now, for any $(x,x') \in C \times C \subset [a,b] \times [a,b]$, we have $\psi(x,x') \geq \psi(b,a) = \epsilon$, which implies (A.10). □

From (A.5) and Lemmas A.4 and A.5, we obtain Lemma A.3, and hence part (a) of Theorem 3.1. Parts (b) and (c) of the theorem follow from part (a) and the following.

Lemma A.6. *Suppose that Q is order constricting. If Q is either downward reaching or upward reaching, then Q is order reversing.*

Proof. Suppose that Q is order constricting. Fix $x, x' \in \mathcal{X}$ with $x \geq x'$, and let $\{X_t\}_{t=0}^{\infty}$ and $\{X'_t\}_{t=0}^{\infty}$ be independent Markov-(Q, x) and Markov-(Q, x') processes, respectively. Define

$$s := \sup_{t \geq 0} \mathbb{E} V(X'_t) = \sup_{t \geq 0} (Q^t V)(x') < \infty,$$

where the inequality holds by Lemma A.2. Let $C := \{y \in \mathcal{X} : V(y) \leq s\}$. By definition of s, we have $\mathbb{P}\{V(X'_t) \leq s\} = \mathbb{P}\{X'_t \in C\} > 0$ for all $t \geq 0$. Since V has order bounded sublevel sets, there exists a pair $a, b \in \mathcal{X}$ such that $C \subset [a, b]$. Hence

$$\mathbb{P}\{X'_t \in [a, b]\} \geq \mathbb{P}\{X'_t \in C\} > 0 \qquad \forall\, t \geq 0. \tag{A.12}$$

Now, suppose that Q is downward reaching. Then there exists an $n \in \mathbb{N}$ such that $Q^n(x, (-\infty, a]) > 0$. From (A.12) we have $Q^n(x', [a, \infty)) > 0$. Thus Remark 3.1 with $c = a$ implies that Q is order reversing. The case in which Q is downward reaching is similar. ☐

A.3 Proof of Theorem 3.2

Let Assumptions A.1, A.2, 3.1, and 3.2 hold. Then Q is Feller and increasing. By hypothesis, Q is order constricting. In view of Theorem 3.1, therefore, it suffices to prove that conditions (a), (b), and (c) imply that Q is order reversing, downward reaching, and upward reaching, respectively.

Assume condition (a). Fix $x, x' \in \mathcal{X}$ with $x \geq x'$. Then there exist finite $\mathcal{Z}$-valued sequences $\{z_t\}_{t=1}^{n}$ and $\{z'_t\}_{t=1}^{n}$ such that

$$F^n(x, \{z_t\}_{t=1}^{n}) \ll F^n(x', \{z'_t\}_{t=1}^{n}). \tag{A.13}$$

Let $\{Z_t\}_{t=1}^{\infty}$ and $\{Z'_t\}_{t=1}^{\infty}$ be independent, IID shock processes with common distribution ϕ. Define

$$\gamma := \mathbb{P}\{F^n(x, \{Z_t\}_{t=1}^n) \ll F^n(x', \{Z'_t\}_{t=1}^n)\}. \tag{A.14}$$

Evidently $(Q \times Q)^n((x, x'), \mathbb{G}) \geq \gamma$, and hence we need only show that $\gamma > 0$.

By (A.13) and continuity of F, there exist finite sequences $\{N_t\}_{t=1}^n$ and $\{N'_t\}_{t=1}^n$ of open neighborhoods of z_t and z'_t, respectively, such that

$$\forall t \in \{1, \ldots, n\}, \tilde{z}_t \in N_t, \tilde{z}'_t \in N'_t \implies F^n(x, \{\tilde{z}_t\}_{t=1}^n) \ll F^n(x', \{\tilde{z}'_t\}_{t=1}).$$

This leads to the estimate

$$\gamma \geq \mathbb{P} \cap_{t=1}^n \{Z_t \in N_t, Z'_t \in N'_t\} = \prod_{t=1}^n \phi(N_t)\phi(N'_t),$$

where the equality holds by independence. Since $\mathcal{Z}$ is the support of ϕ, the rightmost side is strictly positive, and $\gamma > 0$. The proofs for conditions (b) and (c) are similar.[37]

B Proofs of Section 5.2 Claims

This appendix proves inequality (5.8) and its implication that the stochastic kernel Q induced by the model in Section 5.2 is order constricting. To simplify notation, given any bounded or nonnegative measurable function $h : \mathcal{Z} \to \mathbb{R}$, we define

$$\mathbb{E}_z h(z) := \int h(z)\phi(dz).$$

B.1 Proof of (5.8)

Let $x, z, s > 0$ with $z \in \mathcal{Z}$. Note that $k(x, z) + \rho s \in \Gamma(x + s, z)$, i.e., it is feasible to choose $k_{t+1} = k(x, z) + \rho s$ when $k_t = x + s$.[38] Let $a(x, z, s)$ be the

[37] For example, regarding condition (b), replace the right-hand sides of (A.13) and (A.14) with c, and consider only $\{z_t\}$, $\{Z_t\}$, and $\{N_t\}$.

[38] This particular way of perturbing $k(x, z)$ and the inequality in (B.6) are the key steps in this proof.

corresponding level of consumption:

$$a(x,z,s) := zf(x+s) + \rho(x+s) - [k(x,z) + \rho s] \tag{B.1}$$
$$= zf(x+s) + \rho x - [zf(x) + \rho x - c(x,z)] \tag{B.2}$$
$$= zf(x+s) - zf(x) + c(x,z). \tag{B.3}$$

For simplicity, let $x' := k(x,z)$. By the Bellman equation (5.7),

$$v(x+s,z) \geq u(a(x,z,s)) + \delta \mathbb{E}_{z'} v(x' + \rho s, z').$$

Subtracting $v(x,z) = u(c(x,z)) + \delta \mathbb{E}_{z'} v(x',z')$, we have

$$v(x+s,z) - v(x,z) \tag{B.4}$$
$$\geq u(a(x,z,s)) - u(c(x,z)) + \delta \mathbb{E}_{z'}[v(x' + \rho s, z') - v(x',z')] \tag{B.5}$$
$$\geq u(a(x,z,s)) - u(c(x,z)), \tag{B.6}$$

where the second inequality holds since $v(\cdot, z')$ is increasing.

Since $u'(0) = \infty$, we have $c(x,z) > 0$, so that $x' + s \in \Gamma(x,z)$ if s is sufficiently small. For such an $s > 0$, by the Bellman equation (5.7),

$$u(c(x,z) - s) + \delta \mathbb{E}_{z'} v(x' + s, z') \leq u(c(x,z)) + \delta \mathbb{E}_{z'} v(x',z').$$

Rearranging, we have

$$\begin{aligned} u(c(x,z)) - u(c(x,z) - s) &\geq \delta \mathbb{E}_{z'}[v(x' + s, z') - v(x',z')] \\ &\geq \delta \mathbb{E}_{z'}[u(a(x',z',s)) - u(c(x',z'))], \end{aligned}$$

where the second inequality uses (B.4)–(B.6) (which hold for any $x, z, s > 0$ with $z \in \mathcal{Z}$). Dividing through by s, recalling (B.1)–(B.3), and letting $s \downarrow 0$ yields (5.8).

B.2 Proof that Q is Order Constricting

We verify our claim that the function $V : \mathbb{R}_{++} \to \mathbb{R}_+$ given by (5.9) satisfies the drift condition (3.4) and has order bounded sublevel sets. To this end, let

$x, z > 0$ with $z \in \mathcal{Z}$.[39] To simplify notation, let $c := c(x,z), x' := k(x,z)$, and $c' := c(x', z')$. By (5.2), (5.8), and the Cauchy-Schwarz inequality,[40] we have

$$\infty > u'(c)^{1/2}[\mathbb{E}_{z'}\{\delta z' f'(x')\}^{-1}]^{1/2} \tag{B.7}$$
$$\geq [\delta \mathbb{E}_{z'} u'(c') z' f'(x')]^{1/2} [\mathbb{E}_{z'}\{\delta z' f'(x')\}^{-1}]^{1/2} \tag{B.8}$$
$$\geq \mathbb{E}_{z'}[\delta u'(c') z' f'(x') \{\delta z' f'(x')\}^{-1}]^{1/2} \tag{B.9}$$
$$= \mathbb{E}_{z'} u'(c')^{1/2}. \tag{B.10}$$

Raising to the power of $1/2$ and integrating over z with respect to ϕ, we obtain

$$\mathbb{E}_z\{\mathbb{E}_{z'} u'(c')^{1/2}\}^{1/2} \leq \mathbb{E}_z\{u'(c)^{1/2}[\mathbb{E}_{z'}\{\delta z' f'(x')\}^{-1}]^{1/2}\}^{1/2}.$$

Applying again the Cauchy-Schwarz inequality to the right-hand side, and then using Jensen's inequality, we have

$$\mathbb{E}_z\{\mathbb{E}_{z'} u'(c')^{1/2}\}^{1/2} \tag{B.11}$$
$$\leq \{\mathbb{E}_z u'(c)^{1/2}\}^{1/2}\{\mathbb{E}_z[\mathbb{E}_{z'}\{\delta z' f'(x')\}^{-1}]^{1/2}\}^{1/2} \tag{B.12}$$
$$\leq \{\mathbb{E}_z u'(c)^{1/2}\}^{1/2}\{[\mathbb{E}_z \mathbb{E}_{z'}\{\delta z' f'(x')\}^{-1}]^{1/2}\}^{1/2}. \tag{B.13}$$

For $x > 0$, define

$$w_1(x) := \{\mathbb{E}_z u'(c(x,z))^{1/2}\}^{1/2},$$

which is finite for any $x > 0$ by (B.7)–(B.10).[41] Fix $x > 0$ for the moment. Since

$$w_1(k(x,z)) = \{\mathbb{E}_{z'} u'(c(k(x,z), z'))^{1/2}\}^{1/2} = \{\mathbb{E}_{z'} u'(c')^{1/2}\}^{1/2}$$

for any $z > 0$ with $z \in \mathcal{Z}$, it follows from (B.11)–(B.13) that

$$E_z w_1(k(x,z)) \leq w_1(x)[\mathbb{E}_z \mathbb{E}_{z'}\{\delta z' f'(k(x,z))\}^{-1}]^{1/4}. \tag{B.14}$$

[39] The following argument is similar to the proof of Kamihigashi (2007, Theorem 3.1), but is more involved in that there are two state variables in the Bellman equation (5.7) while there is only one in the setting of Section 5.1.

[40] A version of the inequality that we use here is the following: for any nonnegative random variables X and Y, we have $\mathbb{E}\{XY\}^{1/2} \leq \{\mathbb{E}X\}^{1/2}\{\mathbb{E}Y\}^{1/2}$ (see Folland, 1999, p. 182).

[41] To see this, note that, for any $x, z > 0$ with $z \in \mathcal{Z}$, we have $\mathbb{E}_{z'} u'(c(k(x,z), z'))^{1/2} < \infty$ by (B.7)–(B.10). Let $\tilde{x} > 0$. Since $k(x,z) \to 0$ as $x \to 0$, we have $k(x,z) < \tilde{x}$ if x is sufficiently small. For such an $x > 0$, we have $\mathbb{E}_{z'} u'(c(\tilde{x}, z'))^{1/2} \leq \mathbb{E}_{z'} u'(c(k(x,z), z'))^{1/2} < \infty$ since $c(\cdot, z')$ is increasing. Because $\tilde{x}$ was arbitrary, it follows that $w_1(\tilde{x}) < \infty$ for any $\tilde{x} > 0$.

To estimate the right-hand side, note that

$$\mathbb{E}_z \mathbb{E}_{z'} \{\delta z' f'(k(x,z))\}^{-1} = \mathbb{E}_{z'} \frac{1}{z'} \mathbb{E}_z \frac{1}{\delta f'(k(x,z))}. \tag{B.15}$$

Since $k(x,z)$ is increasing in x, the right-hand side of (B.15) is increasing in x, and strictly less than one for $x > 0$ sufficiently small by (5.2) and the monotone convergence theorem. Thus there exists an $\bar{x} > 0$ such that

$$\alpha_1 := [\mathbb{E}_z \mathbb{E}_{z'} \{\delta z' f'(k(\bar{x},z))\}^{-1}]^{1/4} < 1.$$

Since $f'(k(x,z))$ is decreasing in x, this together with (B.14) implies that

$$\mathbb{E}_z w_1(k(x,z)) \le \alpha_1 w_1(x) \qquad \forall x \in (0, \bar{x}]. \tag{B.16}$$

Since $c(x,z)$ is increasing in x, $w_1(\cdot)$ is decreasing. Hence $w_1(x) \le \beta_1 := w_1(\bar{x})$ for all $x \ge \bar{x}$. Combining this inequality with (B.16), we obtain

$$\mathbb{E}_z w_1(k(x,z)) \le \alpha_1 w_1(x) + \beta_1 \qquad \forall x > 0.$$

Define $w_2(x) := x$ for $x > 0$. By concavity of f and the assumption that $f'(\infty) = 0$, there exists an $\alpha_2 \in (0,1)$ and a $\beta_2 > 0$ such that $\mathbb{E}_z z f(x) + \rho x \le \alpha_2 x + \beta_2$ for all $x > 0$. Thus, for any $x > 0$,

$$\begin{aligned} \mathbb{E}_z w_2(k(x,z)) = \mathbb{E}_z k(x,z) &\le \mathbb{E}_z z f(x) + \rho x \\ &\le \alpha_2 x + \beta_2 = \alpha_2 w_2(x) + \beta_2. \end{aligned}$$

Now define $V(x) := w_1(x) + w_2(x)$ for $x > 0$. This is the function given by (5.9). The function V satisfies the drift condition (3.4) with $\alpha := \max\{\alpha_1, \alpha_2\}$ and $\beta := \beta_1 + \beta_2$, and has order bounded sublevel sets because $V(x) \to \infty$ as $x \downarrow 0$ or $x \uparrow \infty$. This completes the proof.

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[42] This manuscript has never been considered for publication anywhere, and is no longer intended for publication.

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