The Spirit of Capitalism, Stock Market Bubbles, and Output Fluctuations*

Takashi Kamihigashi†

October 5, 2007

Abstract

This paper presents a representative agent model in which stock market bubbles cause output fluctuations. Assuming that utility depends directly on wealth, we show that stock market bubbles arise if the marginal utility of wealth does not decline to zero as wealth goes to infinity. Bubbles may affect output positively or negative depending on whether the production function exhibits increasing or decreasing returns to scale. In sunspot equilibria, the bursting of a bubble is followed by a sharp decline in output one period later. Various numerical examples are given to illustrate the behavior of stochastic bubbles and the relationship between bubbles and output.

Keywords: Spirit of capitalism, stock market bubbles, output fluctuations, wealth in utility, sunspot equilibria

JEL Classification: E20, E32

*This paper is dedicated to late Koji Shimomura. The memory of various discussions with him continues to be a great source of inspiration in many different ways. Earlier versions of the paper were presented at the ISER, Osaka University, the Workshop of Macroeconomics, Osaka, and the Department of Economics, Hitotsubashi University. Comments from participants are gratefully acknowledged.

†RIEB, Kobe University, Rokkodai, Nada, Kobe 657-8501 JAPAN. Email: tkamihig@rieb.kobe-u.ac.jp. Tel/Fax: +81-78-803-7015.

1 Introduction

It is popularly believed that the bursting of a stock market bubble is often followed by a severe recession. Famous examples include the Great Depression after the Wall Street Crash of 1929, and Japan's "lost decade" after the bursting of the Japanese bubble in the early 1990s.[1] While there is a large empirical literature on the relationship between stock prices and real activity,[2] there has been little theoretical research on the effects of stock market bubbles on output fluctuations. The purpose of this paper is to construct a simple general equilibrium model in which stock market bubbles affect real activity, giving rise to output fluctuations.

The biggest difficulty in constructing such a model is that there are very few general equilibrium models in which stock market bubbles arise under the hypotheses of rational expectations and symmetric information. It is known that in deterministic representative agent models, bubbles are simply impossible (Kamihigashi, 2001). Even in stochastic representative models, bubbles are impossible except under rather pathological specifications (Kamihigashi, 1998; Montrucchio and Privileggi, 2001). For general overlapping generations models with complete markets, it has been known since Wilson (1981) that bubbles are impossible if the value of aggregate wealth is finite. This result also holds for general overlapping generations models with incomplete markets (Santos and Woodford, 1997). The value of aggregate wealth is finite if for example there is a traded stock whose dividend stream is larger than a fixed fraction of the aggregate endowment stream. Therefore, if we are to analyze a general equilibrium model in which the "Lucas tree" is traded, some deviation from standard assumptions is necessary.

As surveyed in Brunnermeier (2007) and Iraola and Santos (2007), bubbles are possible in models with asymmetric information, heterogeneous beliefs, limited rationality, limited arbitrage, or agency problems. While such settings offer attractive explanations for bubbles, they are rather difficult to formulate in an aggregative framework in which the effects of stock market bubbles on output fluctuations can be analyzed relatively easily. In this paper we propose an alternative explanation for bubbles based on "the spirit

[1]See, e.g., Allen and Gale (1999) for other historical examples. This paper focuses on stock market bubbles though bubbles on other assets such as real estate are often discussed together in the literature.

[2]See, e.g., Hassapis and Kalyvistis (2002), Binswanger (2004), and the references therein.

of capitalism" (Weber, 1905).

According to Zou (1995, p. 132), the spirit of capitalism in the sense of Weber (1905) "motivates the continual accumulation of wealth not only for the material reward that it brings, but also for its own sake," and "even before Weber, Adam Smith, Nassau Senior, and Karl Marx expressed similar views, and subsequently Werner Sombart, Joseph Schumpeter, John Maynard Keynes and Gustav Cassel argued essentially the same hypothesis."[3] In the recent literature, Zou (1994, 1995) and Bakshi and Chen (1996) among others formulated the spirit of capitalism by assuming a utility function that depends not only on consumption but also on wealth itself.[4] Following this approach we show that stock market bubbles are possible when the spirit of capitalism, or the marginal utility of wealth, does not decline to zero as wealth increases to infinity.

Though this paper seems the first to show the possibility of bubbles in a wealth-in-the-utility-function model, it has been known since Obstfeld and Rogoff (1986) that (deflationary) bubbles are possible in a money-in-the-utility-function model.[5] In both cases bubbles are possible for the same reason: if the marginal utility of wealth (or real balances) does not decline fast enough as wealth increases to infinity, the associated transversality condition is satisfied for divergent paths. In this paper, however, showing the possibility of bubbles is only an initial step toward the analysis of the effects of stock market bubbles on output.

The relationship between bubbles and growth has already been addressed in the literature. Grossman and Yanagawa (1993) and Futagami and Shibata (2000) analyzed endogenous growth models with overlapping generations, showing that bubbles on intrinsically useless assets negatively affect the growth rate of output by crowding out productive investment.[6] By contrast Olivier (2000) showed that bubbles on stocks in newly created firms (or technologies) positively affect growth by encouraging creation of new firms.[7]

[3] See Zou (1994, 1995) for numerous historical quotes.

[4] This approach was pioneered by Kurz (1968). Recent studies based on the wealth-in-the-utility-function approach include Smith (2001), Gong and Zou (2002), Nakajima (2003), and Chang et al. (2004).

[5] See Buiter and Sibert (2007) for a recent treatment of deflationary bubbles in a money-in-the-utility-function model.

[6] See Tirole (1985) for related arguments in a neoclassical overlapping generations model.

[7] In his model, however, economically identical stocks are required to have different prices in order to prevent the aggregate bubble from outgrowing the economy.

In this paper we show that stock market bubbles can have a positive or negative effect on output (or the capital stock) depending on whether the production function exhibits increasing or decreasing returns to scale. We consider a one-sector growth model with homogeneous consumers who derive utility from consumption and wealth. Both the capital stock and stocks in homogeneous firms are traded. Since firms are homogeneous, all stocks have the same price. When the stock price rises, the marginal utility of wealth declines, so that the incentive to hold capital as wealth decreases. This means that the supply curve of capital, which is upward sloping, shifts to the left. If the production function exhibits decreasing returns to scale, the demand curve for capital is downward sloping. Hence a rise in stock price results in a fall in capital. If on the other hand the production function exhibits increasing returns (or, more precisely, the private marginal product of capital is increasing in the overall capital stock), then the demand curve for capital is upward sloping and a rise in stock price may entail a rise in capital. We show that there is an equilibrium in which bubbles always affect the capital stock positively in the case of increasing returns.

Because a current change in stock price affects investment for the next period, a large drop in stock price caused by a sunspot shock is followed by a large drop in output in the next period. In other words, the bursting of a bubble is followed by a sharp decline in output one period later. This feature is not captured in standard real business cycle models, where productivity shocks affect output and stock prices simultaneously (e.g., Rebelo, 2005).

The rest of the paper is organized as follows. Section 2 studies an exchange economy to show that a strong spirit of capitalism implies the possibility of bubbles. We assume that utility is additively separable in consumption and wealth, and show the existence of a unique steady state. We completely characterize the equilibria, showing that there exists a continuum of equilibria with bubbles. Then we introduce sunspot shocks to the stock market, giving several numerical examples of stochastic equilibria. We observe that bubbles decay to zero asymptotically if the sunspot shocks are sufficiently volatile. Section 3 introduces a typical neoclassical production function to the exchange economy, deriving conditions characterizing the equilibria. Section 4 considers the special case of this production economy in which utility is linear in wealth. We show that the equilibrium capital path is determined independently of the stock price path, i.e., bubbles do not affect real activity. Section 5 considers the special case of the production economy in which utility is linear in consumption. We show the existence of a unique steady state and characterize equilibria (from initial capital stocks at least as large as the steady state capital stock). We show that there exists a continuum of equilibria with bubbles as in the exchange economy, but that in equilibria with bubbles, the stock price keeps rising, while the capital stock keeps declining. Section 6 assumes that the production function exhibits increasing returns to scale at the social level but decreasing returns at the private level. We establish the existence of a steady state and the existence of a continuum of equilibria with bubbles. In these equilibria, the stock price and the capital stock both increase over time. Then we introduce sunspot shocks to the stock market, illustrating how a change in stock price is followed by a similar change in output one period later. Section 7 concludes the paper.

2 An Exchange Economy

Our model here is similar to the Lucas (1978) asset pricing model except that utility depends on wealth in addition to consumption. The dependence of utility on wealth captures the idea of the spirit of capitalism. Consider an economy in which there is only one asset, or "stock," and there are many homogeneous consumers each of whom faces the following maximization problem:

$$\max_{\{c_t, w_t, s_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(w_t)] \tag{2.1}$$

$$\text{s.t.} \quad c_t + w_t = (p_t + d)s_{t-1}, \tag{2.2}$$

$$w_t = p_t s_t, \tag{2.3}$$

$$c_t, s_t \geq 0, \tag{2.4}$$

$$s_{-1} = 1 \text{ given,} \tag{2.5}$$

where c_t is consumption in period t, w_t is wealth in period t, p_t is the price of the stock in period t, s_t is shares of the stock held at the end of period t, and $d > 0$ is the dividend per share, which is assumed to be constant over time. We assume the following.

Assumption 2.1. $u, v : \mathbb{R}_+ \to [-\infty, \infty)$ are continuously differentiable on $\mathbb{R}_{++}$, continuous, concave, and strictly increasing.

The Euler equation and the transversality condition are given by

$$[u'(c_t) - v'(w_t)]p_t = \beta u'(c_{t+1})(p_{t+1} + d), \tag{2.6}$$

$$\lim_{t\to\infty} \beta^t [u'(c_t) - v'(w_t)]p_t s_t = 0. \tag{2.7}$$

The above conditions are sufficient for optimality by a standard argument (e.g., Stokey and Lucas, 1989). The Euler equation is necessary for interior solutions, while the transversality condition is necessary under additional conditions (e.g, Kamihigashi, 2001, 2003).[8]

An equilibrium of this economy is a set of nonnegative paths $\{p_t, c_t, w_t, s_t\}_{t=0}^{\infty}$ such that (i) given $\{p_t\}$, $\{c_t, w_t, s_t\}$ solves the maximization problem (2.1)–(2.5); and (ii) the good and stock markets clear:

$$c_t = d, \tag{2.8}$$

$$s_t = 1. \tag{2.9}$$

If $p_t \leq 0$, consumers wish to increase s_t indefinitely in order to receive an indefinite amount of dividend payment in period $t + 1$.[9] Since this cannot happen in equilibrium, we have in equilibrium

$$p_t > 0. \tag{2.10}$$

We may normalize $u'(d)$ to one without loss of generality.

Assumption 2.2. $u'(d) = 1$.

We assume that the marginal utility of wealth does not decline to zero as wealth goes to infinity.

Assumption 2.3. $\lim_{w\to\infty} v'(w) \equiv \nu > 0$.

This assumption means that the spirit of capitalism stays strong even when wealth is very large.[10] Our analysis goes through as long as v' declines sufficiently slowly to zero, as in Obstfeld and Rogoff (1986, p. 356).

Though we assume that the marginal utility of wealth does not go to zero, we assume that it becomes sufficiently small as wealth increases to infinity:

Assumption 2.4. $1 - \nu > \beta$, where ν is given by Assumption 2.3.

This assumption is needed to ensure the existence of a steady state.

[8] In this paper the necessity of the transversality condition is not required.

[9] Strictly speaking, before this argument is applied to the case in which v is unbounded below, the concept of optimality must be extended properly.

[10] For a money-in-the-utility-function model, Ono (2001) justified the same assumption using Keynes's notion of liquidity preference.

2.1 Bubbles

By Assumption 2.2 and equilibrium conditions (2.8) and (2.9), the Euler equation (2.6) and the transversality condition (2.7) reduce to

$$[1 - v'(p_t)]p_t = \beta(p_{t+1} + d), \tag{2.11}$$

$$\lim_{t\to\infty} \beta^t [1 - v'(p_t)]p_t = 0. \tag{2.12}$$

Equation (2.11) can be rewritten as

$$p_{t+1} = \frac{1 - v'(p_t)}{\beta} p_t - d. \tag{2.13}$$

Define

$$R(p_t) = \frac{1 - v'(p_t)}{\beta}, \tag{2.14}$$

$$q_{t+1} = \frac{1}{R(p_t)}. \tag{2.15}$$

Note that $R(p_t)$ is the (implicit) gross interest rate, and q_{t+1} is its inverse. From (2.13), p_t can be written successively as

$$
\begin{aligned}
p_t &= q_{t+1}d + q_{t+1}p_{t+1} && (2.16)\\
&= q_{t+1}d + q_{t+1}q_{t+2}d + q_{t+1}q_{t+2}p_{t+2} && (2.17)\\
&= q_{t+1}d + q_{t+1}q_{t+2}d + q_{t+1}q_{t+2}q_{t+3}d + q_{t+1}q_{t+2}q_{t+3}p_{t+3} && (2.18)\\
&\vdots && (2.19)\\
&= \sum_{j=1}^{J} \left(\prod_{i=1}^{j} q_{t+i} \right) d + \left(\prod_{i=1}^{J} q_{t+i} \right) p_{t+J} && (2.20)\\
&= \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} q_{t+i} \right) d + \lim_{J\to\infty} \left(\prod_{i=1}^{J} q_{t+i} \right) p_{t+J}. && (2.21)
\end{aligned}
$$

The above limit exists since the sum in (2.20) is increasing in J and thus the second term in (2.20) is decreasing in J. Define

$$p_t^f = \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} q_{t+i} \right) d, \tag{2.22}$$

$$p_t^b = \lim_{J\to\infty} \left(\prod_{i=1}^{J} q_{t+i} \right) p_{t+J}. \tag{2.23}$$

Note that p_t^f is the present discounted value of the dividend stream. We call p_t^f the fundamental value of the stock, and p_t^b the bubble component of p_t. Now p_t can be written as

$$p_t = p_t^f + p_t^b. \tag{2.24}$$

Recall from (2.15) that q_{t+1} depends on p_t. Thus the fundamental value p_t^f itself depends on the price path $\{p_t\}$. Hence the fundamental value cannot be expressed in terms of fundamentals here, in sharp contrast to the case in which utility does not depend on wealth.

2.2 Steady States

Let us consider steady states, where p_t is constant over time. It follows from (2.13) that steady state prices are characterized by

$$p = R(p)p - d, \tag{2.25}$$

which rearranges to

$$\frac{d}{p} = R(p) - 1. \tag{2.26}$$

As p varies from zero to infinity, d/p decreases from infinity to zero, while $[R(p)-1]$ increases from a value no greater than $(1-\nu)/\beta-1$ to $(1-\nu)/\beta-1 > 0$, where the inequality uses Assumption 2.4. It follows that (2.26) has a unique solution in p. Since the transversality condition (2.12) is trivially satisfied if p_t is constant over time, we obtain the following.

Proposition 2.1. *There exists a unique steady state.*

Let p^* and q^* be the steady state values of p_t and q_t, respectively. Note from (2.15) and (2.26) that

$$\frac{1}{q^*} = R(p^*) = \frac{d}{p^*} + 1 > 1. \tag{2.27}$$

Thus from (2.26) and (2.15),

$$p^* = \frac{d}{R(p^*) - 1} = \frac{q^*}{1 - q^*} d = \sum_{i=1}^{\infty} {q^*}^i d, \tag{2.28}$$

i.e., p^* equals the fundamental value of the stock:

Proposition 2.2. *The steady state price p^* contains no bubble.*

2.3 Dynamics

It follows from (2.13) and (2.14) that

$$p_{t+1} = R(p_t)p_t - d. \tag{2.29}$$

To characterize the equilibrium dynamics, let us prepare the following lemma.

Lemma 2.1. *A strictly positive sequence $\{p_t\}$ is an equilibrium price path if and only if it satisfies (2.29) for all $t \geq 0$.*

Proof. The "only if" part follows from the fact that (2.29), which is equivalent to the Euler equation (2.11), is a necessary condition for optimality. To show the "if" part, let $\{p_t\}$ be a strictly positive sequence satisfying (2.29) for all $t \geq 0$. It suffices to verify the transversality condition (2.12). By (2.29) and Assumption 2.3, $p_{t+1}/p_t \leq (1-\nu)/\beta < 1/\beta$. Thus $\beta^t p_t \to 0$ as $t \to \infty$. Since $0 < 1 - v'(p_t) \leq 1 - \nu$ by (2.11) and Assumption 2.3, the transversality condition (2.12) holds. □

The following result characterizes all equilibrium price paths.

Proposition 2.3. *There exists a continuum of equilibria. In particular, for each $p \geq p^*$, there exists a unique equilibrium price path $\{p_t\}$ with $p_0 = p$. If $p_0 = p^*$, then $p_t = p^*$ for all $t \geq 0$. If $p_0 > p^*$, then $\{p_t\}$ is strictly increasing and satisfies $\lim_{t\to\infty} p_t = \infty$. There exists no equilibrium price path $\{p_t\}$ with $p_0 < p^*$.*

Proof. Let $\{p_t\}$ be a strictly positive sequence satisfying (2.29). To show that there is no equilibrium price path with $p_0 < p^*$, let us begin by making two observations. First, if $p_t < p^*$ for some $t \geq 0$, then from (2.29), (2.14), and (2.27),

$$\frac{p_{t+1}}{p_t} = R(p_t) - \frac{d}{p_t} < R(p^*) - \frac{d}{p^*} = 1. \tag{2.30}$$

Second, for p_t sufficiently close to zero,

$$\frac{p_{t+1}}{p_t} = R(p_t) - \frac{d}{p_t} \leq R(p^*) - \frac{d}{p_t} < 0. \tag{2.31}$$

Suppose $p_0 < p^*$. Then since there is no steady state between 0 and p^*, by (2.30), $\lim_{t\uparrow\infty} p_t = 0$. But this contradicts the strict positivity of $\{p_t\}$ by (2.31). Hence there is no equilibrium with $p_0 < p^*$ by Lemma 2.1.

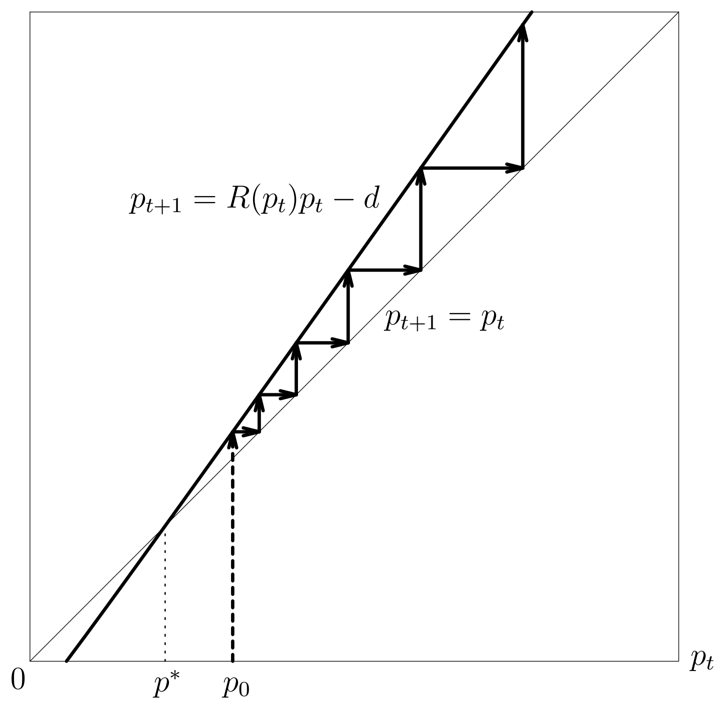


Figure 1: Equilibrium price dynamics

If $p_0 = p^*$, then the unique sequence $\{p_t\}$ satisfying (2.29) is given by $p_t = p^*$ for all $t \geq 0$. If $p_0 > p^*$, then the unique sequence $\{p_t\}$ satisfying (2.29) is strictly increasing with $\lim_{t\uparrow\infty} p_t = \infty$ since $R(p) \geq R(p^*) > 1$ for $p \geq p^*$ by (2.14) and (2.27). Now the proposition follows by Lemma 2.1. □

See Figure 1 for an example of an equilibrium price path. One can also see from the figure that the path satisfying (2.29) from $p_0 < p^*$ cannot remain strictly positive forever and thus cannot be an equilibrium.

Recall that the steady state price p^* contains no bubble by Proposition 2.2. How about other equilibrium price paths? To answer this question, suppose $p_0 > p^*$. Then p_t grows unboundedly by Proposition 2.3. On the other hand, the fundamental value p_t^f is bounded above by p^*. To see this, note from (2.15) that

$$q_{t+1} = \frac{1}{R(p_t)} \leq \frac{1}{R(p^*)} = q^* < 1, \tag{2.32}$$

where the last inequality holds by (2.27). In view of (2.22) and (2.28), we see that $p_t^f \leq p^*$.[11] Since $p_t \geq p_0 > p^*$ for all $t \geq 0$, we must have $p_t^b > 0$ for all $t \geq 0$. The following proposition summarizes the preceding argument.

Proposition 2.4. *For any equilibrium price path $\{p_t\}$ with $p_0 > p^*$, we have $p_t^b > 0$ and $p_t^f \leq p^*$ for all $t \geq 0$.*

Therefore, all strictly increasing equilibrium price paths contain strictly positive bubbles, which reduce the fundamental value while increasing the stock price in each period.

2.4 Sunspot Equilibria

Now consider an economy in which the equilibrium price path $\{p_t\}$ is allowed to be a stochastic process. In this case, the objective function in (2.1) must be replaced with

$$E_0 \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(w_t)], \tag{2.33}$$

where E_t denotes expectation conditional on the information available in period t. Since $\{p_t\}$ is stochastic, $\{c_t\}$, $\{w_t\}$, and $\{s_t\}$ are all allowed to be

[11] This inequality becomes strict if we assume that v is strictly concave.

stochastic for individual consumers, though $\{c_t\}$ and $\{s_t\}$ are deterministic in equilibrium.

The Euler equation and the transversality condition are given by

$$[u'(c_t) - v'(w_t)]p_t = \beta E_t u'(c_t)(p_{t+1} + d), \tag{2.34}$$

$$\lim_{t\to\infty} \beta^t E_0[u'(c_t) - v'(w_t)]p_t s_t = 0. \tag{2.35}$$

As in the deterministic case, these conditions are sufficient for optimality. Since there is no fundamental uncertainty, the market-clearing conditions (2.8) and (2.9) remain the same.

By (2.8) and Assumption 2.2, the stochastic Euler equation (2.34) reduces in equilibrium to

$$E_t p_{t+1} = R(p_t)p_t - d, \tag{2.36}$$

where $R(\cdot)$ is as defined in (2.14). It can be shown as in the deterministic case that any strictly positive stochastic process $\{p_t\}$ satisfying (2.36) for all $t \geq 0$ is an equilibrium.

We now wish to simulate (2.36) in order to understand the behavior of stochastic bubbles. For this purpose it would be natural to rewrite the Euler equation (2.36) as

$$p_{t+1} = R(p_t)p_t - d + \epsilon_{t+1}, \tag{2.37}$$

where ϵ_{t+1} is an expectational error satisfying $E_t\epsilon_{t+1} = 0$. However it is easy to see that in order to ensure $p_t \geq 0$ for all $t \geq 0$, we must have $p_t \geq p^*$ for all $t \geq 0$. This means that ϵ_{t+1} cannot be an i.i.d. shock; in particular, the support of ϵ_{t+1} must be adjusted according to the distance between p^* and $R(p_t)p_t - d$.

A more convenient representation of (2.36) for simulation purposes is

$$p_{t+1} - p^* = [R(p_t)p_t - d - p^*](1 + \epsilon_{t+1}), \tag{2.38}$$

where once again $E_t\epsilon_{t+1} = 0$. Applying E_t to (2.38), one can see that (2.38) is equivalent to (2.36); furthermore, $p_{t+1} \geq p^*$ as long as $p_t \geq p^*$ and the support of ϵ_{t+1} is a subset of $[-1, \infty)$. For example, if $\{e_t\}$ is an i.i.d. process with mean zero and support $(-1, 1)$, then ϵ_{t+1} may be of the form

$$\epsilon_{t+1} = \mu e_{t+1} \tag{2.39}$$

for any $\mu \in [0, 1)$. In what follows we use this specification of sunspot shocks. To distinguish between ϵ_t and e_t, we call ϵ_t the sunspot shock and e_t the primitive sunspot shock in period t.

Figure 2 shows an example of an i.i.d. sequence of primitive sunspot shocks. In the figure, the e_t are approximately normally distributed but with support $(-1, 1)$. Figure 3 shows examples of equilibrium price paths with the same primitive sunspot shocks but with different values of μ. These examples assume $\beta = 0.98, v(w) = 10w^{0.9} + 0.01w$, and $p_0 = p^* + 1$.[12]

Figure 3(a) shows that if there is no sunspot shock, p_t grows exponentially. Figure 3(b) suggests that if the price process is subject to small sunspot shocks, a realized price path more or less resembles the deterministic path. However, Figure 3(c) suggests that if the sunspot shocks are not small in magnitude, a realized path does not resemble the deterministic path, and it appears that the bubble disappears asymptotically. Figure 3(d) suggests that if the sunspot shocks are even larger, the bubble decays more quickly.

Figures 3(c) and 3(d) seem to suggest that bubbles decay exponentially if the sunspot shocks are sufficiently large. Figure 4 shows the details of the path in Figure 3(d) for $t = 800, \ldots, 1000$. It appears there also that the bubble decays exponentially. As a matter of fact, the exponential decay of bubbles can be proved formally using the results in Kamihigashi (2006).[13] Figures 3 and 4 illustrate the implication of the results in Kamihigashi (2006) that "explosive" bubbles do not always appear explosive: they decay to zero if the sunspot shocks are sufficiently large in magnitude.[14]

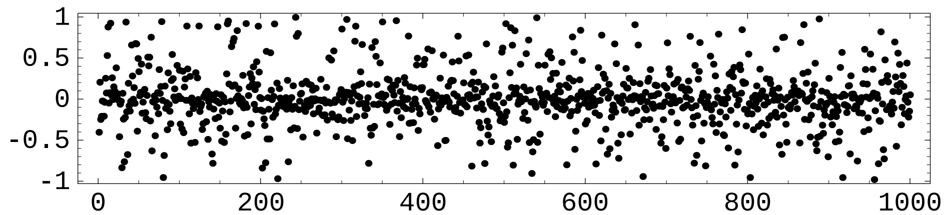
3 A Production Economy with Decreasing Returns

Now we consider a production economy to study how bubbles affect real activity. The model here is similar to a decentralized version of the Ramsey

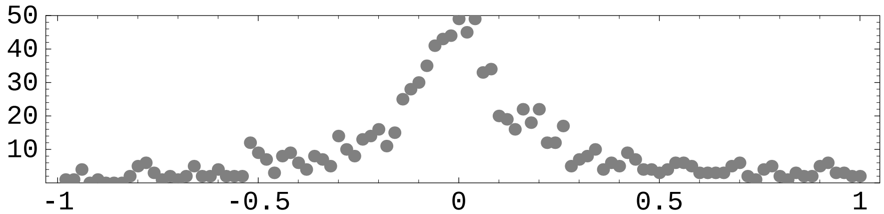
[12] In this case, p^* is approximately equal to 273. These parameter values are not intended to be empirically plausible. We are only interested in qualitative characteristics here.

[13] In fact the results in Kamihigashi (2006) were initially motivated by the numerical examples here.

[14] Salge (1997, p. 155) noted a similar phenomenon for a special parametric model. Nevertheless the exponential decay of bubbles seems to be worth pointing out since it is still widely believed that "in most models bubbles burst, while in reality bubbles seem to deflate over several weeks or even months" (Brunnermeier, 2007).



(a) e_t for $t = 1, \dots, 1000$



(b) Histogram of e_t

Figure 2: Primitive sunspot shocks e_t

model, except that utility depends on wealth in addition to consumption, and that the shares in firms are traded in the stock market as in Section 2. Each consumer in this economy faces the following maximization problem:

$$\max_{\{c_t, w_t, s_t, x_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t [u(c_t) + v(w_t)] \tag{3.1}$$

$$\text{s.t.} \quad c_t + w_t = (p_t + d_t)s_{t-1} + R_t x_{t-1}, \tag{3.2}$$

$$w_t = p_t s_t + x_t, \tag{3.3}$$

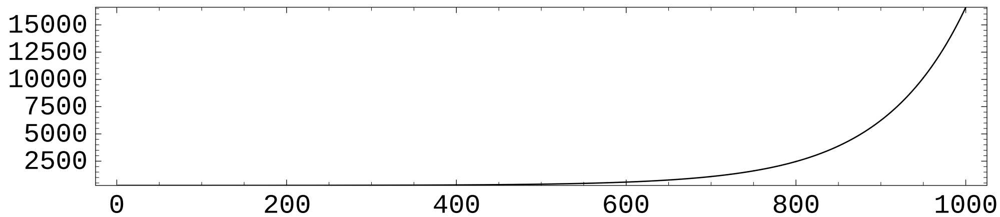
$$c_t, s_t, x_t \geq 0, \tag{3.4}$$

$$s_{-1} = 1, x_{-1} > 0 \text{ given}, \tag{3.5}$$

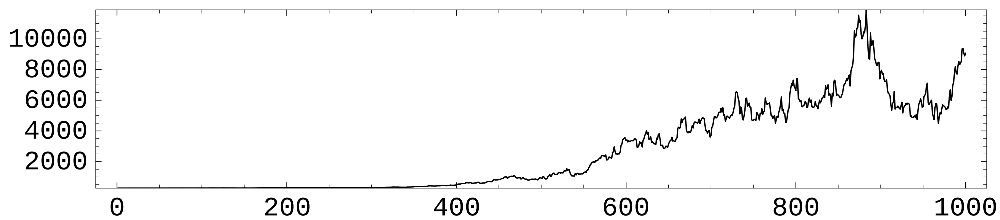
where x_t is the capital stock at the end of period t, and R_t is the gross rental rate on capital in period t. Equation (3.3) means that wealth consists of stock holdings and physical capital. Other notation is as in Section 2.

There are many homogeneous firms in this economy. In each period t, they maximize their profits, i.e., solve the following maximization problem:

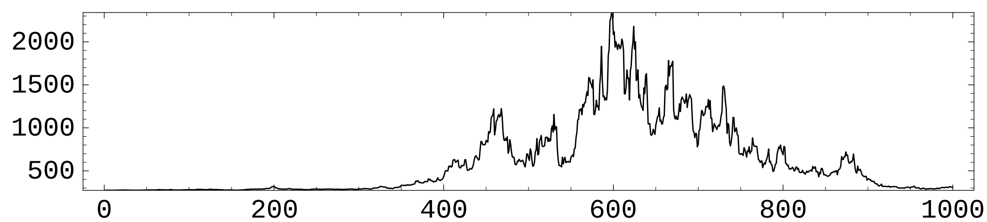
$$\max_{k_t \geq 0} [f(k_t) - R_t k_t], \tag{3.6}$$



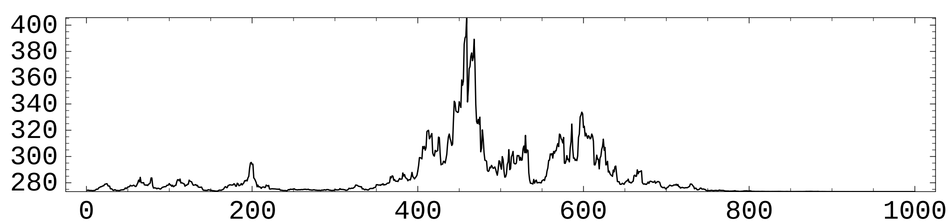
(a) $\epsilon_t = 0$



(b) $\epsilon_t = 0.2e_t$

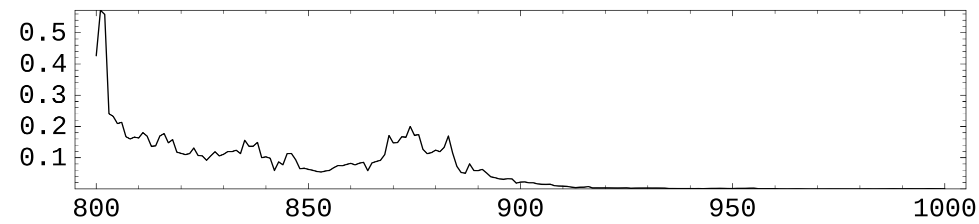


(c) $\epsilon_t = 0.4e_t$

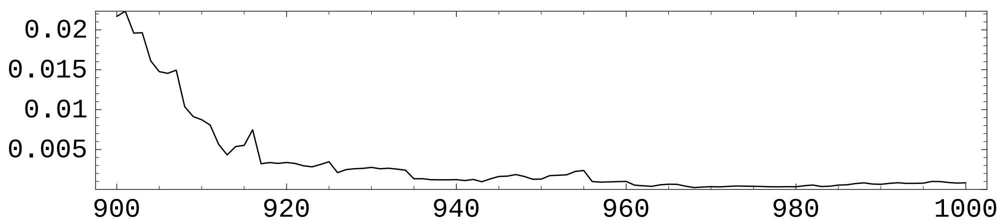


(d) $\epsilon_t = 0.6e_t$

Figure 3: p_t for $t = 1, \ldots, 1000$ with $p_0 = p^* + 1$



(a) $p_t - p^*$ for $t = 800, \ldots, 1000$



(b) $p_t - p^*$ for $t = 900, \ldots, 1000$

Figure 4: Details of Figure 3(d) for $t = 800, \ldots, 1000$

where f is the production function, and k_t is capital rented from consumers. The profits of firms are paid out to share holders as dividends:

$$d_t = f(k_t) - R_t k_t. \tag{3.7}$$

We maintain Assumption 2.1. In addition we assume standard properties on the production function.

Assumption 3.1. $f : \mathbb{R}_+ \to \mathbb{R}_+$ is continuously differentiable on $\mathbb{R}_{++}$, continuous, strictly increasing, and strictly concave.

An equilibrium of this economy is a set of paths $\{p_t, R_t, c_t, w_t, s_t, x_t\}_{t=0}^{\infty}$ such that (i) given $\{p_t, R_t\}$ and (3.7), $\{c_t, w_t, s_t, x_t\}$ solves the consumers' problem (3.1)–(3.5); (ii) given R_t, the solution to the firms' problem (3.6) is given by

$$k_t = x_{t-1}, \tag{3.8}$$

i.e., the capital market clears; and (iii) the good and stock markets clear:

$$c_t = f(k_t) - x_t. \tag{3.9}$$

$$s_t = 1. \tag{3.10}$$

For the rest of the paper we focus on interior equilibria, i.e., the equilibria where the nonnegativity constraints in (3.4) and (3.6) are never binding.[15] The firms' problem (3.6) and (3.8) imply

$$R_t = f'(x_{t-1}). \tag{3.11}$$

Thus $d_t = d(x_{t-1})$, where

$$d(x) = f(x) - f'(x)x. \tag{3.12}$$

It is easy to see (graphically) that

$$d(\cdot) \text{ is strictly increasing.} \tag{3.13}$$

In conjunction with (3.11) and (3.12), the Euler equations associated with s_t and x_t are written as

$$[u'(c_t) - v'(w_t)]p_t = \beta u'(c_{t+1})[p_{t+1} + d(x_t)], \tag{3.14}$$

$$u'(c_t) - v'(w_t) = \beta u'(c_{t+1}) f'(x_t). \tag{3.15}$$

The corresponding transversality conditions in equilibrium are

$$\lim_{t\to\infty} \beta^t [u'(c_t) - v'(w_t)]p_t = 0, \tag{3.16}$$

$$\lim_{t\to\infty} \beta^t [u'(c_t) - v'(w_t)]x_t = 0. \tag{3.17}$$

It follows from from (3.14) and (3.15) that

$$f'(x_t)p_t = p_{t+1} + d(x_t). \tag{3.18}$$

We define the fundamental value p_t^f of the stock as in Subsection 2.1:

$$p_t^f = \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} \frac{1}{f'(x_{t+i})} \right) d(x_{t+i}). \tag{3.19}$$

The bubble component p_t^b is given by $p_t^b = p_t - p_t^f$.

Since the dynamical system given by (3.14) and (3.15) is not easily tractable, we consider two tractable cases separately to better understand the equilibrium dynamics. The first case assumes that v is linear. The second case assumes that u is linear.

[15] Our assumptions do not immediately rule out non-interior equilibria, but note that we never assume the existence of an interior equilibrium.

4 The Case of Linear Utility of Wealth

In addition to Assumptions 2.1 and 3.1, this section assumes the following.

Assumption 4.1. v is linear, i.e., $v'(w)$ is constant for all $w > 0$. Without loss of generality, $v'(w) = 1$ for all $w > 0$.

Under this assumption, the Euler equations (3.14) and (3.15) as well as the transversality conditions (3.16) and (3.17) are independent of w_t:

$$[u'(c_t) - 1]p_t = \beta u'(c_{t+1})[p_{t+1} + d(x_t)], \tag{4.1}$$

$$u'(c_t) - 1 = \beta u'(c_{t+1}) f'(x_t), \tag{4.2}$$

$$\lim_{t\to\infty} \beta^t [u'(c_t) - 1]p_t = 0, \tag{4.3}$$

$$\lim_{t\to\infty} \beta^t [u'(c_t) - 1]x_t = 0. \tag{4.4}$$

Since (4.2) and (4.4) do not depend on the price path $\{p_t\}$, we may conclude that the capital path $\{x_t\}$ is determined independently of $\{p_t\}$. To see this more clearly, consider the following maximization problem:

$$\max_{\{c_t, x_t\}_{t=0}^{\infty}} \quad \sum_{t=0}^{\infty} \beta^t [u(c_t) + x_t] \tag{4.5}$$

$$\text{s.t.} \quad c_t + x_t = f(x_{t-1}), \tag{4.6}$$

$$x_t \geq 0, \tag{4.7}$$

$$x_{-1} > 0 \text{ given.} \tag{4.8}$$

This is a special case of the discrete-time version of Kurz's (1968) "capital-in-the-utility-function" model. Note that (4.2) and (4.4) are exactly the Euler equation and the transversality condition for the above maximization problem. Hence it completely characterizes the equilibrium capital paths, i.e., the "real" side of the economy is unaffected whether bubbles arise or not.[16] This indicates that the marginal utility of wealth plays a key role if bubbles affect real activity. We now turn to such a special case.

[16] Under additional conditions one can easily show that there are equilibria with bubbles.

5 The Case of Linear Utility of Consumption

In this section we maintain Assumptions 2.1, 2.3, 2.4, and 3.1. In addition we assume the following.

Assumption 5.1. u is linear, i.e., $u'(c)$ is constant for all $c > 0$. Without loss of generality, $u'(c) = 1$ for all $c > 0$. Furthermore, v is strictly concave.

Assumption 5.2. $\lim_{x\downarrow 0} f'(x) > (1-\nu)/\beta$ and $\lim_{x\uparrow\infty} f'(x) < 1$.

The last assumption is needed to ensure the existence of a steady state. Under the above assumptions, the Euler equations (3.14) and (3.15) and the transversality conditions (3.16) and (3.17) reduce to

$$[1 - v'(p_t + x_t)]p_t = \beta[p_{t+1} + d(x_t)], \tag{5.1}$$

$$1 - v'(p_t + x_t) = \beta f'(x_t), \tag{5.2}$$

$$\lim_{t\to\infty} \beta^t [1 - v'(p_t + x_t)]p_t = 0, \tag{5.3}$$

$$\lim_{t\to\infty} \beta^t [1 - v'(p_t + x_t)]x_t = 0. \tag{5.4}$$

5.1 Steady States

It follows from (5.1), (5.2), and (3.9) that steady state prices and capital stocks are characterized by the following three conditions:

$$1 - v'(p + x) = \beta f'(x), \tag{5.5}$$

$$p = \frac{d(x)}{f'(x) - 1}, \tag{5.6}$$

$$f(x) - x \geq 0. \tag{5.7}$$

Since (5.5) and (5.6) do not imply positive consumption, (5.7) must be included here. It is easy to see from (5.6) and (3.19) that there is no bubble in any steady state. The following result establishes the existence of a unique steady state.

Proposition 5.1. *There exists a unique steady state.*

Proof. By Assumptions 5.2, 2.3, and 2.4, there exist $\underline{x}, \overline{x} > 0$ such that

$$f'(\underline{x}) = \frac{1-\nu}{\beta} > 1 = f'(\overline{x}). \tag{5.8}$$

By strict concavity of f, we have $\underline{x} < \overline{x}$. Note from (5.5) and (5.6) that there is no (nonzero) steady state capital stock outside $(\underline{x}, \overline{x})$. For $x \in (\underline{x}, \overline{x})$, define

$$\phi(x) = 1 - v'\left(\frac{d(x)}{f'(x) - 1} + x\right) - \beta f'(x). \tag{5.9}$$

This is obtained by substituting (5.6) into (5.5). It follows that $x \in (\underline{x}, \overline{x})$ is a steady state capital stock if and only if $\phi(x) = 0$ and (5.7) holds. By (3.13) and strict concavity of f and v, $\phi(x)$ is strictly increasing. By (5.8),

$$\phi(\underline{x}) = -v'\left(\frac{d(\underline{x})}{f'(\underline{x}) - 1} + \underline{x}\right) + \nu < 0, \tag{5.10}$$

$$\lim_{x \uparrow \overline{x}} \phi(x) = 1 - \nu - \beta > 0, \tag{5.11}$$

where the inequalities hold by Assumptions 2.3 and 2.4. It follows that there is a unique $x^* \in (\underline{x}, \overline{x})$ satisfying $\phi(x^*) = 0$. Let $p^* = d(x^*)/(f'(x^*) - 1)$. Since $x^* < \overline{x}$, we have $f'(x^*) > 1$. Thus $f(x^*) - x^* > 0$ by strict concavity. Hence (5.7) holds. It follows that (p^*, x^*) is the unique steady state. □

5.2 Dynamics

The Euler equations (5.2) and (5.1) can be written as

$$1 - v'(p_t + x_t) = \beta f'(x_t), \tag{5.12}$$

$$\frac{p_{t+1} + d(x_t)}{p_t} = f'(x_t). \tag{5.13}$$

Equation (5.1) can also be written as

$$p_{t+1} = \frac{1 - v'(p_t + x_t)}{\beta} p_t - d(x_t). \tag{5.14}$$

As in the proof of Proposition 5.1, let p^* and x^* be the steady state price and capital stock, respectively. The following result characterizes the equilibrium paths with the initial capital stock at least as large as x^*. We focus on such equilibria for simplicity though other equilibria can be treated similarly.[17]

[17] In fact, since utility is linear in consumption, the initial capital stock x_{-1} is irrelevant as long as $f(x_{-1}) - x_0 \geq 0$, where x_0 is as in the proof of Proposition 5.2.

Proposition 5.2. *Suppose $x_{-1} \geq x^*$. There exists a continuum of equilibria. In particular, for any $p \geq p^*$, there exists a unique equilibrium $\{p_t, x_t\}$ with $p_0 = p$. If $p_0 = p^*$, then $p_t = p^*$ and $x_t = x^*$ for all $t \geq 0$. If $p_0 > p^*$, then $x_0 < x^*$, x_t is strictly decreasing and converges to $\underline{x}$ (recall (5.8)), and p_t is strictly increasing and goes to infinity. There exists no equilibrium with $p_0 < p^*$.*

Proof. The case $p_0 = p^*$ is straightforward. To handle the other cases, we first show the following.

Claim: Let $(p_{t-1}, x_{t-1}) \gg 0$ satisfy (5.12) with $t-1$ replacing t. Let $p_t > 0$. There exists a unique $(x_t, p_{t+1}) \in (\underline{x}, \infty) \times \mathbb{R}$ satisfying (5.12) and (5.13). If $p_t > p_{t-1} \geq p^*$ and $x_{t-1} \leq x^*$, then $x_t < x_{t-1}$ and $p_{t+1} > p_t$. Likewise, if $p_t < p_{t-1} \leq p^*$ and $x_{t-1} \geq x^*$, then $x_t > x_{t-1}$ and $p_{t+1} < p_t$.

To prove this claim, let $(p_{t-1}, x_{t-1}) \gg 0$ satisfy (5.12), and let $p_t > 0$. Note first that there is a unique $x_t > \underline{x}$ satisfying (5.12). Clearly there is a unique $p_{t+1} \in \mathbb{R}$ satisfying (5.13). Suppose $p_t > p_{t-1} \geq p^*$ and $x_{t-1} \leq x^*$. We have $x_t < x_{t-1}$ by strict concavity of v and f. Suppose $p_{t+1} \leq p_t$. Then by (5.13), (3.13), and (5.6),

$$f'(x_t) = \frac{p_{t+1} + d(x_t)}{p_t} \leq 1 + \frac{d(x_t)}{p_t} < 1 + \frac{d(x^*)}{p^*} = f'(x^*) < f'(x_t), \tag{5.15}$$

a contradiction. Thus $p_{t+1} > p_t$. The other case is similar. This completes the proof of the claim.

Let $p > p^*$. We construct an equilibrium $\{p_t, x_t\}$ with $p_0 = p$. By the above claim with $p_{-1} = p^*$ and $x_{-1} = x^*$, there is a unique $(x_0, p_1) \in (\underline{x}, \infty) \times \mathbb{R}$ satisfying (5.12) and (5.13). By the claim, $x_0 < x^*$ and $p_1 > p_0$. Constructing the entire sequence $\{p_t, x_t\}$ by repeated application of the claim, we see that $\{p_t\}$ is strictly increasing and $\{x_t\}$ is strictly decreasing. Since (p^*, x^*) is the unique steady state by Proposition 5.1, $p_t \uparrow \infty$ and $x_t \downarrow \underline{x}$ as $t \uparrow \infty$.[18] The associated consumption path is strictly positive since $f(x) > x$ for all $x \in (0, x^*]$. The proof of Lemma 2.1 still shows that the transversality condition (5.3) holds; (5.4) also holds since $\{x_t\}$ is bounded. It follows that $\{p_t, x_t\}$ is the unique equilibrium with $p_0 = p$.

Let $p \in (0, p^*)$. We show that there is no equilibrium $\{p_t, x_t\}$ with $p_0 = p$. Suppose $\{p_t, x_t\}$ is an equilibrium with $p_0 = p$. Then it follows from the above

[18] In more detail, if p_t converges to some $p \in (p^*, \infty)$, then x_t converges to some $x > \underline{x}$ by (5.12). Likewise, if x_t converges to some $x > \underline{x}$, then p_t converges to some $p \in (p^*, \infty)$. In either case, $(p, x) \neq (p^*, x^*)$ must be a steady state, contradicting uniqueness.

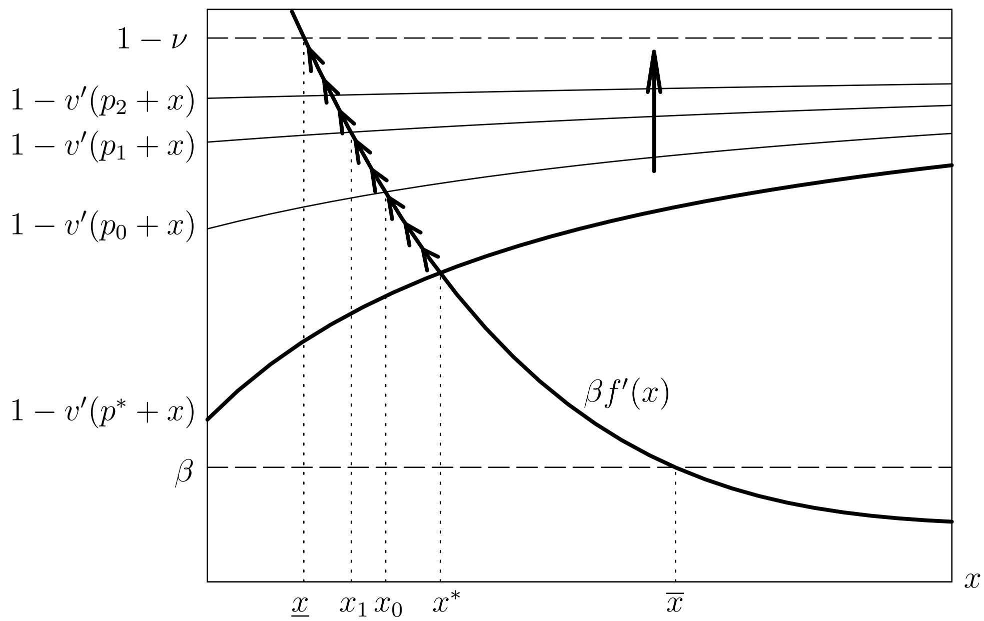


Figure 5: Equilibrium dynamics under decreasing returns

claim that $p_t \downarrow 0$ and $x_t \uparrow \infty$ since (p^*, x^*) is the unique steady state. This contradicts (5.14) since it implies that $p_{t+1} < 0$ for p_t close enough to 0. $\square$

Proposition 5.2 shows that there is a continuum of equilibria in which the capital stock keeps declining toward $\underline{x}$, while the stock price keeps rising indefinitely. We see from (3.19), (3.13), and the strict concavity of f that the fundamental value p_t^f keeps declining toward $d(\underline{x})/(f'(\underline{x})-1)$. Hence a bubble lowers the fundamental value as well as the capital stock, while increasing the stock price.

One can easily see from (5.12) why the stock price and the capital stock move in opposite directions. The left-hand side of (5.12) is the supply curve of capital given a stock price p_t. This curve is upward sloping. The right-hand side of (5.12) is the demand curve for capital, which is downward sloping. See Figure 5. If the stock price rises, the marginal utility of wealth declines, so that the incentive to hold capital as wealth decreases, i.e., the supply curve of capital shifts to the left, or upwards, resulting in a lower capital stock. The proof of Proposition 5.2 shows that a rise in stock price in one period implies another rise in the next period. Hence if $p_0 > p^*$, the stock price keeps increasing indefinitely, i.e., the supply curve of capital keeps shifting upwards, and converges to the horizontal line at $1 - \nu$ (because there is no steady state price other than p^*). This means that the capital stock converges to $\underline{x}$. See Figure 5 again.

Unfortunately, this inverse relationship between the stock price and the capital stock is contrary to conventional wisdom. However, it can be reversed if the demand curve of capital is upward sloping. The next section studies a model with this feature.

6 A Production Economy with Increasing Social Returns

Now we assume increasing social returns and externalities. The consumers' side of the model remains the same. The firms' problem is modified as follows:

$$\max_{k_t \geq 0} f(k_t, K_t) - R_t k_t, \tag{6.1}$$

where K_t is the social capital stock, which individual firms take as given. In equilibrium, k_t and K_t coincide:

$$k_t = K_t \ (= x_{t-1}). \tag{6.2}$$

We maintain Assumptions 2.1, 2.3, 2.4, and 5.1. In addition we assume the following.

Assumption 6.1. $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ is continuously differentiable on $\mathbb{R}^2_{++}$ and continuous, satisfying $f_1 > 0$ and $f_1 + f_2 > 0$ on $\mathbb{R}^2_{++}$. Furthermore, $f_1(x, y)$ is strictly decreasing in $x > 0$ for each $y > 0$, and $f_1(x, x)$ is strictly increasing in $x > 0$.

This assumption says that the production function is strictly increasing and strictly concave at the private level and strictly increasing in the overall capital stock; furthermore the private marginal product of capital is strictly increasing in the overall capital stock.

In order to ensure the existence of a steady state, we also assume that as capital increases from zero, the private marginal product of capital increases from a relatively small level to a relatively large level:

Assumption 6.2. $\lim_{x \downarrow 0} f_1(x,x) < 1$ and $\lim_{x \uparrow \infty} f_1(x,x) > (1-\nu)/\beta$.

The definition of equilibrium remains the same as in Section 3 except that the firms' problem is given by (6.1) and the rational expectations condition (6.2) is imposed as an equilibrium condition.

The Euler equations (5.1) and (5.2) remain the same except that $f'(x_t)$ is replaced with $f_1(x_t, x_t)$ and that $d(\cdot)$ is redefined as

$$d(x) = f(x,x) - f_1(x,x)x. \tag{6.3}$$

The transversality conditions (5.3) and (5.4) remain identical. The equations corresponding to (5.12) and (5.13) are

$$1 - v'(p_t + x_t) = \beta f_1(x_t, x_t), \tag{6.4}$$

$$\frac{p_{t+1} + d(x_t)}{p_t} = f_1(x_t, x_t). \tag{6.5}$$

Since $f(x,y)$ is strictly concave in x, we have $f(x,x) > f_1(x,x)x$, i.e., $d(x) > 0$ for $x > 0$.

We define the fundamental value p_t^f of the stock as in the case of decreasing returns:

$$p_t^f = \sum_{j=1}^{\infty} \left(\prod_{i=1}^{j} \frac{1}{f_1(x_{t+i}, x_{t+i})} \right) d(x_{t+i}). \tag{6.6}$$

The bubble component p_t^b is given by $p_t^b = p_t - p_t^f$ once again.

6.1 Steady States

The following three conditions, which correspond to (5.5)–(5.7), characterize the steady state prices and capital stocks:

$$1 - v'(p + x) = \beta f_1(x,x) \tag{6.7}$$

$$p = \frac{d(x)}{f_1(x,x) - 1}, \tag{6.8}$$

$$f(x,x) - x \geq 0. \tag{6.9}$$

It is easy to see from (6.8) and (6.6) that there is no bubble in any steady state, as in the case of decreasing returns. The following result establishes the existence of a steady state without asserting uniqueness.

Proposition 6.1. *There exists a steady state.*

Proof. By Assumption 6.2, there exist $\underline{x}, \overline{x} > 0$ such that

$$f_1(\underline{x}) = 1 < \frac{1-\nu}{\beta} = f_1(\overline{x}), \tag{6.10}$$

where $f_1(x) = f_1(x,x)$ for simplicity. By Assumption 6.1, $\underline{x} < \overline{x}$. By (6.7) and (6.8), there is no steady state capital stock outside $(\underline{x}, \overline{x})$. For $x \in (\underline{x}, \overline{x})$, define

$$\phi(x) = 1 - v'\left(\frac{d(x)}{f_1(x)-1} + x\right) - \beta f_1(x). \tag{6.11}$$

Note that $x \in (\underline{x}, \overline{x})$ is a steady state capital stock if and only if $\phi(x) = 0$ and (6.9) holds. We have

$$\lim_{x \downarrow \underline{x}} \phi(x) = 1 - \nu - \beta > 0, \tag{6.12}$$

$$\phi(\overline{x}) = -v'\left(\frac{d(\overline{x})}{f_1(\overline{x})-1} + \overline{x}\right) + \nu < 0. \tag{6.13}$$

Thus there is $\hat{x} \in (\underline{x}, \overline{x})$ (not necessarily unique) satisfying $\phi(\hat{x}) = 0$. Let $\hat{p} = d(\hat{x})/[f_1(\hat{x}) - 1]$. It remains to verify (6.9). Since $f(x,y)$ is strictly concave in x, we have $f(\hat{x},\hat{x}) > f_1(\hat{x},\hat{x})\hat{x} > \hat{x}$, where the last inequality holds since $\hat{x} > \underline{x}$. Now (6.9) follows. □

6.2 Dynamics

Let x^* be the largest steady state capital stock and p^* the corresponding stock price. For the rest of this section, we focus on equilibria with the initial capital stock at least as large as x^*. We assume the following.

Assumption 6.3. $\overline{x} \leq f(x^*, x^*)$.

This assumption means that $\overline{x}$ is technologically feasible from x^* in one step. While (6.4) and (6.5) do not prevent consumption from becoming negative, Assumption 6.3 ensures that consumption is strictly positive whenever the capital stock moves within $[x^*, \overline{x})$. This extra assumption is needed here since contrary to the case of decreasing returns, the capital stock may increase over time in equilibria with bubbles.

Proposition 6.2. *Suppose $x_{-1} \geq x^*$. For any $p \geq p^*$, there exists an equilibrium $\{p_t, x_t\}$ with $p_0 = p$. In particular, the constant sequence with $(p_0, x_0) = (p^*, x^*)$ is an equilibrium, and for $p > p^*$, there exists an equilibrium with $p_0 = p$ such that $x_0 > x^*$, x_t is strictly increasing and converges to $\overline{x}$ (recall (6.10)), and p_t is strictly increasing and goes to ∞.*

Proof. The case $p_0 = p^*$ is straightforward. To handle the other case, we show the following.

Claim: Let $p_{t-1} \geq p^*$ and $x_{t-1} \geq x^*$ satisfy (6.4) with $t-1$ replacing t. Let $p_t > p_{t-1}$. Then there is $(x_t, p_{t+1}) \in (x_{t-1}, \overline{x}) \times (p_t, \infty)$ satisfying (6.4) and (6.5) such that $x_t > x_{t-1}$, $f(x_{t-1}, x_{t-1}) - x_t > 0$, and $p_{t+1} > p_t$.

To prove this claim, let $p_{t-1} \geq p^*$ and $x_{t-1} \geq x^*$ satisfy (6.4). Let $p_t > p_{t-1}$. Note that

$$1 - v'(p_t + x_{t-1}) > 1 - v'(p_{t-1} + x_{t-1}) = \beta f_1(x_{t-1}), \tag{6.14}$$

$$1 - v'(p_t + \overline{x}) < 1 - \nu = \beta f_1(\overline{x}). \tag{6.15}$$

Thus there exists $x_t \in (x_{t-1}, \overline{x})$ satisfying (6.4). We have $f(x_{t-1}, x_{t-1}) \geq f(x^*, x^*) > x_t$ by Assumption 6.3. Let $p_{t+1} \in \mathbb{R}$ be given by (6.5). Suppose $p_{t+1} \leq p_t$. Then by (6.5),

$$p_t = \frac{d(x_t)}{f_1(x_t) - \frac{p_{t+1}}{p_t}} \leq \frac{d(x_t)}{f_1(x_t) - 1}, \tag{6.16}$$

where $f_1(x_t) > 1$ since $x_t > x_{t-1} \geq x^* > \underline{x}$ (recall (6.10)). By (6.4) and (6.16),

$$0 = 1 - v'(p_t + x_t) - \beta f_1(x_t) \tag{6.17}$$

$$\leq 1 - v'\left(\frac{d(x_t)}{f_1(x_t) - 1} + x_t\right) - \beta f_1(x_t) = \phi(x_t), \tag{6.18}$$

where ϕ is as defined in (6.11). This together with (6.13) implies that there is a steady state capital stock $x \in [x_t, \overline{x})$, which is a contradiction since x^* is the largest steady state capital stock. Hence $p_{t+1} > p_t$. This completes the proof of the claim.

Let $p > p^*$. We construct an equilibrium $\{p_t, x_t\}$ with $p_0 = p$. By the above claim with $p_{-1} = p^*$ and $x_{-1} = x^*$, there is $(x_0, p_1) \in (x^*, \overline{x}) \times (p_0, \infty)$ satisfying (6.4) and (6.5) such that $x_0 > x^*$, $f(x^*, x^*) - x_0 > 0$, and $p_1 > p_0$. Constructing the entire sequence $\{p_t, x_t\}$ by repeated application of the claim,

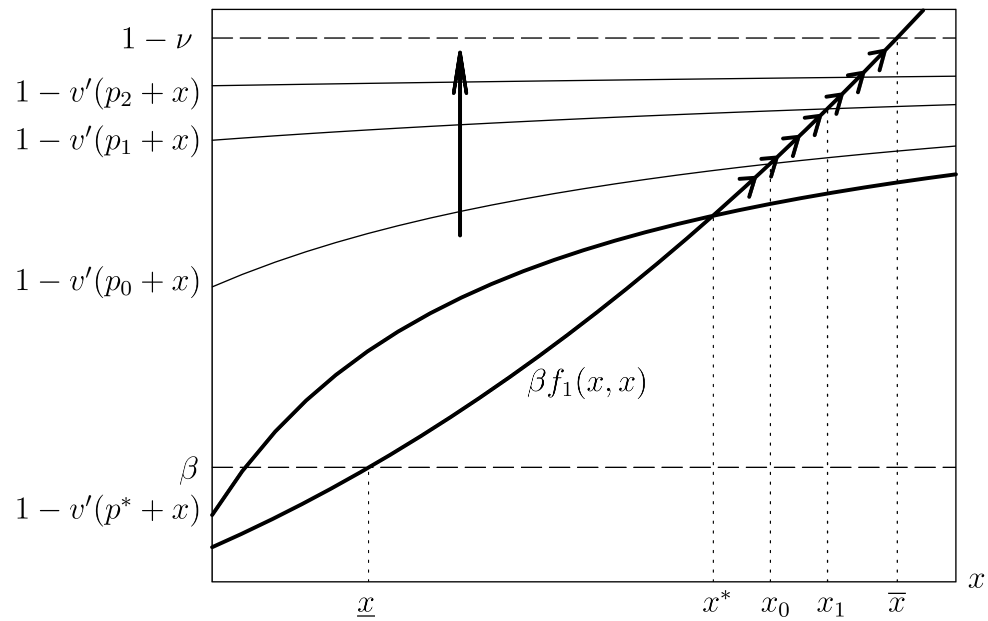


Figure 6: Equilibrium dynamics under increasing returns

we see that both $\{p_t\}$ and $\{x_t\}$ are strictly increasing and that the associated consumption path is strictly positive. Since there is no steady state (p, x) such that $(p, x) \gg (p^*, x^*)$, we have $\lim_{t\uparrow\infty} p_t = \infty$ and $\lim_{t\uparrow\infty} x_t = \overline{x}$. As in the proof of Proposition 5.2, $\{p_t, x_t\}$ satisfies the transversality conditions (5.3) and (5.4). Hence it is an equilibrium. $\square$

Proposition 6.2 shows that there is a continuum of equilibria in which both the stock price and the capital stock grow over time. Contrary to the case of decreasing returns, the dynamic behavior of the fundamental value is not immediately clear. On the other hand, it follows from (6.6) that for the equilibria described in Proposition 6.2, the fundamental value is bounded above:

$$p_t^f \leq \frac{\max_{x^* \leq x \leq \overline{x}} d(x)}{f_1(x^*, x^*) - 1}. \tag{6.19}$$

This implies that the bubble component p_t^b is always strictly positive whenever the stock price p_t grows unboundedly.

As in the case of decreasing returns, the left-hand side of (6.4) is the supply curve of capital given a stock price p_t, while the right-hand side is the demand curve for capital. If the stock price rises, the marginal utility of wealth declines, so that the incentive to hold capital as wealth decreases, i.e., the supply curve of capital shifts to the left, or upwards, as in the case of decreasing returns. Here, however, the demand curve for capital is upward sloping and crosses the supply curve from below.[19] See Figure 6. Therefore, a higher stock price implies a higher capital stock. In other words, bubbles affect output positively.

The proof of Proposition 6.2 shows that a rise in stock price in one period implies another rise in the next period. Hence if $p_0 > p^*$, the stock price grows indefinitely, i.e., the supply curve of capital keeps shifting upwards and converges to the horizontal line at $1 - \nu$ (because there is no steady state price strictly greater than p^*). This means that the capital stock converges to $\overline{x}$. See Figure 6 again.

Unlike Proposition 5.2, however, Proposition 6.2 does not characterize all equilibria. In particular, it says nothing about equilibria with $p_0 < p^*$. Such equilibria may or may not exist depending on the number of steady states and the shapes of the functions involved in (6.4) and (6.5).

6.3 Sunspot Equilibria

Sunspot equilibria can be constructed as in the exchange economy. In particular, the stochastic version of (6.5) can be written as

$$E_t p_{t+1} = f_1(x_t, x_t) p_t - d(x_t). \tag{6.20}$$

As in (2.38), this can be expressed equivalently as

$$p_{t+1} - p^* = [f_1(x_t, x_t) p_t - d(x_t) - p^*](1 + \epsilon_{t+1}), \tag{6.21}$$

where ϵ_{t+1} is a sunspot shock satisfying $E_t \epsilon_{t+1} = 0$.

Figure 8 shows examples of sunspot equilibria with common primitive sunspot shocks, which are plotted in Figure 7. The primitive sunspot shocks e_t here are different from those in Figure 2, but they are drawn from the same

[19] The demand curve crosses the supply curve from below if there is only one intersection, or at least at the rightmost intersection if there are multiple intersections. This is because $1 - v'(p + \overline{x}) < 1 - \nu = \beta f_1(\overline{x}, \overline{x})$ for any $p > 0$. It follows that if the supply curve shifts to the left, we can always find a higher capital stock equating demand and supply.

distribution. The sunspot shocks ϵ_t are given by (2.39) for each value of μ. The examples in Figure 8 assume $\beta = 0.98, v(w) = 207984w^{0.1} + 0.01w$, and $f(k, K) = 0.01k^{0.7}K^{0.4} + 0.96k$. These parameter values are chosen in such a way that all the relevant assumptions are satisfied; these values are not intended to be empirically plausible. Output is given by $y_t = 0.01k^{0.7}K^{0.4}$.

Figure 8 shows four pairs of $(p_t - p^*)/(p_1 - p^*)$ and $(y_t - y^*)/(y_1 - y^*)$, where y^* is the steady state level of output. We normalize p_t and y_t this way so that the relationship between p_t and y_t can easily be seen.[20] Figure 8(a) illustrates how p_t and y_t grow deterministically if there is no sunspot shock. By contrast, in Figures 8(b), 8(c), and 8(d), the bubble collapses around period 25 due to a series of negative sunspot shocks. It is important to notice that a change in stock price is followed by a similar change in output one period later. This is because a current change in stock price affects investment for the next period. Hence in our model, the bursting of a bubble is necessarily followed by a sharp decline in output one period later.

Though the first halves of Figures 8(b), 8(c), and 8(d) are quite similar, the second halves are strikingly different. In particular, in Figure 8(b), both p_t and y_t follow an upward trend after period 40, while such a trend is not visible in Figures 8(c) and 8(d). Note also that the "post-bubble recession" is more severe the larger the sunspot shocks are in magnitude.[21] This is because as discussed in Section 2.4, bubbles decay to zero asymptotically if the sunspot shocks are sufficiently large (or μ is sufficiently large), while a stochastic equilibrium resembles its deterministic counterpart when the sunspot shocks are sufficiently small (or μ is sufficiently small).

Though the bubbles in these examples collapse due to a series of negative sunspot shocks, such bad luck is not necessary for bubbles to disappear asymptotically. Once again, this is because bubbles decay to zero asymptotically if the sunspot shocks are sufficiently large. Figures 3(c) and 3(d) show such examples. Similar patterns will be observed for both p_t and y_t if the primitive sunspot shocks in Figure 2 are used here.

[20] However, the variabilities of p_t and y_t differ considerably. In all four cases, $p^* \approx 7.45\times10^8$, $x^* \approx 3.49\times10^8$, $y^* \approx 2.50\times10^7$, $p_0 = 2p^*$, $x_0 = x^*$, and $y_1 \approx 2.53\times10^7$, which is known in period 0 given p_0 and x_0. We have $p_1 \approx 1.50\times10^9, 1.52\times10^9, 1.54\times10^9, 1.55\times10^9$ in cases (a)–(d), respectively.

[21] In standard macroeconomic models, fluctuations occur around a steady state, so that a recession occurs below a steady state. In our examples, by contrast, all fluctuations occur above the steady state, which is the worst possible state of the economy here. Hence we interpret as a recession a period in which output is close to the steady state level.

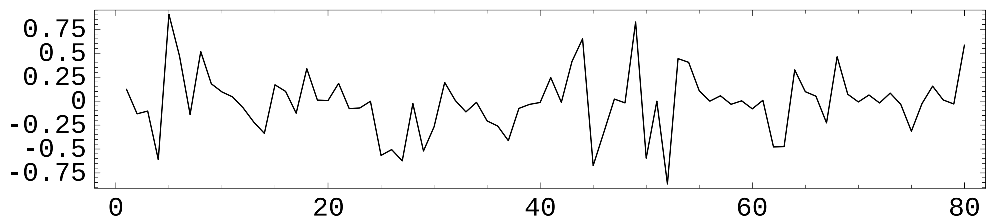
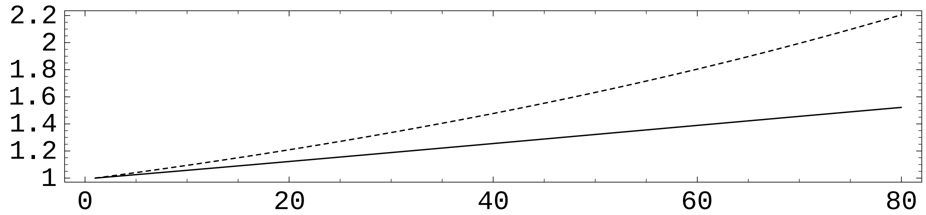


Figure 7: Primitive sunspot shocks e_t for $t = 1, \ldots, 80$

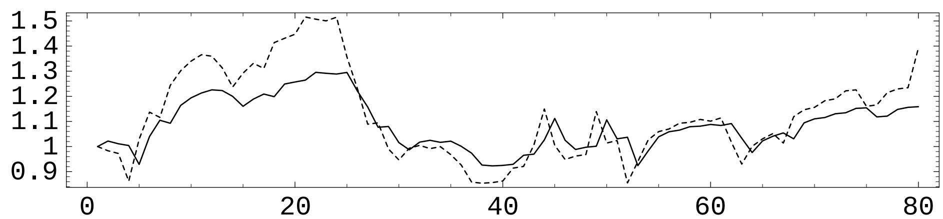
7 Concluding Remarks

In this paper we have constructed a representative agent model in which stock market bubbles affect real activity, giving rise to output fluctuations. Following the literature on the spirit of capitalism, we have assumed that utility depends directly on wealth. We have shown that stock market bubbles arise if the marginal utility of wealth does not decline to zero as wealth goes to infinity. We have also shown that bubbles may affect output positively or negatively depending on whether the production function exhibits increasing or decreasing returns to scale. In our mode, a rise in stock price reduces the incentive to hold capital as wealth; thus the supply curve of capital, which is upward sloping, shifts to the left. In the case of decreasing returns, the demand curve for capital is downward sloping, so that the capital stock falls in response to a fall in stock price. In the case of increasing returns, on the other hand, the demand curve for capital is upward sloping, so that the capital stock may rise in response to a fall in stock price. Since a current change in stock price affects investment for the next period, the bursting of a bubble is followed by a sharp decline in output one period later. Using numerical examples of sunspot equilibria, we have shown that bubbles decay to zero asymptotically if the sunspot shocks are sufficiently large, and that fluctuations in stock price are followed closely by fluctuations in output one period later.

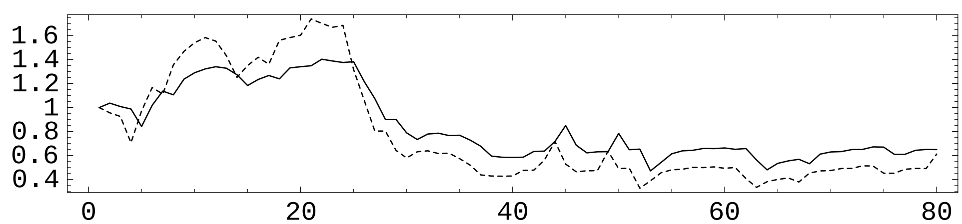
We have established various analytical results at the cost of some restrictive assumptions. For example, we have assumed throughout the paper that utility is additively separable in consumption and wealth. We have also assumed that utility is linear in consumption in the main part of our analysis. We have shown however that bubbles do not affect real activity at all if utility is strictly concave in consumption but is linear in wealth. This suggests that



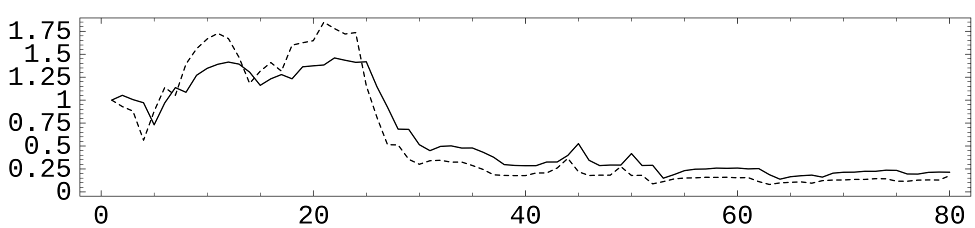
(a) $\epsilon_t = 0$



(b) $\epsilon_t = 0.2e_t$



(c) $\epsilon_t = 0.4e_t$



(d) $\epsilon_t = 0.6e_t$

Figure 8: $(p_t - p^*)/(p_1 - p^*)$ (dashed line) and $(y_t - y^*)/(y_1 - y^*)$ (solid line) for $t = 1, \ldots, 80$

the real effects of bubbles may depend on the relative degree of concavity in terms of consumption and wealth. An investigation of this issue and an examination of the effects of bubbles induced by other mechanisms are left for future research.

References

Allen, F., and D. Gale (1999), "Bubbles, crises, and policy," *Oxford Review of Economic Policy* 15, 9–18.

Bakshi, G.S., and Z. Chen (1996), "The spirit of capitalism and stock-market prices," *American Economic Review* 86, 133–157.

Binswanger, M., (2004), "Stock returns and real activity in the G-7 countries: did the relationship change during the 1980s?" *Quarterly Review of Economics and Fiance* 44, 237–252.

Brock, W.A. (1974), "Money and growth: the case of long run perfect foresight," *International Economic Review* 15, 750–777.

Brunnermeier, M.K. (2007), "Bubbles," S. Durlauf and L. Blume, eds., *New Palgrave Dictionary of Economics,* 2nd Edition, forthcoming, London: Palgrave Macmillan.

Buiter, W.H., and A.C. Sibert (2007), "Deflationary bubbles," *Macroeconomic Dynamics,* forthcoming.

Chang, W.-Y., H.-F. Tsai, and C.-C. Lai (2004), "Taxation, growth, and the spirit of capitalism," *European Journal of Political Economy* 20, 1011–1025.

Futagami, K., and A. Shibata (2000), "Growth effects of bubbles in an endogenous growth model," *Japanese Economic Review* 51, 221–235.

Gong, L., and H.-F. Zou (2002), "Direct preferences for wealth, the risk premium puzzle, growth, and policy effectiveness," *Journal of Economic Dynamics and Control* 26, 247–270.

Grossman, G.M., and N. Yanagawa (1993), "Asset bubbles and endogenous growth," *Journal of Monetary Economics* 31, 3–19.

Hassapis, C., and S. Kalyvitis (2002), "Investigating the links between growth and real stock changes with empirical evidence from the G-7 economies," *Quarterly Review of Economics and Finance* 42, 543–575.

Iraola, M.A., and M.S. Santos (2007), "Speculative bubbles," S. Durlauf and L. Blume, eds., *New Palgrave Dictionary of Economics,* 2nd Edition, forthcoming, London: Palgrave Macmillan.

Kamihigashi, T. (1998), "Uniqueness of asset prices in an exchange economy with unbounded utility," *Economic Theory* 12, 103–122.

Kamihigashi, T. (2001), "Necessity of transversality conditions for infinite horizon problems," *Econometrica* 69, 995–1012.

Kamihigashi, T. (2003), "Necessity of transversality conditions for stochastic problems," *Journal of Economic Theory* 109, 140–149.

Kamihigashi, T. (2006), "Almost sure convergence to zero in stochastic growth models," *Economic Theory* 29, 231–237.

Kurz, M. (1968), "Optimal economic growth and wealth effects," *International Economic Review* 9, 348–357.

Lucas, R.E., Jr. (1978), "Asset prices in an exchange economy," *Econometrica* 46, 1429–1445.

Montrucchio, L., and F. Privileggi (2001), "On fragility of bubbles in equilibrium asset pricing models of Lucas-type," *Journal of Economic Theory* 101, 158–188.

Nakajima, T. (2003), "Sunspot fluctuations in asset prices and business cycles in Japan over 1986–1999," *Japanese Economic Review* 54, 253–274.

Obstfel, M., and K. Rogoff (1986), "Ruling out divergent speculative bubbles," *Journal of Monetary Economics* 17, 349–362.

Olivier, J. (2000), "Growth-enhancing bubbles," *International Economic Review* 41, 133–151.

Ono, Y. (2001), "A reinterpretation of Chapter 17 of Keynes's General Theory: effective demand shortage under dynamic optimization," *International Economic Review* 42, 207–236.

Rebelo, S. (2005), "Real business cycle models: past, present and future," *Scandinavian Journal of Economics* 107, 217–238.

Salge, M. (1997), *Rational Bubbles: Theoretical Basis, Economic Relevance, and Empirical Evidence with a Special Emphasis on the German Stock Market.* Springer-Verlag: Berlin.

Santos, M., and M. Woodford (1997), "Rational asset pricing bubbles," *Econometrica* 65, 19–57.

Smith, W.T. (2001), "How does the spirit of capitalism affect stock market prices?" *Review of Financial Studies* 14, 1215–1232.

Tirole, J. (1985), "Asset Bubbles and Overlapping Generations," *Econometrica* 53, 1499–1528.

Weber, M. (1905), *The Protestant Ethic and the Spirit of Capitalism.* Charles Scribner's Sons: New York.

Zou, H.-F. (1994), "The spirit of capitalism' and long-run growth," *European Journal of Political Economy* 10, 279–293.

Zou, H.-F. (1995), "The spirit of capitalism and savings behavior," *Journal of Economic Behavior and Organization* 28, 131–143.