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Abstract

We establish a simple no-bubble theorem that applies to a wide range of deterministic sequential economies with infinitely lived agents. In particular, we show that asset bubbles never arise if at least one agent can reduce his asset holdings permanently from some period onward. Our no-bubble theorem is based on the optimal behavior of a single agent, requiring virtually no assumption beyond the strict monotonicity of preferences. The theorem is a substantial generalization of Kocherlakota's (1992, *Journal of Economic Theory* 57, 245–256) result on asset bubbles and short sales constraints.

Keywords: Asset bubbles; no-bubble theorem; sequential budget constraints; optimality; binary relation

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1 Introduction

Since the global financial crisis of 2007-2008, there has been a surge of interest in rational asset pricing bubbles, or simply "asset bubbles." Numerous economic mechanisms that give rise to asset bubbles are still being proposed. In constructing models of asset bubbles, it is important to understand the conditions under which asset bubbles do or do no exist. While the conditions for the existence of bubbles are mostly restricted to specific models, some general conditions for nonexistence are known.

Most of the results on the nonexistence of bubbles, or no-bubble theorems, for general equilibrium models can be grouped into two categories. A no-bubble theorem of the first category typically states that asset bubbles never arise if the present value of the aggregate endowment process is finite. Santos and Woodford (1997, Theorems 3.1, 3.3) established seminal no-bubble theorems of the first category.[1] These results were extended by Huang and Werner (2000, Theorem 6.1) and Werner (2014, Remark 1, Theorem 1) to different settings.

Unlike these results, no-bubble theorems of the second category are mostly based on the optimal behavior of a single agent without relying on the present value of the aggregate endowment process. For example, in a deterministic economy with finitely many agents, Kocherlakota (1992, Proposition 3) showed that asset bubbles can be ruled out if at least one agent can reduce his asset holdings permanently from some period onward. Obstfeld and Rogoff (1986) used a similar idea earlier to rule out deflationary equilibria in a money-in-the-utility-function model.[2] These results rely on the necessity of a transversality condition,[3] and a fairly general no-bubble theorem based on the necessity of a transversality condition was shown by Kamihigashi (2001, p. 1007) for deterministic representative-agent models under the assumption that instantaneous utility functions are differentiable and strictly increasing.

In this paper we establish a simple no-bubble theorem of the second category that can be used to rule out asset bubbles in an extremely wide range of deterministic models. We consider the problem of a single agent who faces sequential budget constraints and has strictly monotone preferences.

[1] Wilson's (1981, Theorem 2) result on the existence of a competitive equilibrium can be viewed as an earlier example of a no-bubble theorem of the first category.

[2] See Kamihigashi (2008a, 2008b) for results on asset bubbles in related models.

[3] Kamihigashi (2001, 2002, 2003, 2005) established various results on necessity of transversality conditions.

We show that asset bubbles never arise if the agent can reduce his asset holdings permanently from some period onward. This result is a substantial generalization of Proposition 3 in Kocherlakota (1992). Our contribution is to show that this no-bubble theorem holds under extremely general conditions.

The rest of the paper is organized as follows. In Section 2 we present a single agent's problem along with necessary assumptions, and formally define asset bubbles. In Section 3 we offer several examples satisfying our assumptions. In Section 4 we state our no-bubble theorem and show several consequences. In Section 5 we offer some concluding comments. Longer proofs are relegated to the appendices.

2 The General Framework

2.1 Feasibility and Optimality

Time is discrete and denoted by $t \in \mathbb{Z}_+$. There is one consumption good and one asset that pays a dividend of d_t units of the consumption good in each period $t \in \mathbb{Z}_+$. Let p_t be the price of the asset in period $t \in \mathbb{Z}_+$. Consider an infinitely lived agent who faces the following constraints:

$$c_t + p_t s_t = y_t + (p_t + d_t)s_{t-1},\ c_t \geq 0, \quad \forall t \in \mathbb{Z}_+, \tag{2.1}$$

$$s \in \mathcal{S}(s_{-1}, y, p, d), \tag{2.2}$$

where c_t is consumption in period t, $y_t \in \mathbb{R}$ is (net) income in period t, s_t is asset holdings at the end of period t with s_{-1} historically given, and $\mathcal{S}(s_{-1}, y, p, d)$ is a set of sequences in $\mathbb{R}$ with $s = \{s_t\}_{t=0}^\infty$, $y = \{y_t\}_{t=0}^\infty$, $p = \{p_t\}_{t=0}^\infty$, and $d = \{d_t\}_{t=0}^\infty$, We offer several examples of (2.2) in Subsection 3.1.

Although we consider a single agent problem and assume only one asset, it should be clear that the results we develop here apply to a general or partial equilibrium model with at least one agent and at least one asset.

Let $\mathcal{C}$ be the set of sequences $\{c_t\}_{t=0}^\infty$ in $\mathbb{R}_+$. For any $c \in \mathcal{C}$, we let $\{c_t\}_{t=0}^\infty$ denote the sequence representation of c, and vice versa. We therefore use c and $\{c_t\}_{t=0}^\infty$ interchangeably; likewise we use s and $\{s_t\}_{t=0}^\infty$ interchangeably, and so on. The inequalities $<$ and $\leq$ on the set of sequences in $\mathbb{R}$ (which includes $\mathcal{C}$) are defined as follows:

$$c \leq c' \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_t,\ c_t \leq c'_t, \tag{2.3}$$

$$c < c' \quad \Leftrightarrow \quad c \leq c' \text{ and } \exists t \in \mathbb{Z}_+, c_t < c'_t. \tag{2.4}$$

The agent's preferences are represented by a binary relation $\prec$ on $\mathcal{C}$. More concretely, for any $c, c' \in \mathcal{C}$, the agent strictly prefers c' to c if and only if $c \prec c'$. The assumptions stated in this section are maintained throughout the paper unless otherwise noted.

Assumption 2.1. $d_t \geq 0$ and $p_t \geq 0$ for all $t \in \mathbb{Z}_+$.

Assumption 2.2. $p_t > 0$ for all $t \in \mathbb{Z}_+$.

While we use the second assumption for most of our results, we are unable to use it in one important case. In particular, if the asset is intrinsically useless, that is, if $d_t = 0$ for all $t \in \mathbb{Z}_+$, then it is necessary to consider the possibility that $p_t = 0$ for all $t \in \mathbb{Z}_+$. One of our results deals with this particular case without applying Assumption 2.2; see Proposition 4.2.

We say that a pair of sequences $c = \{c_t\}_{t=0}^{\infty}$ and $s = \{s_t\}_{t=0}^{\infty}$ in $\mathbb{R}$ is a *plan*; that a plan (c, s) is *feasible* if it satisfies (2.1) and (2.2); and that a feasible plan (c^*, s^*) is *optimal* if there exists no feasible plan (c, s) such that $c^* \prec c$. Whenever we take an optimal plan (c^*, s^*) as given, we assume the following.

Assumption 2.3. For any $c \in C$ with $c^* < c$, we have $c^* \prec c$.

This assumption holds if $\prec$ is strictly monotone in the sense that for any $c, c' \in \mathcal{C}$ with $c < c'$, we have $c \prec c'$. While this latter requirement may seem reasonable, it remains unsatisfied in one important case; see Example 3.1.

2.2 Asset Bubbles

In this subsection we define the fundamental value of the asset and the bubble component of the asset price in period $t \in \mathbb{Z}_+$ using the period t prices of the consumption goods in periods $t, t+1, \ldots$. To be more concrete, let q_0^t be the period 0 price of the consumption good in period $t \in \mathbb{Z}_+$. It is well known (e.g., Huang and Werner, 2000, (8)) that the absence of arbitrage implies that there exists a price sequence $\{q_0^t\}$ such that

$$\forall t \in \mathbb{Z}_+, \quad q_0^t p_t = q_0^{t+1}(p_{t+1} + d_{t+1}), \tag{2.5}$$
$$\forall t \in \mathbb{N}, \quad q_0^t > 0, \tag{2.6}$$
$$q_0 = 1. \tag{2.7}$$

Under Assumption 2.2, conditions (2.5) and (2.7) uniquely determine the price sequence $\{q_0^t\}$. For the rest of the paper, we take the price sequence $\{q_0^t\}$ given by (2.5) and (2.7).

For $t \in \mathbb{N}$ and $i \in \mathbb{Z}_+$, we define

$$q_t^i = q_0^{t+i}/q_0^t, \tag{2.8}$$

which is the period t price of consumption in period $t + i$. Note that

$$\forall i, j, t \in \mathbb{Z}_+, \quad q_t^i q_{t+i}^j = \frac{q_0^{t+i}}{q_0^t}\frac{q_0^{t+i+j}}{q_0^{t+i}} = q_t^{i+j}. \tag{2.9}$$

Let $t \in \mathbb{Z}_+$. Equqations (2.5) and (2.8) give us $p_t = q_t^1(p_{t+1} + d_{t+1})$. By repeatedly applying this equality and (2.9), we obtain

$$\begin{aligned}
p_t &= q_t^1 d_{t+1} + q_t^1 p_{t+1} && (2.10)\\
&= q_t^1 d_{t+1} + q_t^1 q_{t+1}^1 (p_{t+2} + d_{t+2}) && (2.11)\\
&= q_t^1 d_{t+1} + q_t^2 d_{t+2} + q_t^2 p_{t+2} && (2.12)\\
&\vdots && (2.13)\\
&= \sum_{i=1}^{n} q_t^i d_{t+i} + q_t^n p_{t+n}, \quad \forall n \in \mathbb{N}. && (2.14)
\end{aligned}$$

Since the above finite sum is increasing in $n \in \mathbb{N}$,[4] it follows that

$$p_t = \sum_{i=1}^{\infty} q_t^i d_{t+i} + \lim_{n\uparrow\infty} q_t^n p_{t+n}. \tag{2.15}$$

As is commonly done in the literature, we define the *fundamental value* of the asset in period t as the present discounted value of the dividend stream

[4]In this paper, "increasing" means "nondecreasing," and "decreasing" means "nonincreasing."

from period $t+1$ onward:

$$f_t = \sum_{i=1}^{\infty} q_t^i d_{t+i}. \tag{2.16}$$

The *bubble* component of the asset price in period t is the part of p_t that is not accounted for by the fundamental value:

$$b_t = p_t - f_t. \tag{2.17}$$

It follows from (2.15)–(2.17) that

$$b_t = \lim_{n \uparrow \infty} q_t^n p_{t+n}. \tag{2.18}$$

Using (2.9) we see that

$$q_0^t \lim_{n \uparrow \infty} q_t^n p_{t+n} = \lim_{n \uparrow \infty} q_0^{t+n} p_{t+n} = \lim_{i \uparrow \infty} q_0^i p_i. \tag{2.19}$$

Hence, (2.18) and (2.6) give us

$$\lim_{i \uparrow \infty} q_0^i p_i = 0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ b_t = 0. \tag{2.20}$$

3 Examples

In this section we present several examples of (2.2) as well as examples of preferences that satisfy Assumption 2.3. Some of the examples are used in Section 4.

3.1 Constraints on Asset Holdings

The simplest example of (2.2) would be the following:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq 0. \tag{3.1}$$

This constraint is often used in representative-agent models; see, e.g., Lucas (1978) and Kamihigashi (1998).

Kocherlakota (1992) uses a more general version of (3.1):

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma, \tag{3.2}$$

where $\sigma \in \mathbb{R}$. If $\sigma < 0$, then the above constraint is called a short sales constraint. The following constraint is even more general:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma_t, \tag{3.3}$$

where $\sigma_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Note that (3.2) is a special case of (3.3) with $\sigma_t = \sigma$ for all $t \in \mathbb{Z}_+$.

Santos and Woodford (1997, p. 24) consider a (state-dependent) borrowing constraint that reduces to the following in our single-asset setting:

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t \geq -\xi_t, \tag{3.4}$$

where $\xi_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. This constraint is a special case of (3.3) with $\sigma_t = -\xi_t/p_t$.

The (state-dependent) debt constraint considered by Werner (2014) and LeRoy and Werner (2014, p. 313) reduces to the following in our setting:

$$\forall t \in \mathbb{Z}_+, \quad (p_{t+1} + d_{t+1}) s_t \geq -\xi_{t+1}. \tag{3.5}$$

This constraint is another special case of (3.3) with

$$\sigma_t = -\xi_{t+1}/(p_{t+1} + d_{t+1}). \tag{3.6}$$

In addition to (3.2), Kocherlakota (1992) considers the following wealth constraint:

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t + \sum_{i=1}^{\infty} q_t^i y_{t+i} \geq 0, \tag{3.7}$$

which is another example of (2.2). The left-hand side above is the period t value of the agent's current asset holdings and future income. Note that (3.7) is yet another special case of (3.3) with

$$\forall t \in \mathbb{Z}_+, \quad \sigma_t = -\sum_{i=1}^{\infty} q_t^i y_{t+i}/p_t. \tag{3.8}$$

See Wright (1987) and Huang and Werner (2000) for related discussion.

3.2 Preferences

Example 3.1. A typical objective function in an agent's maximization problem takes the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{3.9}$$

where $\beta \in (0,1)$ and $u : \mathbb{R}_+ \to [-\infty, \infty)$ is a strictly increasing function. Suppose further that u is bounded, and define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad \sum_{t=0}^{\infty} \beta^t u(c_t) < \sum_{t=0}^{\infty} \beta^t u(c_t'). \tag{3.10}$$

Then $\prec$ clearly satisfies Assumption 2.3.

If u is unbounded below, i.e., if $u(0) = -\infty$, then the above definition of $\prec$ may not satisfy Assumption 2.3. In particular, given $c^*, c \in \mathcal{C}$ with $c^* < c$, we do not have $c^* \prec c$ if $c_t^* = c_t = 0$ for some $t \in \mathbb{Z}_+$ and if u is bounded above. Indeed, in this case,

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) = \sum_{t=0}^{\infty} \beta^t u(c_t) = -\infty. \tag{3.11}$$

Hence the inequality in (3.10) does not hold.

Example 3.2. The above problem with unbounded utility can be avoided by using an alternative optimality criterion. To be specific, let $u_t : \mathbb{R}_+ \to [-\infty, \infty)$ be a strictly increasing function for $t \in \mathbb{Z}_+$ as above. In this case, the infinite sum $\sum_{t=0}^{\infty} u_t(c_t)$ may not be well defined. Even if it is always well defined, it may not be strictly increasing, as discussed above. To deal with these problems, consider the binary relation $\prec$ defined by

$$c \prec c' \quad \Leftrightarrow \quad \varliminf_{n \uparrow \infty} \sum_{t=0}^{n} [u_t(c_t) - u_t(c_t')] < 0, \tag{3.12}$$

where we follow the convention that $(-\infty) - (-\infty) = 0$; see Dana and Le Van (2006) for related optimality criteria. The binary relation $\prec$ defined above clearly satisfies Assumption 2.3.

Continuing with this example, suppose that (2.2) is given by (3.1). Suppose further that each u_t is differentiable on $\mathbb{R}_{++}$, and that there exists an optimal plan (c^*, s^*) such that

$$\forall t \in \mathbb{Z}_+, \quad c_t^* > 0, \; s_t^* = 1. \tag{3.13}$$

Then the standard Euler equation holds:

$$u_t'(c_t^*)p_t = u_{t+1}'(c_{t+1}^*)(p_{t+1} + d_{t+1}), \quad \forall t \in \mathbb{Z}_+. \tag{3.14}$$

In view of (2.5), the price sequence $\{q_0^t\}$ is given by

$$q_0^t = \frac{u_t'(c_t^*)}{u_0'(c_0^*)}, \quad \forall t \in \mathbb{Z}_+. \tag{3.15}$$

The fundamental value f_t takes the familiar form:

$$f_t = \sum_{i=1}^{\infty} \frac{u_{t+i}'(c_{t+i}^*)}{u_t'(c_t^*)} d_{t+i}, \quad \forall t \in \mathbb{Z}_+. \tag{3.16}$$

Example 3.3. Let $v : \mathcal{C} \to \mathbb{R}$ be a strictly increasing function. Define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad v(c_0, c_1, c_2, \ldots) < v(c_0', c_1', c_2', \ldots). \tag{3.17}$$

Note that (3.17) satisfies Assumption 2.3 without any additional condition on v. For example, v can be a recursive utility function.

4 Implications of Feasibility and Optimality

4.1 A No-Bubble Theorem

To state our no-bubble theorem, we need to introduce some notation. Given any sequence $\{s_t^*\}_{t=0}^{\infty}$ in $\mathbb{R}$, $\tau \in \mathbb{Z}_+$, and $\epsilon > 0$, let $\mathcal{S}_{\tau,\epsilon}(s^*)$ be the set of sequences $\{s_t\}_{t=0}^{\infty}$ in $\mathbb{R}$ such that

$$s_t \begin{cases} = s_t^* & \text{if } t < \tau, \\ \geq s_t^* - \epsilon & \text{if } t \geq \tau. \end{cases} \tag{4.1}$$

In other words, a sequence $\{s_t\}$ in $\mathcal{S}_{\tau,\epsilon}(s^*)$ coincides with $\{s_t^*\}$ up to period $\tau - 1$ and is only required to satisfy the lower bound $s_t^* - \epsilon$ from period τ onward. Now we are ready to state the main result of this paper.

Theorem 4.1. *Let (c^*, s^*) be an optimal plan. Suppose that there exist $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ such that*

$$\mathcal{S}_{\tau,\epsilon}(s^*) \subset \mathcal{S}(s_{-1}, y, p, d). \tag{4.2}$$

Then $b_t = 0$ for all $t \in \mathbb{Z}_+$.

Proof. See Appendix A. □

It seems remarkable that asset bubbles can be ruled out by a simple condition such as (4.2). No explicit utility function is assumed, and the only requirement on the binary relation $\prec$ is Assumption 2.3, which merely requires strict monotonicity at the given optimal consumption plan c^*.

The proof of Theorem 4.1 is based on a simple idea. If the left equality in (2.20) is violated, we can construct the following alternative plan. Let $\delta > 0$, and let $s_\tau = s^*_\tau - \delta$ and $c_\tau = c^*_\tau + p_\tau \delta$, where τ is given by the statement of the theorem. For $t \neq \tau$, let s_t be determined by the budget constraint (2.1) with $c_t = c^*_t$. This alternative plan provides the same consumption sequence except in period τ, where consumption is increased by $p_\tau \delta > 0$. The plan, therefore, is strictly preferred over the original plan (c^*, s^*). We derive a contradiction by showing that the alternative plan is feasible for a sufficiently small $\delta > 0$.

Huang and Werner (2000, Theorems 5.1, 6.1) use similar constructions as "Ponzi schemes," but their constructions are not directly linked to the nonexistence of asset bubbles.

4.2 Consequences of Theorem 4.1

In this subsection we provide fairly simple consequences of Theorem 4.1. Throughout this subsection we take an optimal plan (c^*, s^*) as given. We start with a simple result assuming that the feasibility constraint on asset holdings (2.2) is given by a sequence of constraints of the form (3.3). As discussed in Subsection 3.1, this simple form covers various constraints on borrowing, debt, and wealth,

Corollary 4.1. *Suppose that (2.2) is given by (3.3) with $\sigma_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Suppose further that*

$$\varliminf_{t \uparrow \infty} (s^*_t - \sigma_t) > 0. \tag{4.3}$$

Then the conclusion of Theorem 4.1 holds.

Proof. Assume (4.3). Let $\epsilon \in (0, \underline{\lim}_{t\uparrow\infty}(s_t^* - \sigma_t))$. Then there exists $\tau \in \mathbb{Z}_+$ such that $s_t^* - \sigma_t \geq \epsilon$, or $s_t^* - \epsilon \geq \sigma_t$, for all $t \geq \tau$. This implies (4.2). The conclusion of Theorem 4.1 therefore holds. □

If there is a constant lower bound on asset holdings s_t, the above result reduces to the following.

Corollary 4.2. *Suppose that (2.2) is given by (3.2) for some $\sigma \in \mathbb{R}$. Suppose further that $\underline{\lim}_{t\uparrow\infty} s_t^* > \sigma$. Then the conclusion of Theorem 4.1 holds.*

Kocherlakota (1992, Proposition 3) in effect shows a special case of the above result under the following additional assumptions: (i) the binary relation $\prec$ is represented by (3.10); (ii) u is continuously differentiable on $\mathbb{R}_{++}$, strictly increasing, concave, and bounded above or below; and (iii) the optimal plan (c^*, s^*) satisfies

$$\forall t \in \mathbb{Z}_+, \quad c_t^* > 0, \tag{4.4}$$

$$\left| \sum_{t=0}^{\infty} \beta^t u(c_t^*) \right| < \infty. \tag{4.5}$$

Since Corollary 4.2 requires none of these additional assumptions, it is a substantial generalization of Proposition 3 in Kocherlakota (1992). He uses the extra assumptions mostly to derive a transversality condition, which is crucial to his approach. By contrast, Corollary 4.2 is based on our Theorem 4.1, which is proved by an elementary perturbation argument that fully exploits the structure of sequential budget constraints. Since Corollary 4.1 and Theorem 4.1 are even more general, they can also be viewed as generalizations of Kocherlakota's result.

Next we present two results that apply to representative-agent models.

Corollary 4.3. *Suppose that (2.2) is given by (3.1). Suppose further that*

$$\forall t \in \mathbb{Z}_+, \quad s_t^* = 1. \tag{4.6}$$

Then the conclusion of Theorem 4.1 holds.

Proof. Note that (4.6) and (3.1) imply (4.2) with $\tau = 0$ and $\epsilon = 1$. The conclusion of Theorem 4.1 therefore holds. □

The following proposition is immediate from the above result and (3.16).

Proposition 4.1. *In the setup for Example 3.2 (including (3.14)–(3.16)), we have*

$$\forall t \in \mathbb{Z}_+, \quad p_t = \sum_{i=1}^{\infty} \frac{u'_{t+i}(c^*_{t+i})}{u'_t(c^*_t)} d_{t+i}. \tag{4.7}$$

Kamihigashi (2001, Section 4.2.1) shows a similar result for a continuous-time model with a nonlinear constraint. It is known that a stochastic version of Proposition 4.1 requires additional assumptions; see Kamihigashi (1998) and Montrucchio and Privileggi (2001).[5]

Finally we consider the case of fiat money, or an asset with no dividend payment. Since the fundamental value of fiat money is zero, its price must also be zero in the absence of an asset bubble. Hence the case of fiat money is not directly covered by Theorem 4.1, which requires Assumption 2.2,

Proposition 4.2. *Drop Assumption 2.2 and (2.5)–(2.7) (but maintain Assumptions 2.1 and 2.3). Suppose that there exist $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ satisfying (4.2). Suppose further that*

$$\forall t \geq \tau + 1, \quad d_t = 0. \tag{4.8}$$

Then

$$\forall t \geq \tau, \quad p_t = 0. \tag{4.9}$$

Proof. See Appendix B. □

5 Concluding Comments

In this paper we established a simple no-bubble theorem that applies to a wide range of deterministic economies with infinitely lived agents facing sequential budget constraints. In particular, we showed that asset bubbles can be ruled out if at least one agent can reduce his asset holdings permanently from some period onward. This is a substantial generalization of Kocherlakota's (1992) result on asset bubbles and short sales constraints.

Our no-bubble theorem is based on the optimal behavior of a single agent, requiring virtually no assumption beyond the strict monotonicity of preferences. Although we showed several useful consequences of the theorem, additional results can be shown by using the results presented here in conjunction with other arguments based on market-clearing and aggregation.

[5] See Kamihigashi (2011) for sample-path properties of stochastic asset bubbles.

Appendix A Proof of Theorem 4.1

Let (c^*, s^*) be an optimal plan. It suffices to verify that

$$\lim_{i \uparrow \infty} q_0^i p_i = 0, \tag{A.1}$$

which implies the desired conclusion by (2.20). Suppose by way of contradiction that

$$\lim_{i \uparrow \infty} q_0^i p_i > 0. \tag{A.2}$$

Then since $q_0^i p_i > 0$ for all $i \in \mathbb{Z}_+$ by Assumption 2.2 and (2.6), it follows that there exists $\underline{b} > 0$ such that

$$\forall i \in \mathbb{Z}_+, \quad q_0^i p_i \geq \underline{b}. \tag{A.3}$$

Equivalently, we have $1/p_i \leq q_0^i/\underline{b}$ for all $i \in \mathbb{Z}_+$. Multiplying through by d_i and summing over $i \in \mathbb{N}$, we obtain

$$\sum_{i=1}^{\infty} \frac{d_i}{p_i} \leq \sum_{i=1}^{\infty} \frac{q_0^i d_i}{\underline{b}} = \frac{f_0}{\underline{b}} < \infty, \tag{A.4}$$

where the equality uses (2.16).[6]

Let $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ be given by (4.2). For each $\delta \in (0, \epsilon)$ we construct an alternative plan (c^δ, s^δ) as follows:

$$c_t^\delta = \begin{cases} c_t^* & \text{if } t \neq \tau, \\ c_\tau^* + p_\tau \delta & \text{if } t = \tau, \end{cases} \tag{A.5}$$

$$s_t^\delta = \begin{cases} s_t^* & \text{if } t \leq \tau - 1, \\ s_\tau^* - \delta & \text{if } t = \tau, \\ [y_t + (p_t + d_t) s_{t-1}^\delta - c_t^*]/p_t & \text{if } \tau \geq \tau + 1. \end{cases} \tag{A.6}$$

It suffices to show that (c^δ, s^δ) is feasible for $\delta > 0$ sufficiently small; for then, we have $c^* \prec c^\delta$ by (A.5) and Assumption 2.3, contradicting the optimality of (c^*, s^*).

[6] An arguments similar to (A.4) is used by Montrucchio (2004, Theorem 2).

Note that (c^δ, s^δ) satisfies (2.1) by construction. Hence by (2.1) we have

$$\forall t \geq \tau + 1, \quad p_t(s_t^* - s_t^\delta) = (p_t + d_t)(s_{t-1}^* - s_{t-1}^\delta). \tag{A.7}$$

For $t \geq \tau$ define

$$\delta_t = s_t^* - s_t^\delta. \tag{A.8}$$

Note that $\delta_\tau = \delta$ by (A.6). We have $p_t \delta_t = (p_t + d_t)\delta_{t-1}$ for all $t > \tau$ by (A.7). Thus for any $t > \tau$ we have

$$\delta_t = \frac{p_t + d_t}{p_t}\delta_{t-1} = \frac{p_t + d_t}{p_t}\frac{p_{t-1} + d_{t-1}}{p_{t-1}}\delta_{t-2} = \cdots \tag{A.9}$$

$$= \delta \prod_{i=\tau+1}^{t} \frac{p_i + d_i}{p_i} \leq \delta \prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}, \tag{A.10}$$

where the equality in (A.10) holds since $\delta_\tau = \delta$, and the inequality in (A.10) holds since $d_t \geq 0$ for all $t \in \mathbb{Z}_+$ by Assumption 2.1.[7]

To show that (c^δ, s^δ) is feasible, it suffices to verify that $\delta_t \leq \epsilon$ for all $t \geq \tau$; for then, we have $s \in \mathcal{S}(s_{-1}, y, p, d)$ by (4.2) and (A.8). For this purpose, note from (A.4) that

$$\frac{f_0}{\underline{b}} \geq \sum_{i=1}^{\infty} \frac{d_i}{p_i} \geq \sum_{i=1}^{\infty} \ln\left(1 + \frac{d_i}{p_i}\right) \tag{A.11}$$

$$= \sum_{i=1}^{\infty} \ln\left(\frac{p_i + d_i}{p_i}\right) = \ln\left(\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}\right). \tag{A.12}$$

It follows that

$$\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i} < \infty. \tag{A.13}$$

Using this and recalling (A.9)–(A.10), we can choose $\delta > 0$ small enough that $\delta_t \leq \epsilon$ for all $t \geq \tau$. For such δ, (c^δ, s^δ) is feasible, contradicting the optimality of (c^*, s^*). We have verified (A.1), which implies the conclusion of the theorem.

[7] An argument similar to (A.9)–(A.10) is used by Bosi et al. (2014).

Appendix B Proof of Proposition 4.2

Let $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ be as in (4.2). Suppose by way of contraction that $p_{\tau'} > 0$ for some $\tau' \geq \tau$. Without loss of generality, we assume that $\tau' = \tau = 0$;[8] i.e.,

$$p_0 > 0. \tag{B.1}$$

First suppose that

$$\forall t \in \mathbb{N}, \quad p_t > 0. \tag{B.2}$$

Then Assumption 2.2 holds. We construct $\{q_0^t\}_{t=0}^\infty$ by (2.5) with $q_0^0 = 1$. Then (2.5)–(2.7) hold. Since Assumptions 2.1–2.3 and (2.5)–(2.7) now hold, Theorem 4.1 applies. But note from (4.8) and (2.16) that

$$\forall t \in \mathbb{Z}_+, \quad f_t = 0. \tag{B.3}$$

Hence by Theorem 4.1, we have $b_t = 0$, i.e., $p_t = f_t = 0$, for all $t \in \mathbb{Z}_+$. This contradicts (B.2).

We have shown that (B.2) cannot be true. In other words, there must be $t \in \mathbb{N}$ such that $p_t = 0$. Let T be the first $T \in \mathbb{Z}_+$ with

$$p_T > 0, \quad p_{T+1} = 0. \tag{B.4}$$

Such T must exist by (B.1). We construct an alternative plan (c, s) as follows:

$$c_t = \begin{cases} c_t^* & \text{if } t \neq T, \\ c_T^* + p_T\epsilon & \text{if } t = T, \end{cases} \tag{B.5}$$

$$s_t = \begin{cases} s_t^* & \text{if } t \neq T, \\ s_T^* - \epsilon & \text{if } t = T. \end{cases} \tag{B.6}$$

According to this plan, the agent sells the asset when its price is strictly positive, and buys it back when it is free. It is easy to see from (4.2), (2.1), and (B.4) that (c, s) is feasible. But then we have $c^* \prec c$ by (B.5) and Assumption 2.3, contradicting the optimality of (c^*, s^*).

We have shown that we reach a contradiction whether (B.2) holds or not; thus we must have $p_0 = 0$.

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[8] It is no loss of generality to assume that $\tau' = \tau = 0$ since we only consider variables in and after period τ'.

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