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Abstract

We show a simple no-bubble theorem that applies to a wide range of deterministic sequential economies with infinitely lived agents. In particular, we show that asset bubbles never arise if there is at least one agent who can reduce his asset holdings permanently from some period onward. This is a substantial generalization of Kocherlakota's (1992, *Journal of Economic Theory* 57, 245–256) result on asset bubbles and short sales constraints. Our no-bubble theorem requires virtually no assumption except for the strict monotonicity of preferences.

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1 Introduction

Since the global financial crisis of 2007-2008, there has been a surge of interest in rational asset pricing bubbles, or simply "asset bubbles." Numerous economic mechanisms that give rise to asset bubbles are still being proposed, and the implications of asset bubbles on various economic issues are actively discussed in the current literature; we refer the reader to Miao (2014) for a short survey on recent developments.

In constructing models of asset bubbles, it is important to understand conditions under which asset bubbles exist or not. While conditions for existence are mostly restricted to specific models, some general conditions for nonexistence are known. In fact, there are various versions of the no-bubble theorem that essentially states that asset bubbles never arise if the present value of the aggregate endowment process is finite.

Wilson's (1981, Theorem 2) result on the existence of a competitive equilibrium in a deterministic economy with infinitely many agents can be viewed as an early version of this no-bubble theorem. Santos and Woodford (1997, Theorems 3.1, 3.3) established a weak and a strong version of the no-bubble theorem for an incomplete market economy with infinitely many agents each of whom may be finitely or infinitely lived. While the weak version shows the existence of state prices for which no bubble arises on any asset in positive net supply, the strong version shows that no bubble arises on any such asset for any state prices under an additional assumption of uniformly impatient preferences. Huang and Werner (2000, Theorem 6.1) established a version of the no-bubble theorem applicable to an asset in zero net supply for a deterministic economy with finitely many agents whose preferences are represented by explicit utility functions. Recently, Werner (2014, Remark 1, Theorem 1) extended Santos and Woodford's (1997) weak version to a complete market economy with debt constraints (instead of borrowing constraints) as well as their strong version to an incomplete market economy with debt constraints under the additional assumption of uniformly impatient preferences.

While these results are based on equilibrium prices and allocations, there are closely related results based mostly on the optimal behavior of a single agent. For example, in a deterministic economy with finitely many agents, Kocherlakota (1992, Proposition 3) showed that in an equilibrium with a positive asset bubble, the short sales constraints of all agents must be asymptotically binding; equivalently, asset bubbles can be ruled out if there is at least one agent whose asset holdings can be lowered permanently from some period onward. A similar idea was used earlier by Obstfeld and Rogoff (1986) to rule out deflationary equilibria in a money-in-the-utility-function model.[1]

The results mentioned in the preceding paragraph rely on the necessity of a transversality condition,[2] and a general no-bubble result based on the necessity of a transversality condition was shown in Kamihigashi (2001, p. 1007) for deterministic representative-agent models. Basically this result only requires the differentiability and strict monotonicity of instantaneous utility functions; thus it can be used to rule out asset bubbles in various representative-agent models.

In this paper we establish a simple no-bubble theorem that can be used to rule out asset bubbles in a considerably broader range of deterministic models. More specifically, we consider the problem of a single agent facing sequential budget constraints and having strictly monotone preferences. We show that asset bubbles never arise if the agent can reduce his asset holdings permanently from some period onward. This result uses the same idea as those based on transversality conditions mentioned above; the contribution of this paper is to show that the result holds true under extremely general conditions.

To demonstrate the applicability of our no-bubble theorem to general equilibrium models, we consider a general equilibrium model with multiple agents and multiple assets. Using our no-bubble theorem, we show substantial generalizations of Proposition 3 in Kocherlakotoa (1992).

The rest of the paper is organized as follows. In Section 2 we present a single agent's problem along with necessary assumptions, and formally define asset bubbles. In Section 3 we offer several examples satisfying our assumptions. In Section 4 we state our no-bubble theorem and show some immediate consequences. In Section 5 we present a general equilibrium model. In Section 6 we show several results on asset bubbles in general equilibrium. In Section 7 we offer some concluding comments. Some of the proofs are included in the appendices.

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[1]See Kamihigashi (2008a, 2008b) for results on asset bubbles in related models.

[2]Various results on necessity of transversality conditions were established in Kamihigashi (2001, 2002, 2003, 2005).

2 Single-Agent/Single-Asset Framework

2.1 Feasibility and Optimality

Time is discrete and denoted by $t \in \mathbb{Z}_+$. In this section we assume that there are one consumption good and one asset that pays a dividend of d_t units of the consumption good in each period $t \in \mathbb{Z}_+$. Let p_t be the price of the asset in period $t \in \mathbb{Z}_+$. Consider an infinitely lived agent who faces the following constraints:

$$c_t + p_t s_t = y_t + (p_t + d_t)s_{t-1},\ c_t \geq 0, \quad \forall t \in \mathbb{Z}_+, \tag{2.1}$$

$$s \in \mathcal{S}(s_{-1}, y, p, d), \tag{2.2}$$

where c_t is consumption in period t, $y_t \in \mathbb{R}$ is (net) income in period t, s_t is asset holdings at the end of period t with s_{-1} historically given, and $\mathcal{S}(s_{-1}, y, p, d)$ is a set of sequences in $\mathbb{R}$ with $s = \{s_t\}_{t=0}^\infty$, $y = \{y_t\}_{t=0}^\infty$, $p = \{p_t\}_{t=0}^\infty$, and $d = \{d_t\}_{t=0}^\infty$, We present several examples of (2.2) in Subsection 3.1.

Although we consider a single agent's problem and assume that there is only one asset here, our results developed within this framework apply to a general equilibrium model with many agents and many assets, as shown in Section 6.

Let $\mathcal{C}$ be the set of sequences $\{c_t\}_{t=0}^\infty$ in $\mathbb{R}_+$. For any $c \in \mathcal{C}$, we let $\{c_t\}_{t=0}^\infty$ denote the sequence representation of c, and vice versa. In other words, we use c and $\{c_t\}_{t=0}^\infty$ interchangeably; likewise, we use s and $\{s_t\}_{t=0}^\infty$ interchangeably, and so on. We define the inequalities $<$ and $\leq$ on the set of sequences in $\mathbb{R}$ (which includes $\mathcal{C}$) as follows:

$$c \leq c' \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_t,\ c_t \leq c'_t, \tag{2.3}$$

$$c < c' \quad \Leftrightarrow \quad c \leq c' \text{ and } \exists t \in \mathbb{Z}_+, c_t < c'_t. \tag{2.4}$$

The agent's preferences are represented by a binary relation $\prec$ on $\mathcal{C}$. More concretely, for any $c, c' \in \mathcal{C}$, the agent strictly prefers c' to c if and only if $c \prec c'$. The assumptions stated in this section are maintained until the end of Section 4 unless otherwise noted.

Assumption 2.1. $d_t \geq 0$ and $p_t \geq 0$ for all $t \in \mathbb{Z}_+$.

Assumption 2.2. $p_t > 0$ for all $t \in \mathbb{Z}_+$.

Although the second assumption is used for most of our results, there is an important case in which it cannot be used. In particular, if the asset is intrinsically useless—i.e., $d_t = 0$ for all $t \in \mathbb{Z}_+$—then it is more than natural to consider the possibility that $p_t = 0$ for all $t \in \mathbb{Z}_+$. One of our results deals with this particular case without assuming Assumption 2.2; see Proposition 4.2.

We say that a pair of sequences $c = \{c_t\}_{t=0}^{\infty}$ and $s = \{s_t\}_{t=0}^{\infty}$ in $\mathbb{R}$ is a *plan*; a plan (c, s) is *feasible* if it satisfies (2.1) and (2.2); and a feasible plan (c^*, s^*) is *optimal* if there exists no feasible plan (c, s) such that $c^* \prec c$. Whenever we take an optimal plan (c^*, s^*) as given, we assume the following.

Assumption 2.3. For any $c \in C$ with $c^* < c$, we have $c^* \prec c$.

This assumption is satisfied if $\prec$ is strictly monotone in the sense that for any $c, c' \in \mathcal{C}$ with $c < c'$, we have $c \prec c'$. Although this latter requirement may seem reasonable, there is an important case in which it is not satisfied. Such a case and other examples of preferences satisfying Assumption 2.3 are discussed in Subsection 3.2.

2.2 Asset Bubbles

In this subsection we define the fundamental value of the asset and the bubble component of the asset price in period $t \in \mathbb{Z}_+$ using the period t prices of the consumption goods in periods $t, t+1, \ldots$. To be more concrete, let q_0^t be the period 0 price of the consumption good in period $t \in \mathbb{Z}_+$. It is well known (e.g., Huang and Werner, 2000, (8)) that the absence of arbitrage implies that there exists a price sequence $\{q_0^t\}$ such that

$$\forall t \in \mathbb{Z}_+, \quad q_0^t p_t = q_0^{t+1}(p_{t+1} + d_{t+1}), \tag{2.5}$$

$$\forall t \in \mathbb{N}, \quad q_0^t > 0, \tag{2.6}$$

$$q_0 = 1. \tag{2.7}$$

In the current setting with a single asset, under Assumption 2.2, conditions (2.5) and (2.7) uniquely determine the price sequence $\{q_0^t\}$.

For $t \in \mathbb{N}$ and $i \in \mathbb{Z}_+$, we define

$$q_t^i = q_0^{t+i}/q_0^t, \tag{2.8}$$

which is the period t price of consumption in period $t+i$. Note that

$$\forall i, j, t \in \mathbb{Z}_+, \quad q_t^i q_{t+i}^j = \frac{q_0^{t+i}}{q_0^t} \frac{q_0^{t+i+j}}{q_0^{t+i}} = q_t^{i+j}. \tag{2.9}$$

Let $t \in \mathbb{Z}_+$. By (2.5) and (2.8) we have $p_t = q_t^1(p_{t+1} + d_{t+1})$. By repeated application of this equality and (2.9), we have

$$\begin{aligned} p_t &= q_t^1 d_{t+1} + q_t^1 p_{t+1} && (2.10)\\ &= q_t^1 d_{t+1} + q_t^1 q_{t+1}^1 (p_{t+2} + d_{t+2}) && (2.11)\\ &= q_t^1 d_{t+1} + q_t^2 d_{t+2} + q_t^2 p_{t+2} && (2.12)\\ &\vdots && (2.13)\\ &= \sum_{i=1}^{n} q_t^i d_{t+i} + q_t^n p_{t+n}, \quad \forall n \in \mathbb{N}. && (2.14) \end{aligned}$$

Since the above finite sum is increasing in $n \in \mathbb{N}$,[3] it follows that

$$p_t = \sum_{i=1}^{\infty} q_t^i d_{t+i} + \lim_{n \uparrow \infty} q_t^n p_{t+n}. \tag{2.15}$$

As is common in the literature, we define the *fundamental value* of the asset in period t as the present discounted value of the dividend stream from period $t+1$ onward:

$$f_t = \sum_{i=1}^{\infty} q_t^i d_{t+i}. \tag{2.16}$$

The *bubble* component of the asset price in period t is the part of p_t that is not accounted for by the fundamental value:

$$b_t = p_t - f_t. \tag{2.17}$$

It follows from (2.15)–(2.17) that

$$b_t = \lim_{n \uparrow \infty} q_t^n p_{t+n}. \tag{2.18}$$

[3]In this paper, "increasing" means "nondecreasing," and "decreasing" means "nonincreasing."

Using (2.9) we see that

$$q_0^t \lim_{n \uparrow \infty} q_t^n p_{t+n} = \lim_{n \uparrow \infty} q_0^{t+n} p_{t+n} = \lim_{i \uparrow \infty} q_0^i p_i. \tag{2.19}$$

Hence by (2.18) and (2.6) we have

$$\lim_{i \uparrow \infty} q_0^i p_i = 0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ b_t = 0. \tag{2.20}$$

3 Examples

In this section we present several examples of (2.2) as well as some examples of preferences that satisfy Assumption 2.3. Some of these examples are used in Section 4.

3.1 Constraints on Asset Holdings

The simplest example of (2.2) would be the following:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq 0. \tag{3.1}$$

This constraint is often used in representative-agent models; see, e.g., Lucas (1978) and Kamihigashi (1998).

Kocherlakota (1992) uses a more general version of (3.1):

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma, \tag{3.2}$$

where $\sigma \in \mathbb{R}$. If $\sigma < 0$, then the above constraint is called a short sales constraint.

The following constraint is even more general:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma_t, \tag{3.3}$$

where $\sigma_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Note that (3.2) is a special case of (3.3) with $\sigma_t = \sigma$ for all $t \in \mathbb{Z}_+$.

Santos and Woodford (1997, p. 24) consider a (state-dependent) borrowing constraint that reduces in our single-asset setting to

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t \geq -\xi_t, \tag{3.4}$$

where $\xi_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. This constraint is a special case of (3.3) with $\sigma_t = -\xi_t/p_t$.

The (state-dependent) debt constraint considered by Werner (2014) and LeRoy and Werner (2014, p. 313) reduces in our current setting to

$$\forall t \in \mathbb{Z}_+, \quad (p_{t+1} + d_{t+1})s_t \geq -\xi_{t+1}. \tag{3.5}$$

This constraint is another special case of (3.3) with

$$\sigma_t = -\xi_{t+1}/(p_{t+1} + d_{t+1}). \tag{3.6}$$

In addition to (3.2), Kocherlakota (1992) considers the following wealth constraint:

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t + \sum_{i=1}^{\infty} q_t^i y_{t+i} \geq 0, \tag{3.7}$$

which is another example of (2.2). The left-hand side above is the period t value of the agent's current asset holdings and future income. Note that (3.7) is yet another special case of (3.3) with

$$\forall t \in \mathbb{Z}_+, \quad \sigma_t = -\sum_{i=1}^{\infty} q_t^i y_{t+i}/p_t. \tag{3.8}$$

See Wright (1987) and Huang and Werner (2000) for related discussion.

3.2 Preferences

Example 3.1. A typical objective function in an agent's maximization problem takes the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{3.9}$$

where $\beta \in (0, 1)$ and $u : \mathbb{R}_+ \to [-\infty, \infty)$ is a strictly increasing function. Suppose further that u is bounded, and define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad \sum_{t=0}^{\infty} \beta^t u(c_t) < \sum_{t=0}^{\infty} \beta^t u(c'_t). \tag{3.10}$$

Then $\prec$ clearly satisfies Assumption 2.3.

If u is unbounded below, i.e., if $u(0) = -\infty$, then the above definition of $\prec$ may not satisfy Assumption 2.3. In particular, given $c^*, c \in \mathcal{C}$ with $c^* < c$, we do not have $c^* \prec c$ if $c_t^* = c_t = 0$ for some $t \in \mathbb{Z}_+$ and if u is bounded above. Indeed, in this case,

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) = \sum_{t=0}^{\infty} \beta^t u(c_t) = -\infty. \tag{3.11}$$

Hence the inequality in (3.10) does not hold.

Example 3.2. The above problem with unbounded utility can be avoided by using an alternative optimality criterion. To be specific, let $u_t : \mathbb{R}_+ \to [-\infty, \infty)$ be a strictly increasing function for $t \in \mathbb{Z}_+$ as above. In this case, the infinite sum $\sum_{t=0}^{\infty} u_t(c_t)$ may not be well defined. Even if it is always well defined, it may not be strictly increasing, as discussed above. To deal with these problems, consider the binary relation $\prec$ defined by

$$c \prec c' \quad \Leftrightarrow \quad \varliminf_{n\uparrow\infty} \sum_{t=0}^{n} [u_t(c_t) - u_t(c_t')] < 0, \tag{3.12}$$

where we follow the convention that $(-\infty) - (-\infty) = 0$; see Dana and Le Van (2006) for related optimality criteria. It is easy to see that the binary relation $\prec$ defined above satisfies Assumption 2.3.

Continuing with this example, suppose that (2.2) is given by (3.1). Suppose further that each u_t is differentiable on $\mathbb{R}_{++}$, and that there exists an optimal plan (c^*, s^*) such that

$$\forall t \in \mathbb{Z}_+, \quad c_t^* > 0, \ s_t^* = 1. \tag{3.13}$$

Then the standard Euler equation holds:

$$u_t'(c_t^*) p_t = u_{t+1}'(c_{t+1}^*)(p_{t+1} + d_{t+1}), \quad \forall t \in \mathbb{Z}_+. \tag{3.14}$$

In view of (2.5), the price sequence $\{q_0^t\}$ is given by

$$q_0^t = \frac{u_t'(c_t^*)}{u_0'(c_0^*)}, \quad \forall t \in \mathbb{Z}_+. \tag{3.15}$$

The fundamental value f_t takes the familiar form:

$$f_t = \sum_{i=1}^{\infty} \frac{u_{t+i}'(c_{t+i}^*)}{u_t'(c_t^*)} d_{t+i}, \quad \forall t \in \mathbb{Z}_+. \tag{3.16}$$

Example 3.3. Let $v : \mathcal{C} \to \mathbb{R}$ be a strictly increasing function. Define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad v(c_0, c_1, c_2, \dots) < v(c'_0, c'_1, c'_2, \dots). \tag{3.17}$$

Note that (3.17) satisfies Assumption 2.3 without any additional condition on v. For example, v can be a recursive utility function.

4 Implications of Feasibility and Optimality

4.1 No-Bubble Theorem

To state our no-bubble theorem, we need to introduce some notation. Given any sequence $\{s_t^*\}_{t=0}^\infty$ in $\mathbb{R}$, $\tau \in \mathbb{Z}_+$, and $\epsilon > 0$, let $\mathcal{S}_{\tau,\epsilon}(s^*)$ be the set of sequences $\{s_t\}_{t=0}^\infty$ in $\mathbb{R}$ such that

$$s_t \begin{cases} = s_t^* & \text{if } t < \tau, \\ \geq s_t^* - \epsilon & \text{if } t \geq \tau. \end{cases} \tag{4.1}$$

In other words, a sequence $\{s_t\}$ in $\mathcal{S}_{\tau,\epsilon}(s^*)$ coincides with $\{s_t^*\}$ up to period $\tau - 1$ and is only required to satisfy the lower bound $s_t^* - \epsilon$ from period τ onward. We are ready to state the main result of this paper.

Theorem 4.1. *Let (c^*, s^*) be an optimal plan. Suppose that there exist $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ such that*

$$\mathcal{S}_{\tau,\epsilon}(s^*) \subset \mathcal{S}(s_{-1}, y, p, d). \tag{4.2}$$

Then $b_t = 0$ for all $t \in \mathbb{Z}_+$.

Proof. See Appendix A. □

It seems remarkable that asset bubbles can be ruled out by a simple condition such as (4.2) alone. In particular, no explicit utility function is assumed, and the only requirement on the binary relation $\prec$ is Assumption 2.3, which merely requires strict monotonicity at the given optimal consumption plan c^*.

The idea of the proof of Theorem 4.1 is simple. In the proof, assuming that the left equality in (2.20) is violated, we construct an alternative plan as follows. Let $\delta > 0$, and let $s_\tau = s_\tau^* - \delta$ and $c_\tau = c_\tau^* + p_\tau \delta$, where τ is given by the statement of the theorem. For $t \neq \tau$, let s_t be determined by the budget constraint (2.1) with $c_t = c_t^*$. This alternative plan gives the same consumption sequence except in period τ, where consumption is increased by $p_\tau \delta > 0$. Hence this plan is strictly preferred to the original plan (c^*, s^*). We derive a contradiction by showing that the alternative plan is feasible for sufficiently small $\delta > 0$.

Similar constructions are used as "Ponzi schemes" by Huang and Werner (2000, Theorems 5.1, 6.1), but they are not directly linked to the nonexistence of asset bubbles.

4.2 Consequences of Theorem 4.1

In this subsection we provide fairly simple consequences of Theorem 4.1 in the current single-agent framework. Throughout this subsection we take an optimal plan (c^*, s^*) as given. We start with a simple result assuming that the feasibility constraint on asset holdings (2.2) is given by a sequence of constraints of the form (3.3). As discussed in Subsection 3.1, this simple form covers various constraints on borrowing, debt, and wealth,

Corollary 4.1. *Suppose that (2.2) is given by (3.3) with $\sigma_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Suppose further that*

$$\varliminf_{t \uparrow \infty} (s_t^* - \sigma_t) > 0. \tag{4.3}$$

Then the conclusion of Theorem 4.1 holds.

Proof. Assume (4.3). Let $\epsilon \in (0, \varliminf_{t \uparrow \infty} (s_t^* - \sigma_t))$. Then there exists $\tau \in \mathbb{Z}_+$ such that $s_t^* - \sigma_t \geq \epsilon$, or $s_t^* - \epsilon \geq \sigma_t$, for all $t \geq \tau$. This implies (4.2). Hence the conclusion of Theorem 4.1 holds. □

If there is a constant lower bound on asset holdings s_t, the above result reduces to the following.

Corollary 4.2. *Suppose that (2.2) is given by (3.2) for some $\sigma \in \mathbb{R}$. Suppose further that $\varliminf_{t \uparrow \infty} s_t^* > \sigma$. Then the conclusion of Theorem 4.1 holds.*

In Section 6 we present some consequences of the above two results in the context of general equilibrium and discuss them in relation to Proposition 3 in Kocherlakota (1992).

Next we present two results that apply to representative-agent models.

Corollary 4.3. *Suppose that (2.2) is given by (3.1). Suppose that*

$$\forall t \in \mathbb{Z}_+, \quad s_t^* = 1. \tag{4.4}$$

Then the conclusion of Theorem 4.1 holds.

Proof. Note that (4.4) and (3.1) imply (4.2) with $\tau = 0$ and $\epsilon = 1$. Thus the conclusion of Theorem 4.1 holds. □

The following proposition is immediate from the above result and (3.16).

Proposition 4.1. *In the setup of Example 3.2 (including (3.14)–(3.16)), we have*

$$\forall t \in \mathbb{Z}_+, \quad p_t = \sum_{i=1}^{\infty} \frac{u'_{t+i}(c^*_{t+i})}{u'_t(c^*_t)} d_{t+i}. \tag{4.5}$$

A similar result is shown in Kamihigashi (2001, Section 4.2.1) for a continuous-time model with a nonlinear constraint. It is known that a stochastic version of Proposition 4.1 requires additional assumptions; see Kamihigashi (1998) and Montrucchio and Privileggi (2001).[4]

Finally we consider the case of fiat money, or an asset with no dividend payment. Since the fundamental value of fiat money is zero, its price must also be zero if there is no asset bubble. Hence the case of fiat money is not directly covered by Theorem 4.1, which requires Assumption 2.2,

Proposition 4.2. *Drop Assumption 2.2 and (2.5)–(2.7) (but maintain Assumptions 2.1 and 2.3). Suppose that there exist $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ satisfying (4.2). Suppose further that*

$$\forall t \geq \tau + 1, \quad d_t = 0. \tag{4.6}$$

Then

$$\forall t \geq \tau, \quad p_t = 0. \tag{4.7}$$

Proof. See Appendix B. □

[4]See Kamihigashi (2011) for sample-path properties of stochastic asset bubbles.

5 General Equilibrium with Multiple Agents and Multiple Assets

5.1 Feasibility, Optimality, and Equilibrium

Consider an exchange economy with countably many infinitely lived agents indexed by $a \in A$, where $A = \{1, 2, \dots, \overline{a}\}$ with $\overline{a} \in \mathbb{N} \cup \{\infty\}$. There are countably many assets indexed by $k \in K$, where $K = \{1, 2, \dots, \overline{k}\}$ with $\overline{k} \in \mathbb{N} \cup \{\infty\}$. Agent $a \in A$ faces the following constraints:

$$c_t^a + \sum_{k \in K} p_{k,t} s_{k,t}^a = y_t^a + \sum_{k \in K} (p_{k,t} + d_{k,t}) s_{k,t-1}^a, \ c_t^a \geq 0, \quad \forall t \in \mathbb{Z}_+, \tag{5.1}$$

$$s^a \in \mathcal{S}^a(s_{-1}^a, y^a, p, d), \tag{5.2}$$

where c_t^a and y_t^a are agent a's consumption and endowment in period t; for each $k \in K$, $s_{k,t}^a$ is agent a's holdings of asset k at the end of period t, $p_{k,t}$ is the price of asset k in period t, and $d_{k,t}$ is the dividend payment of asset k in period t. In (5.2), $s_{-1}^a = (s_{k,-1}^a)_{k \in K}$ is agent a's initial portfolio of all assets $k \in K$, which are historically given, and $\mathcal{S}^a(s_{-1}^a, y^a, p, d)$ is a set of sequences in $\mathbb{R}^{\overline{k}}$ with $s^a = \{(s_{k,t}^a)_{k \in K}\}_{t=0}^\infty$, $y^a = \{y_t^a\}_{t=0}^\infty$, $p = \{(p_{k,t})_{k \in K}\}_{t=0}^\infty$, and $d = \{(d_{k,t})_{k \in K}\}_{t=0}^\infty$.

The supply of each asset $k \in K$ is given by $\overline{s}_k \geq 0$ and is constant over time. We assume the following for the rest of the paper.

Assumption 5.1. For any $k \in K$ and $t \in \mathbb{Z}_+$, we have $d_{k,t} \geq 0$. Furthermore, for each $k \in K$ we have

$$\sum_{a \in A} s_{k,-1}^a = \overline{s}_k. \tag{5.3}$$

Agent a's preferences are represented by a binary relation $\prec^a$ on $\mathcal{C}$. We say that a pair of sequences $c^a = \{c_t^a\}_{t=0}^\infty$ and $s^a = \{(s_{k,t}^a)_{k \in K}\}_{t=0}^\infty$ in $\mathbb{R}$ and $\mathbb{R}^{\overline{k}}$, respectively, is a *plan*; a plan (c^a, s^a) is *feasible* for agent a if it satisfies (5.1) and (5.2); and a feasible plan $(\hat{c}^a, \hat{s}^a)$ is *optimal* for agent a if there exists no feasible plan (c^a, s^a) for agent a such that $\hat{c}^a \prec c^a$.

An *equilibrium* of this economy is a set of sequences $(p, \{c^a, s^a\}_{a \in A})$ such that (i) (c^a, s^a) is optimal for each agent $a \in A$, (ii) for each $k \in K$ and $t \in \mathbb{Z}_+$, we have $p_{k,t} \geq 0$, and (iii) the asset and good markets clear in all periods:

$$\sum_{a \in A} s^a_{k,t} = \overline{s}_k, \quad \forall k \in K, \forall t \in \mathbb{Z}_+, \tag{5.4}$$

$$\sum_{a \in A} c^a_t = \sum_{a \in A} y^a_t + \sum_{k \in K} \overline{s}_k d_{k,t}, \quad \forall t \in \mathbb{Z}_+. \tag{5.5}$$

Whenever we take an equilibrium $(p, \{c^a, s^a\}_{a \in A})$ as given, we assume the following.

Assumption 5.2. For any $a \in A$ and $\tilde{c}^a \in C$ with $c^a < \tilde{c}^a$, we have $c^a \prec \tilde{c}^a$.

5.2 Asset Bubbles

As in Subsection 2.2 we define the fundamental value of an asset and the bubble component of the asset price in period t using the period t prices of the consumption goods in periods $t+1, t+2, \ldots$. As in Subsection 2.2 we let q^t_o be the period 0 price of the consumption good in period t for $t \in \mathbb{Z}_+$.

It is well-known (Santos and Woodoford, 1997, (2.1); Werner (2014, (7)) that under borrowing or debt constraints (see (5.13) and (5.16)), the absence of arbitrage implies that there exists a price sequence $\{q^t_0\}$ satisfying (2.6) and (2.7) such that for all $k \in K$,

$$\forall t \in \mathbb{Z}_+, \quad q^t_0 p_{k,t} = q^{t+1}_0 (p_{k,t+1} + d_{k,t+1}). \tag{5.6}$$

Although this need not be true for all assets simultaneously under short sales constraints, for each asset $k \in K$ there exists a price sequence $\{q^t_0\}$ satisfying (5.6) (as in Subsection 2.2). To simplify the exposition, for the rest of the paper, we take as given an asset $h \in K$ satisfying the following.

Assumption 5.3. There exists an asset $h \in K$ such that $p_{h,t} > 0$ for all $t \in \mathbb{Z}_+$.

This assumption uniquely determines the price sequence $\{q^t_0\}$ satisfying (5.6) and (2.7) (i.e., $q^0_0 = 1$). As in Subsection 2.2, we define q^i_t for all $t \in \mathbb{N}$ and $i \in \mathbb{Z}_+$ using (2.8). We use these sequences to define the fundamental value of asset h and the bubble component of $p_{h,t}$ as in Subsection 2.2:

$$f_{h,t} = \sum_{i=1}^{\infty} q^i_t d_{h,t+i}, \tag{5.7}$$

$$b_{h,t} = p_{h,t} - f_{h,t}. \tag{5.8}$$

Following the arguments for (2.10)–(2.20), we have

$$\lim_{i\uparrow\infty} q_0^i p_{h,i} = 0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ b_{h,t} = 0. \tag{5.9}$$

5.3 Examples of Constraints on Asset Holdings

In this subsection we fix an agent $a \in A$ and present some examples of (5.2). As in Subsection 3.1, the simplest example would be the following:

$$\forall t \in \mathbb{Z}_+, \forall k \in K, \quad s_{k,t}^a \geq 0. \tag{5.10}$$

To discuss more general constraints, for $t \in \mathbb{Z}_+$ and $k' \in K$, we define

$$s_{K\setminus k,t}^a = (s_{k',t}^a)_{k' \in K\setminus\{k\}}. \tag{5.11}$$

Consider the following constraint:

$$\forall t \in \mathbb{Z}_+, \forall k \in K, \quad s_{k,t}^a \geq \sigma_{k,t}^a(s_{K\setminus k,t}^a), \tag{5.12}$$

where $\sigma_{k,t}^a : \mathbb{R}^{\bar{k}-1} \to \mathbb{R} \cup \{-\infty\}$ for all $k \in K$ and $t \in \mathbb{Z}_+$. Note that (5.10) is a special case of (5.12) with $\sigma_{k,t}^a(s_{K\setminus k,t}^a) = 0$ for all $k \in K, s_{K\setminus k,t} \in \mathbb{R}^{\bar{k}-1}$, and $t \in \mathbb{Z}_+$.

The (state-dependent) borrowing constraint considered by Santos and Woodford (1997, p. 24) can be written in the current setting as

$$\forall t \in \mathbb{Z}_+, \quad \sum_{k \in K} p_{k,t} s_{k,t}^a \geq -\xi_t^a, \tag{5.13}$$

where $\xi_t^a \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. This constraint is a special case of (5.12) with

$$\sigma_{k,t}^a(s_{K\setminus k,t}^a) = \begin{cases} -\dfrac{\xi_t^a + \sum_{k' \in K\setminus\{h\}} p_{k',t} s_{k',t}^a}{p_{k,t}} & \text{if } p_{k,t} > 0, \\ -\infty & \text{otherwise.} \end{cases} \tag{5.14}$$

In fact, with $\sigma_{k,t}^a$ defined as above, (5.13) is equivalent to

$$\forall t \in \mathbb{Z}_+, \quad s_{h,t}^a \geq \sigma_{h,t}^a(s_{K\setminus h,t}^a), \tag{5.15}$$

since $p_{h,t} > 0$ for all $t \in \mathbb{Z}_+$ by Assumption 5.3. In other words, it suffices to impose (5.15) only for one asset $h \in K$; the other constraints given by (5.12) are redundant.

The (state-dependent) debt constraint considered by Werner (2014) and LeRoy and Werner (2014, p. 313) can be written in the current setting as

$$\forall t \in \mathbb{Z}_+, \quad \sum_{k \in K} (p_{k,t+1} + d_{k,t+1}) s_t^a \geq -\xi_{t+1}^a. \tag{5.16}$$

Given Assumption 5.3, this constraint is another special case of (5.15) with

$$\sigma_{h,t}^a(s_{K\setminus h,t}) = -\frac{\xi_{t+1}^a + \sum_{k \in K \setminus \{h\}} (p_{k,t+1} + d_{k,t+1}) s_{k,t}^a}{p_{h,t+1} + d_{h,t+1}}. \tag{5.17}$$

A multi-asset version of the wealth constraint considered by Kocherlakota (1992) can be written as

$$\forall t \in \mathbb{Z}_+, \quad \sum_{k \in K} p_{k,t} s_{k,t}^a + \sum_{i=1}^{\infty} q_t^i y_{t+i} \geq 0. \tag{5.18}$$

This is yet another special case of (5.15) with

$$\sigma_{h,t}^a(s_{K\setminus h,t}^a) = -\frac{\sum_{i=1}^{\infty} q_t^i y_{t+i} + \sum_{k \in K \setminus \{h\}} p_{k,t} s_{k,t}^a}{p_{h,t}}. \tag{5.19}$$

We should mention that (5.16) and (5.18) are also special cases of (5.12) with $\sigma_{k,t}^a$ defined similarly to (5.14).

6 General Equilibrium Results

In this section we take an equilibrium $(p, \{c^a, s^a\}_{a \in A})$ as given, and consider conditions to rule out asset bubbles in the general equilibrium setting introduced in the previous section.

We start by extending Theorem 4.1 to the current general equilibrium setting. For this purpose, we need additional notation. Given any sequence $s^a = \{(s_{k,t}^a)_{k \in K}\}_{t=0}^{\infty}$ in $\mathbb{R}^{\overline{k}}$, $\tau \in \mathbb{Z}_+$, and $\epsilon > 0$, let $\mathcal{S}_{h,\tau,\epsilon}(s^a)$ be the set of sequences $\{(s_{k,t})_{k \in K}\}_{t=0}^{\infty}$ in $\mathbb{R}^{\overline{k}}$ such that

$$s_{k,t} = s_{k,t}^a, \quad \forall t \in \mathbb{Z}_+ \text{ if } k \neq h, \tag{6.1}$$

$$s_{h,t} \begin{cases} = s_{h,t}^a & \text{if } t < \tau, \\ \geq s_{h,t}^a - \epsilon & \text{if } t \geq \tau. \end{cases} \tag{6.2}$$

Note that (6.2) takes the same form as (4.1).

Theorem 6.1. *Suppose that there exists an agent $a \in A$ such that for some $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$, we have*

$$\mathcal{S}_{h,\tau,\epsilon}(s^a) \subset \mathcal{S}^a(s^a_{-1}, y^a, p, d). \tag{6.3}$$

Then $b_{h,t} = 0$ for all $t \in \mathbb{Z}_+$.

Proof. Let $p_t = p_{h,t}, d_t = d_{h,t}$, and $b_t = b_{h,t}$ for all $t \in \mathbb{Z}_+$. Then (2.5)–(2.7) hold, and Assumptions 2.1–2.3 follow from Assumptions 5.1–5.3. Since (c^a, s^a) is optimal for agent a and (6.3) implies (4.2) with $(c^*, s^*) = (c^a, s^a)$, it follows that the conclusion of Theorem 4.1 holds. □

The following result is essentially a restatement of Corollary 4.1 in the current general equilibrium setting with multiple assets.

Proposition 6.1. *Suppose that there exists an agent $a \in A$ such that (5.2) is given by (5.15) for some $\sigma^a_{h,t} : \mathbb{R}^{\bar{k}-1} \to \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Suppose further that*

$$\varliminf_{t \uparrow \infty} (s^a_{h,t} - \sigma^a_{h,t}(s^a_{K\setminus h,t})) > 0. \tag{6.4}$$

Then the conclusion of Theorem 6.1 holds.

Proof. This follows from Corollary 4.1 with $\sigma_t = \sigma^a_{h,t}(s^a_{K\setminus h,t})$ for all $t \in \mathbb{Z}_+$. □

As discussed in Subsection 5.3, various constraints on borrowing, debt, and wealth can be written in the form of (5.15). Hence Proposition 6.1 applies to economies with such constraints.

In what follows, we present some results that can be regarded as generalizations of Proposition 3 in Kocherlakota (1992). We discuss the relation between his and our results after showing our results. For the rest of this section, we maintain the following assumption.

Assumption 6.1. For each agent $a \in A$, there exists a sequence $\{(\eta^a_{k,t})_{k\in K}\}_{t=0}^\infty$ in $\mathbb{R}^{\bar{k}}$ such that given any sequence $s = \{(s_{k,t})_{k\in K}\}_{t=0}^\infty$ in $\mathbb{R}^{\bar{k}}$, we have

$$s_{k,t} \geq \eta^a_{k,t}, \forall k \in K, \forall t \in \mathbb{Z}_+ \quad \Leftrightarrow \quad s \in \mathcal{S}^a(s^a_{-1}, y^a, p, d). \tag{6.5}$$

This assumption means that the feasibility constraint on asset holdings for each agent, (5.2), consists of sequences of constraints of the form (3.3) for all assets. The following result is immediate from Proposition 6.1.

Corollary 6.1. *If there exists an agent $a \in A$ such that*

$$\varliminf_{t\uparrow\infty}(s^a_{h,t} - \eta^a_{h,t}) > 0, \tag{6.6}$$

then the conclusion of Theorem 6.1 hold.

To state the next result, we define the following for each $a \in A$:

$$\overline{\eta}^a_h = \varlimsup_{t\uparrow\infty} \eta^a_{h,t}. \tag{6.7}$$

Corollary 6.2. *If there exists an agent $a \in A$ such that*

$$\varliminf_{t\uparrow\infty} s^a_{h,t} > \overline{\eta}^a_k, \tag{6.8}$$

then the conclusion of Theorem 6.1 holds.

Proof. Let $a \in A$ satisfy (6.8). This strict inequality implies that

$$\varliminf_{t\uparrow\infty} s^a_{h,t} > -\infty, \quad \overline{\eta}^a_{h,t} < \infty. \tag{6.9}$$

Hence

$$\varliminf_{t\uparrow\infty}(s^a_{h,t} - \eta^a_{h,t}) \geq \varliminf_{t\uparrow\infty} s^a_{h,t} - \overline{\eta}^a_{h,t} > 0, \tag{6.10}$$

where the second inequality holds by (6.8). Thus the conclusion of Theorem 6.1 holds by Corollary 6.1. □

If there exists a constant lower bounded on the asset h for each agent $a \in A$, the above result reduces to the following.

Corollary 6.3. *Suppose that*

$$\forall a \in A, \exists \eta^a_h \in \mathbb{R}, \forall t \in \mathbb{Z}_+, \quad \eta^a_{h,t} = \eta^a_h. \tag{6.11}$$

Suppose further that there exists an agent $a \in A$ such that

$$\varliminf_{t\uparrow\infty} s^a_{h,t} > \eta^a_h. \tag{6.12}$$

Then the conclusion of Theorem 6.1 holds.

Kocherlakota (1992, Proposition 3) in effect shows a special case of Corollary 6.3 under the following additional assumptions: (i) there is only one asset (i.,e $\overline{k} = 1$); (ii) the binary relation $\prec^a$ of each agent $a \in A$ is represented by (3.10), where u depends on $a \in A$ and is denoted as $u_a : \mathbb{R}_+ \to [-\infty, \infty)$ but β is common to all agents; (iii) for each $a \in A$, u_a is continuously differentiable on $\mathbb{R}_{++}$, strictly increasing, concave, and bounded above or below by zero; and (iv) the optimal plan (c^a, s^a) of each agent $a \in A$ satisfies

$$\forall t \in \mathbb{Z}_+, \quad c_t^a > 0, \tag{6.13}$$

$$\left| \sum_{t=0}^{\infty} \beta^t u_a(c_t^a) \right| < \infty. \tag{6.14}$$

Corollary 6.3 shows that none of Kocherlakota's additional assumptions is needed under Assumption 5.2, which is implied by his assumptions. Hence Corollary 6.3 is a substantial generalization of Proposition 3 in Kocherlakota (1992). He uses the extra assumptions mostly to derive a transversality condition, which is crucial to his approach. By contrast, our results are based on our simple no-bubble theorem, Theorem 4.1, which can be proved by an elementary perturbation argument that fully exploits the structure of sequential budget constraints.

7 Concluding Comments

In this paper we showed a simple no-bubble theorem that applies to a wide range of deterministic economies with infinitely lived agents facing sequential budget constraints. In particular, we showed that asset bubbles can be ruled out if there is at least one agent who can reduce his asset holdings permanently from some period onward. This is a substantial generalization of Kocherlakota's (1992) result on asset bubbles and short sales constraints; our no-bubble theorem requires virtually no assumption except for the strict monotonicity of preferences.

Although we also developed some results on asset bubbles in a general equilibrium setting, all of them are based on the optimal behavior of a single agent. Additional results can be shown by using our results in conjunction with other arguments based on market-clearing and aggregation.

Appendix A Proof of Theorem 4.1

Let (c^*, s^*) be an optimal plan. It suffices to verify that

$$\lim_{i\uparrow\infty} q_0^i p_i = 0, \tag{A.1}$$

which implies the desired conclusion by (2.20). Suppose by way of contradiction that

$$\lim_{i\uparrow\infty} q_0^i p_i > 0. \tag{A.2}$$

Then since $q_0^i p_i > 0$ for all $i \in \mathbb{Z}_+$ by Assumption 2.2 and (2.6), it follows that there exists $\underline{b} > 0$ such that

$$\forall i \in \mathbb{Z}_+, \quad q_0^i p_i \geq \underline{b}. \tag{A.3}$$

Equivalently, we have $1/p_i \leq q_0^i/\underline{b}$ for all $i \in \mathbb{Z}_+$. Multiplying through by d_i and summing over $i \in \mathbb{N}$, we obtain

$$\sum_{i=1}^{\infty} \frac{d_i}{p_i} \leq \sum_{i=1}^{\infty} \frac{q_0^i d_i}{\underline{b}} = \frac{f_0}{\underline{b}} < \infty, \tag{A.4}$$

where the equality uses (2.16).[5]

Let $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ be given by (4.2). For each $\delta \in (0, \epsilon)$ we construct an alternative plan (c^δ, s^δ) as follows:

$$c_t^\delta = \begin{cases} c_t^* & \text{if } t \neq \tau, \\ c_\tau^* + p_\tau \delta & \text{if } t = \tau, \end{cases} \tag{A.5}$$

$$s_t^\delta = \begin{cases} s_t^* & \text{if } t \leq \tau - 1, \\ s_\tau^* - \delta & \text{if } t = \tau, \\ [y_t + (p_t + d_t)s_{t-1}^\delta - c_t^*]/p_t & \text{if } \tau \geq \tau + 1. \end{cases} \tag{A.6}$$

It suffices to show that (c^δ, s^δ) is feasible for $\delta > 0$ sufficiently small; for then, we have $c^* \prec c^\delta$ by (A.5) and Assumption 2.3, contradicting the optimality of (c^*, s^*).

[5] An arguments similar to (A.4) is used by Montrucchio (2004, Theorem 2).

Note that (c^δ, s^δ) satisfies (2.1) by construction. Hence by (2.1) we have

$$\forall t \geq \tau + 1, \quad p_t(s_t^* - s_t^\delta) = (p_t + d_t)(s_{t-1}^* - s_{t-1}^\delta). \tag{A.7}$$

For $t \geq \tau$ define

$$\delta_t = s_t^* - s_t^\delta. \tag{A.8}$$

Note that $\delta_\tau = \delta$ by (A.6). We have $p_t\delta_t = (p_t + d_t)\delta_{t-1}$ for all $t > \tau$ by (A.7). Thus for any $t > \tau$ we have

$$\begin{aligned} \delta_t &= \frac{p_t + d_t}{p_t}\delta_{t-1} = \frac{p_t + d_t}{p_t}\frac{p_{t-1} + d_{t-1}}{p_{t-1}}\delta_{t-2} = \cdots & \text{(A.9)} \\ &= \delta \prod_{i=\tau+1}^{t} \frac{p_i + d_i}{p_i} \leq \delta \prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}, & \text{(A.10)} \end{aligned}$$

where the equality in (A.10) holds since $\delta_\tau = \delta$, and the inequality in (A.10) holds since $d_t \geq 0$ for all $t \in \mathbb{Z}_+$ by Assumption 2.1.[6]

To show that (c^δ, s^δ) is feasible, it suffices to verify that $\delta_t \leq \epsilon$ for all $t \geq \tau$; for then, we have $s \in \mathcal{S}(s_{-1}, y, p, d)$ by (4.2) and (A.8). For this purpose, note from (A.4) that

$$\begin{aligned} \frac{f_0}{\underline{b}} &\geq \sum_{i=1}^{\infty} \frac{d_i}{p_i} \geq \sum_{i=1}^{\infty} \ln\left(1 + \frac{d_i}{p_i}\right) & \text{(A.11)} \\ &= \sum_{i=1}^{\infty} \ln\left(\frac{p_i + d_i}{p_i}\right) = \ln\left(\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}\right). & \text{(A.12)} \end{aligned}$$

It follows that

$$\prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i} < \infty. \tag{A.13}$$

Using this and recalling (A.9)–(A.10), we can choose $\delta > 0$ small enough that $\delta_t \leq \epsilon$ for all $t \geq \tau$. For such δ, (c^δ, s^δ) is feasible, contradicting the optimality of (c^*, s^*). We have verified (A.1), which implies the conclusion of the theorem.

[6] An argument similar to (A.9)–(A.10) is used by Bosi et al. (2014).

Appendix B Proof of Proposition 4.2

Let $\tau \in \mathbb{Z}_+$ and $\epsilon > 0$ be as in (4.2). Suppose by way of contraction that $p_{\tau'} > 0$ for some $\tau' \geq \tau$. Without loss of generality, we assume that $\tau' = \tau = 0$;[7] i.e.,

$$p_0 > 0. \tag{B.1}$$

First suppose that

$$\forall t \in \mathbb{N}, \quad p_t > 0. \tag{B.2}$$

Then Assumption 2.2 holds. We construct $\{q_0^t\}_{t=0}^\infty$ by (2.5) with $q_0^0 = 1$. Then (2.5)–(2.7) hold. Since Assumptions 2.1–2.3 and (2.5)–(2.7) now hold, Theorem 4.1 applies. But note from (4.6) and (2.16) that

$$\forall t \in \mathbb{Z}_+, \quad f_t = 0. \tag{B.3}$$

Hence by Theorem 4.1, we have $b_t = 0$, i.e., $p_t = f_t = 0$, for all $t \in \mathbb{Z}_+$. This contradicts (B.2).

We have shown that (B.2) cannot be true. In other words, there must be $t \in \mathbb{N}$ such that $p_t = 0$. Let T be the first $T \in \mathbb{Z}_+$ with

$$p_T > 0, \quad p_{T+1} = 0. \tag{B.4}$$

Such T must exist by (B.1). We construct an alternative plan (c, s) as follows:

$$c_t = \begin{cases} c_t^* & \text{if } t \neq T, \\ c_T^* + p_T\epsilon & \text{if } t = T, \end{cases} \tag{B.5}$$

$$s_t = \begin{cases} s_t^* & \text{if } t \neq T, \\ s_T^* - \epsilon & \text{if } t = T. \end{cases} \tag{B.6}$$

According to this plan, the agent sells the asset when its price is strictly positive, and buys it back when it is free. It is easy to see from (4.2), (2.1), and (B.4) that (c, s) is feasible. But then we have $c^* \prec c$ by (B.5) and Assumption 2.3, contradicting the optimality of (c^*, s^*).

We have shown that we reach a contradiction whether (B.2) holds or not; thus we must have $p_0 = 0$.

[7] It is no loss of generality to assume that $\tau' = \tau = 0$ since we only consider variables in and after period τ'.

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