Discussion Paper Series No.144

A Note on Transversality Conditions

Ngo Van Long **McGill University**
Koji Shimomura **Kobe University**

November 2003

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Ngo Van Long[a] **and Koji Shimomura**[b]

We derive a new transversality condition (which we prove to be necessary) for a class of infinite horizon optimal control problems.
a: Department of Economics, McGill University, 855 Sherbrooke Street West, Montreal, Quebec, Canada H3A 2T7. Fax: (514)398-4938, Tel: (514)398-4844. Email: ngo.long@mcgill.ca
b: Research Institute for Economics and Business Administration, Kobe University, 2-1, Rokkodai-cho, Nada-ku, Kobe, Japan, 657-8501. Fax: 81-78-803-7059, Tel: 81-78-803-7002, Email: coe@rieb.kobe-u.ac.jp
Date: 11 September 2003

JEL Classification: C61
Key Words:Transversality condition, optimal control theory

1 Introduction

While transversality conditions for optimal control problems (in continuous time[1]) with a fixed, finite time horizon are parts of the necessary conditions, and are relatively straightforward to derive[2], the same cannot be said for transversality conditions for infinite horizon problems. For example, with a fixed, finite time horizon, when a state variable is *free* (not fixed) at the terminal time T, it is necessary that the associated co-state variable, $\pi(t)$, be zero at T. However, when the time horizon is infinite, it is not true that $\lim_{T\to\infty} \pi(T) = 0$ is a necessary condition. (A counterexample, due to Halkin, is reported in Arrow and Kurz (1970, p. 46), and discussed in Takayama (1985, p. 625)).

For infinite time horizon problems, various transversality conditions have been stated as parts of the sufficient conditions (see, for example, Seierstad and Sydsaeter, 1977, Leonard and Long, 1992) for an optimal solution, or as necessary conditions for certain problems with a special structure. For example, for a special investment problem with adjustment costs, Takayama (1985, p. 699) stated as necessary the condition

$$\lim_{T\to\infty} \lambda(T)e^{-rT} = 0$$

while Arrow (1968) claimed that in the standard one-sector optimal growth model, with a positive discount rate, the following conditions are appropriate:

$$\lim_{T\to\infty} \lambda(T)e^{-rT} \geq 0$$

$$\lim_{T\to\infty} \lambda(T)e^{-rT}k(T) = 0$$

For a macro-economc model with a stock of bonds b, and a capital stock k, Turnovsky (1995, p 236) stated the transversality conditions

$$\lim_{T\to\infty} \lambda(T)e^{-rT}k(T) = 0$$

$$\lim_{T\to\infty} \lambda(T)e^{-rT}b(T) = 0$$

[1] We are not dealing with discrete-time problems. For transversality conditions in discrete-time problems, see Weitzman (1973), Ekeland and Scheinkman (1986), and Michel (1990).

[2] For heuristic derivations, see, for example, Leonard and Long (1992). For rigorous proofs, see Hestenes (1966).

However, there were no proofs that those conditions are necessary conditions for the special problems under considerations.

A number of theorems have been proved on the necessary transversality conditions for various special cases of the *reduced-form model*

$$\max_x \int_0^\infty v(x, \dot{x}, t) dt$$

subject to

$$x(0) = x_0, \qquad (x, \dot{x}) \in W \subseteq (R^N)^2$$

For example, Benveniste and Scheinkman (1982), under the assumptions that v is *non-negative* and that $v(x, \dot{x}, t)$ is integrable, established the necessity of the following standard transversality conditions (STC):

$$\lim_{t \to \infty} [-v_2(x, \dot{x}, t)]\, x = 0 \tag{1}$$

Kamihigashi (2001), by assuming that the boundedness of v_1 and v_2, demonstrated, under certain additional hypotheses, the necessity of a variant of STC:

$$\lim_{t \to \infty} [-v_2(x, \dot{x}, t)]\, x \;\square\; 0$$

Michel (1982) proves another transversality condition: as time tends to infinity, the Hamiltonian tends to zero.

The purpose of this note is to *prove the necessity* of a new transversality condition, for an infinite time horizon problem with n state variables. As we shall see, this new necessary condition implies that two commonly used transversality conditions are in fact equivalent.

2 The theorem

Let x be a vector of n state variables, and $\dot{x}$ denote its derivative with respect to time. Consider the problem

$$\max \int_0^\infty v(x, \dot{x}, t) dt \tag{2}$$

subject to

$$(x(t), \dot{x}(t)) \in W \subseteq (R^n)^2 \text{ for all } t \geq 0$$

$$x(0) = x_0$$

The function v is twice differentiable.

In what follows, x, $\dot{x}, v_x$ and $v_{\dot{x}}$ are column vectors in R^n. If y is a column vector, then y^T denotes y tranposed, a row vector. Let the time path $x^*(.)$ be a solution of the problem (2). We make the following assumptions:

Assumption A1: For all $t \geq 0$

$$(x^*(t), \dot{x}^*(t)) \in \text{int}W$$

Assumption A2: There exists a small $\delta > 0$ such that for all $\alpha \in (1-\delta, 1+\delta)$ the pair $(y, \dot{y})$ generated by $y(0) = x_0$ and $\dot{y}(t) = \alpha \dot{x}^*(t)$ has the property that

$$(y(t), \dot{y}(t)) \in \text{int}W \text{ for all } t \geq 0 \tag{3}$$

Remark: Assumption 1 does not ensure that (3) is satisfied. Consider the following counter-example.Let

$$W = \left\{(a, b) \in R^2 : a \in [-1, 1] \text{ and } -(1-a) \;\square\; b \;\square\; 2(1-a)\right\}$$

Consider the problem

$$\max_x \int_0^\infty \dot{x}(t)dt$$

subject to

$$(x, \dot{x}) \in W$$

and $x(0) = 0$. Thus, it is required that $x \in [-1, 1]$ and $-(1-x) \;\square\; \dot{x} \;\square\; 2(1-x)$. Clearly, given $x(0) = 0$, $x(t) < 1$ for all finite t. A solution for this problem is $x^*(t) = 1 - e^{-t}$ which is less than 1 for all finite t, and $\lim_{t\to\infty} x^*(t) = 1$. Yet, with $y(0) = x_0$ and $\dot{y}(t) = \alpha\dot{x}^*(t)$, we have

$$y(t) = \alpha\left[x^*(t) - x_0\right] + x_0 = \alpha x^*(t) \text{ for } x_0 = 0$$

and

$$\lim_{t\to\infty} y(t) > 1 \text{ for any } \alpha > 1,$$

which is not feasible. In other words, we cannot have $\dot{y} = \alpha\dot{x}^*$ for $\alpha > 1$.

Theorem: Under Assumptions A1 and A2, the following transversality condition is a necessary condition:

$$\lim_{t\to\infty} \left[x^*(t) - x_0\right]^T v_{\dot{x}}\left(x^*(t), \dot{x}^*(t), t\right) = 0 \tag{4}$$

Corollary: If, in addition to assumptions A1 and A2,the vector $v_{\dot{x}}\left(x^*(t), \dot{x}^*(t), t\right)$ (evaluated along the optimal path) is non-positive, and x_0 is strictly positive, then the following two conditions are equivalent:

$$\lim_{t\to\infty} v_{\dot{x}}\left(x^*(t), \dot{x}^*(t), t\right) = 0 \tag{5}$$

$$\lim_{t\to\infty} \left[x^*(t)\right]^T v_{\dot{x}}\left(x^*(t), \dot{x}^*(t), t\right) = 0 \tag{6}$$

Proof of the theorem: we construct a time path $y(.)$ generated by

$$y(0) = x_0$$

$$\dot{y}(t) = \alpha \dot{x}^*(t)$$

Then

$$y(t) - y(0) = \int_0^t \dot{y}(t)dt = \alpha \int_0^t x^*(t)dt = \alpha x^*(t) - \alpha x_0$$

i.e.

$$y(t) = \alpha \left[x^*(t) - x_0\right] + x_0$$

By Assumption A2, $(\ y(t), \dot{y}(t)) \in \text{int}W$ for α sufficiently close to unity.

Define

$$\psi(\alpha) = \int_0^\infty v(y, \dot{y}, t)\, dt - \int_0^\infty v(x^*, \dot{x}^*, t)\, dt$$

$$= \int_0^\infty v(\alpha x^* + (1-\alpha)x_0, \alpha \dot{x}^*, t)dt - \int_0^\infty v(x^*, \dot{x}^*, t)dt$$

Then, there exists $\delta > 0$ such that for all $\alpha \in (1-\delta, 1+\delta)$, we have $\psi(\alpha)$ □ 0. In particular, $\psi(1) = 0$. In addition, since $x^*(.)$ is an interior solution, it must hold that $\psi'(\alpha) = 0$ at $\alpha = 1$. Thus

$$0 = \frac{d\psi(\alpha)}{d\alpha} \mid_{\alpha=1} = \int_0^\infty \left[(x^* - x_0)^T v_x(x^*, \dot{x}^*, t) + (\dot{x}^*)^T v_{\dot{x}}(x^*, \dot{x}^*, t)\right] dt$$

Now, recall Euler's equation

$$v_x(x^*, \dot{x}^*, t) = \frac{d}{dt}\left[v_{\dot{x}}(x^*, \dot{x}^*, t)\right]$$

Thus

$$0 = \int_0^\infty \{(x^*)^T \frac{d}{dt}\left[v_{\dot{x}}(x^*, \dot{x}^*, t)\right] + \left(\frac{d\dot{x}^*}{dt}\right)^T v_{\dot{x}}(x^*, \dot{x}^*, t)\} dt$$

$$-\int_0^\infty x_0^T \frac{d}{dt}\left[v_{\dot{x}}(x^*, \dot{x}^*, t)\right] dt$$

$$= \int_0^\infty \frac{d}{dt}\{(x^*)^T v_{\dot{x}}(x^*, \dot{x}^*, t)\} dt - \int_0^\infty x_0^T \frac{d}{dt}\left[v_{\dot{x}}(x^*, \dot{x}^*, t)\right] dt$$

$$= \lim_{t\to\infty} (x^*)^T v_{\dot{x}}(x^*, \dot{x}^*, t) - (x_0^*)^T v_{\dot{x}}(x_0^*, \dot{x}^*(0), 0)$$

$$-x_0^T \lim_{t\to\infty} v_{\dot{x}}(x^*, \dot{x}^*, t) + (x_0^*)^T v_{\dot{x}}(x_0^*, \dot{x}^*(0), 0)$$

Thus we obtain equation (4) as a necessary condition.

Proof of the corollary: This follows from (4) and the assumptions that the vector $v_{\dot{x}}\left(x^*(t), \dot{x}^*(t), t\right)$ (evaluated along the optimal path) is non-positive, and x_0 is strictly positive.

3 Discussion

Our theorem implies that the following path cannot be optimal:

$$x(t) = e^{rt} \text{ with } v_{\dot{x}} = -e^{-rt}\ ,\ r > 0.$$

Concerning the example by Halkin (mentioned in the introduction), which violates our transversality condition (4), we note that he imposes the constraint

$$-(1-x) \square\ \dot{x} \square\ 1 - x$$

which violates Assumption A2.

References

Arrow, K. J. and M. Kurz, 1970, Public Investment, the Rate of Return, and Optimal Fiscal Policy, Johns Hopkins Press, Baltimore, Md.

Benveniste, L. M. and J. A. Sheinkman, 1982, Duality Theory for Dynamic Optimization Models of Economics: The Continuous Time Case, Journal of Economic Theory, 27, 1-19.

Ekeland, I., and J. A. Sheinkman, 1986, Transversality Conditions for some Infinite Horizon Discrete Time Optimization Problems, Mathematics of Operations Research, 11, 216-229.

Hestenes, Magnus R.,1966, Calculus of Variations and Optimal Control Theory, New York, Wiley.

Kamihigashi, Takashi, 2001, Necessity of Transversality Conditions for Infinite-Horizon Problems, Econometrica, 69, 995-1012.

Leonard, Daniel, and Ngo Van Long, 1992, Optimal Control Theory and Static Optimization in Economics, Cambridge University Press, Cambridge and New York.

Michel, P., 1982, On the Transversality Condition in Infinite-Horizon Problems, Econometrica, 50, 975-985.

Michel, P.,1990, Some Clarifications on the Transversality Conditions, Econometrica, 58, 705-723.

Takayama, A., 1985, Mathematical Economics, Cambridge University Press, Cambridge, U.K

Turnovsky, S., 1995, Methods of Macroeconomic Dynamics, MIT Press, Cambridge, MA.

Weitzman, M., 1973, Duality Theory for Infinite Horizon Convex Models, Management Science, 19, 783-789.