Transversality Conditions and Dynamic Economic Behavior*

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Abstract

Transversality conditions are optimality conditions often used along with Euler equations to characterize the optimal paths of dynamic economic models. This article explains the foundations of transversality conditions using a geometric example, a finite horizon problem, and an infinite horizon problem. Their relationships to asset bubbles, hyperdeflations, and no-Ponzi-game conditions are also discussed.

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Transversality conditions and dynamic economic behavior

Transversality conditions are optimality conditions often used along with Euler equations to characterize the optimal paths (plans, programs, trajectories, etc) of dynamic economic models.

An Euler equation is a local condition that no gain be achieved by slightly deviating from an optimal path for a short period of time. In many cases an Euler equation is equivalent to the property that no gain be achieved by deviating from an optimal path and eventually returning to it. If the terminal (or initial) point is not fixed, there may be many paths satisfying the Euler equation. A transversality condition enables one to single out the optimal path among those satisfying the Euler equation, or at least to rule out some non-optimal paths. Along with the Euler equation, it requires that no gain be achieved by deviating from an optimal path and never returning to it. Such deviations are possible only if the terminal point is not fixed.

A simple geometric example best illustrates the roles of an Euler equation and a transversality condition. What is the shortest path from a point A to a straight line L infinitely long in both directions? The answer is of course the straight line from point A to line L that is perpendicular to line L. There are two conditions involved here. The first condition is that the shortest path be a straight line: one cannot make the path shorter by deviating from it and eventually returning to it. This is the implication of the Euler equation for this problem. But there are infinitely many straight lines from point A to line L. In fact, a straight line from point A to line L can be arbitrarily long, so that even very bad choices satisfy the Euler equation. This is why one needs the second condition, that the shortest path be perpendicular to line L. This additional condition ensures that one cannot make the path shorter by deviating from it and never returning to it.

The condition of perpendicularity in this example and similar conditions on end points in other problems are called transversality conditions in dynamic optimization theory (Hestenes, 1966, p. 87). According to Bolza (1904, p. 106), the term was first introduced by Kneser (1900).

The finite horizon case

Though both Euler equations and transversality conditions were initially developed for continuous time models (more precisely, calculus of variations problems), the basic arguments can be better understood in discrete time models, to which we restrict ourselves. All arguments below apply to both cases.

Consider the following maximization problem:

$$\max_{\{x_t\}_{t=0}^{T+1}} \sum_{t=0}^{T} \beta^t v(x_t, x_{t+1}) \tag{1}$$

$$\text{s.t. } x_0 = \overline{x}_0, \quad x_{t+1} \geq 0,\ t = 0, 1, 2, \ldots, T, \tag{2}$$

where $\beta \in (0, 1)$ is called the discount factor, v is called the return function, and $\overline{x}_0$ is a given initial condition. To be concrete, we interpret x_t as the stock of wealth (or capital) at the beginning of period t.

There may be other constraints, but we assume that they are not binding at the optimum. We assume for simplicity that the nonnegativity constraint is not binding at the optimum except for x_{T+1}. This can be ensured by assuming the Inada condition $v_1(0, x) = \infty$ for all x. Note that x_{T+1} is free except for the nonnegativity constraint. We also assume that the return function v is differentiable and concave. It may be allowed to depend on t though we do not assume so here for notational simplicity.

The Euler equation for this problem is simply the first order condition with respect to x_{t+1} for $t < T$:

$$v_2(x_t, x_{t+1}) + \beta v_1(x_{t+1}, x_{t+2}) = 0. \tag{3}$$

This condition means that no gain can be achieved by deviating from an optimal path for one period.

The first order condition with respect to x_{T+1} consists of two cases:

$$\beta^T v_2(x_T, x_{T+1}) = 0 \quad \text{or} \quad \beta^T v_2(x_T, x_{T+1}) \leq 0, x_{t+1} = 0. \tag{4}$$

In most economic problems, it is costly to accumulate wealth. Hence we assume that $v_2(x, y) \leq 0$ for all x, y. Then the two cases in (4) can be combined into

$$\beta^T[-v_2(x_T, x_{T+1})]x_{T+1} = 0. \tag{5}$$

This is the transversality condition for this problem. It means that nothing should be saved in the last period unless it is costless to do so (i.e., $-v_2(x_T, x_{T+1}) = 0$). Alternatively it can be interpreted as saying that the present discounted value of the terminal stock must be zero.

Since the Euler equation (3) and the transversality condition (5) are first order conditions, they are necessary for optimality. It is easy to verify that they are sufficient as well by concavity of the return function v. There is no technical issue in the finite horizon case.

Equation (5) is a typical transversality condition in economics, but it is not the only possibility in other models. The proper choice of transversality condition depends on the exact constraint on the terminal stock.

The infinite horizon case

Let us now consider the infinite horizon case. The maximization problem is

$$\max_{\{x_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t v(x_t, x_{t+1}) \tag{6}$$

$$\text{s.t. } x_0 = \overline{x}_0, \quad x_{t+1} \geq 0,\ t = 0, 1, 2, \ldots . \tag{7}$$

We assume for simplicity that the nonnegativity constraint is never binding at the optimum. This can be ensured by the Inada condition mentioned above.

The first order condition with respect to x_{t+1} remains the same, so that the Euler equation remains the same. Unfortunately, unlike in the finite horizon case, one cannot directly derive the transversality condition here. Instead one needs to derive the transversality condition for the finite horizon case first ((5) in this case) and then take the limit:

$$\lim_{T \to \infty} \beta^T [-v_2(x_T, x_{T+1})] x_{T+1} = 0. \tag{8}$$

This can be interpreted as saying that the present discounted value of wealth at infinity must be zero, or wealth (x_{T+1}) should not grow too fast compared to its marginal value ($\beta^T[-v_2(x_T, x_{T+1})]$). In other words, the transversality condition (8) rules out overaccumulation of wealth. The idea is that if one saves too much and spends too little forever, then one is not behaving optimally.

There is an alternative transversality condition that is often used:

$$\lim_{T\to\infty} \beta^T v_1(x_T, x_{T+1})x_T = 0. \tag{9}$$

Though this condition is equivalent to (8) under the Euler equation (3), it has no counterpart in the finite horizon case and in the continuous time case.

It is well known (and easy to show) that the Euler equation (3) and the transversality condition (8) are sufficient for optimality (e.g., Stokey and Lucas, 1989, p. 89). This result is often credited to Mangasarian (1966), who showed the finite horizon version of the result for a continuous time model.

Since the Euler equation is simply the first order condition with respect to x_{t+1}, it remains to be a necessary condition in the infinite horizon case. On the other hand, necessity of the transversality condition in the infinite horizon case is often considered to be a difficult issue. But there are two simple ways to prove it if the objective function is assumed to be finite for all feasible paths (Kamihigashi, 2002, 2005). If this assumption is not assured, one can try the following test. Shift the entire optimal path downward by a small fixed proportion. Does it reduce the value of the objective function by only a finite amount? If so, the transversality condition is necessary. See Kamihigashi (2001, 2003) for precise assumptions and statements. See Weitzman (1973), Benveniste and Scheinkman (1982), and Michele (1982) for earlier results and arguments, and Kamihigashi (2001) for a literature review.

Asset bubbles and transversality conditions

Transversality conditions are often used to rule out asset bubbles. To be specific, consider a deterministic version of the Lucas (1978) asset pricing model. There are many homogeneous agents, a single good, and a single asset that pays a dividend of d_t units of the good in each period t. The population of agents is normalized to one; so is the supply of the asset. Each agent solves

$$\max_{\{c_t,x_t\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{10}$$

$$\text{s.t. } x_0 = 1, \tag{11}$$

$$c_t + p_t x_{t+1} = (p_t + d_t)x_t,\ x_{t+1} \geq 0,\ t = 0, 1, 2, \ldots, \tag{12}$$

where c_t is consumption, p_t is the price of the asset, and x_t is shares in the asset at the beginning of period t. In equilibrium, $c_t = d_t$ and $x_t = 1$. We assume that the utility function u is concave, differentiable, and strictly increasing. The Euler equation and the transversality condition in equilibrium are

$$u'(d_t)p_t = \beta u'(d_{t+1})(p_{t+1} + d_{t+1}), \tag{13}$$

$$\lim_{T\to\infty} \beta^T u'(d_T)p_T = 0. \tag{14}$$

It is easy to see that the sequence $\{p_t^*\}$ given by

$$p_t^* = \sum_{i=1}^{\infty} \beta^i \frac{u'(d_{t+i})}{u'(d_t)} d_{t+i} \tag{15}$$

satisfies the Euler equation (13). The right-hand side of (15) is called the fundamental value of the asset. Let $\{b_t\}$ be any nonnegative sequence satisfying

$$u'(d_t)b_t = \beta u'(d_{t+1})b_{t+1}. \tag{16}$$

Then the sequence $\{p_t^* + b_t\}$ also satisfies the Euler equation. Hence there are infinitely many paths satisfying the Euler equation. The extra component b_t, which grows at a gross rate of $u'(d_t)/[\beta u'(d_{t+1})]$, is interpreted as a bubble.

Notice that the bubble component b_t, if positive, violates the transversality condition (14) (with $p_T = b_T$). Therefore, if the transversality condition is necessary, the bubble component must vanish, so that the price must always be equal to the fundamental value. This is indeed the case here (Kamihigashi, 2001, p. 1007).

In stochastic models, bubbles can be ruled out under standard assumptions, but there are pathological cases in which bubbles are possible (Kamihigashi, 1998; Montrucchio and Privileggi, 2001).

Hyperdeflations and transversality conditions

Transversality conditions are often used to rule out hyperdeflationary paths in money-in-the-utility-function models of the type studied by Brock (1974) and Obstfeld and Rogoff (1986). In these models, agents derive utility from real money balances in addition to consumption. As in the Lucas asset pricing model, there are many paths satisfying the Euler equation. A solution to the

Euler equation with a positive bubble is often called a hyperdeflationay path, in which the nominal price keeps declining toward zero. If the nominal price falls, the value of real balances rises. Hence in a hyperdeflationary path, the value of real balances grows unboundedly. Under reasonable assumptions, such paths are ruled out by an appropriate transversality condition, which once again rules out overaccumulation of wealth.

However, there are cases in which the transversality condition does not rule out hyperdeflationary paths (Obstfeld and Rogoff, 1986, p. 356). This is because agents, who derive utility from real balances, directly benefit from accumulating wealth.

No-Ponzi-game conditions and transversality conditions

In formulating a consumer's problem, one must include some constraint on debt, since otherwise the consumer would never pay back his debt, letting it grow unboundedly. One way to rule out this behavior is to prohibit debt entirely, i.e., to require wealth to be always nonnegative. A more lenient way is to require only the present discounted value of wealth at infinity to be nonnegative. This type of condition is known as a no-Ponzi-game condition (Blanchard and Fischer, 1989, p. 49), but often called a transversality condition as well. A no-Ponzi-game condition is a constraint that prevents overaccumulation of debt, while a transversality condition is an optimality condition that rules out overaccumulation of wealth. They place opposite restrictions, and should not be confused.

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