Discussion Paper Series No.149

Indeterminacy in a Two-sector Endogenous Growth Model with Productive Government Spending

Yunfang Hu Kobe University
Ryoji Ohdoi Osaka University
Koji Shimomura Kobe University

March 2004

Indeterminacy in a Two-sector Endogenous Growth Model with Productive Government Spending*

Yunfang Hu
GSE, Kobe University

Ryoji Ohdoi
GSE, Osaka University

Koji Shimomura†
RIEB, Kobe University

Abstract: We extend the Barro (1990) model of endogenous growth to a two-sector one which consists of pure consumption and investment goods. It is possible that the extended version has a unique balanced growth rate such that for given initial values of state variables, (i) the extended model economy grows at the unique rate right from the beginning or (ii) it has a continuum of equilibrium paths whose growth rates commonly converge to the balanced growth rate. That is, unlike the original one-sector model, it has transitional dynamics in case (ii). We also show that the effects of small changes in some parameters on the balanced growth rate and the price of the consumption good in terms of the investment good are opposite between (i) and (ii).

Keywords and Phrases: Public services, Indeterminacy, A continuum of equilibrium paths, Two-sector endogenous growth.

JEL Classification Numbers: O41, E62, E32.

1 Introduction

Indeterminacy is one of the recent topics that have been paid much attention in macroeconomic theory[1]. However, while there have appeared many papers which explore this topic in the framework of exogenous growth models, the literature on the topic in the context of multisector endogenous growth is limited only to the models in which human capital is the

---

*We thank Koichi Futagami, Tatsuro Iwaisako, Kazuo Mino and the seminar participants at the universities of Kobe and Osaka for their valuable comments.

†Corresponding author: RIEB, Kobe University, Rokkodai-cho, Nada-ku, Kobe, 657-8501, Japan. Tel/Fax: 81-78-803-7002. E-mail: simomura@rieb.kobe-u.ac.jp

[1]See Benhabib and Farmer (1999) for a survey on this literature.

engine of persistent growth.[2] This paper shows that indeterminacy is possible in a multisector model of endogenous growth in which government expenditure on a public intermediate good is a source of persistent growth.[3]

However, the main purpose of this paper is *not* to derive indeterminacy in an endogenous growth model, but to make clear the implication of indeterminacy for the dependence of the balanced growth path on some parameters.

In static equilibrium models, Walrasian or Marshallian stability conditions provide useful information concerning comparative statical analysis. Likewise, the conditions for saddlepoint stability are useful for examining the effects of changes in parameters on the steady state in dynamic equilibrium models. However, it is well known that saddlepoint stability is not the unique stability concept in dynamic equilibrium models. It is possible that the characteristic equations of some of the latter models have their roots such that the number of the roots with the negative real parts, say n, is larger than the number of the state values, say m. In that case it is possible that there is a continuum of equilibrium paths starting form the same initial condition and converging to a common steady state. Thus, while there is no mechanism that determines which equilibrium path is realized, the steady state itself is stable, which implies that we can study the effects of parameter changes on the long-run equilibrium for dynamic equilibrium models in which indeterminacy takes place.

Moreover, the effects can be opposite between the cases of saddlepoint stability ($n = m$) and indeterminacy ($n > m$). To illustrate, consider a simplest possible dynamic equilibrium model which consists of one state variable, say x, and one jump variable, say y :

$$\begin{aligned} \dot{x} &= f(x, y) - a \\ \dot{y} &= h(x, y) \end{aligned} \tag{1}$$

where a is a parameter. Assuming that the steady state $(\bar{x}(a), \bar{y}(a))$ uniquely exists for any given a, and totally differentiating

$$\begin{aligned} 0 &= f(\bar{x}(a), \bar{y}(a)) - a \\ 0 &= h(\bar{x}(a), \bar{y}(a)) \end{aligned} \tag{2}$$

with respect to $\bar{x}(a)$, $\bar{y}(a)$ and a, we have

$$\begin{bmatrix} f_x & f_y \\ h_x & h_y \end{bmatrix} \begin{bmatrix} d\bar{x}(a)/da \\ d\bar{y}(a)/da \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{3}$$

where $f_x \equiv \frac{\partial}{\partial x} f$, $f_y \equiv \frac{\partial}{\partial y} f$, $h_x \equiv \frac{\partial}{\partial x} h$, and $h_y \equiv \frac{\partial}{\partial y} h$, all of which are evaluated at the

[2]See, among others, Benhabib, Meng and Nishimura (2000), Bond, Wang and Yip (1996), Mino (2002).

[3]This type of endogenous growth models is Barro's. Ohdoi (2002) firstly extends the Barro one-sector model to a second-sector one for examining growth patterns and international specialization in a small-open economy, in which prices are given and constant over time.

steady state. It follows from (3) that

$$sign[d\bar{x}(a)/da] = sign[h_y]sign[\det \begin{bmatrix} f_x & f_y \\ h_x & h_y \end{bmatrix}] \tag{4a}$$

$$sign[d\bar{y}(a)/da] = -sign[h_x]sign[\det \begin{bmatrix} f_x & f_y \\ h_x & h_y \end{bmatrix}] \tag{4b}$$

We see that (4) means that the effects of a small change in parameter a on the steady-state values depend on the sign of the determinant of the coefficient matrix in (3). Since the characteristic equation is formulated as

$$\Gamma(\lambda) \equiv \det \begin{bmatrix} f_x - \lambda & f_y \\ h_x & h_y - \lambda \end{bmatrix} = 0,$$

it is clear that while the determinant is negative if the steady state is saddlepoint-stable, it should be positive if indeterminacy takes place. Therefore, the effects of a small change in parameter a on the steady state are completely opposite between saddlepoint stability and indeterminacy.

The foregoing argument sheds light on an aspect of indeterminacy that is left unexplored. One may then ask whether the aforementioned opposite long-run effects can be derived in a standard endogenous growth model. Our answer is "yes": The main purpose of the rest of this paper is to show that the opposite long-run effects are possible in a two-sector model of endogenous growth which is a straightforward extension of Barro (1990).

As a by-product, there is another implication of indeterminacy for the Barro model. While the economy is on a steady state right from the beginning in the case of saddlepoint stability, *transitional dynamics* is possible when indeterminacy takes place.

This paper is organized as follows. Section 2 presents the model. Section 3 makes clear under what conditions a balanced growth path does exist. Section 4 provides a sufficient condition for saddlepoint stability and that for indeterminacy. Section 5 derives the above opposite effects of a small change in each parameter on the balanced growth path. Section 6 provides some concluding remarks about it.

2 The Model

We consider a closed economy with competitive markets which consists of households, firms and the government. Following Barro (1990), we assume that the government imposes income tax on households and use the tax revenues to provide nontrivial and nonexcludable public services to enhance private labor efficiency. Taking the public services as given, firms produces two goods, pure consumption and investment goods, by using capital and labor as inputs.

2.1 Firms

Let $G(t)$, $K_i(t)$, and $L_i(t)$ be public services flow, capital and labor inputs to the production of good i, respectively. The production function of good i is denoted as $Y_i = F_i(\eta K_i, GL_i)$. Here $i = 1$ (resp. 2) corresponds to the consumption (resp. investment) good. Public services G enter the production functions as a labor-augmenting factor. It is this labor-augmenting effect that makes possible permanent growth. η is a positive parameter.

Denote the aggregate capital stock and labor supply by K and $\bar{L}$. The aggregate labor supply is assumed to be constant over time. We set $\bar{L} = 1$ for simplicity thereafter. Thus full employment conditions are $K = K_1 + K_2$ and $1 = L_1 + L_2$.

Let r and w represent the rental rate and the wage rate respectively. Then, $v \equiv w/G$ can be interpreted as the wage rate paid for employing an efficiency unit of labor. Similarly, let us denote r/η by R. Due to constant returns to scale concerning private factors of production, we see that competitive markets and profit maximization lead to

$$p = \phi^1(R, v), \tag{5a}$$

$$1 = \phi^2(R, v), \tag{5b}$$

where $\phi^i(R, v)$ represents the unit cost function of good i and p is the price of consumption good in terms of the investment good, which serves as the numeraire in this paper.

We make the following assumptions concerning the unit cost functions.

ASSUMPTION 1 (COST FUNCTIONS): *(i) Each unit cost function is increasing and strictly concave in each variable: For any positive r and v,*

$$\phi^i_x(R, v) \equiv \frac{\partial}{\partial x}\phi^i(R, v) > 0 \quad \textit{and} \quad \phi^i_{xx}(R, v) \equiv \frac{\partial^2}{\partial x^2}\phi^i(R, v) < 0,$$

for $x = R, v$, and $i = 1, 2$. (ii) Each unit cost function is linearly homogeneous and quasi-concave in R and v. (iii) The Inada conditions hold:

$$\lim_{x\to 0}\phi^i_x(R, v) = \infty \quad \text{and} \quad \lim_{x\to\infty}\phi^i_x(R, v) = 0, \quad x = R, v, \quad i = 1, 2.$$

REMARK 1: Assumption 1 implies that for any $(R, v) > (0, 0)$ and for $i = 1, 2$,

$$\lim_{y\to 0}\phi^i(R, y) = \lim_{x\to 0}\phi^i(x, v) = 0,$$

$$\lim_{y\to\infty}\phi^i(R, y) = \lim_{x\to\infty}\phi^i(x, v) = \infty.$$

For the proof, see Barro and Sala-i-Martin (Chapter 1, 1995).

ASSUMPTION 2 (FACTOR-INTENSITY RANKING): *For any $p > 0$, (5a) and (5b) uniquely determine a positive pair $(R(p), v(p))$. The consumption good (good 1) is* *more labor-intensive than the investment good (good 2) in the sense that for any* $p > 0$

$$\frac{\phi^1_R(R(p), v(p))}{\eta\phi^1_v(R(p), v(p))} < \frac{\phi^2_R(R(p), v(p))}{\eta\phi^2_v(R(p), v(p))}. \tag{6}$$

REMARK 2: Denote the iso-cost curve (5b) by $R \equiv \zeta(v)$. ASSUMPTION 1 implies that $\zeta(v)$ is a decreasing and strictly convex function of v. Moreover, we can prove that $\lim_{v\to 0}\zeta(v) = \infty$ and $\lim_{v\to\infty}\zeta(v) = 0$[4]. It follows from ASSUMPTION 2 that

$$\lim_{p\to 0} v(p) = 0,\ \lim_{p\to\infty} v(p) = \infty,\ \lim_{p\to 0} R(p) = \infty,\ \lim_{p\to\infty} R(p) = 0. \tag{7}$$

Recall that $\phi^i_R/\eta\phi^i_v$ is the capital/(efficiency) labor ratio in sector i. The factor-intensity ranking (6) seems to be plausible. An example satisfying ASSUMPTIONS 1 and 2 is Cobb-Douglas technologies, $\phi^i(r/\eta, v) = (r/\eta)^{\theta_i} v^{1-\theta_i}$, $i = 1, 2$, where $0 < \theta_1 < \theta_2 < 1$.

Note that the rental rate is $r(p; \eta) \equiv \eta R(p)$. Defining $\varepsilon_r(p) \equiv \frac{p r_p(p;\eta)}{r(p)}$ and $\varepsilon_v(p) \equiv \frac{p v_p(p)}{v(p)}$, we have, from ASSUMPTION 2, $\varepsilon_r(p) = \frac{p R_p(p)}{R(p)} < 0$ and $\varepsilon_v(p) > 1$, which is referred to as the "magnification effects" in the Stolper-Samuelson Theorem. It is clear that $\varepsilon_r(p)$ and $\varepsilon_v(p)$ are independent of the value of η. In the special case of Cobb-Douglas technology ε_r and ε_v are constant. Henceforth, we assume that $-\infty < \varepsilon_r(p) < 0$ and $\infty > \varepsilon_v(p) > 1$ hold for any $p \geq 0$.

2.2 The government and market-clearing condition

Following Barro (1990), we assume that the government finances its spending through a proportional tax at a constant rate τ imposed on the aggregate national income. Then the budget constraint of the government is $G = \tau[v(p)G + \eta R(p)K]$, from which we obtain the government-spending/capital ratio

$$\frac{G}{K} = \frac{\tau\eta R(p)}{1 - \tau v(p)} \tag{8}$$

It is well known in trade theory[5] that, as far as the two goods are both produced, the partial derivative of the national income $v(p)G + \eta R(p)K$ with respect to the price of the consumption good is positive and equal to the supply of the consumption good. Therefore, we obtain the market-clearing condition

$$C = \eta R_p(p)K + v_p(p)G, \tag{9}$$

[4]Let us prove $\lim_{v\to 0}\zeta(v) = \infty$. Suppose not. If $\lim_{v\to 0}\zeta(v) \equiv \bar{R}$ were positive and finite, it would contradict $\lim_{y\to 0}\phi^2(\bar{R}, y) = 0$ in Remark 1. One possibility is that there exists a positive $\underline{v}$ such that $\lim_{v\to\underline{v}}\zeta(v) = \infty$. However, this possibility contradicts Remark 1 again, since it and Remark 1 together would imply

$$\infty = \lim_{x\to\infty}\phi^2(x, \underline{v}) \leq \lim_{v\to\underline{v}}\phi^2(\zeta(v), v) = 1$$

We can prove $\lim_{v\to\infty}\zeta(v) = 0$ in a similar way.

[5]See, for example, Wong (Chapter 2, 1995).

where C is the demand for the consumption good[6]. Combining (8) and (9), we have the consumption/capital ratio

$$\frac{C}{K} = \Big[\eta R_p(p) + v_p(p)\frac{\tau\eta R(p)}{1-\tau v(p)}\Big]. \tag{10}$$

2.3 Households

Each infinitely lived household owns one unit of labor and capital K, and earns income by supplying them to firms as production inputs. Each household spends the after-tax income in consumption goods purchasing, pC, and capital stock investing, $\dot{K}$. The flow budget constraint she is facing is $(1-\tau)[\eta R(p)K + v(p)G] = pC + \dot{K}$. Under this constraint, she seeks to maximize the life-time utility

$$\int_0^\infty \Big[\frac{C^{1-\sigma}-1}{1-\sigma}\Big]e^{-\rho t}dt,$$

for given $\{\eta R(t)\}_0^\infty$, $\{v(t)\}_0^\infty$ and $\{G(t)\}_0^\infty$. Here $\sigma \in (0,1)\cup(1,\infty)$ is the inverse of the constant intertemporal elasticity of substitution, and $\rho > 0$ is the rate of time preference.

Associated with the current value Hamiltonian

$$\mathcal{H} = \frac{C^{1-\sigma}-1}{1-\sigma} + \lambda\{(1-\tau)[\eta R(p)K + v(p)G] - pC\},$$

we obtain the first-order conditions for optimality

$$C^{-\sigma} = \lambda p, \tag{11a}$$

$$\dot{\lambda} = \lambda[\rho - (1-\tau)\eta R(p)], \tag{11b}$$

$$\dot{K} = (1-\tau)[\eta R(p)K + v(p)G] - pC, \tag{11c}$$

and the transversality condition (TVC)

$$\lim_{t\to\infty} \lambda(t)e^{-\rho t}K(t) = 0. \tag{12}$$

2.4 The two-sector endogenous growth model

Based on the foregoing argument, we now present the two-sector model. Letting $Y \equiv p^{1/\sigma}C$ and considering (11a) and (11b), we can derive the Keynes-Ramsey rule,

$$\dot{Y} = \frac{Y}{\sigma}\Big[(1-\tau)\eta R(p) - \rho\Big]. \tag{13}$$

Furthermore, combining the two budget constraints (8) and (11c), we obtain

$$\dot{K} = (1-\tau)\Big[\eta R(p) + \frac{\tau\eta R(p)v(p)}{1-\tau v(p)}\Big]K - p^{\frac{\sigma-1}{\sigma}}Y. \tag{14}$$

[6]Equation (9) means that the government purchases the investment good by the tax revenue and transforms it to the public services.

Finally, substituting Y into the market-clearing condition (10), we get

$$\frac{p^{-1/\sigma}Y}{K} = \Big[\eta R_p(p) + \frac{\tau\eta R(p)v_p(p)}{1-\tau v(p)}\Big]. \tag{15}$$

Let $x \equiv Y/K$, combining (13) and (14), and considering (15), we derive a two-sector version of the Barro model of endogenous growth as follows:

$$\frac{\dot{x}}{x} = \frac{\eta R(p)}{(1-\tau v(p))\sigma}\Big[\sigma\Theta(p) - \Phi(p;\eta)\Big], \tag{16}$$

$$x = p^{1/\sigma}\Big[\eta R_p(p) + \frac{\tau\eta R(p)v_p(p)}{1-\tau v(p)}\Big] \equiv \Lambda(p). \tag{17}$$

where

$$\Theta(p) \equiv [1-\tau v(p)]\varepsilon_r(p) + \tau v(p)\varepsilon_v(p) - (1-\tau), \tag{18}$$

$$\Phi(p;\eta) \equiv [1-\tau v(p)]\Big\{\frac{\rho}{\eta R(p)} - (1-\tau)\Big\}. \tag{19}$$

3 The condition to be satisfied along a balanced growth path

Let us make some preparations for deriving our main result. A balanced growth path of the above system (16) and (17) is defined as an equilibrium path such that p and x are constants over time, therefore K, C grow with a positive rate at BGP. In order for a pair (p^e, x^e) to be a balanced growth path for a given tax rate $\tau \in (0,1)$, the following conditions for balanced growth and equilibrium have to be satisfied.

(C1) $\sigma\Theta(p^e) = \Phi(p^e;\eta)$

(C2) $1-\tau v(p^e) > 0$

(C3) $(1-\tau)\eta R(p^e) - \rho > 0$

(C4) $\rho - (1-\tau)(1-\sigma)\eta R(p^e) > 0$

(C5) $-(1-\tau) < \Theta(p^e) < 0$.

Considering (16), we see that unless (C1) were satisfied, x and p could not remain to be constant over time. (C2) is necessary for G/K to be positive. As is clear from (13), the balanced growth rate is not positive unless (C3) is satisfied. (C4) is the transversality condition (12).

Considering the properties of $R(p)$ and $v(p)$ in (7), we see that there exist p_2, p_3, and p_4 respectively such that

- $1-\tau v(p) \gtreqless 0$, according as $p \lesseqgtr p_2$.

- $(1-\tau)\eta R(p) - \rho \gtreqless 0$, according as $p \lesseqgtr p_3$.
- $\rho - (1-\tau)(1-\sigma)\eta R(p^e) \gtreqless 0$, according as $p_4 \lesseqgtr p$.

Note that $p_4 < p_3$.

Next, let us discuss what the two inequalities in (C5) mean. Recalling the definitions of $\varepsilon_r(p^e)$ and $\varepsilon_v(p^e)$, we see, from (10),

$$\begin{aligned}
&[1-\tau v(p^e)]\varepsilon_r(p^e) + \tau v(p^e)\varepsilon_v(p^e) \\
&= \frac{p^e(1-\tau v(p^e))}{\eta R(p^e)}\Big[\eta R_p(p^e) + \frac{\tau\eta R(p^e)v_p(p^e)}{1-\tau v(p^e)}\Big] \\
&= \frac{p^e(1-\tau v(p^e))}{\eta R(p^e)}\frac{C}{K}
\end{aligned}$$

Thus, the first inequality in (C5), $-(1-\tau) < \Theta(p^e)$, means that the consumption good is produced along a balanced growth path. Next, from (14) and (15),

$$\begin{aligned}
\dot{K}/K &= (1-\tau)\Big[\eta R(p) + \frac{\tau\eta R(p)v(p)}{1-\tau v(p)}\Big] - p^{\frac{\sigma-1}{\sigma}}Y/K \\
&= \frac{\eta R(p^e)}{(1-\tau v(p^e))}\Big[(1-\tau) - \{(1-\tau v(p^e))\varepsilon_r(p^e) + \tau v(p^e)\varepsilon_v(p^e)\}\Big],
\end{aligned}$$

Thus, under (C2), the second inequality in (C5), $\Theta(p^e) < 0$, means that the investment good is also produced. Using a term in trade theory, the two inequalities ensures us that production in the economy is *incompletely specialized.*

LEMMA 1: *Under (C2) and (C5),* $\Theta(p)$ *is increasing in* p.

Proof: Differentiating $\Theta(p)$ with respect to p, we have

$$\begin{aligned}
\frac{d\Theta(p)}{dp} &= \Big(\frac{1-\tau v(p)}{\eta R(p)}\Big)[1-\varepsilon_r(p)]\Big[\eta R_p(p) + \frac{\tau\eta R(p)v_p(p)}{1-\tau v(p)}\Big] \\
&\quad + \frac{p[1-\tau v(p)]}{\eta R(p)}\Big[\eta R_{pp}(p) + \frac{\tau\eta R(p)v_{pp}(p)}{1-\tau v(p)}\Big].
\end{aligned}$$

Note that if production is incompletely specialized, the second term is the slope of the supply curve of the consumption good, which is known to be positive in trade theory[7]. Since (C2) means that the first term is also positive, the slope of $\Theta(p)$ is positive, as was to be proved. ■

Note that by definition the function $\Theta(p)$ is continuous in p. Since $\Theta(0) = \varepsilon_r(0) - (1-\tau) < -(1-\tau)$ and $\Theta(p_2) = \varepsilon_v(p_2) - (1-\tau) > 0$ due to the "magnification" effects, it follows from Lemma 1 that the interval of incomplete specialization (p_*, p^*) is strictly within $(0, p_2)$.

Based on the foregoing argument, we have the following condition to be satisfied along a balanced growth path.

[7] See, for example, Wong (Chapter 2, 1995).

THE BGP CONDITION: *Let p^e be a price that satisfies $\sigma\Theta(p) = \Phi(p;\eta)$. In order that p^e be a BGP price, the following inequalities have to hold:*

$$\max[p_*, p_4] < p^e < \min[p_3, p^*]. \tag{BGP}$$

Finally, in order to discuss about indeterminacy, the dynamic system (16) and (17) have to satisfy *causality* (Burmeister and Dobell (Chapter 4, 1970)).

LEMMA 2: *Under (C2) and (C5), $\Lambda(p)$ in (17) is increasing in p.*

Proof: This can be seen easily from the following:

$$\frac{d\Lambda}{dp} = \frac{1}{\sigma}p^{\frac{1}{\sigma}}\frac{C}{K} + p^{\frac{1}{\sigma}}\left[\eta R_{pp}(p) + \frac{\tau\eta R(p)}{1-\tau v(p)}v_{pp}\right] + p^{\frac{1}{\sigma}}\left(\frac{\tau v_p(p)}{1-\tau v(p)}\right)\frac{C}{K},$$

and along with the same reasoning as in proving Lemma 1. ■

The last lemma guarantees that for a given $x(t)$, $\dot{x}(t)$ is uniquely determined by (16) and (17), i.e., causality is guaranteed.[8]

4 The indeterminacy result

We shall diagrammatically obtain the main result in this paper.

First, choose the value of η, say η_0, so that the following equality holds.

$$(1-\tau)\eta_0 R(p^*) - \rho = 0$$

(Recall that p^* is independent of η.) See Figure 1. If σ is sufficiently small, the graph of $\sigma\Theta(p)$ is depicted like a relatively flat curve $AEMB$, while the graph of $\Phi(p;\eta_0)$ is like a dotted curve CMD.

Now, let us slightly reduce the value of η, from η_0 to $\eta_0 - \Delta\eta$. The graph of $\Phi(p;\eta_0 - \Delta\eta)$ is like the solid curve FED. It is clear from inspecting Figure 1 that as far as $\Delta\eta$ and σ are sufficiently small positive values, the intersection of $\sigma\Theta(p)$ and $\Phi(p;\eta_0 - \Delta\eta)$, p^e, satisfies the BGP Condition.

Moreover, as long as σ is small, the slope of $\sigma\Theta(p)$ is smaller than that of $\Phi(p;\eta_0 - \Delta\eta)$ at p^e. Since Lemma 2 ensures us that $\Lambda_p(p^e) > 0$, considering (17) and differentiating (16) with respect to x at p^e, we have

$$\frac{d}{dx}\left(\frac{\dot{x}}{x}\right) = \frac{(\eta_0 - \Delta\eta)R(p^e)}{(1-\tau v(p^e))\sigma}\left[\sigma\Theta_p(p^e) - \Phi_p(p^e;\eta_0 - \Delta\eta)\right]\frac{1}{\Lambda_p(p^e)} < 0,$$

which means that the dynamic system, (16) and (17), is locally stable. We obtain the following proposition.

[8] It is worthnoting that the estiblishment of this causality is based on Assumption 2, since when consumption good is more capital intensive than the capital one, the monotonity of $\Lambda(p)$ may loss.

PROPOSITION 1: *Take any given τ in the interval $(0, 1)$ and any positive ρ. If σ is sufficiently close to zero, then there is a parameter space for η such that there exists a balanced growth path to which a continuum of equilibrium paths starting from the same initial capital stock converges. The balanced growth path is indeterminate in the sense that we cannot specify which equilibrium path in the continuum is realized in a decentralized market economy. We call this kind of balanced growth path "Indeterminate BGP".*

Making a parallel argument but assuming that σ is sufficiently large, we obtain Figure 2. Increasing the value of η slightly, from $\bar{\eta}_0$ to $\bar{\eta}_0 + \Delta\eta$, we have the balanced growth path E'. However, this time the slope of $\sigma\Theta(p)$ is larger than that of $\Phi(p; \bar{\eta}_0 + \Delta\eta)$, which means that the dynamic system, (16) and (17), is locally unstable. Since p is a jump variable, it follows that the balanced growth path is the only equilibrium path.

PROPOSITION 2: *If σ is sufficiently large, for a certain set of parameter η, the economy is on a balanced growth path right from the beginning. We call the balanced growth path "Determinate BGP".*

5 An implication of indeterminacy for comparative statics

In this section we point out an important implication of indeterminacy for comparative statics. Consider the BGP condition $\sigma\Theta(p^e) = \Phi(p^e; \eta, \rho)$. Apparently, the BGP price of the consumption good depends on the parameters σ, η and ρ. To see how the BGP price depends on them, let us totally differentiate the BGP condition with respect to p^e, σ, η and ρ, we have

$$dp^e = \frac{\frac{\partial\Phi}{\partial\eta}d\eta + \frac{\partial\Phi}{\partial\rho}d\rho - \Theta(p^e)d\sigma}{\frac{\partial}{\partial p^e}[\sigma\Theta(p^e) - \Phi(p^e; \eta, \rho)]}, \tag{20}$$

where

$$\begin{aligned}
&\frac{\partial\Phi}{\partial\eta} = -\frac{\rho(1 - \tau v(p^e))}{\eta^2 R(p^e)} < 0, \\
&\frac{\partial\Phi}{\partial\rho} = \frac{(1 - \tau v(p^e))}{\eta R(p^e)} > 0, \\
&\Theta(p^e) < 0 \quad (\because \text{ (C5)}).
\end{aligned}$$

Note that these signs hold whether the BGP is indeterminate or not.

Therefore, the effects of small changes in the three parameters are determined once the sign of the denominator in (20) is made clear. As we have already shown in the previous section, the sign of the denominator is opposite between a determinate and indeterminate BGP. Thus, we now arrive at the results shown in Table 1.

Now let us turn to another important endogenous variable, the balanced growth path, g_e. From (13)

$$g_e = \frac{1}{\sigma}\Big[(1-\tau)\eta R(p^e) - \rho\Big].$$

Taking partial differentiation with respect to η, ρ and σ, yields

$$\frac{\partial g_e}{\partial \eta} = \frac{1}{\sigma}\Big[(1-\tau)R(p^e) + (1-\tau)\eta R_p(p^e)\frac{\partial p^e}{\partial \eta}\Big], \tag{21}$$

$$\frac{\partial g_e}{\partial \rho} = \frac{1}{\sigma}\Big[(1-\tau)\eta R_p(p^e)\frac{\partial p^e}{\partial \rho} - 1\Big], \tag{22}$$

$$\frac{\partial g_e}{\partial \sigma} = (1-\tau)\eta R_p(p^e)\frac{\partial p^e}{\partial \sigma}\frac{1}{\sigma} - \frac{1}{\sigma^2}[(1-\tau)\eta R(p^e) - \rho]. \tag{23}$$

The results in Table 1 and Assumpation 2 imply that, when the BGP is indeterminate, the signs of (21)-(23) are

$$\frac{\partial g_e}{\partial \eta} > 0, \quad \frac{\partial g_e}{\partial \rho} < 0, \quad \frac{\partial g_e}{\partial \sigma} < 0.$$

Next let us examine the effects of changes in η, ρ and σ on g_e in the case of determinate BGP. Define a pair $(\tilde{\eta}, \tilde{p})$ as satisfying $\Phi(p, \eta) = 0$ and $\Theta(p) = 0$ simultanously for any given $\rho > 0$ and $\tau \in (0, 1)$. For such pair, define $\tilde{\sigma}$ as

$$\tilde{\sigma} \equiv \frac{\partial \Phi(p, \tilde{\eta})/\partial p}{\partial \Theta(p)/\partial p}\Big|_{p=\tilde{p}}$$

Then for a σ larger but sufficiently close to $\tilde{\sigma}$, and from the construction of BGP in Section 4, we see that this σ together with some η, which is larger but sufficiently close to $\tilde{\eta}$, corresponds to a determinate BGP, p^e. That is

$$\frac{\partial[\sigma\Theta - \Phi]}{\partial p^e} > 0. \tag{24}$$

Note that, (24) is close to zero because σ is close to $\tilde{\sigma}$. Based on the foregoing results, we can derive the signs of (21)-(23) as follows.

First, combining (20) and (21), we have

$$\frac{\partial g_e}{\partial \eta} = \frac{1}{\sigma}\frac{(1-\tau)R(p^e)}{\partial[\sigma\Theta - \Phi]/\partial p^e}\Big[\frac{\partial[\sigma\Theta - \Phi]}{\partial p^e} + \eta\frac{R_p(p^e)}{R(p^e)}\frac{\partial[\sigma\Theta - \Phi]}{\partial \eta}\Big]$$

Since the first term in the square bracket is close to zero, it is dominated by the second one. Hence $\partial g_e/\partial \eta < 0$. Making a parallel argument, we obtain $\partial g_e/\partial \rho > 0$.

Finally, inspecting (23) we see that $\partial g_e/\partial \sigma$ is positive for a pair (σ, η) which is larger but sufficiently close to $(\tilde{\sigma}, \tilde{\eta})$.

Table 2 summarizes the effects of changes in η, ρ and σ on g_e for both indeterminate and determinate BGPs. As in the case of p^e, comparative-statical results are completely opposite between indeterminate and determinate BGPs.[9]

[9]The comparative statics of p^e and g_e with respect to τ's change is generally ambiguous. Since the polar

6 Concluding remarks

To our knowledge, this paper is the first to illustrate an indeterminate balanced growth path in a two-sector version of the Barro-type endogenous growth model under the factor-intensity ranking such that the investment good is more capital-intensive than the consumption good.

The opposite long-run effects of a small change in each parameter between determinate and indeterminate equilibria indicate a theoretical possibility that indeterminacy provides a new dimension for the correspondence principle and, therefore, makes it possible for applied dynamic equilibrium theorists to pursue comparative statical results that cannot be derived under saddlepoint stability. For example, dynamic trade theorists may want to explore whether the effects of parametric changes in preferences, technologies, factor endowments and commercial policies in trading countries on long-run trade and production structures in each country can be different between determinate and indeterminate equilibrium cases.

References

[1] Barro, R. J., 1990, Government spending in a simple model of endogenous growth, *Journal of Political Economy* **98**, 103-125.

[2] Barro, R. J., and Xavier Sala-i-Martin, 1995, *Economic Growth* (MacGraw-Hill, Inc.).

[3] Benhabib, J., and Farmer, R., 1999, Indeterminacy and sunspots in macroeconomics, *Handbook of Macroeconomics* Vol.1 (Elsevier Science B.V.), 387-448

[4] Benhabib, J., Meng, Q., and Nishimura, K., 2000, Indeterminacy under constant returns to scale in multisector economies, *Econometrica* **68**, 1541-1548.

[5] Bond, E., Wang, P., and Yip, C., 1996, A general two-sector model of endogenous growth with human and physical capital: Balanced growth and transitional dynamics, *Journal of Economic Theory* **68**, 149-173.

[6] Burmeister, E. and A. R. Dobell, 1970, *Mathematical Theories of Economic Growth* (Macmillan, London).

[7] Mino, K., 2002, Preference structure and indeterminacy in two-sector models of endogenous growth, to appear in: M. Boldrin, B-L Chen and P. Wang eds., *The Development Process of Rapidly Growing Economies: From Theory to Empirics* (Edward Elgar Publishing Inc., Cheltenham, UK).

cases of $\tau = 0$ and $\tau = 1$ make permanent growth impossible in the present setting, the balanced growth rate, g_e, shall exhibit a inverse U shape in relation to $\tau \in (0, 1)$, no matter how many equilibrium paths exist in the economy for a given initial point. Different properties can be predicted for the two kinds of BGP when τ takes a medium value, although singling out those differences analytically is generally a tough work.

[8] Ohdoi, R., 2002, Government spending, patterns of specialization and economic growth in a small-open economy, Mimeo. (Osaka University, Osaka).

[9] Wong, K.-Y., 1995, *International Trade in Goods and Factor Mobility* (MIT Press, Cambridge, MA.)

Table 1: Comparative statics of p^e

$\partial p^e/\partial$	ρ	η	σ
Indeterminate BGP	$+$	$-$	$+$
Determinate BGP	$-$	$+$	$-$

Table 2: Comparative statics of g_e

$\partial g_e/\partial$	ρ	η	σ
Indeterminate BGP	$-$	$+$	$-$
Determinate BGP	$+$	$-$	$+$

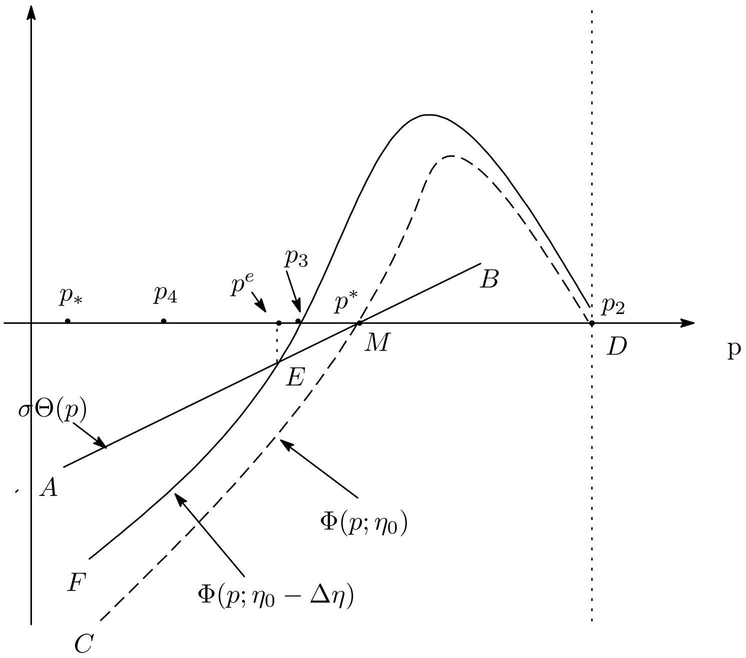

Figure 1: An indeterminate BGP

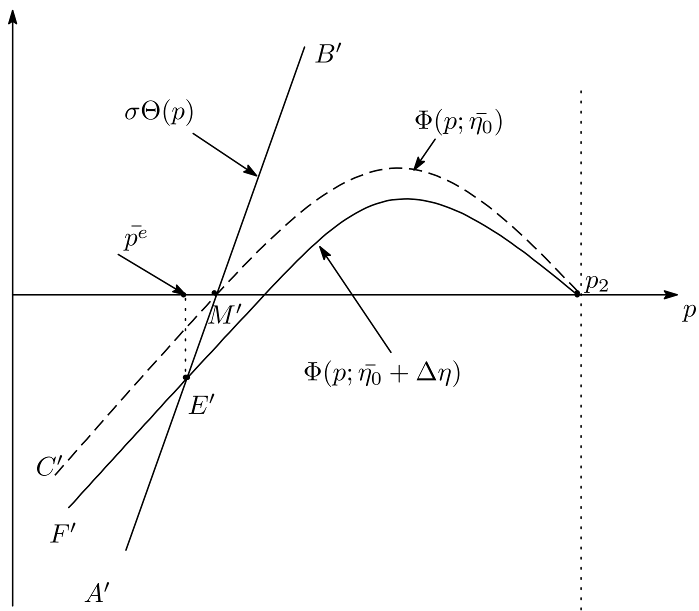

Figure 2: A determinate BGP