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RECURSIVE UTILITY AND THE SOLUTION TO THE BELLMAN EQUATION

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Abstract. This study infinite-horizon deterministic dynamic programming problems based on recursive utility in discrete time. Under a small number of conditions, we show that the Bellman operator has a fixed point using Knaster–Tarski's fixed point theorem. We also show the fixed point of the Bellman operator can be computed by iteration from the initial function between the lower boundary and the fixed point. To show the convergence theorem, we use Tarski–Kantorovitch's fixed point theorem.

1. Introduction

Dynamic programming has become an important tool in economic theory, particularly since Stokey and Lucas (1989). Due to technical reasons, many dynamic economic models are based on time additive separable utility with a constant discount rate. However, these models have been criticized with respect to consistency with economic situations. Becker (1980) examines the long-run behavior of economy in a one-good model based on time additive separable utility with a constant discount rate of dynamic equilibrium with heterogeneous households. He discussed that unless all of the households have the same discount rate, the household which has the lowest discount rate owns all the capital in the long-run and all the others consume nothing using their labor income to service their dept. So, Lucas and Stokey (1984) tackles this problem using recursive utility. Recently, recursive utility has attracted attention, replacing time additive utility with a constant discount rate. Recursive utility comprises a wide class of representations of utility including time additive utility with a constant discount rate. [1]

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[1]The significance of recursive utility is discussed in Section 1.3 "Why Study Recursive utility?" in Becker and Boyed (1997) with various economical examples.

Recursive utility as introduced by Koopmans (1960) formalizes a method used by Fisher (1907). Many researchers have proposed solutions to the problem of dynamic programming on recursive utility, often using an aggregator function constructed from recursive utility functions to solve the problem. The aggregator method was first proposed by Lucas and Stoky (1984). Important studies with the aggregator function are the method based on the contraction mapping theorem used by Lucas and Stokey (1984), the weighted contraction method used by Boyd (1990), the partial sum method used for all aggregators by Le Van and Vailakis (2005), k-contraction used by Ricón-Zapatero and Rodríguez-Palmero (2007) and Biconvergence condition used by Streufert (1990, 1992). Becker and Boyd (1997) and Boyd (2006) give a good exposition of these studies. With the exception of Streufert's method, these approaches assume a variant of the Lipschitz condition to the aggregator function. Streufert's (1990, 1992) method is unique compared to the other approaches, as it does not use any fixed point theorem. Streufert (1998) provides a detailed exposition of these results and methods.

However, these contributions heavily rely on topological assumptions, such as continuity of a utility function and continuity of a feasible correspondence. Kamihigashi (2014) establishes some elementary results on fixed points of the Bellman operator without topological assumptions in a dynamic economic model based on time additive separable utility with a constant discount rate.

This paper is motivated by the method of Streufert (1992) and the idea of Kamihigashi (2014). The goal is to obtain some results on the solution to the Bellman equation – or fixed points of the Bellman operator – without topological assumptions in the dynamic economic model based on recursive utility.

As in Kamihigashi's (2014) approach, we use a fixed point theorem of an ordered space instead of a variant of the contraction mapping theorem to solve the problem. Kamihigashi (2014) uses the Knaster–Tarski fixed point theorem, which is a sort of a fixed point theorem of an ordered space. However, the Knaster–Tarski fixed point theorem only shows existence of a fixed point; it does not show that some sequence converges to the fixed point. We show a fixed point theorem for the Bellman operator using Tarski-Kantrovitch's fixed point theorem[2] [3],. Using the theorem, we show that a fixed point of the Bellman operator can be computed by iteration starting from some boundary function [4].

Our main results are as follows. Given order interval of functions that is mapped into itself by the Bellman operator, if the aggregator has certain properties, then (a) the Bellman operator has a fixed point in the order interval and (b) the fixed point can be computed iteratively starting from the lower boundary of the order interval. Under some

[2]In some research field, the fixed point theorem is called Kleene's fixed point theorem, for example see Baranga (1991)

[3]Knaster–Tarski fixed point theorem is sometimes used in macro-economic theory. See, for example, Datta and Reffett (2006).

[4]Kamihigashi, Reffet and Yao (2015), Kamihigashi and Yao (2015a) and Kamihigashi and Yao (2015b) use similar approach of this paper in the model of time addtive separable utility.

topological assumptions, if an upper boundary with certain properties exists, then (i) the Bellman operator has a fixed point in the order interval, (ii) this fixed point can be computed iteratively starting from the upper boundary of the order interval. Using this result, we can construct an alternative proof of Theorem 2.3 (a) in Streufert (1998).

The remainder of this article is organized as follows. In Section 2, we prepare some mathematical tools. In Section 3, we describe the model based on Streufert's (1992) without topological assumptions of the state space and find a fixed point of Bellman operator. In Section 4, we show that the fixed point operator can be computed iteratively starting from the upper boundary of the order interval with some assumption about continuity of the Bellman operator or topological assumptions. In Appendix A, we state proofs on our results.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{Z}_+$ the set of nonnegative integers, by $\mathbb{R}$ the set of real numbers and by $\mathbb{R}_+$ the set of positive real numbers.

A partially ordered set is pair $(P, \leq)$, where P is nonempty set and $\leq$ is a relation in P which is reflexive ($p \leq p$ for all $p \in P$), antisymmetric (for $p, q \in P$, $p \leq q$ and $q \leq p$ implies $p = q$) and transitive (for $p, q, r \in P$, $p \leq$ and $q \leq r$ implies $p \leq r$). Let $(P; \leq)$ be a partially ordered set and $M \subset P$ be a nonempty subset. An upper (resp. lower) bound for M is an element $p \in P$ with $m \leq p$ (resp. $m \leq p$) for each $m \in M$. The least (resp. greatest) element of $M \subset P$ is an element $p \in M$ satisfying $p \leq m$ (resp. $m \leq p$) forall $m \in M$. The supremum (resp. infimum) of $M \subset P$ denoted as $\sup M$ (resp. $\inf M$), is the least upper bound (resp. greatest lower bound) of M. A partially ordered set $(P; \leq)$ is complete lattice if every subset M of P has both the supremum and infimum in $(P, \leq)$. $M \subset P$ is said to be a chain if for $p, q \in M$, either $p \leq q$ or $q \leq p$. A sequence $\{p_n\}_{n \in \mathbb{N}}$ is called increasing (resp. decreasing) if $p_n \leq p_{n+1}$ (resp. $p_{n+1} \leq p_n$) for all $n \in \mathbb{N}$. A mapping $F : P \to P$ is said to be monotone if for $p, q \in P$, $p \leq q$ implies $F(p) \leq F(q)$.

Theorem 2.1 (Knaster–Tarski)**.** *Let $(P, \leq)$ be a partially ordered set and $T : P \to P$ monotone. Assume that there is a $p_0 \in P$ such that*

(i) $p_0 \leq T(p_0)$*; and*
(ii) *every chain in $P_0 \equiv \{p \in P \colon p_0 \leq p\}$ has a supremum.*

Then T has a fixed point in P_0.

Proof. See Aliprantis and Border (2006), p16. □

Using the inverse ordering and infimum, we have the following dual version of Theorem 2.1.

Corollary 2.1. *Let $(P, \leq)$ be a partially ordered set and $T : P \to P$ monotone. Assume that there is a $p_0 \in P$ such that*

(i) $T(p_0) \leq p_0$; *and*

(ii) *every chain in* $P_0 \equiv \{p \in P \colon p \leq p_0\}$ *has a infimum.*

Then T *has a fixed point in* P_0.

A mapping F from a partially ordered set $(P, \leq)$ into itself is said to be s-order continuous (resp. i-order continuous) if for every countable chain $C \subset P$ having a supremum (resp. infimum), the image $F(C)$ has a supremum (resp. infimum) and

$$\sup F(C) = F(\sup C) \quad (\text{resp. } \inf F(C) = F(\inf C)). \tag{1}$$

It is clear that s-order continuous (resp. i-order continuous) is monotone.

Theorem 2.2 (Tarski–Kantorovitch)**.** *Let* $(P, \leq)$ *be a partially ordered set and* $T : P \to P$ *s–order continuous. Assume that there is a* $p_0 \in P$ *such that*

(i) $p_0 \leq T(p_0)$; *and*

(ii) *every countable chain in* $P_0 \equiv \{p \in P \colon p_0 \leq p\}$ *has a supremum.*

Then T *has a fixed point* $p^* = \sup_{n\in\mathbb{N}} T^n p_0$ *and* p^* *is the infimum of the set of the fixed points of* T *in* P_0.

Proof. See Granas and Dugundji (2003), p26. □

Using the inverse ordering and infimum, we have the following dual version of Theorem 2.2.

Corollary 2.2. *Let* $(P, \leq)$ *be a partially ordered set and* $T : P \to P$ *i–order continuous. Assume that there is a* $p_0 \in P$ *such that*

(i) $T(p_0) \leq p_0$; *and*

(ii) *every countable chain in* $P_0 \equiv \{p \in P \colon p \leq p_0\}$ *has a infimum.*

Then T *has a fixed point* $p^* = \inf_{n\in\mathbb{N}} T^n p_0$ *and* p^* *is the supremum of the set of the fixed points of* T *in* P_0.

The following result is from Kamihigashi (2008).

Lemma 2.1. *Let* Y *and* Z *be sets. Let* $\Omega \subset Y \times Z$, *and let* $f; \Omega \to \overline{\mathbb{R}}$. *For any* $y \in Y$ *and* $z \in Z$, *define*

$$\Omega_y = \{z \in Z \colon (y, z) \in \Omega\}, \tag{2}$$

$$\Omega_z = \{y \in Y \colon (y, z) \in \Omega\}. \tag{3}$$

Then $\sup_{(y,z)\in\Omega} f(y, z) = \sup_{y\in Y} \sup_{z\in\Omega_y} f(y, z) = \sup_{y\in Z} \sup_{y\in\Omega_z} f(y, z)$.

Proof. See Lemma 1 in Kamihigashi (2008). □

The following results are elementary but plays an important role in this paper.

Lemma 2.2. *Let* f *be a monotone and lower semicontinuous function from* $\overline{\mathbb{R}}$ *into itself and let* $A \subset \overline{\mathbb{R}}$. *Then, we have*

$$\sup f(A) = f(\sup A). \tag{4}$$

Proof. See Appendix A. □

By similar proof, we have the following dual version of Lemma 2.2

Corollary 2.3. *Let f be a monotone and upper semicontinuous function from $\overline{\mathbb{R}}$ into itself and let $A \subset \overline{\mathbb{R}}$. Then, we have*

$$\inf f(A) = f(\inf A). \tag{5}$$

3. The Model and Results

Let X be a nonempty set. An infinite sequence of elements of X will be denoted by $\{x_t\}_{t=0}^{\infty}$, with $x \in X$ for all $t \in \mathbb{Z}_+$. The space of these sequence is denoted by X^{∞}. Let Γ be a nonempty correspondence from X to X. Let $U : X^{\infty} \to \overline{\mathbb{R}}$ where $\overline{\mathbb{R}} = [-\infty, \infty]$.

In the optimization problem introduced below, $t \in \mathbb{Z}_+$ is a discrete time period, X is a state space, Γ is the feasible correspondence and U is a utility function.

Example 3.1. *In the one–sector growth model, today's state (consuming and saving) $x = (c, s) \in \mathbb{R}_+^2$ is feasible if $c + s$ does not exceed the income $F(s_0)$ determined by yesterday's saving s_0. There we define $X = \mathbb{R}_+^2$ with $x = (c, s)$, assume that $F : \mathbb{R}_+ \to \mathbb{R}_+$, and define $\Gamma : X \to 2^X$ by $\Gamma(x_0) = \{x = (c, s) \in X : c + s \leq F(x_0)\}$.*

Let S denote the shift operator defined by $S(\{x_t\}_{t=0}^{\infty}) = \{x_t\}_{t=1}^{\infty}$.

We assume that there exists an aggregator $W : X \times \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ defined by

(A1) $\forall \{x_t\}_{t=0}^{\infty} \in X^{\infty}$,

$$U(\{x_t\}_{t=0}^{\infty}) = W(x_0, U(S(\{x_t\}_{t=0}^{\infty}))) \tag{6}$$

We call such U a recursive utility function. We also assume that

(A2) W is weakly increasing in its second argument, that is, for all $x \in X$ and all $y, z \in \overline{\mathbb{R}}$ with $y \leq z$, $W(x, y) \leq W(x, z)$.

The followings are the examples of a recursive utility function in one–sector growth model.

Example 3.2 (the discounted TAS utility)**.** *A simple example of a recursive utility function is the discounted time additive separable (TAS) form*

$$U(\{c_t\}_{t=0}^{\infty}) = \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{7}$$

which has an aggregator $W(c_0, U(\{c_t\}_{t=1}^{\infty})) = u(c_0) + \beta \sum_{t=1}^{\infty} \beta^{t-1} u(c_t)$, a return function $u : \mathbb{R}_+ \to [-\infty, \infty)$ and a discount factor $\beta \in (0, 1)$.

Example 3.3 (the CEIS aggregator, the additive CRRA utility)**.** *The aggregator W exhibits constant elasticity of intertemporal substitution (CEIS) if*

$$W(x, y) = (x^{1-1/\gamma} + \beta y^{1-1/\gamma})^{\frac{1}{1-1/\gamma}}, \tag{8}$$

where $\gamma > 0$ is the elasticity of intertemporal substitution and $\beta > 0$ is the discount factor. If the return function $u(c) = c$ and $\gamma = 1/\delta$, we have the standard additive CRRA utility function as follows:

$$U(\{c_t\}_{t=0}^{\infty}) = (\sum_{t=0}^{\infty} \beta^t c_t^{1-\delta})^{\frac{1}{1-\delta}}. \tag{9}$$

Example 3.4 (the EHU aggregator). *A non-TAS utility is given by* [5]

$$U(\{c_t\}_{t=0}^{\infty}) = -\sum_{t=0}^{\infty} \exp(-[\sum_{\tau=0}^{t} u(c_\tau)]). \tag{10}$$

This function has return function u and aggregator

$$W(x, y) = (-1 + y)\exp(-u(x)). \tag{11}$$

We refer to it Epstein-Hynes-Uzawa (EHU) aggregator.

Let Π and $\Pi(x_0)$ denote the set of feasible paths and that the set of feasible paths from an initial state $x_0 \in X$ respectively.

$$\Pi = \{\{x_t\}_{t=0}^{\infty} \in X^{\infty} : x_{t+1} \in \Gamma(x_t)\ (\forall t \in \mathbb{Z}_+)\}, \tag{12}$$

$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X^{\infty} : \{x_t\}_{t=0}^{\infty} \in \Pi\},\ \forall x_0 \in X. \tag{13}$$

Given $x_0 \in X$, consider the following optimization problem:

$$\sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} U(\{x_t\}_{t=0}^{\infty}). \tag{14}$$

The value function $v^* : X \to \overline{\mathbb{R}}$ is defined by

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} U(\{x_t\}_{t=0}^{\infty}), \quad \forall x_0 \in X. \tag{15}$$

Let V be the set of functions $v : X \to \overline{\mathbb{R}}$, and let $v, w \in V$. We define the partial order $\leq$ on V in the usual way:

$$v \leq v' \Rightarrow v(x) \leq v'(x), \quad \forall x \in X. \tag{16}$$

If $v \leq v'$, we define the order interval $[v, v']$ by

$$[v, v'] = \{f \in V : v \leq f \leq v'\}. \tag{17}$$

The order interval $[v, v^{\infty}]$ means $[v(x), v^{\infty}(x)]$ with $v^{\infty}(x) = \infty$ for all $x \in X$.

The Bellman operator B from V into itself is defined by

$$Bv(x) = \sup_{y \in \Gamma(x)} W(x, v(y)), \forall x \in X, v \in V. \tag{18}$$

From assumption (A2) and (18), B is a monotone operator:

$$v \leq v' \Rightarrow Bv \leq Bv' \tag{19}$$

[5]The utility is discrete-time version of the modified Uzawa (1964) utility by Epstein and Hynes (1983).

We now present some fixed point theorems for the Bellman operator.

Proposition 3.1. *Assume that there exists $v \in V$ such that*

$$v \leq Bv \tag{20}$$

Then, there exist a fixed point of B in $[v, v^{\infty}]$.

Proof. See the Appendix A. □

As a direct consequence of Proposition 3.1 and Corollary 2.1, we have the following corollary:

Corollary 3.1. *Assume that there exists $v \in V$ such that*

$$Bv \leq v \tag{21}$$

Then, there exist a fixed point of B in $[v^{-\infty}, v]$.

By Proposition 3.2 and Corollary 2.2, we have the following corollary.

Corollary 3.2. *Assume that there exists $\underline{v}, \overline{v} \in V$ such that*

$$\underline{v} \leq B\underline{v} \tag{22}$$

$$B\overline{v} \leq \overline{v} \tag{23}$$

$$\underline{v} \leq \overline{v}. \tag{24}$$

Then, there exist a fixed point of B in $[\underline{v}, \overline{v}]$.

The following proposition shows that the value function v^* is fixed point of B in V.

Proposition 3.2. *Assume that*

(LC) *W is lower semicontinuous in its second argument.*

Then $v^* \in V$ is a fixed point of B.

Proof. See the Appendix A. □

As stated above, we know that the Bellman operator has a fixed point if there exists some boundary function and the value function is the fixed point of the Bellman operator. These are merely the discussion about the existence of a fixed point. From a practical perspective, we need a method to calculate the fixed point of the Bellman operator. In the next result, we solve the problem to add some assumption to aggregator.

Theorem 3.1. *Assume (LC), and that there exists $\underline{v} \in V$ such that*

$$\underline{v} \leq B\underline{v}. \tag{25}$$

Then the following conclusions hold:

(a) $\underline{v}^* \equiv \sup_{n\in\mathbb{N}}(B^n\underline{v})$ *is the least fixed point of B in $[\underline{v}, v^{\infty}]$.*

(b) *The increasing sequence $\{B^n\underline{v}\}_{n\in\mathbb{N}}$ convergences to $\underline{v}^*$ pointwise.*

Proof. See the Appendix A. □

As a direct consequence of Theorem 3.1, we have the following corollary:

Corollary 3.3. *Assume (LC), and that there exist* $\underline{v}, \overline{v} \in V$ *such that*

$$\underline{v} \leq B\underline{v}, \tag{26}$$

$$B\overline{v} \leq \overline{v}, \tag{27}$$

$$\underline{v} \leq \overline{v}. \tag{28}$$

Then the following conclusions hold:

(a) $\underline{v}^* \equiv \sup_{n\in\mathbb{N}}(B^n\underline{v})$ *is the least fixed point of* B *in* $[\underline{v}, \overline{v}]$.
(b) *The increasing sequence* $\{B^n\underline{v}\}_{n\in\mathbb{N}}$ *convergences to* $\underline{v}^*$ *pointwise.*

Theorem 3.1 and Corollary 3.3 say that the fixed point of the Bellman operator can be computed by iteration starting from the lower boundary $\underline{v}$ which satisfies suitable conditions. This approach has the applicability to a wide class of the dynamic optimization problem. However, there is a problem that how should we find such boundary. The next result is a partial answer of the problem.

Proposition 3.3. *Assume (LC), and that there exists* $\underline{v} \in V$ *such that*

$$\underline{v} \leq B\underline{v}. \tag{29}$$

Then the following conclusions hold:

(a) $\underline{v}^* \equiv \sup_{n\in\mathbb{N}}(B^n\underline{v})$ *is the least fixed point of* B *in* $[\underline{v}, v^\infty]$.
(b') *For all* $v_0 \in [\underline{v}, \underline{v}^*]$, *the increasing sequence* $\{B^n v_0\}_{n\in\mathbb{N}}$ *convergences to* $\underline{v}^*$ *pointwise.*

Proof. See the Appendix A. □

With respect to the precondition and the conclusion (a), Proposition 3.3 is the same as Theorem 3.1. Conclusion (b') in Proposition 3.3 say that an arbitrary function v_0 between the boundary function $\underline{v}$ and the least fixed point $\underline{v}^*$ converges to $\underline{v}^*$ pointwise. If there exists the boundary function, it means that the initial function that you start iteration can be chosen freely to some extent.

4. The convergence from the upper boundary

In the previous section, we show that the fixed point of the Bellman operator can be computed by iteration starting from the lower boundary. The natural question is, can the fixed point be computed by iteration starting from the upper boundary of the order interval just as in Theorem 3.1. The answer is no under our assumptions, even if the aggregator function W has some assumption as (AS). In the standard case of the bounded returns, the fixed point of the Bellman equation can be computed by iteration starting from any function. But, in this case, that sup and inf cannot be replaced in general is the cause of iteration starting from the upper boundary of the order interval not holding. In this section, we tackle the problem.

At first, we assume that B is i–order continuous. Then we have the following result.

Theorem 4.1. *Assume that there exists $\overline{v} \in V$ such that*

$$B\overline{v} \leq \overline{v}. \tag{30}$$

and that B is i–order continuous on $[v^{-\infty}, \overline{v}]$. Then the following conclusions hold:

(a) *$\overline{v}^* \equiv \inf_{n\in\mathbb{N}}(B^n \underline{v})$ is the greatest fixed point of B in $[v^{-\infty}, \overline{v}]$.*
(b) *The decreasing sequence $\{B^n\overline{v}\}_{n\in\mathbb{N}}$ convergences to $\overline{v}^*$ pointwise.*

Proof. See the Appendix A. □

The conclusions in Theorem 4.1 may be regarded as a dual version of the conclusion in Theorem 3.1. However, i-order continuity of the Bellman operator does not hold in general since it means that for all $x \in X$ and all $\{v_n\}_{n\in\mathbb{N}} \subset [v^{-\infty}, \overline{v}]$,

$$\inf_{n\in\mathbb{N}} \sup_{y\in\Gamma(x)} W(x, v_n(y)) = \sup_{y\in\Gamma(x)} \inf_{n\in\mathbb{N}} W(x, v_n(y)). \tag{31}$$

That is, the cause that the dual version of Theorem 3.1 does not hold is due to the fact that the Bellman operator is defined by supremum.

We make the following topological assumptions:

(i) the state space is a topological space;
(ii) the feasible correspondence is upper hemicontinuous and compact-valued; and
(iii) the aggregator function is upper semicontinuous for both arguments.

That is,

Assumption 4.1. *X is a topological space, Γ is upper hemicontinuous and compact-valued and W is upper semicontinuous for both arguments.*

To show the next theorem, we prepare the following two lemmas. The first lemma is standard result.

Lemma 4.1. *Let v be any upper semicontinuous function in V. Then, the following conditions hold under assumption 4.1:*

(i) *for all $x \in X$, $\max_{y\in\Gamma(x)} W(x, v(y))$ exists;*
(ii) *Bv is upper semicontinuous and $Bv \in V$; and*
(iii) *for all $n \in \mathbb{N}$, B^n is upper semicontinuous and $B^n \in V$.*

Proof. See the Appendix A. □

The following lemma is a variant of Sion's minimax theorem (Sion (1958)) and the proof is inspired by Komiya's elementary proof of Sion's minimax theorem (Komiya (1988)).

Lemma 4.2. *Let C be a nonempty compact subset of a topological space, and let $\{f_n\}_{n\in\mathbb{N}}$ be a wealy decreasing sequence of an upper semicontinuous functions. Then*

$$\max_{x\in C}\inf_{n\in\mathbb{N}} f_n(x) = \inf_{n\in\mathbb{N}}\max_{x\in C} f_n(x) \tag{32}$$

Proof. See the Appendix A.[6] □

Let V^u denote the set of upper semicontinous functions from X into $\overline{\mathbb{R}}$ where X is a topological space. It is clear that $V^u \subset V$. Using above lemmas, we can prove the following fixed point theorem.

Theorem 4.2. *Under Assumption 4.1, assume that there exists $\overline{v} \in V^u$ such that*

$$B\overline{v} \leq \overline{v}. \tag{33}$$

Then, the following conditions hold:

(a) $\overline{v}^* \equiv \inf_{n\in\mathbb{N}}(B^n\underline{v})$ *is the greatest fixed point of B in $[v^{-\infty}, \overline{v}]$ and $\overline{v}^*$ is upper semicontinuous.*

(b) *The decreasing sequence $\{B^n\overline{v}\}_{n\in\mathbb{N}}$ convergences to $\overline{v}^*$ pointwise.*

Proof. See the Appendix A. □

Using Theorem 4.1, we can construct an an alternative proof of Theorem 2.3 (a) in Streufert (1998).

Appendix A. Proofs

A.1. **Proof of Lemma 2.2.** At first we show that $\sup f(A) \leq f(\sup A)$. For all $a \in A$, $a \leq \sup A$. By monotonicity of f, we have $f(x) \leq f(\sup A)$. Then we have $\sup f(A) \leq f(\sup A)$.

To show that the inverse inequality, we proceed by cases.

Case 1: $\sup A \in \mathbb{R}$.
Let $\alpha = \sup A$. For all $n \in \mathbb{N}$, there exists $a_n \in A$ such that

$$a_n > \alpha + 1/n. \tag{34}$$

By definition of supremum, $a_n \leq \alpha$. Then, $a_n \to \alpha$ as $n \to \infty$. So,

$$f(\sup A) = f(\alpha) \tag{35}$$

$$\leq \liminf_{n\to\infty} f(a_n) \tag{36}$$

$$\leq \sup_{n\in\mathbb{N}} f(a_n) \tag{37}$$

$$\leq \sup f(A), \tag{38}$$

[6]After completing proof, we find similar result and it is often called Dini–Cartan theorem. See Dellacherie and Meyer (2011), p98.

where (36) holds since f is a lower semicontinuous function. Then we have $f(\sup A) \leq \sup f(A)$.

Case 2: $\sup A = \infty$. Let $\alpha = \sup A$. For all $n \in \mathbb{N}$, there exists $a_n \in A$ such that

$$a_n > \max\{n, a_{n-1}\}.$$

Then $a_n \to \alpha$ as $n \to \infty$. By the same argument of the Case 1, we have $f(\sup A) \leq \sup f(A)$.

Case 3: $\sup A = -\infty$. Then, $a = -\infty$ for all $a \in A$. Form the left hand side of (4), we have

$$\sup f(A) = \sup_{x \in A} f(x) = f(-\infty). \tag{39}$$

Form the right hand side of (4), we have

$$f(\sup A) = f(-\infty). \tag{40}$$

By (39) and (40), we obtain (4).

A.2. **Proof of Proposition 3.1.** Since $[v, v^\infty]$ is complete lattice, the subset has suprermum.in $[v, v^\infty]$. By the monotonicity of B and (20), B maps $[v, v^\infty]$ into itself. Then $\{B^n v\}$ is the countable chain in $[v, v^\infty]$. Thus, $[v, v^\infty]$ has a at most one chain and every chain has suprermum in $[v, v^\infty]$. Therefore, B has fixed points in $[v, v^\infty]$ by Theorem 2.1.

A.3. **Proof of Proposition 3.2.** Let $A \subset \overline{\mathbb{R}}$. For all $b \in A$, $b \leq \sup_{a \in A} a$. By the monotonicity of B, we have $W(x, b) \leq W(x, \sup_{a \in A} a)$ for all $x \in X$. Then, we obtain $\sup_{a \in A} W(x, a) \leq W(x, \sup_{a \in A} a)$. By (LC), we have

$$\sup_{a \in A} W(x, a) = W(x, \sup_{a \in A} a). \tag{41}$$

For all $x_0 \in X$, we have

$$\begin{align}
Bv^*(x_0) &= \sup_{x_1 \in \Gamma(x_0)} W(x_0, v^*(x_1)) \tag{42}\\
&= \sup_{x_1 \in \Gamma(x_0)} W(x_0, \sup_{\{x_t\}_{t=2}^\infty \in \Pi(x_1)} U(\{x_t\}_{t=1}^\infty)) \tag{43}\\
&= \sup_{x_1 \in \Gamma(x_0)} \sup_{\{x_t\}_{t=2}^\infty \in \Pi(x_1)} W(x_0, U(\{x_t\}_{t=1}^\infty)) \tag{44}\\
&= \sup_{\{x_t\}_{t=1}^\infty \in \Pi(x_0)} W(x_0, U(\{x_t\}_{t=1}^\infty)) \tag{45}\\
&= v^*(x_0) \tag{46}
\end{align}$$

where (43) follows by definition of V^*, (44) follows by (41) and that x_1 is independent by $\{x_t\}_{t=2}^\infty$ and (45) follows by Lemma 2.1. Since x_0 is arbitrary in X, $Bv^* = v^*$. That is, v^* is fixed point of B in V.

A.4. **Proof of Theorem 3.1.** By the monotonicity of B and (25), B maps $[\underline{v}, v^\infty]$ into itself. We apply Theorem 2.2 to $B : [\underline{v}, v^\infty] \to [\underline{v}, v^\infty]$. For this purpose, it suffices to show that every countable chain in $[\underline{v}, v^\infty]$ has superemum and that B is s–order continuous on $[\underline{v}, v^\infty]$.

Since $[\underline{v}, v^\infty]$ is complete lattice, every subset of $[\underline{v}, v^\infty]$ has supremum. Then To see that every countable chain in $[\underline{v}, v^\infty]$ has superemum.

To see that B is s–order continuous, let $\{v_n\}_{n\in\mathbb{N}}$ be a countable chain in $[\underline{v}, v^\infty]$. Let $x \in X$. By the monotonicity of B, we have $W(x, v_n(y)) \leq W(x, \sup_{n\in\mathbb{N}} v_n(y))$ for all $n \in \mathbb{N}$ and all $y \in X$. Then, we obtain $\sup_{n\in\mathbb{N}} W(x, \sup_{n\in\mathbb{N}} v_n(y)) \leq W(x, \sup_{n\in\mathbb{N}} v_n(y))$. By (LC), we have

$$\sup_{n\in\mathbb{N}} W(x, v_n(y)) = W(x, \sup_{n\in\mathbb{N}} v_n(y)). \tag{47}$$

Then, We have

$$[B(\sup_{n\in\mathbb{N}} v_n)](x) = \sup_{y\in\Gamma(x)} W(x, (\sup_{n\in\mathbb{N}} v_n)(y)) \tag{48}$$

$$= \sup_{y\in\Gamma(x)} \sup_{n\in\mathbb{N}} W(x, v_n(y)) \tag{49}$$

$$= \sup_{n\in\mathbb{N}} \sup_{y\in\Gamma(x)} W(x, v_n(y)) \tag{50}$$

$$= [\sup_{n\in\mathbb{N}} (Bv_n)](x) \tag{51}$$

where (49) follows by (47), and (50) follows by Lemma 2.1. Since x is arbitrary, it follows that $B \sup_{n\in\mathbb{N}} v_n = \sup_{n\in\mathbb{N}} Bv_n$. Thus, B is s–order continuous on $[\underline{v}, v^\infty]$.

Now by Theorem 2.2, conclusion (a) follows. To see (b), by the monotonicity of B and (29), we have $\underline{v} \leq B\underline{v} \leq B^2\underline{v} \leq \cdots$. Then $\{B^n\underline{v}\}_{n\in\mathbb{N}}$ is increasing. By Theorem 2.2, $\sup_{n\in\mathbb{N}} B^n\underline{v} = \underline{v}^*$. Therefore, we have the desired result.

A.5. **Proof of Proposition 3.3.** To see (b'), let $v_0 \in [\underline{v}, \underline{v}^*]$. That is, $\underline{v} \leq v_0 \leq \underline{v}^*$. By the monotonicity of B, $B\underline{v} \leq Bv_0 \leq B\underline{v}^* = \underline{v}^*$, and by induction $B^n\underline{v} \leq B^n v_0 \leq \underline{v}^*$. Since $\{B^n\underline{v}\}_{n\in\mathbb{N}}$ converges to $\underline{v}^*$ pointwise, $\{B^n v_0\}_{n\in\mathbb{N}}$ also converges to $\underline{v}^*$ pointwise by the squeeze theorem.

A.6. **Proof of Theorem 4.1.** By the monotonicity of B and (30), B maps $[v^{-\infty}, \overline{v}]$ into itself. By the assumption, B is i–order continous. Using similar argument in the proof of Theorem 3.1, we have that every countable chain in $[v^{-\infty}, \overline{v}]$ has infimum Now by Corollary 2.2, conclusion (a) follows. To see (b), by the monotonicity of B and (30), we have $\overline{v} \geq B\overline{v} \geq B^2\overline{v} \geq \cdots$. Then $\{B^n\overline{v}\}_{n\in\mathbb{N}}$ is decreasing. By Corollary 2.2, $\inf_{n\in\mathbb{N}} B^n\overline{v} = \overline{v}^*$.

A.7. **Proof of Lemma 4.1.**

(i) By Assumption 4.1, W is upper semicontinuous for both arguments and Γ is nonempty, compact-valued. Then $\max_{y\in\Gamma(x)} W(x, v(y))$ exists for all $x \in X$ by the Weierstrass maximization theorem.

(ii) By (i), for all $x \in X$,

$$(Bv)(x) = \max_{y\in\Gamma(x)} W(x, v(y)). \tag{52}$$

Using the Berge maximization theorem, $(Bv)(x)$ is upper semicontinuous for all $x \in X$. Since x is arbitrary in X, Bv is upper semicontinuous. By the definition of V, $Bv \in V$.

(iii) By the definition of V and the results of (i) and (ii), we have the desired result.

A.8. **Proof of Lemma 4.2.** It is obvious that $\max_{x\in C}\inf_{n\in\mathbb{N}} f_n(x) \le \inf_{n\in\mathbb{N}}\max_{x\in C} f_n(x)$. Hence, we show the reverse inequality. Let $\alpha \in \overline{\mathbb{R}}$ with $\alpha > \max_{x\in C}\inf_{n\in\mathbb{N}} f_n(x)$. We define $C_n = \{x \in C\colon f_n(x) \ge \alpha\}$ for all $n \in \mathbb{N}$. By upper semicontinuity of f_n, C_n is compact. Then $\bigcap_{n\in\mathbb{N}} C_n = \emptyset$, and hence there exist $n_1, \cdots, n_m \in \mathbb{N}$ such that $\bigcap_{i=1}^m C_{n_i} = \emptyset$. Thus we have $\alpha > \max_{x\in C}\min_{1\le i\le m} f(x, n_i)$. Without loss of generality, $n_1 \ge \cdots \ge n_m$. Since f is weakly decreasing in its second argument, $f(x, n_1) \ge \cdots \ge f(x, n_m)$ for all $x \in C$. Let $n_m = n_0$. We have $f(x, n_0) = \min_{1\le i\le m} f(x, n_i)$ for all $x \in C$. Then, there exists $n_0 \in \mathbb{N}$ with $\alpha > \max_{x\in C} f(x, n_0)$. Thus we obtain $\alpha > \inf_{n\in\mathbb{N}}\max_{x\in C} f(x, n)$. Therefore we have

$$\max_{x\in C}\inf_{n\in\mathbb{N}} f(x, n) \ge \inf_{n\in\mathbb{N}}\max_{x\in C} f(x, n). \tag{53}$$

A.9. **Proof of Theorem 4.2.** By the monotonicity of B and (33), B maps $[v^{-\infty}, \overline{v}]$ into itself. We apply Theorem 2.2 to $B : [v^{-\infty}, \overline{v}] \to [v^{-\infty}, \overline{v}]$. For this purpose, it suffices to show that every countable chain in $[v^{-\infty}, \overline{v}]$ has infimum and that B is i–order continuous on $[v^{-\infty}, \overline{v}]$.

Since $[v^{-\infty}, \overline{v}]$ is complete lattice, every subset of $[v^{-\infty}, \overline{v}]$ has supremum. Then to see that every countable chain in $[v^{-\infty}, \overline{v}]$ has superemum.

To see that B is i–order continuous, let $\{v_n\}_{n\in\mathbb{N}}$ be a countable chain in $[v^{-\infty}, \overline{v}]$. Since $B(v_n) \subset [v^{-\infty}, \overline{v}]$, $\inf_{n\in\mathbb{N}} B(v_n) \in [v^{-\infty}, \overline{v}]$. Let $x \in X$. By Lemma 4.1 and 4.2, we have

$$\inf_{n\in\mathbb{N}}[B(v_n)](x) = \inf_{n\in\mathbb{N}}\max_{y\in\Gamma(x)}\{W(x, v_n(y))\} \tag{54}$$

$$= \max_{y\in\Gamma(x)}\{W(x, \inf_{n\in\mathbb{N}} v_n(y))\} \tag{55}$$

$$= [B(\inf_{n\in\mathbb{N}} v_n)](x). \tag{56}$$

Since x is arbitrary, it follows that $B\inf_{n\in\mathbb{N}} v_n = \inf_{n\in\mathbb{N}} Bv_n$. Thus, B is i–order continuous on $[v^{-\infty}, \overline{v}]$.

Now by Corollary 2.2, conclusion (a) follows. To see (b), by the monotonicity of B and (33), we have $\overline{v} \ge B\overline{v} \ge B^2\overline{v} \ge \cdots$. Then $\{B^n\overline{v}\}_{n\in\mathbb{N}}$ is decreasing. By Corollary 2.2, $\inf_{n\in\mathbb{N}} B^n\overline{v} = \overline{v}^*$. Therefore, we have the desired result.

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