Trade and indeterminacy in a dynamic general equilibrium model[*]

Kazuo Nishimura
KIER, Kyoto University
Yoshida-Honmachi, Sakyo-ku
Kyoto 606-8501, Japan
E-mail: nishimura@kier.kyoto-u.ac.jp
Fax: +81-75-753-7198

Koji Shimomura
RIEB, Kobe University
2-1 Rokkodai-cho, Nada-ku
Kobe 657, Japan
E-mail: simomura@rieb.kobe-u.ac.jp
Fax: +81-78-861-6434

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Abstract

This paper introduces sector-specific externalities in the Heckscher-Ohlin two-country dynamic general equilibrium model to show that indeterminacy of the equilibrium path in the world market can occur. Under certain conditions in terms of factor intensities, there are multiple equilibrium paths from the same initial distribution of capital in the world market, and the distribution of capital in the limit differs among equilibrium paths. One equilibrium path converges to a long-run equilibrium in which the international ranking of factor endowment ratios differs from the initial ranking; another equilibrium path maintains the initial ranking and converges to another long-run equilibrium. Since the path realized is indeterminate, so is the long-run trade pattern. Therefore, the Long-Run Heckscher–Ohlin prediction is vulnerable to the introduction of externality.

1 Introduction

This study investigates the dynamic behavior of multiple countries' economic activities in a two-good, two-factor model in which factors are internationally immobile and countries' technologies are

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Correspondence to: Kazuo Nishimura, Institute of Economic Research, Kyoto University, Yoshida-Honmachi, Sakyo-ku, Kyoto 606-8501, Japan
E-mail: nishimura@kier.kyoto-u.ac.jp
Fax: +81-75-753-7198

subject to sector-specific external effects.

It has been demonstrated that in a perfect foresight model with many consumers, a competitive equilibrium path behaves like an optimal growth path; see Becker (1980), Bewley (1982), Yano (1984), and Epstein (1987). As these results suggest, a perfect foresight equilibrium path may exhibit the same behavior as in a single consumer model even in a many consumer model such as a large-country trade model. Nishimura and Yano (1993a, b) studied the interlinkage of business cycles between large countries in the discrete time model. However, indeterminacy in such a setting has not been characterized in the existing literature. Indeterminacy means that there exists a continuum of equilibria starting from the same initial condition, all of which converge to a steady state.

Recently there has been a growing literature on the existence of indeterminate equilibria in dynamic general equilibrium economies. While the earlier results on indeterminacy relied on relatively large increasing returns, Benhabib and Nishimura (1998) and Benhabib, Meng and Nishimura (2000) showed that in multisector models indeterminacy can arise with constant social returns to scale if there is a small wedge between private and social returns.

The present paper extends the Heckscher-Ohlin (H-O) model by introducing sector-specific externalities. Given a two (-country) by two (-good) by two (-factor) model of international trade in which production technologies are subject to constant returns to scale and preferences are homothetic, and in which the difference between the two countries is only in the factor endowment ratio, the H-O Theorem tells us that each country exports such a good that the abundantly endowed factor of production is intensively used for producing it.

The H-O Theorem is a result in a static framework. Formulating a two-sector dynamic general equilibrium model in which capital accumulation is taken into account, Chen (1992) studied the dynamic version of the H-O model.

In the present paper we introduce sector-specific externalities in the dynamic general equilibrium model and show that indeterminacy of the equilibrium path in the world market can occur. It follows that there are multiple equilibrium paths from the same initial distribution of capital in the world market, and the distribution of capital in the limit differs among equilibrium paths, and one equilibrium path converges to a long-run equilibrium in which the international ranking of factor endowment ratios differs from the initial ranking, whereas another equilibrium path maintains the initial ranking and converges to another long-run equilibrium. Since the path realized is indeterminate, so is the long-run trade pattern. Therefore the Long-Run H-O prediction is vulnerable to the introduction of externality.

In Section 2 we present the dynamic H-O model in a continuous time model with factor-generated externalities. In Section 3 we show the existence of a continuum of long-run equilibria. In Section 4 we derive indeterminacy results and discuss implications for the distribution of capital in the long-run equilibrium. Section 5 concludes.

2 The Dynamic Two-Country Model

We shall formulate the continuous-time version of the dynamic two-country model.

2.1 The production side

Two goods, a consumption good and an investment good, are produced using two factors of production, capital and labor. The home and the foreign countries are endowed with the same fixed amount of labor l and capital stocks k and k^*, respectively. The first good is the investment good and the second good is the consumption good.

Following Benhabib, Meng and Nishimura (2000), we assume Cobb-Douglas technologies which are constant returns to scale from the social perspective but decreasing returns to scale from the private perspective due to factor-generated externalities. That is, the production function of good $i, i = 1, 2,$ is

$$y_i = l_i^{a_i} k_i^{b_i} \bar{l}_i^{\alpha_i} \bar{k}_i^{\beta_i}, \tag{1}$$

where $a_i + \alpha_i + b_i + \beta_i = 1$ and all the parameters are positive. $\bar{l}_i^{\alpha_i} \bar{k}_i^{\beta_i}$, $i = 1, 2,$ are externality terms. Defining $\theta_i \equiv a_i + \alpha_i$, profit maximization of each firm implies

$$w = a_1 l_1^{\theta_1 - 1} k_1^{1-\theta_1}; \quad r = b_1 l_1^{\theta_1} k_1^{-\theta_1} \tag{2a}$$

$$w = p a_2 l_1^{\theta_2 - 1} k_1^{1-\theta_2}; \quad r = p b_2 l_2^{\theta_2} k_2^{-\theta_2} \tag{2b}$$

Let

$$W_i \equiv \frac{\theta_i w}{a_i} \text{ and } R_i \equiv \frac{(1-\theta_i) r}{b_i} \quad i = 1, 2 \tag{3}$$

From (2) and (3) we can derive the "virtual" average cost = price condition for each good.

$$1 = \bar{c}^1(W_1, R_1) \tag{4a}$$

$$p = \bar{c}^2(W_2, R_2) \tag{4b}$$

$$\bar{c}^1_W(W_1, R_1) y_1 + \bar{c}^2_W(W_2, R_2) y_2 = l \tag{4c}$$

$$\bar{c}^1_R(W_1, R_1) y_1 + \bar{c}^2_R(W_2, R_2) y_2 = k \tag{4d}$$

where $\bar{c}^i_W(W_i, R_i) \equiv \frac{\partial}{\partial W_i} \bar{c}^i(W_i, R_i)$ and $\bar{c}^i_R(W_i, R_i) \equiv \frac{\partial}{\partial R_i} \bar{c}^i(W_i, R_i)$, $i = 1, 2$. Note that $\frac{W_i \bar{c}^i_W(W_i, R_i)}{\bar{c}^i(W_i, R_i)} = \theta_i$ and $\frac{R_i \bar{c}^i_R(W_i, R_i)}{\bar{c}^i(W_i, R_i)} = 1 - \theta_i$. (3) and (4) determine w, r, y_1 and y_2 for given p and k. Under Cobb-Douglas technologies and substituting (3) into (4), we see that, as far as $\theta_1 \neq \theta_2$, (4a) and (4b) have a unique pair $(w(p), r(p))$ for any given p. Logarithmically differentiating the pair with respect to p, we obtain the Stolper-Samuelson properties

$$\frac{p w'(p)}{w(p)} = -\frac{1-\theta_1}{\theta_1 - \theta_2} \quad \text{and} \quad \frac{p r'(p)}{r(p)} = \frac{\theta_1}{\theta_1 - \theta_2}, \tag{5}$$

where $w'(p) \equiv \frac{d}{dp}w(p)$ and $r'(p) \equiv \frac{d}{dp}r(p)$. If $\theta_1 > (<)\theta_2$, we say that the consumption good is capital (labor) intensive from the social perspective. On the other hand, if $\Delta \equiv a_1b_2 - a_2b_1 > (<)0$, then we say that the consumption good is capital (labor) intensive from the private perspective. The model becomes the standard dynamic Heckscher-Ohlin model if $\alpha_i = \beta_i = 0$, $i = 1, 2$.

Remark 1: As technologies are internationally identical, the same domestic price $(w(p), r(p))$ prevails in the foreign country as well. Thus, factor price equalization still holds in a model with externalities as long as the social returns are constant. This is an extension of factor price eqaulization in the literature[1].

2.2 The consumption side

The demand for the consumption good, z, is based on the dynamic optimization problem the representative household in each country faces. The home household is assumed to maximize the discounted sum of its utilities

$$\int_0^\infty \frac{z^{1-\eta}}{1-\eta}e^{-\rho t}dt, \quad 0 < \eta < 1, \rho > 0 \tag{6}$$

Note that in the present case the total factor income $wl + rk$ is not necessarily equal to $y_1 + py_2$, because of the presence of externalities. The gap $\pi \equiv (y_1 + py_2) - (wl + rk)$ can be interpreted as profits or the remuneration for sector-specific factor of production.[2] Thus, the home household maximizes the discounted sum of its utility (6) subject to the flow budget constraint

$$\begin{aligned} \dot{k} &= y_1 + py_2 - pz - \delta k \\ &= (wl + rk) + \pi - pz - \delta k, \end{aligned} \tag{7}$$

where $\delta (> 0)$ is the rate of capital depreciation. The Hamiltonian associated with the problem is

$$H = \frac{z^{1-\eta}}{1-\eta} + \lambda[wl + rk + \pi - pz - \delta k] \tag{8}$$

The necessary conditions for optimality are the first-order condition

$$z^{-\eta} = \lambda p, \tag{9}$$

the differential equation of the co-state variable

$$\dot{\lambda} = \lambda[\rho + \delta - r(p)]; \tag{10}$$

[1] We thank Murray Kemp for pointing this out to us. See Kemp and Okawa (1998) concerning recent issues on factor price equalization.

[2] Benahabib and Nishimura (1998) assume a fixed cost of entry which makes profits possible. Alternatively, Nishimura and Shimomura (2000) assume that there exist sector-specific factors of production in both sectors and externalities may be negative and study the case that production technology in each industry is subject to constant returns to scale from both the private and social perspectives. Under the latter assumption, π is interpreted as the remuneration of sector-specific factors of production, compatible with free entry and exit.

and the transversality condition

$$\lim_{t\to\infty} k(t)\lambda(t)e^{-\rho t} = 0 \tag{11}$$

Note that the transversality condition is a necessary condition in the present model.[3] We assume that both countries have the same technologies and preferences. Thus, the foreign household also faces the same dynamic optimization problem, and we obtain

$$\begin{aligned} \dot{k}^* &= y_1^* + py_2^* - pz^* - \delta k^* \\ &= w(p)l^* + r(p)k^* + \Pi^* - pz^* - \delta k^* \qquad (12a) \\ [z^*]^{-b} &= \lambda^* p \qquad (12b) \\ \dot{\lambda}^* &= \lambda^*[\rho + \delta - r(p)] \qquad (12c) \\ \lim_{t\to\infty} k^*(t)\lambda^*(t)e^{-\rho t} &= 0 \qquad (12d) \end{aligned}$$

The asterisk (*) indicates the variables belonging to the foreign country.

Finally, the world market-clearing condition for the consumption good is

$$z + z^* = y_2 + y_2^* \tag{13}$$

The market-clearing condition for the investment good is obtained from (7), (12a) and (13).

3 A Long-Run Equilibrium under Incomplete Specialization

We shall prove that the stationary states of the system, (7), (9), (10), (11), (12) and (13), exist under incomplete specialization. For this purpose we will assume the following.

Assumption 1: The consumption good is labor intensive from the social perspective but capital intensive from the private perspective, i.e., $\theta_1 < \theta_2$ and $\Delta > 0$:

Assumption 2: $1/\eta > \max[1; 1/\hat{\eta}]$; where $\hat{\eta}^{-1} \equiv \frac{\theta_1 a_1 b_2(\rho+\delta) + (1-\theta_1)b_1\{\rho a_2 + \delta a_1 b_2 + (1-b_1)a_2\delta\}}{\Delta(\theta_2-\theta_1)\{\rho+\delta(1-b_1)\}}$

Benhabib and Nishimura (1998) derived indeterminacy in their closed economy model under Assumption 1. Note that Assumption 2 is satisfied for a sufficiently small η:

Since the pair of factor prices $(w(p); r(p))$ is established in both countries, we also see from (4c) and (4d) that as long as production is incompletely specialized,[4]

[3] See Kamihigashi(2001).

[4] See Appendix 1 for the derivation of (14).

$$
\begin{aligned}
y_1 + py_2 &= \frac{r(p)k(a_1 - a_2) - w(p)l(b_1 - b_2)}{\Delta}; && (14a)\\
y_1^* + py_2^* &= \frac{r(p)k^*(a_1 - a_2) - w(p)l(b_1 - b_2)}{\Delta} && (14b)\\
y_2 &= \frac{r(p)a_1k - w(p)b_1l}{p\Delta}; && (14c)\\
y_2^* &= \frac{r(p)a_1k^* - w(p)b_1l}{p\Delta} && (14d)
\end{aligned}
$$

Making use of (14), the dynamic general equilibrium model can be written as

$$
\begin{aligned}
\dot{k} &= \frac{r(p)k(a_1 - a_2) - w(p)l(b_1 - b_2)}{\Delta} - p[\lambda p]^{-\frac{1}{\sigma}} - \delta k && (15a)\\
\dot{k}^* &= \frac{r(p)k^*(a_1 - a_2) - w(p)l(b_1 - b_2)}{\Delta} - p[\lambda^* p]^{-\frac{1}{\sigma}} - \delta k^* && (15b)\\
\dot{\lambda} &= \lambda[\rho + \delta - r(p)] && (15c)\\
\dot{\lambda}^* &= \lambda^*[\rho + \delta - r(p)] && (15d)\\
[\lambda p]^{-\frac{1}{\sigma}} + [\lambda^* p]^{-\frac{1}{\sigma}} &= \frac{r(p)a_1(k + k^*) - 2lw(p)b_1}{p\Delta} && (15e)
\end{aligned}
$$

If a solution path $(k(t); k^*(t); \lambda(t); \lambda^*(t); p(t))$ of (15) satisfies the transversality conditions (11) and (12d), we say that $(k(t); k^*(t))$ is called an equilibrium from $(k(0); k^*(0))$: We shall analyze the system in the following way. First, let $K \equiv k + k^*$ and $L \equiv 2l$: Summing (15a) and (15b), we have

$$
\dot{K} = \frac{b_2 w(p)L - a_2 r(p)K}{\Delta} - \delta K \qquad (16)
$$

(15c) and (15d) mean that $\frac{\lambda^*}{\lambda} \equiv m$ is constant over time. Therefore, for any given positive m; the solution to the system that is composed of (15a), (15c), (15e), $K = k + k^*$; (16) and $\lambda^* = m\lambda$ is equivalent to the system (15).

Theorem 1 (i) In the long-run equilibrium the total capital and the price $(K^e; p^e)$ are uniquely determined. (ii) In the long-run equilibrium there exists a continuum of countries' capital $(k^e; k^{e*})$ at which both economies are incompletely specialized.

Proof. (i) Consider the system of equations

$$
\begin{aligned}
\frac{b_2 w(p)L - a_2 r(p)K}{\Delta} - \delta K &= 0 && (17a)\\
\rho + \delta &= r(p) && (17b)
\end{aligned}
$$

First, as far as $\theta_1 \neq \theta_2$, for any positive ρ and δ, (17b) uniquely determines the stationary-state p, say p^e. Second, substituting p^e into (17a) and solving for K, we obtain the unique stationary-state K, K^e. That is,

$$p^e = r^{-1}(\rho + \delta), \tag{18a}$$

where $r^{-1}(.)$ is the inverse function of $r(.)$; and

$$K^e = \frac{b_2 w(p^e) L}{\rho a_2 + \delta a_1 b_2 + (1 - b_1) a_2 \delta} > 0 \tag{18b}$$

The long-run equilibrium in a world trade market is independent of m.
(ii) Given $m > 0$, (15a) and (15e) yield the following

$$\{(\rho + \delta)(a_1 - a_2) - \delta\Delta\} k = w(p) l (b_1 - b_2) + p[\lambda p]^{-\frac{1}{\eta}} \Delta \tag{19a}$$

$$[\lambda p]^{-\frac{1}{\eta}} + [m\lambda p]^{-\frac{1}{\eta}} = \frac{r(p) a_1 K - w(p) b_1 L}{p\Delta} \tag{19b}$$

By substituting (18) into (19), we have

$$k^e = \frac{w(p^e) l (b_1 - b_2) + p^e[\lambda^e p^e]^{-\frac{1}{\eta}} \Delta}{(\rho + \delta)(a_1 - a_2) - \delta\Delta} \tag{20}$$

$$\begin{aligned} [\lambda^e p^e]^{-\frac{1}{\eta}}(1 + m^{-\frac{1}{\eta}}) &= \frac{r(p^e) a_1 K^e - w(p^e) b_1 L}{p^e \Delta} \\ &= \frac{w(p^e) L \{\rho + \delta(1 - b_1)\}}{p^e \{\rho a_2 + \delta a_1 b_2 + (1 - b_1) a_2 \delta\}} \end{aligned} \tag{21}$$

Also, from $\lambda^* = m\lambda$ and $k^{e*} = K^e - k^e$

$$k^{e*} = \frac{w(p^e) l (b_1 - b_2) + p^e[m\lambda^e p^e]^{-\frac{1}{\eta}} \Delta}{(\rho + \delta)(a_1 - a_2) - \delta\Delta} \tag{22}$$

Fact 1: If $m = 1$,

$$k^e = k^{e*} = \frac{b_2 w(p^e) l.}{\rho a_2 + \delta a_1 b_2 + (1 - b_1) a_2 \delta} \tag{23}$$

(See Appendix 2 for the proof.)
From Fact 1, (20) and (22) the following Fact holds, and it completes the proof.
Fact 2: There exists $\overline{m} > 1$. Such that for $\overline{m}^{-1} < m < \overline{m}$, the stationary state under incomplete specialization $(K^e, k^e, \lambda^e, p^e)$ uniquely exists. ■

Note that λ^e and k^e do depend on the value of m. Subtracting (22) from (20), we have

$$k^e - k^{e*} = \frac{(\lambda^e)^{-\frac{1}{\eta}} (p^e)^{1-\frac{1}{\eta}} [1 - m^{-\frac{1}{\eta}}] \Delta}{(\rho + \delta)(a_1 - a_2) - \delta\Delta}$$

It follows from Assumption 1 that if $(\rho + \delta)(a_1 - a_2) - \delta\Delta <$ (resp. $>$)0;

$$k^e \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} k^{e*} \text{ if and only if } m \begin{Bmatrix} > \text{(resp. } <) \\ = \\ < \text{(resp. } >) \end{Bmatrix} 1 \tag{24}$$

See Figure 1. (24) means that under $(\rho + \delta)(a_1 - a_2) - \delta\Delta < 0$; if $m =$ (resp. $>$ and $<$)1; then the stationary state is on (resp. below and above) the $45^\circ-$line.

4 Indeterminacy

Under Assumptions 1 and 2, we can obtain the local indeterminacy result.

Theorem 2 There exists a neighborhood of a long-run equilibrium such that from any initial distribution of capital $(k(0); k^*(0))$ in that neighborhood there exists a continuum of equilibrium paths. Moreover, different equilibrium paths converge to different long-run equilibria.

We consider the dynamic general equilibrium model.

$$\dot{K} = \frac{b_2 w(p)L - a_2 r(p)K}{\Delta} - \delta K \tag{25a}$$

$$\dot{\lambda} = \lambda[\rho + \delta - r(p)] \tag{25b}$$

$$\dot{k} = \frac{r(p)k(a_1 - a_2) - w(p)l(b_1 - b_2)}{\Delta} - p[\lambda p]^{-\frac{1}{\theta}} - \delta k \tag{25c}$$

$$1 + m^{-\frac{1}{\theta}} = \frac{(r(p)a_1 K - w(p)b_1 L)p^{\frac{1-\theta}{\theta}}\lambda^{\frac{1}{\theta}}}{\Delta}; \tag{25d}$$

We first prove that for any m in $\left(\overline{m}^{-1}; \overline{m}\right)$ the long-run equilibrium of the system (25) is locally stable.

Lemma 1 Given $m \in \left(\overline{m}^{-1}; \overline{m}\right)$ there is a two- or three-dimensional stable manifold of the system (25) in a neighborhood of the stationary state $(K^e; k^e; \lambda^e)$ such that the solution to the above system converges to the stationary state, that is, for any initial pair of capital stocks $(K(0); k(0))$ near $(K^e; k^e)$; $\lambda(0)$ is chosen so that $(K(0); k(0); \lambda(0))$ is on the stable manifold.

Proof. First, totally differentiating (15e) with respect to $p; K$ and λ; we have

$$\frac{dp}{\Gamma} = -\frac{r(p^e)a_1(p^e)^{\frac{1}{\theta}}(\lambda^e)^{\frac{1}{\theta}}}{(1 + m^{-\frac{1}{\theta}})\Delta}dK - \frac{p^e}{\theta\lambda^e}d\lambda; \tag{26}$$

where $\Gamma \equiv [\frac{1-\theta}{\theta} + \frac{p^e\{r' a_1 K^e - w' b_1 L\}}{\{r a_1 K^e - w b_1 L\}}]^{-1}$: By substituting (5), (17b) and (18a), we get $\Gamma = [\theta^{-1} - \epsilon^{-1}]^{-1} > 0$ from Assumption 2. Linearizing the above system around the stationary state and making use of (26), we can obtain the characteristic equation:

$$
\begin{aligned}
F(x) &\equiv \begin{vmatrix} x + [\frac{ra_2}{\Delta} + \delta + \frac{-\Omega r a_1 (p^e)^{\frac{1}{\eta}}(\lambda^e)^{\frac{1}{\eta}}}{(1+m^{-\frac{1}{\eta}})\Delta^2}] & \frac{-\Omega p^e}{\eta \lambda^e \Delta} & 0 \\ -\frac{-\lambda^e r' r a_1 (p^e)^{\frac{1}{\eta}}(\lambda^e)^{\frac{1}{\eta}}}{(1+m^{-\frac{1}{\eta}})\Delta} & x - \frac{-r' p^e}{\eta} & 0 \\ * & * & x + \{\delta - \frac{r(a_1 - a_2)}{\Delta}\} \end{vmatrix} \\
&= [x + \frac{\delta\Delta - (\rho + \delta)(a_1 - a_2)}{\Delta}] \\
&\quad \times [x^2 + \{\frac{(\rho + \delta)a_2}{\Delta} + \delta + \frac{-\Omega r a_1 (p^e)^{\frac{1}{\eta}}(\lambda^e)^{\frac{1}{\eta}}}{(1 + m^{-\frac{1}{\eta}})\Delta^2} - \frac{-r' p^e}{\eta}\}x \\
&\quad - \frac{-r' p^e}{\eta}\{\frac{(\rho + \delta)a_2}{\Delta} + \delta\}] \\
&= 0; \qquad (27)
\end{aligned}
$$

where $\Omega \equiv b_2 L w' - a_2 K^e r'$: Since (5) and Assumption 1 imply $r' < 0$ and $\Omega > 0$; we see that

$$
\frac{(\rho + \delta)a_2}{\Delta} + \delta + \frac{-\Omega r a_1 (p^e)^{\frac{1}{\eta}}(\lambda^e)^{\frac{1}{\eta}}}{(1 + m^{-\frac{1}{\eta}})\Delta^2} - \frac{-r' p^e}{\eta} > 0 \qquad (28)
$$

and

$$
\begin{aligned}
& - \frac{-r' p^e}{\eta}\{\frac{(\rho + \delta)a_2}{\Delta} + \delta\} \\
= \; & - \frac{-r' p^e}{\eta \Delta}[\rho a_2 + \delta a_1 b_2 + \delta a_2 (1 - b_1)] > 0 \qquad (29)
\end{aligned}
$$

It follows that the characteristic equation $F(x) = 0$ has at least two roots with negative real parts. This fact implies the Lemma. ■

[Proof of Theorem 2] From Lemma 1, if m is given, then an equilibrium path converges to a long-run equilibrium. If the value of m varies, an equilibrium path from given initial stock $(k(0); k^*(0))$ and, from (20) and (22), the long-run equilibrium $(k^e; k^{e*})$ it converges to also varies. Since m is a ratio of the two jump variables λ and λ^*; there is no economic mechanism in the decentralized world economy which chooses a particular value of m: This completes the proof of Theorem 2.

5 Dynamic Heckscher-Ohlin Theorem

Let us examine the long-run trade pattern in the long-run equilibrium. First, (21) ensures us that if m is larger, λ^e is smaller. Second, differentiating the long-run home demand for the consumption good,

$$D_2 = [\lambda^e p^e]^{-\frac{1}{\eta}}$$

and the stationary-state supply of it

$$S_2 = \frac{r(p^e)a_1k^e - w(p^e)b_1l}{p^e\Delta}$$

with respect to λ^e; and making use of (20), we have

$$\begin{aligned}
&\frac{d}{d\lambda^e}[[\lambda^e p^e]^{-\frac{1}{\eta}} - \frac{(\rho+\delta)a_1k^e - w(p^e)b_1l}{p^e\Delta}] \\
= \quad & (-\frac{1}{\eta})(p^e)^{-\frac{1}{\eta}}(\lambda^e)^{-\frac{1}{\eta}-1} - \frac{(\rho+\delta)a_1}{p^e\Delta}\frac{dk^e}{d\lambda^e} \\
= \quad & \frac{(p^e)^{-\frac{1}{\eta}}(\lambda^e)^{-\frac{1}{\eta}-1}\{\rho a_2 + \delta a_1b_2 + \delta a_2(1-b_1)\}}{\eta\{(\rho+\delta)(a_1-a_2) - \delta\Delta\}}, \qquad (30)
\end{aligned}$$

which is negative (resp. positive) if $(\rho+\delta)(a_1 - a_2) - \delta\Delta <$ (resp. $>$)0.

Lemma 3: If $(\rho+\delta)(a_1 - a_2) - \delta\Delta <$ (resp. $>$)0, in the long-run equilibrium the capital-abundant country exports (resp. imports) the consumption good.

Proof. First, notice that if $m = 1$, the long-run home excess demand for the consumption good, $D_2 - S_2$, is equal to zero. Second, if

$$(\rho+\delta)(a_1 - a_2) - \delta\Delta < 0,$$

then (30) is negative, which means that $D_2 - S_2 <$ (resp. $>$)0 if $m <$ (resp. $>$)1. For, (21) tells us that m is negatively related to λ^e. Finally, note that (24) implies that $m <$(resp. $>$)1 means that $k^e >$ (resp. $<$)k^{e*}, i.e., the home (resp. foreign) country is more capital abundant than the foreign (resp. home) country and exports the consumption good. A parallel argument can be made for the other case $(\rho+\delta)(a_1 - a_2) - \delta\Delta > 0$. ■

Suppose that $(\rho+\delta)(a_1 - a_2) - \delta\Delta < 0$. See Figure 2. If $m = 1$, the stationary state is point E on the line OA with slope $1/2$. Theorem 2 asserts that, starting from any initial point E_0 in a neighborhood of the long-run equilibrium E, the world economy converges to E, in which the two countries have identical factor endowments and no country has comparative advantage to any good. If $m <$ (resp. $>$)1, the stationary state is above (resp. below) the line OA like E_1 (resp. E_2). The world economy starting from any initial point E_0 in a neighborhood of the stationary state, converges to E_1 (resp. E_2), in which the home country is more (resp. less) capital abundant, and, following Lemma 3, it exports (resp. imports) the consumption good. A parallel argument brings a completely opposite long-run trade pattern for the other case $(\rho+\delta)(a_1 - a_2) - \delta\Delta > 0$..

Thus, the long-run trade pattern crucially depends on the sign of $(\rho+\delta)(a_1 - a_2) - \delta\Delta$ and whether m is larger or smaller than 1. However, as we have already mentioned, there is no economic mechanism in the decentralized world economy which chooses a particular value of m. Thus, we arrive at the third theorem.

Theorem 3 There exists a neighborhood of the long-run equilibrium (k^e, k^{*e}) with $m = 1$ such that from any initial distribution of capital $(k(0), k^*(0))$ in that neighborhood there exists an equilibrium path converging to a long-run equilibrium with $m < 1$ and another equilibrium converging to a long-run equilibrium with $m > 1$. It follows that the initial world distribution of capital does not determine the long-run trade pattern.

Remark 2: Using a two-country dynamic Heckscher-Ohlin model without externalities, Chen (1992) showed that the long-run pattern of trade is determined by the initial world distribution of capital. Our results mean that his long-run Heckscher-Ohlin Theorem does not hold if externalities are introduced.

6 Concluding Remarks

Let us add a couple of concluding remarks.

First, the indeterminacy results obtained in this paper crucially depend on the presence of externalities. Suppose that there is no externality. Then, $\Delta = a_1 - a_2 = \theta_1 - \theta_2$, which implies that one root of the characteristic equation, $F(x) = 0$, is $-\frac{\delta\Delta - (\rho+\delta)(a_1 - a_2)}{\Delta} = \rho > 0$, and that $F(0)$ (= (29)) is negative. It follows that the characteristic equation (27) has at most one negative root, which means that for any given positive m the stable manifold is at most one-dimensional. It follows that in order for the solution to the aforementioned dynamic general equilibrium model to converge to a stationary state, we have to choose appropriate $\lambda(0)$ and m. Thus we have a continuum of dynamic general equilibrium paths which converge to different stationary states, where there is no indeterminacy as far as the world initial distribution of capital is given. That is substantially what Chen (1992) and Shimomura (1992) discussed.

Second, a large extent of externalities is not necessary for indeterminacy. Benhabib and Nishimura (p.64, 1998) provides the following numerical example:

$$\begin{aligned} a_1 &= 0.34;\ \alpha_1 = 0;\ b_1 = 0.66;\ \beta_1 = 0 \\ a_2 &= 0.3;\ \alpha_2 = 0.05;\ b_2 = 0.65;\ \beta_2 = 0 \end{aligned}$$

Since $\theta_1 - \theta_2 = -0.01 < 0$ and $\Delta = 0.023 > 0$, this example satisfies Assumption 1. It follows that only a small externality of labor in the production of the consumption good is sufficient for indeterminacy in this paper.

Appendix 1: The Derivation of (14)

Solving (4) for y_1 and y_2; we have

$$y_1 = \frac{l\bar{c}_R^2 - k\bar{c}_W^2}{\bar{c}_W^1\bar{c}_R^2 - \bar{c}_W^2\bar{c}_R^1} \quad \text{and} \quad y_2 = \frac{kc_W^1 - l\bar{c}_R^1}{\bar{c}_W^1\bar{c}_R^2 - \bar{c}_W^2\bar{c}_R^1} \tag{a.1}$$

Using (3), y_1 can be written as follows.

$$\begin{aligned} y_1 &= \frac{l\left(\frac{R_2\bar{c}_R^2}{\bar{c}^2}\right)\left(\frac{1}{R_2}\right) - k\left(\frac{W_2\bar{c}_W^2}{\bar{c}^2}\right)\left(\frac{1}{W_2}\right)}{\left(\frac{W_1\bar{c}_W^1}{\bar{c}^1}\right)\left(\frac{R_2\bar{c}_R^2}{\bar{c}^2}\right)\left(\frac{1}{W_1R_2}\right) - \left(\frac{W_2\bar{c}_W^2}{\bar{c}^2}\right)\left(\frac{R_1\bar{c}_R^1}{\bar{c}^1}\right)\left(\frac{1}{W_2R_1}\right)} \\ &= \frac{\frac{l(1-\mu_2)b_2}{r(1-\mu_2)} - \frac{k\mu_2a_2}{w\mu_2}}{\frac{\mu_1(1-\mu_2)a_1b_2}{\mu_1(1-\mu_2)wr} - \frac{\mu_2(1-\mu_1)a_1b_2}{\mu_2(1-\mu_1)wr}} \\ &= \frac{b_2lw - a_2kr}{\Delta} \end{aligned} \tag{a.2}$$

Making similar calculations, we obtain

$$y_2 = \frac{a_1kr - b_1lw}{p\Delta} \tag{a.3}$$

(14) directly follows from (a.2) and (a.3).

Appendix 2: The Proof of Fact 1

If $m = 1$; (21) becomes

$$[\lambda^e p^e]^{-1} = \frac{\delta w(p^e)l\{\rho + \delta(1-b_1)\}}{p^e\{\rho a_2 + \delta a_1b_2 + (1-b_1)a_2\delta\}} \tag{a.4}$$

The substitution of (a.4) into (20) and (22) yields

$$\begin{aligned} k^e &= k^{*e} = \frac{w(p^e)l(b_1 - b_2) + \frac{w(p^e)l\{\rho+\delta(1-b_1)\}\Delta}{\{\rho a_2 + \delta a_1b_2 + (1-b_1)a_2\delta\}}}{\{(\rho+\delta)(a_1-a_2) - \delta\Delta\}} \\ &= \frac{w(p^e)l[(b_1-b_2)\{\rho a_2 + \delta a_1b_2 + (1-b_1)a_2\delta\} + \{\rho + \delta(1-b_1)\}\Delta]}{\{(\rho+\delta)(a_1-a_2) - \delta\Delta\}\{\rho a_2 + \delta a_1b_2 + (1-b_1)a_2\delta\}} \\ &= \frac{w(p^e)l\Theta}{\{(\rho+\delta)(a_1-a_2) - \delta\Delta\}\{\rho a_2 + \delta a_1b_2 + (1-b_1)a_2\delta\}} \end{aligned}$$

where

$$\begin{aligned} \Theta &\equiv (b_1-b_2)(\rho+\delta)a_2 + (b_1-b_2)\delta\Delta + \{\rho + \delta(1-b_1)\}\Delta \\ &= (b_1-b_2)(\rho+\delta)a_2 - b_2\delta\Delta + (\rho+\delta)\Delta \\ &= (\rho+\delta)\{(b_1-b_2)a_2 + (a_1b_2 - a_2b_1)\} - b_2\delta\Delta \\ &= (\rho+\delta)(a_1-a_2)b_2 - b_2\delta\Delta \\ &= b_2\{(\rho+\delta)(a_1-a_2) - \delta\Delta\} \end{aligned}$$

Therefore, we have

$$k^e = k^{*e} = \frac{b_2 w(p^e) l}{\rho a_2 + \delta a_1 b_2 + (1 - b_1) a_2 \delta};$$

as was to be proved.

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