

Discussion Paper Series

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DP2010-04

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with Additively Separable Payoffs**

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Revised June 8, 2010



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Global Dynamics in Repeated Games with Additively Separable Payoffs*

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June 8, 2010

Abstract

This paper studies the global dynamics of a class of infinitely repeated two-player games in which the action space of each player is an interval, and the one-shot payoff of each player is additively separable in actions. We define an immediately reactive equilibrium (IRE) as a pure-strategy subgame perfect equilibrium such that each player's action is a stationary function of the opponent's last action. We completely characterize IREs and their dynamics in terms of certain indifference curves. Our results are used to show that in a prisoners' dilemma game with mixed strategies, gradual cooperation occurs when the players are sufficiently patient, and that in a certain duopoly game, kinked demand curves emerge naturally.

Keywords: Immediately reactive equilibria; additively separable payoffs; kinked demand; gradual cooperation; prisoners' dilemma

*(Forthcoming in the *Review of Economic Dynamics*) This paper has benefited considerably from the comments, suggestions, and criticisms made by an associate editor of the *Review of Economic Dynamics* and two anonymous referees. Earlier versions of this paper were presented in seminars at Kyoto University, University of Venice, University of Paris 1, and GREQAM. We would like to thank Atsushi Kajii, Tomoyuki Nakajima, Tadashi Sekiguchi, Olivier Tercieux, Julio Davila, Sergio Currarini, and Piero Gottardi for helpful comments and suggestions. We gratefully acknowledge financial support from the Japan Society for the Promotion of Science.

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1 Introduction

In infinitely repeated games with a prisoners' dilemma-like stage game, Nash reversion trigger strategies (Friedman, 1971) are often used to show that cooperation (or collusion) can be sustained by the threat to revert to a non-cooperative Nash equilibrium. In such equilibria, each player continues to cooperate as long as all the other players do so, but will choose to behave selfishly once anyone defects. While Nash reversion equilibria are simple and intuitive, they seem to have two disadvantages. First, small deviations are punished as harshly as large deviations. Second, there are no nontrivial transition dynamics between cooperative and noncooperative states.

This paper studies the global dynamics of a class of infinitely repeated two-player games in which the action space of each player is an interval. We follow Friedman (1968) in focusing on strategies such that each player's action is a stationary function of the opponent's last action. We call such strategies *immediately reactive*, and say that a subgame perfect equilibrium is an *immediately reactive equilibrium (IRE)* if each player chooses an immediately reactive strategy.¹ Unlike Nash reversion equilibria, IREs can be continuous;² their global dynamics are typically nontrivial and can be characterized graphically.

In our framework, interesting dynamics arise naturally. For example, cooperation is achieved gradually in a repeated prisoners' dilemma with mixed strategies if the players are sufficiently patient.³ In a duopoly game, kinked demand curves emerge naturally.⁴ In a collusive steady state, each firm cuts its price if the other firm does so, but neither firm responds if the other firm

¹Friedman (1968) called IREs "reaction function equilibria." We avoid his terminology since it has been used in a broader sense in the subsequent literature. The concept of IRE is related to a few other ones (Friedman and Samuelson, 1994a; Kalai, Samet, and Stanford, 1988; Maskin and Tirole, 1988a, 1988b); see Sections 2.3 and 2.4.

²Existence of nontrivial continuous equilibria has been studied by Samuelson (1987), Friedman and Samuelson (1990, 1994a, 1994b), and Langlois and Sachs (1993).

³Gradual cooperation is known to arise in certain partnership games (e.g., Kranton, 1996; Watson, 1999, 2002; Furusawa and Kawakami, 2006). Our example shows a simplest mechanism of gradual cooperation.

⁴Although there are game-theoretic models of kinked demand in the literature (e.g., Maskin and Tirole, 1988b; Bhaskar, Machin, and Reid, 1991; Radner, 2003; Sen, 2004), they typically require rather specific assumptions. Though our example also requires specific assumptions, it allows one to derive and visualize kinked demand curves in an extremely simple manner.

raises its price. In “inefficient” IREs, the collusive steady state is unstable: a small price cut by either firm triggers price war, eventually leading to a noncollusive steady state. In “efficient” IREs, however, the collusive steady state is stable: a price cut by either firm is matched by a smaller price cut, and the steady state is restored in the long run. All of these dynamic phenomena are properties of *IREs in regular form*, which we define as IREs that are continuous and punish deviations in a minimal way.

To obtain such sharp results, we assume that the one-shot payoff of each player is additively separable in actions.⁵ In addition we assume that each player’s one-shot payoff is continuous, monotone in the opponent’s action, and monotone or unimodal in his own action. These assumptions are satisfied in various games, including the models mentioned above.

As one can easily see, additive separability trivializes the analysis of one-shot games by ruling out strategic interactions. However, it does not trivialize the analysis of repeated games, where various interesting strategic interactions arise, as mentioned above. Those interactions are purely dynamic in nature since they are totally absent in one-shot games. In other words, our framework enables one to concentrate on purely dynamic phenomena.

Given a stage game satisfying the assumptions mentioned above, we show that the set of IREs in the simultaneous move game is identical to that in the alternating move game.⁶ In both games, we completely characterize IREs in terms of indifference curves associated with what we call *effective payoffs*. The effective payoff of a player is the part of his discounted sum of payoffs that is directly affected by his current action.⁷ We show that in any IRE, any equilibrium path stays on the associated indifference curves except for the initial period. By this result, equilibrium dynamics are always characterized by two indifference curves.

Our main results are as follows: First, a pair of indifference curves can be supported by an IRE if and only if it satisfies two certain graphical conditions.

⁵This assumption is also necessary since, as shown by Stanford (1986) and Robson (1986), the only possible IRE is a trivial one in certain duopoly games with additively non-separable payoffs.

⁶Section 2.6 discusses the relationship of this result to Lagunoff and Matsui’s (1997) anti-folk theorem for alternating move games of pure coordination. See Haller and Lagunoff (2000) and Yoon (2001) for further results on alternating move games.

⁷Effective payoffs are similar to what Kamihigashi and Roy (2006) call partial gains in an optimal growth model with linear utility. Equations (2.13)–(2.15) in this paper are similar to (3.7)–(3.9) in Kamihigashi and Roy (2006), but essentially this is the only similarity in analysis between the two papers.

Second, under these conditions, there is an IRE in regular form (which is continuous) supporting the given pair of indifference curves.

All our results depend critically on the simple observation that under additive separability, each player’s dynamic maximization problem given the opponent’s strategy reduces to a static problem. Although we fully exploit this special feature, it is only our point of departure. Our main results, which characterize the entire set of IREs in terms of effective payoffs, still require rather complex reasoning, as is evident in the proofs.

The rest of the paper is organized as follows. Section 2 describes the one-shot game and our assumptions, presents some examples, and introduces the simultaneous and the alternating move games. Section 3 discusses some immediate implications of the simple observation mentioned above. Section 4 examines the dynamics induced by IREs. Section 5 characterizes the entire set of IREs in terms of effective payoffs. Section 6 applies our results to a prisoner’s dilemma and a duopoly game. Section 7 concludes the paper.

2 The Games

2.1 The One-Shot Game

Before introducing repeated games, we describe the underlying one-shot game. There are two players indexed by $i = 1, 2$. Each player’s action space is given by $[0, 1]$. This is merely a normalization, and each player’s action space may be different, and may be any interval. Throughout we follow the following conventions unless otherwise indicated: (a) i and j always belong to $\{1, 2\}$; (b) whenever i (or j) is given, j (or i) denotes the other index; (c) “ $\forall i$ ” means “ $\forall i \in \{1, 2\}$ ”, “ $\forall s_i$ ” means “ $\forall s_i \in [0, 1]$,” and these conventions also apply to other similar expressions.

Let $\pi_i : [0, 1] \times [0, 1] \rightarrow [-\infty, \infty)$ denote player i ’s payoff. The following assumptions are maintained throughout.

Assumption 2.1. For $i = 1, 2$, there exist $u_i : [0, 1] \rightarrow [-\infty, \infty)$ and $v_i : [0, 1] \rightarrow \mathbb{R}$ such that

$$\forall s_i, \forall s_j, \quad \pi_i(s_i, s_j) = u_i(s_i) + v_i(s_j). \quad (2.1)$$

Assumption 2.2. $v_1, v_2 : [0, 1] \rightarrow \mathbb{R}$ are continuous and bounded. Either both are strictly increasing or both are strictly decreasing. Without loss of

generality we assume that both are strictly increasing, and we extend each v_i to a strictly increasing continuous function from \mathbb{R} to \mathbb{R} , denoted v_i again, such that $\lim_{s \downarrow -\infty} v_i(s) = -\infty$ and $\lim_{s \uparrow \infty} v_i(s) = \infty$.

Assumption 2.3. For $i = 1, 2$, u_i is continuous and bounded above.⁸ There exists $\hat{s}_i \in [0, 1]$ such that u_i is strictly increasing on $[0, \hat{s}_i]$ provided $\hat{s}_i > 0$, and strictly decreasing on $[\hat{s}_i, 1]$ provided $\hat{s}_i < 1$.

Assumption 2.1 is crucial to our analysis. Assumption 2.3 implies that \hat{s}_i is player i 's strictly dominant strategy and that (\hat{s}_1, \hat{s}_2) is the unique static Nash equilibrium.

2.2 Examples

Although the above assumptions may appear rather restrictive, they are satisfied in various games. We provide specific examples below.

2.2.1 Tariff War

Consider a game between two countries, 1 and 2, with two goods, 1 and 2. Country i exports good i , imports good j , and imposes a tariff rate s_i on imports of good j . Let $v_i(s_j)$ denote the sum of country i 's producer surplus and consumer surplus for good i . Under standard assumptions, v_i is a strictly decreasing function of s_j . Let $u_i(s_i)$ denote the sum of country i 's tariff revenue, consumer surplus, and producer surplus for good j . Optimal tariff theory suggests that u_i is increasing where s_i is small, and decreasing where s_i is large. For simplicity we assume that u_i has a single peak at \hat{s}_i , and that u_i is strictly increasing for $s_i \leq \hat{s}_i$ and strictly decreasing for $s_i \geq \hat{s}_i$. Country i seeks to maximize its welfare $u_i(s_i) + v_i(s_j)$. This game satisfies our assumptions, and is analyzed in detail in Furusawa and Kamihigashi (2006).⁹

2.2.2 Duopoly

Consider a game played by two firms. Each firm i produces a differentiated product with a constant marginal cost c_i and with no fixed cost. Firm i

⁸We follow the convention that if $u_i(r) = -\infty$ for $r = 0$ or 1 , then continuity of u_i at r means $\lim_{s \rightarrow r} u_i(s) = -\infty$.

⁹A preliminary version of Furusawa and Kamihigashi (2006) contained some of the arguments in this paper, which now appear exclusively in this paper.

faces a demand function $D_i(p_i, p_j)$ that depends on the prices p_i and p_j chosen by the two firms. Firm i 's profit is $D_i(p_i, p_j)(p_i - c_i)$. Suppose D_i is multiplicatively separable: $D_i(p_i, p_j) = d_i^i(p_i)d_i^j(p_j)$ for some functions d_i^i and d_i^j . Then the profit maximization problem of firm i is equivalent to maximizing $u_i(p_i) + v_i(p_j)$, where

$$u_i(p_i) = \ln d_i^i(p_i) + \ln(p_i - c_i), \quad v_i(p_j) = \ln d_i^j(p_j). \quad (2.2)$$

This transformation is innocuous in the one-shot game, and our assumptions are satisfied under reasonable assumptions on d_i^i and d_i^j . In repeated games the above transformation may be justified by assuming that the owners of the firms are “risk averse,” or prefer stable to unstable profit streams.

2.2.3 Prisoner's Dilemma

Though the action spaces are assumed to be intervals in this paper, our framework applies to 2×2 games with mixed strategies as well. A case in point is the prisoner's dilemma game in Figure 1 (with $a, c > 0$), which is a parametrized version of the game discussed by Fudenberg and Tirole (1991, p. 10, p. 111). For $i = 1, 2$, let s_i be player i 's probability of choosing action C. Let $\pi_i(s_i, s_j)$ be player i 's expected payoff:

$$\pi_i(s_i, s_j) = s_i s_j c + s_i(1 - s_j)(-a) + (1 - s_i)s_j(c + a) \quad (2.3)$$

$$= -as_i + (c + a)s_j. \quad (2.4)$$

Our assumptions are clearly satisfied with $\hat{s}_1 = \hat{s}_2 = 0$.¹⁰ In fact our assumptions are satisfied in more general 2×2 games as long as the coefficient of $s_i s_j$ is zero in $\pi_i(s_i, s_j)$.

2.3 The Repeated Game with Simultaneous Moves

Consider the infinitely repeated game in which the stage game is given by the one-shot game described in Section 2.1. For $i = 1, 2$, let $\delta_i \in (0, 1)$ be player i 's discount factor. We restrict ourselves to pure-strategy subgame perfect equilibria in which player i 's action in period t , $s_{i,t}$, is a stationary function f_i of player j 's action in period $t - 1$, $s_{j,t-1}$; i.e., $s_{i,t} = f_i(s_{j,t-1})$. We call such strategies *immediately reactive*.

¹⁰Furusawa and Kawakami (2008) use a payoff function similar to (2.3) to analyze perfect Bayesian equilibria in a model with stochastic outside options.

		Player 2	
		C	D
Player 1	C	c, c	$-a, c + a$
	D	$c + a, -a$	$0, 0$

Figure 1: Prisoner’s dilemma

Friedman (1968) called such strategies “reaction functions.” Immediately reactive strategies are a special case of “single-period-recall strategies” (Friedman and Samuelson, 1994a) and “reactive strategies” (Kalai et al., 1988). Single-period-recall strategies depend only on both players’ last actions, and reactive strategies depend only on the opponent’s past actions. We focus on stationary strategies that depend only on the opponent’s last action.

Let F be the set of all functions from $[0, 1]$ to $[0, 1]$. Taking player j ’s strategy $f_j \in F$ as given, player i faces the following problem:

$$\max_{\{s_{i,t}\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \delta_i^{t-1} [u_i(s_{i,t}) + v_i(s_{j,t})] \tag{2.5}$$

$$\text{s.t. } \forall t \in \mathbb{N}, \quad s_{j,t} = f_j(s_{i,t-1}), \tag{2.6}$$

$$\forall t \in \mathbb{N}, \quad s_{i,t} \in [0, 1]. \tag{2.7}$$

We say that $f_i \in F$ is a *best response to* f_j if for any $(s_{i,0}, s_{j,0}) \in [0, 1]^2$, the above maximization problem has a solution $\{s_{i,t}\}_{t=1}^{\infty}$ such that $s_{i,t} = f_i(s_{j,t-1})$ for all $t \in \mathbb{N}$. We call a strategy profile $(f_1, f_2) \in F^2$ an *immediately reactive equilibrium (IRE)* if f_1 is a best response to f_2 , and vice versa. Note that f_1 and f_2 are not required to be continuous or even measurable, but the maximization problem (2.5)–(2.7) is required to be well defined given f_j .¹¹

¹¹Our results are unaffected even if f_1 and f_2 are required to be continuous or upper semi-continuous. The same remark applies to the alternating move game.

2.4 The Repeated Game with Alternating Moves

Now consider the case of alternating moves. Player 1 updates his action in odd periods, while player 2 updates his action in even periods.¹² Define

$$T_1 = \{1, 3, 5, \dots\}, \quad T_2 = \{2, 4, 6, \dots\}. \quad (2.8)$$

As in the simultaneous move case, we restrict ourselves to subgame perfect equilibria in which each player chooses an immediately reactive strategy, i.e., in each period $t \in T_i$, player i chooses an action $s_{i,t}$ according to a stationary function f_i of player j 's last (or equivalently current) action $s_{j,t-1}$.

Given player j 's strategy $f_j \in F$, player i faces the following problem:

$$\max_{\{s_{i,t}\}_{t=i}^{\infty}} \sum_{t=i}^{\infty} \delta_i^{t-i} [u_i(s_{i,t}) + v_i(s_{j,t})] \quad (2.9)$$

$$\text{s.t. } \forall t \in T_j, \quad s_{j,t} = f_j(s_{i,t-1}), \quad s_{i,t} = s_{i,t-1}, \quad (2.10)$$

$$\forall t \in T_i, \quad s_{i,t} \in [0, 1], \quad s_{j,t} = s_{j,t-1}. \quad (2.11)$$

We say that $f_i \in F$ is a *best response to f_j* if for any $s_{j,i-1} \in [0, 1]$,¹³ the above maximization problem has a solution $\{s_{i,t}\}_{t=i}^{\infty}$ such that $s_{i,t} = f_i(s_{j,t-1})$ for all $t \in T_i$. We call a strategy profile $(f_1, f_2) \in F^2$ an *immediately reactive equilibrium (IRE)* if f_1 is a best response to f_2 , and vice versa. This equilibrium concept is consistent with one definition of ‘‘Markov perfect equilibrium’’ (Maskin and Tirole, 1988b, Section 2), but distinct from another (Maskin and Tirole, 2001) due to additive separability of payoffs.

2.5 Effective Payoffs

We now introduce a function that plays a central role in our analysis. For $i = 1, 2$, define $w_i : [0, 1]^2 \rightarrow [-\infty, \infty)$ by

$$w_i(s_i, s_j) = u_i(s_i) + \delta_i v_i(s_j). \quad (2.12)$$

We call this function player i 's *effective payoff* since in both repeated games, player i in effect seeks to maximize the discounted sum of effective payoffs.

¹²In alternating move games, it is often assumed that the players play simultaneously in the initial period and take turns afterwards. Such an assumption does not affect our analysis, which is concerned only with stationary subgame perfect equilibria.

¹³Notice that for $i = 1, 2$, the first period in which player i plays is period i .

Indeed, in both games, player i 's discounted sum of payoffs from period 1 onward is written as

$$\sum_{t=1}^{\infty} \delta_i^{t-1} [v_i(s_{j,t}) + u_i(s_{i,t})] \quad (2.13)$$

$$= v_i(s_{j,1}) + \sum_{t=1}^{\infty} \delta_i^{t-1} [u_i(s_{i,t}) + \delta_i v_i(s_{j,t+1})] \quad (2.14)$$

$$= v_i(s_{j,1}) + \sum_{t=1}^{\infty} \delta_i^{t-1} w_i(s_{i,t}, s_{j,t+1}). \quad (2.15)$$

In both games, player i has no influence on $s_{j,1}$, so player i 's problem is equivalent to maximizing the discounted sum of effective payoffs. This implies that each player's best responses are characterized by a static maximization problem:

Lemma 2.1. *In both the simultaneous and the alternating move games, for $i = 1, 2$, $f_i \in F$ is a best response to $f_j \in F$ iff*

$$\forall s_j, \quad f_i(s_j) \in \operatorname{argmax}_{s_i \in [0,1]} w_i(s_i, f_j(s_i)). \quad (2.16)$$

In other words, $(f_1, f_2) \in F^2$ is an IRE iff (2.16) holds for $i = 1, 2$.

Proof. See Appendix A. □

Lemma 2.1 implies that the simultaneous and the alternating move games are equivalent as far as IREs are concerned. This allows us to discuss IREs without specifying which repeated game is considered. The differences in dynamics between the two games are discussed in Section 4.

Given an IRE (f_1, f_2) , we say that $(s_1, s_2) \in [0, 1]^2$ is a *steady state* if $s_1 = f_1(s_2)$ and $s_2 = f_2(s_1)$. Needless to say, if the game starts from a steady state (s_1, s_2) , each player i keeps choosing s_i forever according to f_i . Note that any intersection of the graphs of f_1 and f_2 (on the (s_1, s_2) plane) is a steady state.

2.6 Discussions

The equivalence between simultaneous and alternating move games discussed above would appear in sharp contrast to the anti-folk theorem of Lagunoff

and Matsui (1997) for alternating move games of pure coordination. They show that there is a considerable difference between the simultaneous and the alternating move games in the case of pure coordination. If $u_i(s_i) = v_j(s_i)$ and $v_i(s_j) = u_j(s_j)$ for all s_i, s_j and i , then the one-shot game described in Section 2.1 becomes a pure coordination game. Lemma 2.1 of course applies to this case (which is consistent with our assumptions), but does not contradict Lagunoff and Matsui’s result. This is because their result deals with all subgame perfect equilibria, while Lemma 2.1 deals only with IREs.

One may also notice that the property of IREs given by (2.16) is similar to the definition of “conjectural variations equilibrium” (e.g., Figuères et al., 2004, p. 14). The main difference is that a conjectural variations equilibrium consists of an action profile supported by “variational conjectures” as to how each player reacts to an infinitesimal deviation from the action profile by the opponent, while an IRE consists only of two functions that represent how each player optimally reacts to any action by the opponent.

To be more specific, let $(f_1, f_2) \in F^2$ be an IRE, and suppose that it has a steady state $(s_1, s_2) \in [0, 1]^2$. Consider the one-shot game in which player i ’s payoff is given by $w_i(s_i, s_j)$. If both f_1 and f_2 are differentiable and $(s_1, s_2) \in (0, 1)^2$, then (2.16) implies that (s_1, s_2) is a conjectural variations equilibrium with variational conjectures (f'_1, f'_2) for this one-shot game. As we show in Section 5, however, IREs are typically not everywhere differentiable even when u_i and v_i are many times differentiable. Thus not every “interior” steady state can be supported as a conjectural variations equilibrium with variational conjectures (f'_1, f'_2) .¹⁴

On the other hand, any steady state of an IRE can be supported as a more general “conjectural equilibrium” (Figuères et al., 2004, p. 30), which does not require differentiability. In fact (2.16) is very similar to the definition of “consistent conjectural equilibrium.” The main difference is that a consistent conjectural equilibrium is an action profile supported by “consistent conjectures,” while an IRE consists only of two optimal reaction functions.¹⁵

¹⁴This does not mean that the set of steady states of IREs is larger than that of conjectural variations equilibria. On the contrary, under mild regularity conditions, any $(s_1, s_2) \in (0, 1)^2$ is a conjectural variations equilibrium for the one-shot game considered here (Figuères et al., 2004, p. 14). Since Theorem 5.1 indicates that not every $(s_1, s_2) \in (0, 1)^2$ can be supported as a steady state of an IRE, the set of steady states of IREs in $(0, 1)^2$ is typically smaller than that of conjectural variations equilibria.

¹⁵See Sabourian (1992) and Figuères et al. (2004) for detailed discussions on the relation between conjectural variations equilibria and repeated games.

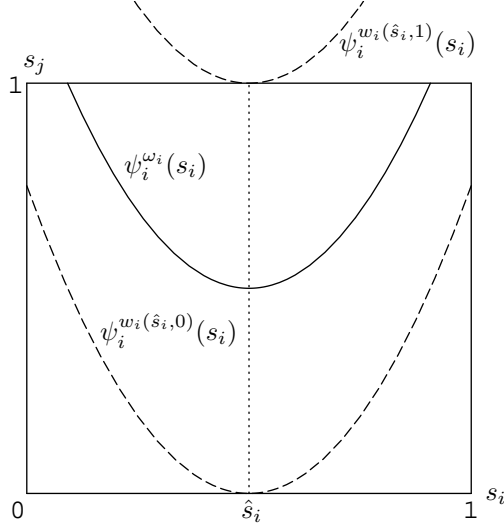


Figure 2: Indifference curve $\psi_i^{\omega_i}$ with $\omega_i \in [w_i(\hat{s}_i, 0), w_i(\hat{s}_i, 1)]$

3 A Preliminary Characterization of IREs

Lemma 2.1 suggests that the indifference curves associated with effective payoffs are closely related with best responses. Since each v_i is strictly increasing and the inverse of v_i is defined over \mathbb{R} by Assumption 2.2, each indifference curve $w_i(s_i, s_j) = \omega_i$ can be expressed as the graph of a function from s_i to s_j . We denote this function by $\psi_i^{\omega_i}$:

$$\forall s_i, \quad \omega_i = w_i(s_i, \psi_i^{\omega_i}(s_i)) = u_i(s_i) + \delta_i v_i(\psi_i^{\omega_i}(s_i)). \quad (3.1)$$

Equivalently,

$$\forall s_i, \quad \psi_i^{\omega_i}(s_i) = v_i^{-1} \left(\frac{\omega_i - u_i(s_i)}{\delta_i} \right). \quad (3.2)$$

Since v_i is strictly increasing, a higher indifference curve is associated with a higher effective payoff. See Figure 2.

Given player j 's strategy $f_j \in F$, let $\omega_i^*|_{f_j}$ denote player i 's maximum effective payoff:

$$\omega_i^*|_{f_j} = \sup_{s_i \in [0,1]} w_i(s_i, f_j(s_i)). \quad (3.3)$$

Assuming that the supremum is attained at some s_i , the associated maxi-

mization problem can be expressed as

$$\max_{s_i, s_j \in [0,1]} w_i(s_i, s_j) \quad \text{s.t. } s_j = f_j(s_i). \quad (3.4)$$

Let $\psi_i^*|_{f_j}$ be player i 's optimal indifference curve given player j 's strategy f_j :

$$\psi_i^*|_{f_j} = \psi_i^{\omega_i^*|_{f_j}}. \quad (3.5)$$

Note that in (3.4), player i takes $s_j = f_j(s_i)$ as a constraint. Thus $\psi_i^*|_{f_j}$ is the highest indifference curve tangent to the graph of f_j . See Figure 3.

The following result provides a necessary and sufficient condition for a given strategy profile to be an IRE:

Proposition 3.1. *A strategy profile $(f_1, f_2) \in F^2$ is an IRE iff*

$$\forall i, \forall s_j, \quad f_j(f_i(s_j)) = \psi_i^*|_{f_j}(f_i(s_j)). \quad (3.6)$$

Proof. For sufficiency, assume (3.6). Fix i and s_j . Let $s_i = f_i(s_j)$. Then $w_i(s_i, f_j(s_i)) = w_i(s_i, \psi_i^*|_{f_j}(s_i)) = \omega_i^*|_{f_j}$. Thus by Lemma 2.1, f_i is a best response to f_j . It follows that (f_1, f_2) is an IRE. Reversing this argument yields the converse. \square

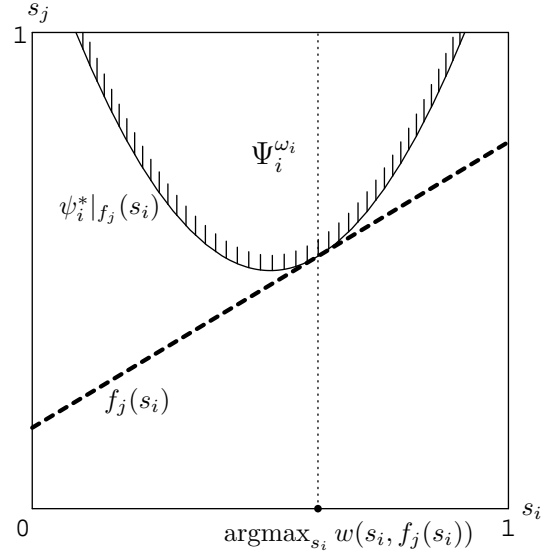
To illustrate Proposition 3.1, let (f_1, f_2) be given by $f_i(s_j) = \hat{s}_i$ for all s_j and i . In other words each player plays his static Nash strategy. See Figure 4. One can easily see that (f_1, f_2) satisfies (3.6), so it is an IRE.

As another example, let (f_1, f_2) be such that $f_i(s_j) = \bar{s}_i$ if $s_j = \bar{s}_j$, and $f_i(s_j) = \hat{s}_i$ otherwise, where \bar{s}_1 and \bar{s}_2 are as in Figure 5. In this case each player ‘‘cooperates’’ as long as the opponent does so, but reverts to the static Nash strategy if the opponent deviates at all. Once again one can easily see that (f_1, f_2) satisfies (3.6), so it is an IRE.

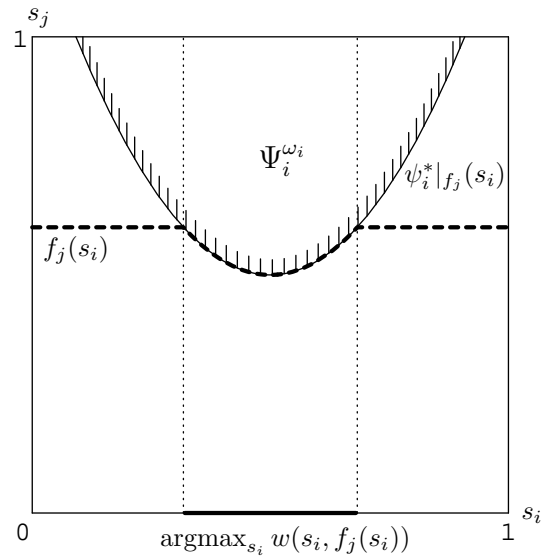
While Proposition 3.1 can be used to construct specific IREs as above, or to check whether a given strategy profile is an IRE, it does not provide the entire picture of how large the set of IREs may be. The main purpose of this paper is to characterize the entire set of IREs. To this end we define an *IRE associated with* $\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$ as an IRE (f_1, f_2) such that

$$\forall i, \quad \omega_i = \omega_i^*|_{f_j}. \quad (3.7)$$

¹⁶From here on we omit the arguments of functions in figures.



(a)



(b)

Figure 3: $\psi_i^* |_{f_j}$ and $\Psi_i^{\omega_i}$ (defined in (5.1))

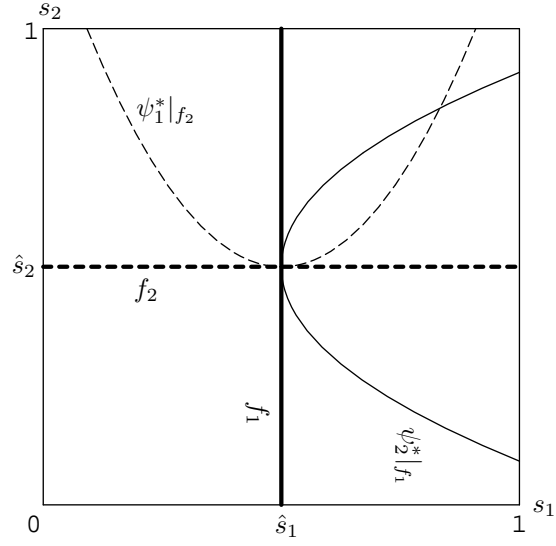


Figure 4: Static Nash equilibrium¹⁶

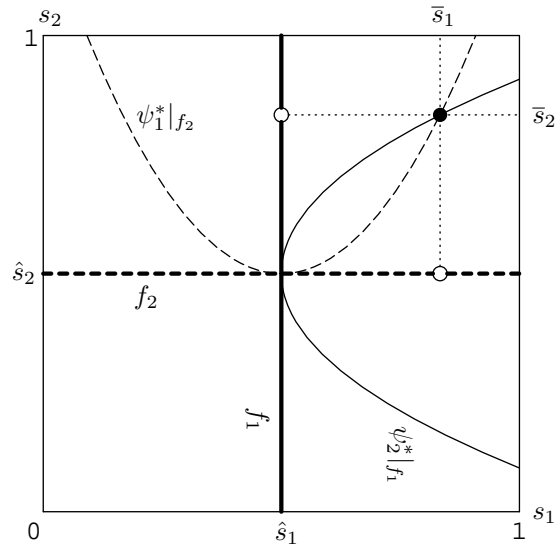


Figure 5: Nash reversion

In other words, in an IRE associated with ω , each player i 's effective payoff is ω_i . Notice that any IRE (f_1, f_2) is associated with $(\omega_1^*|_{f_2}, \omega_2^*|_{f_1})$ by definition. The following result shows one way to construct a nontrivial IRE associated with $\omega \in \mathbb{R}^2$.

Proposition 3.2. *Let $\omega \in \mathbb{R}^2$. Suppose*

$$\forall i, \forall s_i, \quad \psi_i^{\omega_i}(s_i) \in [0, 1]. \quad (3.8)$$

Then $(f_1, f_2) = (\psi_2^{\omega_2}, \psi_1^{\omega_1})$ is an IRE associated with ω .

Proof. Let $(f_1, f_2) = (\psi_2^{\omega_2}, \psi_1^{\omega_1})$. Then for each i ,

$$\forall s_i, \quad w_i(s_i, f_j(s_i)) = w_i(s_i, \psi_i^{\omega_i}(s_i)) = \omega_i, \quad (3.9)$$

so (3.7) trivially holds.¹⁷ It follows that $\psi_i^*|_{f_j} = \psi_i^{\omega_i} = f_j$ for both i . Thus (3.6) holds, and (f_1, f_2) is an IRE by Proposition 3.1. \square

This result can be seen more directly as follows: Notice that if $(f_1, f_2) = (\psi_2^{\omega_2}, \psi_1^{\omega_1})$, each player is entirely indifferent among all possible actions, since whatever player i does, his effective payoff is ω_i by (3.9). This implies that both players are entirely indifferent among all possible strategies; in particular, choosing $f_i = \psi_j^{\omega_j}$ is optimal for each player i , which makes (f_1, f_2) an IRE. See Figure 6 for an example of an IRE satisfying (3.8) and (3.7).

4 Dynamics

Before we turn to our main results, it is useful to have a basic understanding of the dynamics induced by IREs. For this purpose we take an IRE $(f_1, f_2) \in F^2$ as given and study its dynamic properties in this section.

Consider first the alternating move game. Recall that in each period $t \in \mathbb{N}$, player i with $t \in T_i$ updates his action as a function of player j 's last (or current) action. So the “state variable” in each period $t \in T_i$ is player j 's last action $s_{j,t-1}$. Given initial condition $s_{2,0}$, the entire path $\{s_{1,t}, s_{2,t}\}_{t=1}^{\infty}$ (with $s_{2,1} = s_{2,0}$) of the game is uniquely determined by

$$\forall i, \forall t \in T_i, \quad s_{i,t+1} = s_{i,t} = f_i(s_{j,t-1}). \quad (4.1)$$

¹⁷Note that (3.9) implies $\sup_{s_i} w_i(s_i, f_j(s_i)) = \omega_i$.

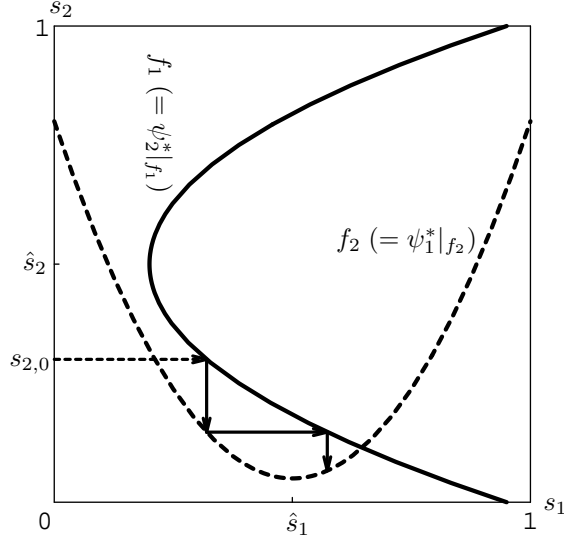


Figure 6: Example of IRE satisfying (3.8) and (3.7).

This means that

$$s_{1,1} = f_1(s_{2,0}), \quad s_{2,2} = f_2(s_{1,1}), \quad s_{1,3} = f_1(s_{2,2}), \quad \dots \quad (4.2)$$

For the alternating move game, we define an *IRE path associated with* (f_1, f_2) as a sequence $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ satisfying (4.1) and $s_{2,1} = s_{2,0}$.¹⁸ See Figure 6 for an example of an IRE path.

Now consider the simultaneous move game. The state variable in each period $t \in \mathbb{N}$ is the pair of both players' last actions $(s_{1,t-1}, s_{2,t-1})$. Given initial condition $(s_{1,0}, s_{2,0})$, the entire path $\{s_{1,t}, s_{2,t}\}_{t=1}^{\infty}$ of the game is uniquely determined by

$$\forall i, \forall t \in \mathbb{N}, \quad s_{i,t} = f_i(s_{j,t-1}). \quad (4.3)$$

For the simultaneous move game, we define an *IRE path associated with* (f_1, f_2) as a sequence $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ satisfying (4.3). Any IRE path can be decoupled into two sequences, one originating from $s_{2,0}$, the other from $s_{1,0}$:

$$s_{1,1} = f_1(s_{2,0}), \quad s_{2,2} = f_2(s_{1,1}), \quad s_{1,3} = f_1(s_{2,2}), \quad \dots, \quad (4.4)$$

$$s_{2,1} = f_2(s_{1,0}), \quad s_{1,2} = f_1(s_{2,1}), \quad s_{2,3} = f_2(s_{1,2}), \quad \dots \quad (4.5)$$

¹⁸Though $s_{1,0}$ is irrelevant for this game, it is included here for notational simplicity.

Obviously, given $s_{2,0}$, the sequences given by (4.2) and (4.4) are identical. The sequence given by (4.5) can be viewed as an IRE path for the alternating move game in which player 2 moves first. Hence an IRE path for the simultaneous move game is equivalent to a pair of IRE paths for the two alternating move games in one of which player 1 moves first and in the other player 2 moves first.

The following result is a simple consequence of Proposition 3.1.

Theorem 4.1. *Any IRE path $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ associated with $(f_1, f_2) \in F^2$ for the simultaneous move game satisfies*

$$\forall t \geq 2, \forall i, \quad s_{i,t} = \psi_j^*|_{f_i}(s_{j,t-1}). \quad (4.6)$$

Furthermore, any IRE path $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ associated with $(f_1, f_2) \in F^2$ for the alternating move game satisfies

$$\forall t \geq 2, \forall i, \quad t \in T_i \Rightarrow s_{i,t} = \psi_j^*|_{f_i}(s_{j,t-1}). \quad (4.7)$$

Proof. Consider the simultaneous move game. Let $\{s_{1,t}, s_{2,t}\}_{t=0}^{\infty}$ be an IRE path associated with (f_1, f_2) . For each i and $t \geq 2$, we have

$$s_{i,t} = f_i(s_{j,t-1}) = f_i(f_j(s_{i,t-2})) = \psi_j^*|_{f_i}(f_j(s_{i,t-2})) = \psi_j^*|_{f_i}(s_{j,t-1}), \quad (4.8)$$

where the third equality uses (3.6) (with i and j interchanged). Now (4.6) follows. The proof for the alternating move game is similar. \square

The above result shows that any IRE path is characterized by the corresponding pair of indifference curves $(\psi_2^*|_{f_1}, \psi_1^*|_{f_2})$ except for the initial period. To better understand this result, consider the alternating move game. The initial period must be excluded in (4.7) because $s_{2,0}$ is an arbitrary initial condition that need not be optimal for player 2 given f_1 , i.e., it need not satisfy $s_{1,1} = \psi_2^*|_{f_1}(s_{2,0})$. Since all subsequent actions must be individually optimal, they must be on the optimal indifference curves. In Figure 6, any IRE path satisfies the equality in (4.7) for all $t \geq 1$. In Figure 4, by contrast, an IRE path (not shown in the figure) violates the equality for $t = 1$ unless $s_{2,0} = \hat{s}_2$, but trivially satisfies it for $t \geq 2$.

Theorem 4.1 also shows that in both games the dynamics of an IRE associated with $\omega \in \mathbb{R}^2$ are essentially characterized by the same dynamical system:

$$\forall t \in T_1, \quad s_{1,t+2} = \psi_2^{\omega_2}(\psi_1^{\omega_1}(s_{1,t})). \quad (4.9)$$

To be precise, the simultaneous move game has another equation, $s_{2,t+2} = \psi_1^{\omega_1}(\psi_2^{\omega_2}(s_{2,t}))$ for $t \in T_1$, but this system is equivalent to (4.9) in terms of dynamics. Hence one can obtain conditions for dynamic properties such as monotonicity and chaos by applying numerous results available on one-dimensional dynamical systems (e.g., Devaney, 1989).¹⁹

5 Main Results

Theorem 4.1 shows that the dynamics of an IRE are characterized by the associated pair of indifference curves. The remaining question then is what pairs of indifference curves are supported by IREs. This section answers this question.

For $i = 1, 2$ and $\omega \in \mathbb{R}^2$, define

$$\Psi_i^{\omega_i} = \{(s_i, s_j) \in [0, 1]^2 \mid w_i(s_i, s_j) \geq \omega_i\} \quad (5.1)$$

$$= \{(s_i, s_j) \in [0, 1]^2 \mid \psi_i^{\omega_i}(s_i) \leq s_j\}. \quad (5.2)$$

The set $\Psi_i^{\omega_i}$ is the collection of all pairs (s_i, s_j) with player i 's effective payoff at least as large as ω_i . On the (s_i, s_j) plane, it is the area on or above the graph $s_j = \psi_i^{\omega_i}(s_i)$; see Figure 3. Provided $\Psi_i^{\omega_i} \cap \Psi_j^{\omega_j} \neq \emptyset$,²⁰ we define

$$\bar{s}_i^\omega = \max\{s_i \in [0, 1] \mid \exists s_j \in [0, 1], (s_i, s_j) \in \Psi_i^{\omega_i} \cap \Psi_j^{\omega_j}\}, \quad (5.3)$$

$$\underline{s}_i^\omega = \min\{s_i \in [0, 1] \mid \psi_i^{\omega_i}(s_i) \leq \bar{s}_j^\omega\}. \quad (5.4)$$

See Figure 7. Note from (5.3) and (5.2) that there is s_i such that $\psi_i^{\omega_i}(s_i) \leq \bar{s}_j^\omega$ and $\psi_j^{\omega_j}(\bar{s}_j^\omega) \leq s_i$; the former inequality assures the existence of \underline{s}_i^ω . In Figure 6, $\bar{s}_1^\omega = \bar{s}_2^\omega = 1$ and $\underline{s}_1^\omega = \underline{s}_2^\omega = 0$.

We are now ready to state our main results.

Theorem 5.1. *Let $\omega \in \mathbb{R}^2$. There exists an IRE associated with ω iff*

$$\Psi_1^{\omega_1} \cap \Psi_2^{\omega_2} \neq \emptyset, \quad (5.5)$$

$$\forall i, \quad \underline{s}_i^\omega \leq \psi_j^{\omega_j}(\hat{s}_j). \quad (5.6)$$

¹⁹See Rand (1978) for an early example of complex dynamics in an ‘‘adaptive’’ dynamic model that has a structure similar to Figure 6. See Rosser (2002) for a recent survey of adaptive duopoly/oligopoly models that generate complex dynamics.

²⁰It is understood that the coordinates of $\Psi_j^{\omega_j}$ (or $\Psi_i^{\omega_i}$) are interchanged so that $\Psi_i^{\omega_i}$ and $\Psi_j^{\omega_j}$ have the same order of coordinates. Similar comments apply to similar expressions below.

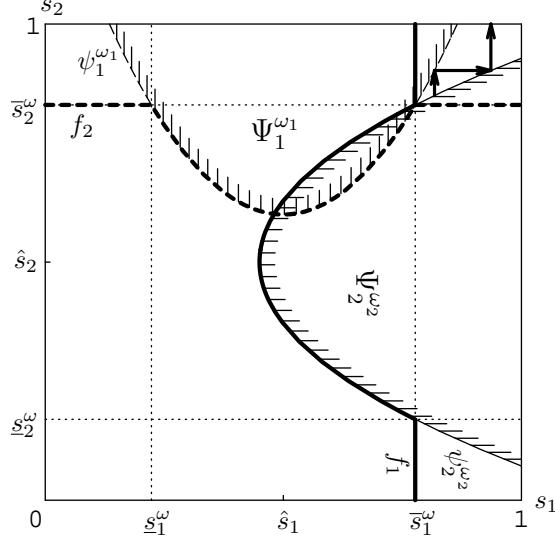


Figure 7: \underline{s}_i , \bar{s}_i , and IRE in regular form

Proof. See Appendix B. □

The sufficiency part of the above result is a consequence of the following:

Theorem 5.2.²¹ *Let $\omega \in \mathbb{R}^2$ satisfy (5.5) and (5.6). Define $(f_1, f_2) \in F^2$ by*

$$\forall s_j, \quad f_i(s_j) = \min\{\psi_j^{\omega_j}(s_j), \bar{s}_i^\omega\}. \quad (5.7)$$

Then (f_1, f_2) is an IRE associated with ω .²² Furthermore, there exists $(s_1, s_2) \in [0, 1]^2$ such that

$$\forall i, \quad (\text{a}) \omega_i = w_i(s_i, s_j), \quad (\text{b}) s_i = f_i(s_j) = \psi_j^{\omega_j}(s_j). \quad (5.8)$$

Proof. See Appendix B.1. □

We say that an IRE satisfying (5.7) is *in regular form*. Note that any IRE in regular form is continuous (i.e, each f_i is continuous). See Figure 7 for an example of an IRE in regular form. By (5.8), any IRE in regular form has a steady state (s_i, s_j) , where the two indifference curves cross each other.

²¹We would like to thank an anonymous referee for his or her suggestion to emphasize this result.

²²By Lemma B.2, $\psi_j^{\omega_j}(\hat{s}_j) \leq \bar{s}_i^\omega$, so $f_i(\hat{s}_j) = \psi_j^{\omega_j}(\hat{s}_j)$.

We call (5.5) the *nonemptiness condition*, and (5.6) the *no-sticking-out condition*. Theorem 5.1 shows that these conditions are necessary and sufficient for an IRE associated with $\omega \in \mathbb{R}^2$ to exist. The nonemptiness condition says that the intersection of the two sets $\Psi_1^{\omega_1}$ and $\Psi_2^{\omega_2}$ must be nonempty. The no-sticking-out condition says that the graph of $\psi_j^{\omega_j}$ must not “stick out” of the straight line $s_i = \underline{s}_i$.

These conditions can be better understood by considering examples where they are violated. In Figure 8(a), the nonemptiness condition (5.5) is violated. In this case, any IRE path for the alternating move game must behave like the path depicted in the figure (except for the initial period) by Theorem 4.1. But since such a path cannot stay on the indifference curves forever, it cannot be an IRE path. In Figure 8(b), the no-sticking-out condition (5.6) is violated for $i = 1$. In this case, if an IRE associated with $\omega \in \mathbb{R}^2$ exists, there is $s_{2,0}$ such that $f_1(s_{2,0}) \leq \psi_2^{\omega_2}(s_{2,0}) < \underline{s}_1^{\omega_1}$.²³ As shown in the figure, the IRE path from such $s_{2,0}$ cannot stay on the indifference curves forever.

We should mention that the IRE in regular form associated with $\omega \in \mathbb{R}^2$ is not the only IRE associated with ω . However, any IRE satisfies one restriction:

Proposition 5.1. *Let (f_1, f_2) be an IRE associated with $\omega \in \mathbb{R}^2$. Then*

$$\forall i, \forall s_j, \quad f_i(s_j) \leq \bar{s}_i^{\omega}. \quad (5.9)$$

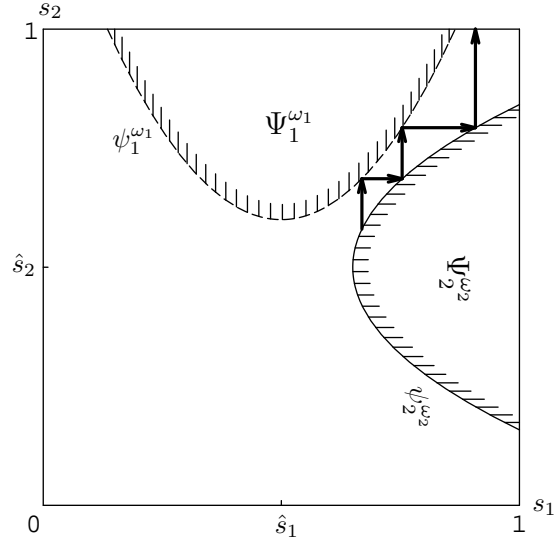
Proof. Immediate from (B.14) and (B.26). \square

To see the idea of this result, suppose the inequality in (5.9) is violated for $i = 1$. Consider the alternating move game. Then for some $s_{2,0}$, we have $s_{1,1} = f_1(s_{2,0}) > \bar{s}_1^{\omega}$. If this path is continued, it behaves like the one depicted in Figure 7 by Theorem 4.1. But such a path cannot be an IRE path since it cannot stay on the indifference curves forever.

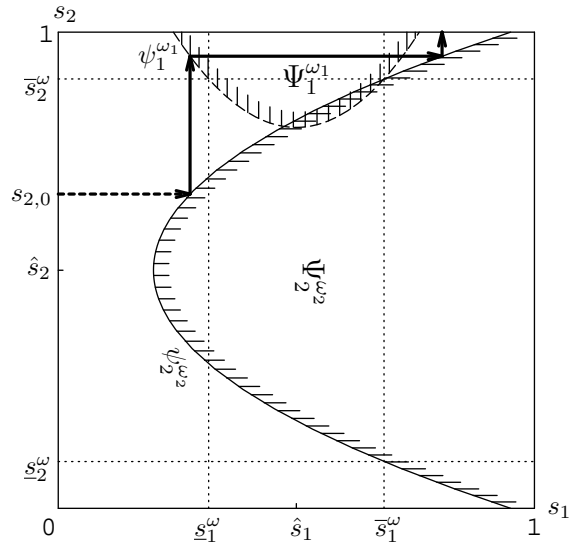
Proposition 5.1 along with (5.7) implies that if (f_1, f_2) is an IRE associated with $\omega \in \mathbb{R}^2$, and if (\bar{f}_1, \bar{f}_2) is the IRE in regular form associated with ω , then $f_i \leq \bar{f}_i$ for both i . In other words, in the IRE (\bar{f}_1, \bar{f}_2) , each player gives the opponent the highest possible effective payoff among all IREs associated with ω in response to any action by the opponent.

In what follows, we say that an IRE (f_1, f_2) is *effectively efficient* if there is no IRE $(\tilde{f}_1, \tilde{f}_2)$ such that $\omega_1^*|_{f_2} \leq \omega_1^*|_{\tilde{f}_2}$ and $\omega_2^*|_{f_1} \leq \omega_2^*|_{\tilde{f}_1}$ with at least

²³The first inequality holds since $\psi_2^{\omega_2} = \psi_2^*|_{f_1}$; recall Figure 3. Figure 8(b) implicitly assumes $f_1(s_{2,0}) = \psi_2^{\omega_2}(s_{2,0})$.



(a)



(b)

Figure 8: Examples with no IRE

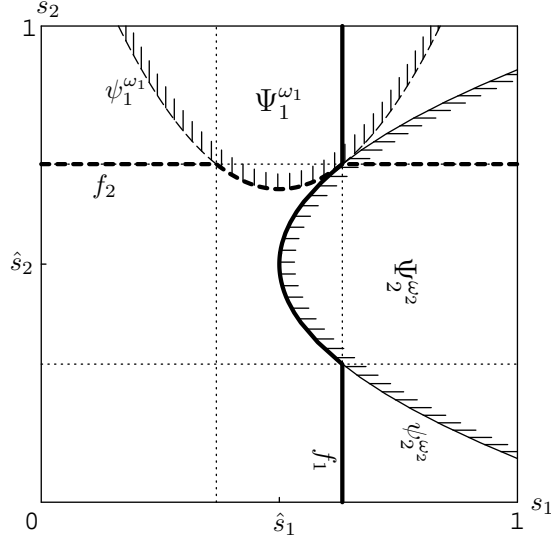


Figure 9: Effectively efficient IRE

one of the inequalities holding strictly. That is, (f_1, f_2) is effectively efficient if it is not Pareto dominated by any other IRE in terms of effective pay-offs. As illustrated in Section 6, effective efficiency has important dynamic implications.

For $i = 1, 2$ and $\omega \in \mathbb{R}^2$, define

$$\tilde{\Psi}_i^{\omega_i} = \{(s_i, s_j) \in [0, 1]^2 \mid w_i(s_i, s_j) > \omega_i\}. \quad (5.10)$$

Since $\psi_i^{\omega_i}$ is strictly increasing in ω_i , it is clear from Theorem 5.1 that an IRE associated with ω is effectively efficient if

$$\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} = \emptyset. \quad (5.11)$$

See Figures 9 and 8(a).

One might conjecture that (5.11) is also necessary for effective efficiency. Unfortunately it is not the case. This is because the no-sticking-out condition (5.6), a necessary condition for an IRE, is not stable under small perturbations to ω . In other words, even when (5.11) does not hold, (5.6) can be violated if ω is slightly changed. For example, when (5.6) holds with equality for $i = 1$, it can be violated after ω_2 is slightly increased, depending on how fast the two sides of the inequality in (5.6) vary with ω_2 .

Even if (5.6) holds with strict inequality, it can be violated after small perturbations to ω , since \underline{s}_i^ω need not be continuous in ω . Figure 10 illustrates this point. There is an IRE in Figure 10(a), but there is no IRE in Figure 10(b) due to violation of (5.6). Note that both \underline{s}_1^ω and \underline{s}_2^ω are discontinuous in this example.²⁴

It turns out, however, that (5.11) is a necessary and sufficient condition for effective efficiency if both players' effective payoffs are higher than the static Nash levels:

Proposition 5.2.²⁵ *Let $\omega \in \mathbb{R}^2$ satisfy*

$$\forall i, \quad \omega_i \geq w_i(\hat{s}_i, \hat{s}_j). \quad (5.12)$$

Suppose an IRE associated with ω exists. Then it is effectively efficient iff (5.11) holds.

Proof. See Appendix C. □

For example, (5.12) trivially holds for any IRE if $\hat{s}_i = 0$ for both i , as in the prisoner's dilemma game in Section 2.2.3.

6 Applications

6.1 Prisoner's Dilemma

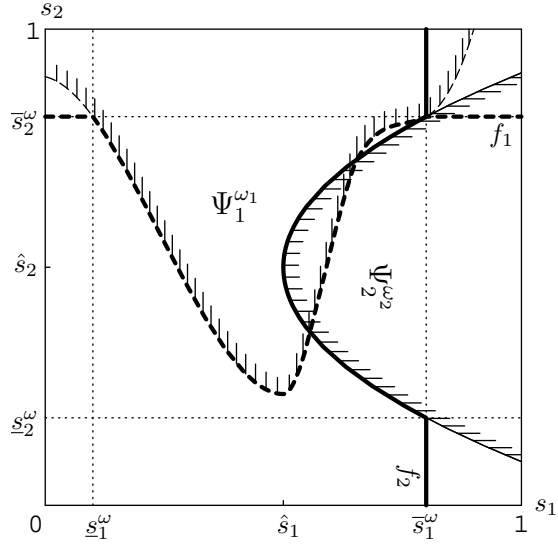
Consider the alternating move game associated with the prisoner's dilemma in Section 2.2.3.²⁶ For simplicity, we assume directly that the one-shot payoff of player i is given by (2.4),²⁷ and that both players have the same discount

²⁴Though Figure 10 only shows that the IRE in (a) is "locally" effectively efficient, it should be clear that one can easily construct a fully specified example of an effectively efficient IRE that violates (5.11).

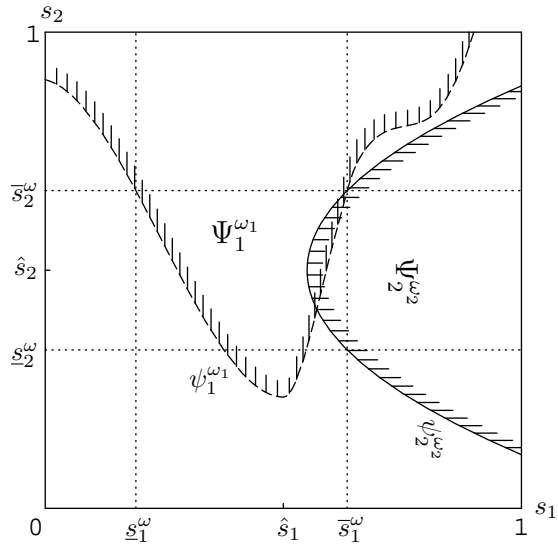
²⁵We would like to thank an anonymous referee for inspiring us to establish this result.

²⁶The simultaneous move game can be analyzed similarly; recall Lemma 2.1 and Section 4.

²⁷Alternatively one may assume that player i 's mixed action in period $t \in T_i$ is observable to player j at the beginning of period $t+1$. In this case, player i 's expected one-shot payoff in period t is $-as_{i,t} + (c+a)r_{j,t-1}$, where $s_{i,t}$ is player i 's probability of choosing C, and $r_{j,t-1}$ is player j 's realized action in period $t-1$. Since $r_{j,t-1}$ does not affect player i 's preferences over his actions from period t onward, all our results hold in this case as well. This argument is unnecessary for the simultaneous move game, where $r_{j,t-1}$ must be replaced by $s_{j,t}$.



(a)



(b)

Figure 10: Effectively efficient IRE violating (5.11)

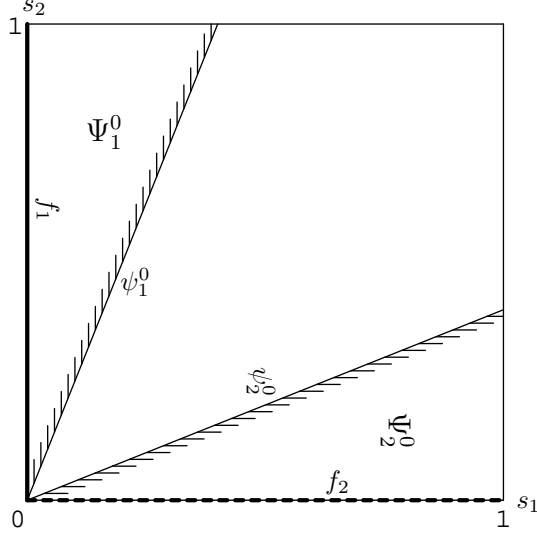


Figure 11: Case $\delta < a/e$: Unique IRE

factor: $\delta_1 = \delta_2 = \delta \in (0, 1)$. Player i 's effective payoff is given by

$$w_i(s_i, s_j) = -as_i + \delta es_j, \quad (6.1)$$

where $e = c + a$. Replacing $w_i(s_i, s_j)$ with ω_i and solving for s_j , we see that the indifference curve associated with $\omega^i \in \mathbb{R}$, or $\psi_i^{\omega^i}$, is linear:

$$\psi_i^{\omega^i}(s_i) = \frac{\omega_i}{\delta e} + \frac{a}{\delta e} s_i. \quad (6.2)$$

Since $\hat{s}_i = 0$ here, Proposition 5.2 applies. We consider three cases separately.

Case 1: $\delta < a/e$. In this case the slope of $\psi_i^{\omega^i}$ is strictly greater than one for any $\omega \in \mathbb{R}^2$. By the no-sticking-out condition (5.6), $\psi_i^{\omega^i}(0) \geq \underline{s}_j^\omega \geq 0$ for both i in any IRE associated with ω . Thus if $\psi_i^{\omega^i}(0) > 0$ for either i , the nonemptiness condition (5.5) is violated; see Figure 11. Hence $\psi_i^{\omega^i}(0) = 0$ and $\omega_i = 0$ for both i . In any IRE (f_1, f_2) associated with $(0, 0)$, therefore, by Proposition 5.1, $f_i(s_j) \leq \bar{s}_i^{(0,0)} = 0$, i.e., $f_i(s_j) = 0$, for all s_j and i . This is the unique IRE here, which corresponds to the static Nash equilibrium; see Figure 11 again. This IRE is effectively efficient by Proposition 5.2 (or simply by uniqueness).

Case 2: $\delta = a/e$. In this knife edge case, the slope of $\psi_i^{\omega^i}$ is equal to one, and the two indifference curves emanating from the origin coincide. The

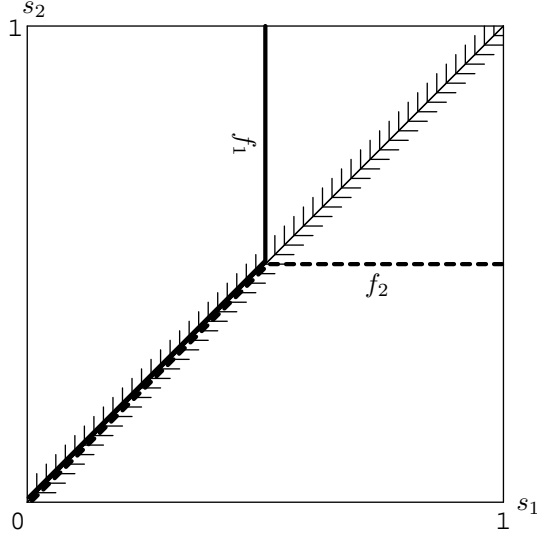


Figure 12: Case $\delta = a/e$: Example of IRE

above argument still shows $\psi_i^{\omega_i}(0) = 0$ and $\omega_i = 0$ for both i . Though, as in the previous case, there is an IRE corresponding to the static Nash equilibrium, there are many other IREs here. Figure 12 depicts one example.

Case 3: $\delta > a/e$. In this case the slope of $\psi_i^{\omega_i}$ is strictly less than one, and there are many pairs of effective payoffs supported by IREs. An IRE satisfying (3.8) is depicted in Figure 13, where there is a unique and globally stable steady state. The existence of a unique and globally stable steady state is a general property of this case by (4.9) and (6.2).

Figure 14 shows a symmetric IRE that is effectively efficient. In this case, gradual cooperation occurs, and full cooperation is achieved in the long run.²⁸ Figure 15 shows an effectively efficient IRE in which uneven gradual cooperation occurs: in the long run, only player 2 fully cooperates, while player 1 enjoys the highest possible effective payoff supported by an IRE.

²⁸Gradual cooperation is known to arise in certain partnership games; see Furusawa and Kawakami (2008) and the references therein.

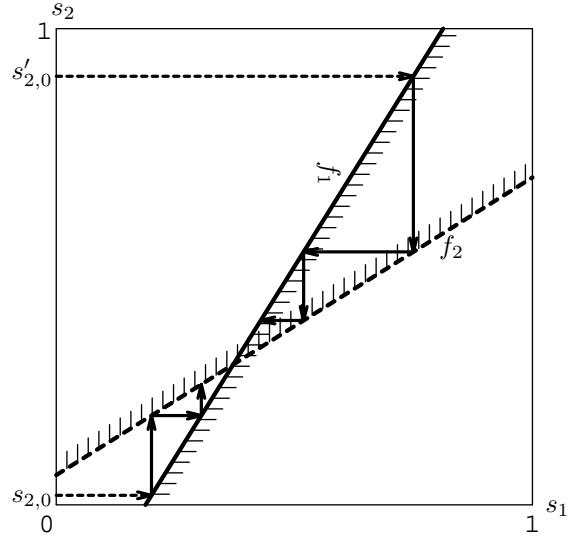


Figure 13: Case $\delta > a/e$: “Typical” IRE

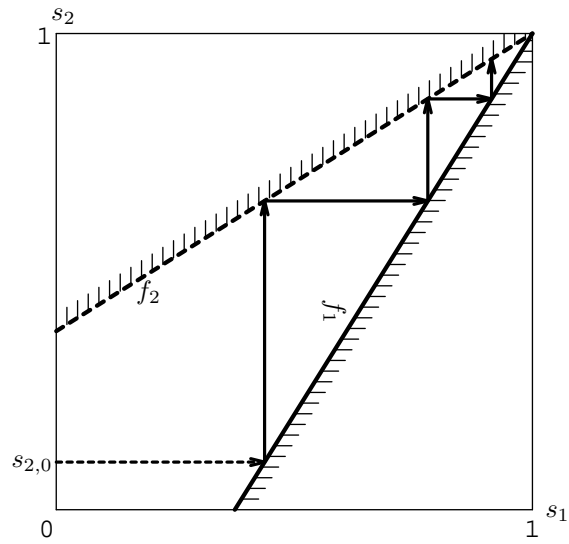


Figure 14: Case $\delta > a/e$: Gradual cooperation

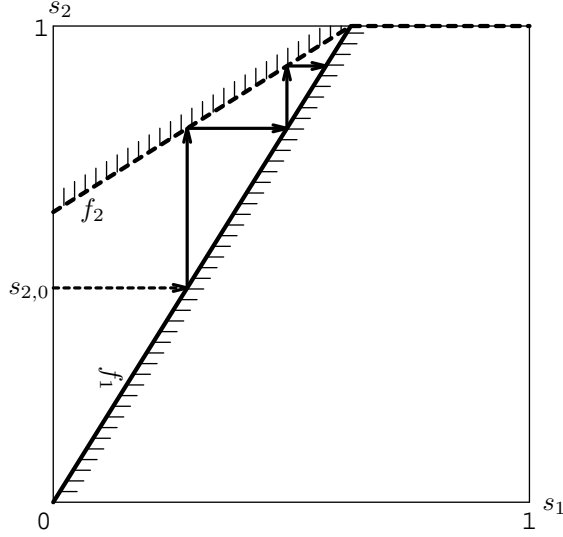


Figure 15: Case $\delta > a/e$: Uneven gradual cooperation

6.2 Duopoly

Consider the alternating move game associated with the duopoly game of Section 2.2.2.²⁹ For simplicity we assume that the firms are symmetric. Let c and δ denote their common marginal cost and discount factor. Recall that firm i 's one-shot profit is given by $D_i(p_i, p_j)(p_i - c)$. We parametrize D_i as follows:

$$D_i(p_i, p_j) = (\bar{p} - p_i)p_j, \quad (6.3)$$

where $\bar{p} > c$. Fix i . Recalling (2.2), we see that firm i 's effective payoff is given by

$$w_i(p_i, p_j) = \ln(\bar{p} - p_i) + \ln(p_i - c) + \delta \ln p_j. \quad (6.4)$$

Replacing $w_i(p_i, p_j)$ with ω_i and solving for p_j , we obtain

$$\psi_i^{\omega_i}(p_i) = \exp[\{\omega_i - \ln(\bar{p} - p_i) - \ln(p_i - c)\}/\delta]. \quad (6.5)$$

Note that $\psi_i^{\omega_i}(c) = \psi_i^{\omega_i}(\bar{p}) = \infty$. Direct calculation of the second derivative shows that $\psi_i^{\omega_i}$ is strictly convex. It is easy to see that given p_j , firm i 's one-shot profit, as well as its effective payoff, is maximized at $p_i = \hat{p} \equiv (c + \bar{p})/2$. This is the price charged by both firms in the unique static Nash equilibrium.

²⁹Once again, the dynamics of the simultaneous move game can be analyzed similarly.

main results show that a given pair of indifference curves is supported by an IRE if and only if it satisfies two certain graphical conditions, and that under these conditions, there is a continuous IRE that punishes deviations in a minimal way.

Although additive separability, which is crucial to our analysis, is rather restrictive, there are various interesting games that satisfy it. We have analyzed two such games and characterized their IREs by applying our general results. We have shown among other things that gradual cooperation arises in an effectively efficient IRE of a prisoners' dilemma game, and that kinked demand curves with stable collusion emerge in an effectively efficient IRE of a duopoly game.

We believe that our results are useful not only in analyzing games that satisfy our assumptions, but also in constructing completely tractable special cases of more general games. Such special cases, whose dynamics can be analyzed explicitly, would enhance the understanding of various interesting problems.

Appendix A Proof of Lemma 2.1

We reproduce (2.16) and define M_i as follows:

$$\forall s_j, \quad f_i(s_j) \in M_i \equiv \operatorname{argmax}_{s_i \in [0,1]} w_i(s_i, f_j(s_i)). \quad (\text{A.1})$$

Consider the simultaneous move game. Fix i and $f_j \in F$. From (2.13)–(2.15) and (2.6), we have

$$\sum_{t=1}^{\infty} \delta_i^{t-1} [u_i(s_{i,t}) + v_i(s_{j,t})] = v_i(s_{j,1}) + \sum_{t=1}^{\infty} \delta_i^{t-1} w_i(s_{i,t}, f_j(s_{i,t})). \quad (\text{A.2})$$

Thus the maximization problem (2.5)–(2.7) is solved iff $s_{i,t} \in M_i$ for all $t \in \mathbb{N}$. Hence if $f_i \in F$ is a best response to f_j , then $f_i(s_{j,0}) = s_{i,1} \in M_i$ for all $s_{j,0}$; thus (A.1) holds. Conversely, if $f_i \in F$ satisfies (A.1), then $s_{i,t} = f_i(s_{j,t-1}) \in M_i$ for all $t \in \mathbb{N}$, so the maximization problem is solved, i.e., f_i is a best response to f_j .

Now consider the alternating move game. From (2.13)–(2.15), (2.10), and

(2.11), we have

$$\sum_{t=i}^{\infty} \delta_i^{t-i} [u_i(s_{i,t}) + v_i(s_{j,t})] \quad (\text{A.3})$$

$$= v_i(s_{j,i}) + \sum_{t \in T_i} \delta_i^{t-i} (1 + \delta_i) w_i(s_{i,t}, s_{j,t+1}) \quad (\text{A.4})$$

$$= v_i(s_{j,i}) + (1 + \delta_i) \sum_{t \in T_i} \delta_i^{t-1} w_i(s_{i,t}, f_j(s_{i,t})). \quad (\text{A.5})$$

Thus the maximization problem (2.9)–(2.11) is solved iff $s_{i,t} \in M_i$ for all $t \in T_i$. Hence the proposition follows as in the simultaneous move case.

Appendix B Proof of Theorem 5.1

We start by preparing some lemmas.

Lemma B.1. *Let $(f_1, f_2) \in F^2$. Recall the definition of M_i in (A.1). For $i = 1, 2$, we have*

$$M_i = \{s_i \in [0, 1] \mid f_j(s_i) = \psi_i^*|_{f_i}(s_i)\}. \quad (\text{B.1})$$

Proof. Fix i . Since $w_i(s_i, s_j)$ is strictly increasing in s_j , we have

$$w_i(s_i, f_j(s_i)) \left\{ \begin{array}{l} = \\ < \end{array} \right\} \omega_i^*|_{f_j} \iff f_j(s_i) \left\{ \begin{array}{l} = \\ < \end{array} \right\} \psi_i^*|_{f_j}(s_i). \quad (\text{B.2})$$

Recall Figure 3. Now (B.1) follows. \square

From here on we fix $\omega \in \mathbb{R}^2$ and omit the superscripts ω_i, ω_j , and ω .

Lemma B.2. *Under (5.5), for $i = 1, 2$,*

$$\psi_i(\bar{s}_i) \leq \bar{s}_j, \quad (\text{B.3})$$

$$\psi_i(\hat{s}_i) \leq \psi_i(\underline{s}_i) \leq \bar{s}_j, \quad (\text{B.4})$$

$$\underline{s}_i \leq \hat{s}_i. \quad (\text{B.5})$$

Proof. To see (B.3), suppose $\psi_i(\bar{s}_i) > \bar{s}_j$. By (5.3) and compactness of $\Psi_i \cap \Psi_j$, there is s_j such that $(\bar{s}_i, s_j) \in \Psi_i \cap \Psi_j$. By (5.2), $\psi_i(\bar{s}_i) \leq s_j$. But then $\bar{s}_j < \psi_i(\bar{s}_i) \leq s_j$, contradicting the definition of \bar{s}_j .

Note from (5.4) that $\psi_i(\underline{s}_i) \leq \bar{s}_j$. We obtain (B.4) since $\psi_i(s_i)$ is minimized at $s_i = \hat{s}_i$ (recall (3.2)). Finally (B.5) follows from (B.4) and (5.4). \square

The sufficiency part of Theorem 5.1 follows from Theorem 5.2, which we prove first.

B.1 Sufficiency: Proof of Theorem 5.2

Assume (5.5) and (5.6). Define (f_1, f_2) by (5.7). To show that (f_1, f_2) is an IRE, it suffices to verify (A.1) for $i = 1, 2$ by Lemma 2.1. Fix i . By (B.4) and (5.7) (with i and j interchanged),

$$\forall s_i, \quad f_j(\hat{s}_i) = \psi_i(\hat{s}_i) \leq f_j(s_i) \leq \psi_i(s_i), \quad (\text{B.6})$$

where the first inequality holds since $\psi_i(s_i)$ is minimized at $s_i = \hat{s}_i$ (recall (3.2)). Since $w_i(s_i, s_j)$ is increasing in s_j , it follows that

$$\forall s_i, \quad w_i(s_i, f_j(s_i)) \leq w_i(s_i, \psi_i(s_i)) = \omega_i \quad (\text{B.7})$$

$$= w_i(\hat{s}_i, \psi_i(\hat{s}_i)) = w_i(\hat{s}_i, f_j(\hat{s}_i)). \quad (\text{B.8})$$

This implies $\omega_i^*|_{f_j} = \omega_i$ and $\psi_i^*|_{f_j} = \psi_i$. Hence by (B.1) and (5.7),

$$M_i = \{s_i \in [0, 1] \mid \psi_i(s_i) \leq \bar{s}_j\}. \quad (\text{B.9})$$

It follows by (B.3) and (B.4) that $\underline{s}_i, \bar{s}_i \in M_i$. Fix s_j . We have

$$\underline{s}_i \leq \psi_j(\hat{s}_j) \leq f_i(s_j) \leq \bar{s}_i \quad (\text{B.10})$$

by (5.6), (B.6) (with i and j interchanged), and (5.7). Since $\underline{s}_i, \bar{s}_i \in M_i$ and M_i is an interval by (B.9), we obtain $f_i(s_j) \in M_i$, i.e., (A.1).

Regarding (5.8), since $f_i(f_j(\cdot))$ is a continuous function from $[0, 1]$ to itself, it has a fixed point s_i (i.e., $s_i = f_i(f_j(s_i))$) by Brouwer's fixed point theorem. Let $s_j = f_j(s_i)$. Then (s_1, s_2) is a steady state and satisfies (5.8)(b) by Theorem 4.1. This in turn implies (5.8)(a).

B.2 Necessity

Let (f_1, f_2) be an IRE associated with ω . We wish to verify (5.5) and (5.6). Although (B.3) implies $(\bar{s}_1, \bar{s}_2) \in \Psi_1 \cap \Psi_2$, we cannot show $\Psi_1 \cap \Psi_2 \neq \emptyset$ by using (\bar{s}_1, \bar{s}_2) , whose existence requires $\Psi_1 \cap \Psi_2 \neq \emptyset$. Thus we need to find an action profile defined in terms of (f_1, f_2) that belongs to $\Psi_1 \cap \Psi_2$.

To this end, note that since (f_1, f_2) is an IRE associated with ω , for $i = 1, 2$,

$$\psi_i (= \psi_i^{\omega_i}) = \psi_i^*|_{f_j}. \quad (\text{B.11})$$

Define

$$D_i = \{s_i \in [0, 1] \mid \psi_i(s_i) \in [0, 1]\}, \quad (\text{B.12})$$

$$R_i = \{s_i \in [0, 1] \mid \exists s_j \in [0, 1], s_i = f_i(s_j)\}, \quad (\text{B.13})$$

$$\bar{r}_i = \sup R_i, \quad (\text{B.14})$$

$$\underline{r}_i = \min\{s_i \in D_i \mid \psi_i(s_i) \leq \bar{r}_j\}. \quad (\text{B.15})$$

Note that D_i is the domain of ψ_i when its range is restricted to $[0, 1]$, R_i is the range of f_i , and \underline{r}_i exists since there is s_i with $f_j(s_i) = \psi_i(s_i)$ by (B.1) and (B.11). The idea here is to use \underline{r}_i and \bar{r}_i as substitutes for \underline{s}_i and \bar{s}_i .

Lemma B.3. *For $i = 1, 2$, we have*

$$(a) \bar{r}_i \in D_i, \quad (b) \psi_i(\bar{r}_i) \in D_j, \quad (c) \psi_j(\psi_i(\bar{r}_i)) \leq \bar{r}_i. \quad (\text{B.16})$$

Proof. Fix i . By (A.1) (which holds by Lemma 2.1), (B.1), and (B.11),

$$R_i \subset M_i \subset D_i. \quad (\text{B.17})$$

Since D_i is compact by continuity of ψ_i ,³⁰ (a) follows from (B.17).

Note from (3.6), (B.11), and (B.17) (with i and j interchanged) that

$$\forall s_i \in R_i, \quad \psi_i(s_i) = f_j(s_i) \in R_j \subset D_j. \quad (\text{B.18})$$

Thus (b) follows by continuity of ψ_i and compactness of D_j .

For (c), note from (B.18) (with i and j interchanged) that $\sup_{s_j \in R_j} \psi_j(s_j) \leq \bar{r}_i$. Since $\psi_i(s_i) \in R_j$ for $s_i \in R_i$ by (B.18),

$$\forall s_i, \quad \psi_j(\psi_i(s_i)) \leq \sup_{s_j \in R_j} \psi_j(s_j) \leq \bar{r}_i. \quad (\text{B.19})$$

Now (c) follows by continuity of ψ_i and ψ_j . □

Lemma B.4. *For $i = 1, 2$,*

$$(\bar{r}_i, \psi_i(\bar{r}_i)) \in \Psi_i \cap \Psi_j, \quad (\text{B.20})$$

$$(a) \hat{s}_j \in D_j, \quad (b) \underline{r}_i \leq \psi_j(\hat{s}_j). \quad (\text{B.21})$$

³⁰Note that R_i need not be compact since f_i need not be continuous.

Proof. Fix i . We have $(\bar{r}_i, \psi_i(\bar{r}_i)) \in \Psi_i$ by (5.2) (with $s_i = \bar{r}_i$ and $s_j = \psi_i(\bar{r}_i)$) and (B.16)(a),(b). By (5.2) (with i and j interchanged) and (B.16), $(\bar{r}_i, \psi_i(\bar{r}_i)) \in \Psi_j$.³¹ Thus (B.20) follows.

For (B.21), note from (B.2) and (B.11) that

$$0 \leq f_i(\hat{s}_j) \leq \psi_j(\hat{s}_j) \leq 1, \quad (\text{B.22})$$

which implies (B.21)(a). By (B.1), (B.11), and (B.14),

$$\forall s_i \in M_i, \quad \psi_i(s_i) = f_j(s_i) \leq \bar{r}_j. \quad (\text{B.23})$$

Hence $M_i \subset \{s_i \in D_i \mid \psi_i(s_i) \leq \bar{r}_j\} \equiv A_i$. Thus by (B.22) and (A.1),

$$\psi_j(\hat{s}_j) \geq f_i(\hat{s}_j) \in M_i \subset A_i. \quad (\text{B.24})$$

Note from (B.15) that

$$\underline{r}_i = \min A_i. \quad (\text{B.25})$$

Thus (B.21)(b) follows. \square

Let us now complete the proof of Theorem 5.1. We have (5.5) by (B.20). Fix i . By (B.20) and (5.3) (both with i and j interchanged),

$$\bar{r}_j \leq \bar{s}_j. \quad (\text{B.26})$$

Thus $A_i \subset \{s_i \in [0, 1] \mid \psi_i(s_i) \leq \bar{s}_j\}$. Recalling (B.25) and (5.4), we have $\underline{r}_j \geq \underline{s}_j$. This together with (B.21) establishes (5.6).

Appendix C Proof of Proposition 5.2

Sufficiency follows from the discussion leading to (5.11). For necessity, fix an IRE (f_1, f_2) associated with $\omega \in \mathbb{R}^2$ satisfying (5.12). This implies

$$\forall i, \quad \psi_i^{\omega_i}(\hat{s}_i) \geq \hat{s}_j. \quad (\text{C.1})$$

Suppose (5.11) does not hold, i.e., $\tilde{\Psi}_1^{\omega_1} \cap \tilde{\Psi}_2^{\omega_2} \neq \emptyset$. Since each $\psi_i^{\omega_i}$ is continuous and strictly increasing in ω_i by (3.2), there is $\omega' \gg \omega$ such that $\tilde{\Psi}_1^{\omega'_1} \cap \tilde{\Psi}_2^{\omega'_2} \neq \emptyset$ and $\psi_i^{\omega'_i}(\hat{s}_i) > \psi_i^{\omega_i}(\hat{s}_i)$ for $i = 1, 2$. This together with (C.1) and (B.5) yields $\psi_i^{\omega'_i}(\hat{s}_i) > \psi_i^{\omega_i}(\hat{s}_i) \geq \hat{s}_j \geq \underline{s}_i^{\omega'}$, which implies (5.6). Thus by Theorem 5.1, there is an IRE associated with ω' , so (f_1, f_2) cannot be effective efficient.

³¹Recall footnote 20.

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