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Abstract: The critical capital stock is a threshold that appears in a nonconcave aggregate growth model such that any optimal capital path from a stock level below the threshold converges to a lower steady state, whereas any optimal capital path from a stock level above the threshold converges to a higher steady state. Unlike a concave model with wealth effect, the threshold is not necessarily an optimal steady state, which makes its characterization difficult. In a continuous–time growth model with a convex–concave production function, we show that: a) the critical capital stock is continuous and strictly increasing in the discount rate; b) as the discount rate increases, it appears at the zero-stock level and disappears at a certain level between the stock levels of the maximum average productivity and the maximum marginal productivity; c) at this upper bound, it merges with the higher steady state; d) once the critical capital stock disappears, the higher steady state is no longer an optimal steady state; and e) the disappearing point can be arbitrarily close to either of the these stock levels, depending on the curvature of the utility function.

Keywords: Continuous–time growth model, convex–concave production function, critical capital stock

Journal of Economic Literature Classification Numbers: C61; D90; O41

1 Introduction

An aggregate growth model with a convex–concave production function explains an interesting and important economic phenomenon: depending on the initial level of capital stock, the economy may advance to the higher steady state or decline to the lower steady state. This history dependence and polarization appears in several economic problems and the model has a wide range of applications, including in economic development (Azariadis and Drazen 1990, Askenazy and Le Van 1999, Hung, Le Van and Michel 2009), firm dynamics (Davidson and Harris 1981, Hartl et al. 2004, Haunschmied et al. 2005, Wagener 2005, Caulkins et al. 2010, 2015), public policy (Brock and Dechert 1983, Caulkins et al. 2001, 2005, 2006, 2007a, 2007b, Feichtinger and Tragler 2002, Feichtinger et al. 2002), international trade (Long et al. 1997, Majumdar and Mitra 1995, Le Van et al. 2010), resource and environmental economics (Clark 1971, Dasgupta and Mäler 2003, Brock and Starrett 2003), and general theoretical studies (Majumdar and Mitra 1982, 1983, Majumdar and Nermuth 1982, Dechert and Nishimura 1983, Amir et al. 1991, Haunschmied et al. 2003, Wagener 2003, 2006, Dockner and Nishimura 2005, Kamihigashi and Roy 2006, 2007, Kiseleva and Wagener 2010). Deissenberg et al. (2004) provides a comprehensive survey of this topic. The convex part of the production function expresses increasing returns to scale, which prevails because of essential infrastructure for economic development, and which requires a large amount of initial investment to enter some industries, and the nonconvexity of nature, such as the Allee effect in biological population dynamics.

The threshold appearing in a nonconcave model is also known as the critical capital stock.[1] The critical capital stock has important economic implications for economies escaping from poverty traps and preventing the ruinous use of environmental assets. Thus, it should be an important research subject to identify its location. However, little is known about this subject. One may expect that the critical capital stock is an unstable steady state of the so-called canonical system and thus the location is identified easily. This could be true if the model was a concave model with wealth effect (Kurz 1968, Wirl and

[1] This threshold is also known as the Skiba or Dechert–Nishimura–Skiba point (Haunschmied et al. 2003), because Skiba (1978) suggested its existence and Dechert and Nishimura (1983) proved that it exists in a certain range of discount rates. Clark's (1971) work on renewable resource management is potentially the earliest analysis of this critical threshold. While Dechert and Nishimura (1983) used a discrete-time model, Askenazy and Le Van (1999) proved its existence in a continuous-time model. See also Long et al. (1997) and Dockner and Nishimura (2005).

Feichtinger 2005), in which the unstable steady state is an *optimal* steady state. However, optimality is hardly expected in a nonconcave model given that the production function is *convex* at the steady state. There is a sufficient condition under which the unstable steady state is *not* an optimal steady state (Dechert and Nishimura 1982, Askenazy and Le Van 1999). In this case, the critical capital stock may be somewhere other than at the unstable steady state.

Unfortunately, even the existence of the critical capital stock is not completely decided. As we prove, it exists if and only if the interior and saddle-point stable steady state is an optimal steady state. The optimality is not trivial because Arrow's sufficiency theorem for optimality is not applicable given the nonconcavity of the production function. It is known that it exists if the discount rate is less than or equal to the maximum average productivity.[2] However, with a larger discount rate, the optimality of the steady state and thus the existence of the critical capital stock are ambiguous. There are only examples in discrete–time models where it is not an optimal steady state (Majumder and Mitra 1982) and where it can be an optimal steady state (Dechert and Nishimura 1983).

As such, in a nonconcave model, the optimality of the steady states of the canonical system is a sensitive problem. This has not been well addressed in many studies, including the classic work by Skiba (1978). As a recent important result, Wagener (2003) showed that the heteroclinic bifurcation of the canonical system implies the existence of a critical capital stock and developed a local criterion of the bifurcation. The subtle point is how to ensure that a steady state of the canonical system is optimal.

In this paper, we investigate these problems. Our main results are as follows. The critical capital stock is continuous and strictly increasing in the discount rate. It starts from the zero-stock level when the discount rate is as low as the marginal productivity at the zero-stock level, and disappears at a certain stock level between the stock levels of the maximum *average* productivity and the maximum *marginal* productivity. The upper bound of the discount rate then lies somewhere between these productivities, and it can be arbitrarily close to either, depending on the curvature of the utility function.

Akao, Kamihigashi, and Nishimura (2012) obtained the continuity and monotonicity results using a discrete–time model known as the Dechert–Nishimura model. However, the proofs are rather different

[2] See Dechert and Nisimura (1983, Lemma 2) in a discrete time model, and Askenazy and Le Van (1999, Proposition 7) in a continuous–time model.

given, e.g., the difference between the Bellman equation in a discrete–time model and the Hamilton–Jacobi–Bellman equation in a continuous–time model. Considering that most economic applications of nonconcave growth models use a continuous–time model, replication in such a model could have its own value. Also, we can strengthen one of their main results: the critical capital stock is *strictly* increasing in the discount rate.

The rest of the paper is structured as follows. Section 2 details the model and the assumptions. Section 3 provides some preliminary results on the optimal paths. Section 4 discusses the results concerning the critical capital stock. Section 5 concludes.

2 Model and Assumptions

Consider a continuous–time optimal growth model:

$$V^*(x) \equiv \max_{c(t)} \int_0^\infty u\left(c(t)\right) e^{-\rho t} dt \tag{2.1}$$

$$\text{subject to } \dot{x}(t) = f\left(x(t)\right) - c(t),\ c(t) \geq 0,\, x(t) \geq 0,\ x(0) = x \geq 0 \text{ given,}$$

where x is the initial capital stock, $x(t)$ is the capital path, $c(t)$ is the consumption path, and $\rho > 0$ is the discount rate. If a path $(x(t), c(t))$ satisfies the state equation and the nonnegativity conditions in (2.1), it is feasible. A feasible path $(x^*(t), c^*(t))$ from $x \geq 0$ is then optimal if there is no feasible path $(x(t), c(t))$ from x that satisfies

$$\int_0^\infty \left[u\left(c(t)\right) - u\left(c^*(t)\right)\right] e^{-\rho t} dt > 0.$$

Throughout this paper, we assume that for each initial stock level $x \geq 0$, there is an optimal path $(x^*(t), c^*(t))$ in which $c^*(t)$ is a piecewise-continuous function. Furthermore, we make the following assumptions:

Assumption 1: The utility function $u : \mathbb{R}_+ \mapsto \mathbb{R} \cup \{-\infty\}$ is a strictly increasing, strictly concave and twice-continuously differentiable function with $\lim_{c \searrow 0} u'(c) = \infty$.

Assumption 2: The production function $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a twice-continuously differentiable function with the following properties: (a) $f(0) = 0$, (b) there is an inflection point x_I such that $f''(x) \gtreqless 0$ for $x \lesseqgtr x_I$, (c) $\lim_{x \searrow 0} f'(x) > 0$, (d) $\lim_{x \searrow 0} f''(x)$ exists, and (e) $\lim_{x \nearrow \infty} f'(x) \leq 0$.

By Assumption 2 (b), the production function is strictly convex on $[0, x_I]$ and strictly concave on $[x_I, \infty)$, and is then the convex–concave production function.

Define the two discount rates ρ_0 and ρ_I by

$$\rho_0 \equiv \lim_{x \searrow 0} f'(x) \text{ and } \rho_I \equiv \max\{f'(x)|x \geq 0\}(= f'(x_I)). \tag{2.2}$$

If $\rho \in (\rho_0, \rho_I)$, there are two positive stock levels that satisfy $f'(x) = \rho$. We denote these with x_s and x^s, where $x_s < x^s$, and refer to them as the lower and the upper stationary capitals, respectively. We denote the stationary capitals also by $x_s(\rho)$ and $x^s(\rho)$ when we highlight the fact that they are functions of ρ. We apply this same convention to the other variables.

The Hamiltonian $H : \mathbb{R}^3_+ \mapsto \mathbb{R} \cup \{-\infty\}$ and the maximized Hamiltonian $H^* : \mathbb{R}^2_{++} \mapsto \mathbb{R}$ associated with the problem (2.1) are defined by

$$H(c, x, q) \equiv u(c) + q(f(x) - c), \tag{2.3}$$

$$\text{and } H^*(x, q) \equiv \max\{H(c, x, q)|c \geq 0\}, \tag{2.4}$$

respectively.

We refer to the following system of differential equations as the canonical system:

$$\dot{x}(t) = \partial H^*(x(t), q(t))/\partial q = f(x(t)) - u'^{-1}(q(t)), \tag{2.5a}$$

$$\dot{q}(t) = \rho q(t) - \partial H^*(x(t), q(t))/\partial x = -[f'(x(t)) - \rho]q(t), \tag{2.5b}$$

where u'^{-1} is the inverse function of u', i.e., $c = u'^{-1}(q) \Longleftrightarrow u'(c) = q$.

Let

$$\sigma(c) \equiv -\frac{cu''(c)}{u'(c)}. \tag{2.6}$$

The system of differential equations:

$$\dot{x}(t) = f(x(t)) - c(t), \tag{2.7a}$$

$$\dot{c}(t) = \frac{c(t)}{\sigma(c(t))}[f'(x(t)) - \rho], \tag{2.7b}$$

is equivalent to the canonical system (2.5). This is the $x - c$ system and the solution of the system is the $x - c$ path.

Let

$$c_s \equiv f(x_s) \text{ and } c^s \equiv f(x^s). \tag{2.8}$$

The interior steady states of the canonical system and the $x - c$ system are $(x_s, u'(c_s)), (x^s, u'(c^s))$ and $(x_s, c_s), (x^s, c^s)$, respectively. Corresponding to the lower and upper stationary capitals, these are the lower and the upper steady states of these systems.

The Jacobian of the $x - c$ system is given by

$$J = \begin{bmatrix} f'(x) & -1 \\ cf''(x)/\sigma(c) & [d\,(c/\sigma(c))\,/dc]\,[f'(x(t)) - \rho] \end{bmatrix}. \tag{2.9}$$

When $\rho \in (\rho_0, \rho_I)$, the eigenvalues associated with the steady states are:

$$\frac{1}{2}\left(\rho \pm \sqrt{\rho^2 - 4cf''(x)/\sigma(c)}\right), \tag{2.10}$$

where $(x, c) = (x_s, c_s), (x^s, c^s)$. Thus (x_s, c_s) is unstable, while (x^s, c^s) is saddle-point stable.

3 Optimal paths

In this section we show some results concerning the canonical system and optimal paths, based on which we characterize the critical capital stock. Throughout the paper, we denote by $(x^*(t), c^*(t))$ an optimal path for the problem (2.1).

Proposition 3.1 *(Interiority) An optimal path* $(x^*(t), c^*(t))$ *starting from a positive capital stock satisfies* $(x^*(t), c^*(t)) \in \mathbb{R}^2_{++}$ *for all* t.

Proof. See the Appendix.[3] ■

Proposition 3.2 *(Monotonicity and convergence) If* $x^*(t)$ *is not constant, then* $x^*(t)$ *monotonically converges to either* 0 *or* x^s*.*[4]

Proof. The proof of the monotonicity is in Askenazy and Le Van (1999, Proposition 4). By Proposition 3.1, the optimal path $(x^*(t), c^*(t))$ is a solution of the $x - c$ system (2.7). Then the monotonicity and Assumption 2 (e) imply that $x^*(t)$ converges to any of the steady states, $0, x_s$, or x^s. But, as seen from (2.10), (x_s, c_s) is an unstable steady state. Thus, $x^*(t)$ cannot converge to x_s. ■

The gain function (Kamihigashi and Roy, 2006, 2007) is a useful tool to characterize an optimal path, as shown below. It is defined by

$$\gamma(x) \equiv f(x) - \rho x. \tag{3.1}$$

When $\rho \in (\rho_0, \rho_I)$, the lower stationary capital x_s is the local minimizer and the upper stationary capital x^s is the local maximizer of $\gamma(x)$. See Figure 1.

<Figure 1>

[3] Although the result is well known, we could not locate the formal proof. Hence, we provide our proof in the Appendix.

[4] The monotonicity property of an optimal capital path is not trivial given the nonconcavity of the production function. To the best of our knowledge, Long, Nishimura, and Shimomura (1997, Lemma 5) provided the first formal proof.

We introduce additional notations. Figure 1 depicts their geometry. Let $\hat{\rho}$ be the discount rate that coincides with the maximum average productivity:

$$\hat{\rho} \equiv \max\{f(x)/x | x \geq 0\}. \tag{3.2}$$

We denote by $\hat{x}$ the capital stock of the maximum average productivity:

$$\hat{x} \equiv \arg\max\{f(x)/x | x \geq 0\}. \tag{3.3}$$

We also define two capital stock levels, $\breve{x}$ and $\underline{x}$. $\breve{x}$ is implicitly defined by

$$\gamma(\breve{x}) = 0, \;\; \breve{x} > 0. \tag{3.4}$$

$\breve{x}$ exists when $\rho \in (\rho_0, \hat{\rho}]$. It holds that $\gamma(x) < 0$ for all $x \in (0, \breve{x})$, while $\gamma(x) > 0$ for all $x \in (\breve{x}, x^s)$, and $\breve{x} = \hat{x} = x^s$ at $\rho = \hat{\rho}$. As easily verified, $\breve{x}(\rho)$ is continuous and strictly increasing. Also, $\lim_{\rho \searrow \rho_0} \breve{x}(\rho) = 0$ and $\lim_{\rho \nearrow \hat{\rho}} \breve{x}(\rho) = x^s(\hat{\rho})$ hold. $\underline{x}$ is implicitly defined by

$$\gamma(\underline{x}) = \gamma(x^s), \;\; \underline{x} \in (0, x^s). \tag{3.5}$$

$\underline{x}$ exists when $\rho \in (\hat{\rho}, \rho_I)$. $\underline{x}$ has the property that $\gamma(x) > \gamma(x^s)$ for all $x \in (0, \underline{x})$, while $\gamma(x) \leq \gamma(x^s)$ for all $x \geq \underline{x}$. As easily verified, $\underline{x}(\rho)$ is continuous, strictly increasing, and satisfies $\lim_{\rho \searrow \hat{\rho}} \underline{x}(\rho) = 0$ and $\lim_{\rho \nearrow \rho_I} \underline{x}(\rho) = \lim_{\rho \nearrow \rho_I} x^s = x_I$.

Lemma 3.1 *Let $(x'(t), c'(t))$ be a nonconstant feasible path such that $\gamma(x'(t)) \leq \gamma(x'(0))$ for all $t > 0$. Then, the constant path $(x(t), c(t)) = (x'(0), f(x'(0)))$ dominates the paths for all $t \geq 0$.*

Proof.

$$\begin{aligned}
\int_0^\infty u(c'(t))e^{-\rho t}dt &< \rho^{-1}u\left(\int_0^\infty c'(t)e^{-\rho t}dt\right) \\
&= \rho^{-1}u\left(\int_0^\infty \rho\gamma(x'(t))e^{-\rho t}dt + \rho x'(0)\right) \\
&\le \rho^{-1}u\left(\int_0^\infty \rho\gamma(x'(0))e^{-\rho t}dt + \rho x'(0)\right) \\
&= \frac{u\left[f(x'(0))\right]}{\rho} = \int_0^\infty u\left[f(x'(0))\right]e^{-\rho t}dt,
\end{aligned} \tag{3.6}$$

where the first line follows from Jensen's inequality, the second line from the integration by parts, and the third line from $\gamma(x'(t)) \le \gamma(x'(0))$. ■

Then we have the following proposition.

Proposition 3.3 *(The stability of the optimal steady states) (i) When $\rho \in (0, \rho_0]$, x^s is an optimal steady state and every optimal capital path from $x > 0$ converges to x^s. (ii) When $\rho \in (\rho_0, \hat{\rho}]$, x^s is an optimal steady state and an optimal capital path from $x \ge \breve{x}$ converges to x^s. (iii) When $\rho \in (\hat{\rho}, \rho_I)$, an optimal capital path from $x \le \underline{x}$ converges to 0. (iv) When $\rho > \rho_0$, there is a capital stock level, from which an optimal capital path converges to 0.*

Proof. Statements (i)–(iii) follow from Proposition 3.2 and Lemma 3.1. We prove (iv) by contradiction. Suppose that any optimal capital path from a positive initial capital stock converges to x^s. Then, for each initial capital stock $x_0 \in (0, x_s)$, there is time $\tau(x_0) > 0$ such that $x^*(\tau) = x_s$. By (2.7b), $c^*(0) > c^*(\tau(x_0))$ holds. As the problem is autonomous, $c^*(\tau(x_0))$ is independent of x_0. For x_0 sufficiently near zero, we have

$$\dot{x}^*(0) = f(x_0) - c^*(0) < f(x_0) - c^*(\tau(x_0)) < 0$$

and thus the optimal capital path cannot approach to x^s, which is a contradiction. ■

We repeatedly use the following three lemmas to identify an optimal path. We also use them to show the uniqueness of an optimal path converging to the origin.[5]

[5] Note that the uniqueness of an optimal path converging to x^s is trivial, despite the nonconcavity of the model, because the convergent point is a saddle point.

Lemma 3.2 *The maximized Hamiltonian is strictly convex in the costate and attains the minimum on the nullcline of* $\dot{x} = 0$*:*

$$\frac{\partial^2 H^*(x,q)}{\partial q^2} < 0, \tag{3.7}$$

$$\min_{q \geq 0} H^*(x,q) = H^*(x, u'(f(x))). \tag{3.8}$$

Proof. A simple calculation yields the results. ■

Lemma 3.3 [6] *Let* $(x(t), q(t))$ *be the solution of the canonical system with* $(x(0), q(0)) = (x_0, q_0)$*. If*

$$\lim_{t \to \infty} H^*(x(t), q(t))e^{-\rho t} = 0, \tag{3.9}$$

then

$$\int_0^\infty u[C(q(t))]e^{-\rho t} dt = \rho^{-1} H^*(x_0, q_0). \tag{3.10}$$

Proof. The proof can be found in Davidson and Harris (1981, Appendix). ■

Lemma 3.4 *An* $x - c$ *path that converges to* (x^s, c^s) *or* $(0,0)$ *satisfies the terminal condition (3.9).*[7]

Proof. It is obvious for an $x - c$ path that converges to (x^s, c^s). Let $(x(t), c(t))$ be an $x - c$ path that converges to $(0,0)$. Denote the associated costate by $q(t) = u'(c(t))$. By the canonical system (2.5), we have

$$q(t) = q(0) \exp\left(\int_0^t \rho - f'(x(s))\, ds \right). \tag{3.11}$$

Then, given $\dot{x}(t) = f[x(t)] - c(t) \to 0$ as $t \to \infty$,

$$\begin{aligned} \lim_{t \to \infty} H^*[x(t), q(t)]e^{-\rho t} &= \lim_{t \to \infty} \left\{ u\left(c(t)\right) e^{-\rho t} + q(0)\dot{x}(t) \exp\left(- \int_0^t f'\left(x(s)\right) ds \right) \right\} \\ &= \lim_{t \to \infty} u\left(c(t)\right) e^{-\rho t} \end{aligned} \tag{3.12}$$

[6] This welfare consequence (3.10) is originally due to Weitzman (1976).

[7] If the path is an optimal path, the result is immediate from Michel (1982, Theorem). However, it may not be an optimal path.

holds. Thus the statement is true if

$$\lim_{t\to\infty} u\left(c(t)\right)e^{-\rho t} = 0. \tag{3.13}$$

(3.13) obviously holds if u is bounded from below. Then, assume that u is not bounded from below. Choose a sufficiently small $x(0)$ such that $u\left(c(t)\right) < 0$ for all $t \geq 0$. Then, with arbitrarily chosen $\hat{c} > 0$, we have:

$$\begin{aligned}
0 &> u\left(c(t)\right)e^{-\rho t} \\
&\geq \{u'\left(c(t)\right)\left(c(t)-\hat{c}\right) + u(\hat{c})\}e^{-\rho t} \\
&= \left[q(t)(c(t)-\hat{c}) + u(\hat{c})\right]e^{-\rho t} \\
&= \left[q(0)\exp\left(\int_0^t \rho - f'\left(x(s)\,ds\right)\right)\left(c(t)-\hat{c}\right) + u(\hat{c})\right]e^{-\rho t} \\
&= q(0)\exp\left(-\int_0^t f'\left(x(s)\,ds\right)\right)\left(c(t)-\hat{c}\right) + u(\hat{c})e^{-\rho t} \to 0 \text{ as } t\to\infty.
\end{aligned}$$

■

Proposition 3.4 *(Uniqueness of an optimal path converging to the origin) An optimal path converging to the origin is unique.*

Proof. We prove by contradiction. Assume that there are different optimal paths $(x_1(t), q_1(t))$ and $(x_2(t), q_2(t))$ from $x = x_1(0) = x_2(0)$ in the canonical system (2.5). Let $q_1(0) < q_2(0)$. Given that both optimal capital paths are strictly decreasing, $q_1(0) < q_2(0) < u'(f(x))$ holds. Then by Lemma 3.2, $H^*(x, q_1(0)) > H^*(x, q_2(0))$, and by Lemmas 3.3 and 3.4, the path $(x_1(t), q_1(t))$ dominates $(x_2(t), q_2(t))$, which contradicts the assumption that both paths are optimal. ■

We then introduce the terms of the ascending and descending paths. We call an $x-c$ path $(x^A(t), c^A(t))$ such that $\dot{x}^A(t) > 0$ for all $t \geq 0$ and $\lim_{t\to\infty} x^A(t) = x^s$ an ascending path. Similarly, we call an $x-c$ path $(x^D(t), c^D(t))$ such that $\dot{x}^D(t) < 0$ for all $t \geq 0$ and $\lim_{t\to\infty} x^D(t) = 0$ a descending path. Note that an ascending or a descending path is not necessarily an optimal path, although a nonconstant optimal path from $x \in (0, x^s)$ is necessarily either an ascending or a descending path. Associated with these paths, define D^A and D^D by:

$$D^A \equiv \{(x,c) \in \mathbb{R}^2_{++} | f(x) - c > 0\},$$

$$D^D \equiv \{(x,c) \in \mathbb{R}^2_{++} | f(x) - c < 0\}.$$

Also define two functions $\xi^A : D^A \times (\rho_0, \rho_I) \mapsto \mathbb{R}$ and $\xi^D : D^D \times (\rho_0, \rho_I) \mapsto \mathbb{R}$ by

$$\xi^j(x,c;\rho) \equiv \frac{c[f'(x) - \rho]}{\sigma(c)\,[f(x) - c]}, \quad j = A, D.$$

ξ^j gives the slope of the vector field of the $x - c$ system in the domain D^j $(j = A, D)$. The following results are immediate:

$$\frac{\partial \xi^A(x,c;\rho)}{\partial \rho} < 0 \text{ and } \frac{\partial \xi^D(x,c;\rho)}{\partial \rho} > 0. \tag{3.14}$$

In what follows, we compare two paths that differ in the discount rates, ρ_i $(i = 1, 2)$. We refer to an optimal path when the discount rate is ρ as a ρ-optimal path. We apply this convention to the other paths and functions such as an ascending path and the value function.

Lemma 3.5 *Let ρ_1 and ρ_2 satisfy $\rho_0 < \rho_1 < \rho_2$. (i) Let $(x_1^A(t), c_1^A(t))$ and $(x_2^A(t), c_2^A(t))$ be ρ_1- and ρ_2-ascending paths from the same initial capital stock. Then $c_1^A(0) < c_2^A(0)$. (ii) Let $(x_1^{D*}(t), c_1^{D*}(t))$ and $(x_2^{D*}(t), c_2^{D*}(t))$ be ρ_1- and ρ_2-optimal and descending paths from the same initial capital stock. Then $c_1^{D*}(0) < c_2^{D*}(0)$.*

Proof. We prove by contradiction. (i) Assume that $c_1^A(0) \geq c_2^A(0)$ holds. Then, the ρ_1-ascending path should cross the ρ_2-ascending path at least once at the intersection point $(x^\#, c^\#)$ such that

$$\xi^A(x^\#, c^\#; \rho_1) \leq \xi^A(x^\#, c^\#; \rho_2), \tag{3.15}$$

because the ascending paths lie in D^A, $x^s(\rho_1) > x^s(\rho_2)$ and

$$\lim_{t\to\infty} c_1^A(t) = f(x^s(\rho_1)) > f(x^s(\rho_2)) = \lim_{t\to\infty} c_2^A(t).$$

But (3.15) contradicts (3.14). (ii) Assume $c_1^{D*}(0) \geq c_2^{D*}(0)$. Let $C_1^*(x)$ and $C_2^*(x)$ be the associated optimal policy, i.e., $C_i^*(x_i^{D*}(t)) = c_i^{D*}(t)$ $(i = 1, 2)$ for all $t \geq 0$. By (3.14), $C_1^*(x) > C_2^*(x)$ holds for all $x \in (0, x_1^D(0))$. Take any $y \in (0, x_1^{D*}(0))$ and consider a ρ_2-$x - c$ path from $(y, C_1^*(y))$. Denote by $C_2^D(x)$ the associated ρ_2-policy function. By (3.14),

$$C_1^*(x) > C_2^D(x) > C_2^*(x) > f(x)$$

holds for all $x \in (0, y)$. Also

$$0 = \lim_{x\to 0} C_1^*(x) \geq \lim_{x\to 0} C_2^D(x) \geq \lim_{x\to 0} C_2^*(x) = 0$$

holds. Therefore, the ρ_2-$x - c$ path from $(y, C_1^*(y))$ converges to the origin. Then, by Lemmas 3.2 - 3.4, the ρ_2-$x - c$ path dominates $(x_2^{D*}(t), c_2^{D*}(t))$, which is a contradiction. ∎

Figure 2 illustrates the ρ_1- and ρ_2-ascending and -descending paths indicated by this lemma.

<Figure 2>

Lemma 3.6 *Let $x_0 > 0$ and ρ_1, ρ_2 satisfy $\rho_0 < \rho_1 < \rho_2$. (i) If the ρ_2-ascending path from x_0 exists, then the ρ_1-ascending path from x_0 exists. (ii) If the ρ_1-descending path from x_0 exists, then the ρ_2-descending path from x_0 exists.*

Proof. (i) Let $\underline{x}_i^A$ be the infimum of the capital stocks from which a ρ_i-ascending path $(x_i^A(t), c_i^A(t))$ starts. Given a ρ_i-ascending path exists if and only if $x_i^A(0) \in (\underline{x}_i^A, x^s(\rho_i))$, the statement is equivalent to $\underline{x}_1^A \leq \underline{x}_2^A$. Suppose that it is false, i.e., $\underline{x}_1^A > \underline{x}_2^A$. Then there is a ρ_2-ascending path $(x_2^A(t), c_2^A(t))$ such that $x_2^A(\tau) = \underline{x}_1^A$ and $c_2^A(\tau) < f(\underline{x}_1^A)$ for some $\tau \in \mathbb{R}$. On the other hand, a ρ_1-ascending path satisfies

$c_1^A(t) \to f(\underline{x}_1^A)$ as $x_1^A(t) \to \underline{x}_1^A$. Then, with a sufficiently small positive number ε, we have ρ_1- and ρ_2-ascending paths from $x_0 = \underline{x}_1^A + \varepsilon$ such that $c_1^A(0) > c_2^A(0)$. But this inequality contradicts Lemma 3.5. (ii) Similarly, let $\bar{x}_i^D$ be the supremum of the capital stocks from which a ρ_i-descending path $(x_i^D(t), c_i^D(t))$ starts. For $i = 1, 2$, by Proposition 3.3 (iv), ρ_i-optimal and descending path $(x_i^{D*}(t), c_i^{D*}(t))$ exists. Although there may be other ρ_i-descending paths, the optimal path is furthest away from the nullcline $c = f(x)$ by Lemmas 3.2 - 3.4. Thus $\bar{x}_i^D$ is the supremum of the capital stock when the optimal and descending path is extended towards the past. We denote this descending path by $(x_i^D(t), c_i^D(t))$. Note that there is time τ_i $(i = 1, 2)$ such that, for $t \geq \tau_i$, $(x_i^D(t), c_i^D(t)) = (x_i^{D*}(t - \tau_i), c_i^{D*}(t - \tau_i))$. Now we assume $\bar{x}_1^D > \bar{x}_2^D$ to derive a contradiction. A ρ_i-descending path exists if and only if $x_i^D(0) \in (0, \bar{x}_i^D)$ and $c_i^D(t) \to f(\bar{x}_i^D)$ as $x_i^D(t) \to \bar{x}_i^D$. Thus $\bar{x}_1^D > \bar{x}_2^D$ implies that when the initial stock is $\bar{x}_2^D + \varepsilon$ for a sufficient small positive number ε, there exist ρ_i-descending paths $(x_i^D(t), c_i^D(t))$, $i = 1, 2$, from $\bar{x}_2^D + \varepsilon$ such that $c_1^D(0) > c_2^D(0)$. By a similar argument in the proof of Lemma 3.5 (ii), the two descending paths never cross. Therefore, we have $c_1^D(\tau_1) > c_2^D(\tau_2)$, and thus $c_1^{D*}(0) > c_2^{D*}(0)$. But this contradicts Lemma 3.5. ■

Then we have the following lemma.

Lemma 3.7 *If $x^s(\rho')$ is not an optimal steady state, then for all $\rho > \rho'$, $x^s(\rho)$ is not an optimal steady state.*

Proof. If $x^s(\rho')$ is not an optimal steady state, the ρ'-optimal path from $x^s(\rho')$ is a ρ'-descending path by Proposition 3.2. For $\rho > \rho'$, by Lemma 3.6, the ρ-descending path $\left(x^D(t;\rho), c^D(t;\rho)\right)$ from $x^D(0;\rho) = x^s(\rho')$ exists. Given $x^s(\rho) < x^s(\rho')$, there is time $\tau > 0$ such that $x^D(\tau;\rho) = x^s(\rho)$. As the path is descending, $c^D(\tau;\rho) > f(x^s(\rho))$ holds. Then, by Lemmas 3.2 - 3.4,

$$H^*(x^s(\rho), u'\left[c^D(\tau;\rho)\right])/\rho > H^*(x^s(\rho), u'\left[f(x^s(\rho))\right]/\rho = u([f(x^s(\rho))]/\rho. \tag{3.16}$$

This shows that the ρ-descending path dominates the path staying at $x^s(\rho)$. That is, $x^s(\rho)$ is not an optimal steady state. ■

This lemma implies the following corollary.

Corollary 3.1 *There exists a discount rate* $\tilde{\rho} \in [\hat{\rho}, \rho_I]$ *such that if* $\rho < \tilde{\rho}$*, then* $x^s(\rho)$ *is an optimal steady state and if* $\rho > \tilde{\rho}$*, then* $x^s(\rho)$ *is not an optimal steady state.*

Proof. Lemma 3.7 implies the existence of $\tilde{\rho}$. By Proposition 3.3, $x^s(\rho)$ is an optimal steady state if $\rho \leq \hat{\rho}$. Therefore, $\hat{\rho} \leq \tilde{\rho}$. Also, $\tilde{\rho} \leq \rho_I$, given $x^s(\rho)$ does not exist for $\rho > \rho_I$. ■

We show the last preliminary result. Take a compact state space $[0, \bar{x}]$, where $\bar{x}$ satisfies $0 < f'(\bar{x}) < \rho_0$. Then, on this state space, define two value functions associated with the ascending and descending paths. Note that if $\rho \geq \rho_0$, an optimal capital path from $x \in [0, \bar{x}]$ stay in the state space by Proposition 3.2.

The ascending value function $V^A : [0, \bar{x}] \mapsto \mathbb{R} \cup \{-\infty\}$ is defined as:

$$V^A(x) \equiv \max \int_0^\infty u(c(t))e^{-\rho t} dt \tag{3.17}$$

$$\text{subject to } \dot{x}(t) = f(x(t)) - c(t),\ c(t) \in [0, f(x(t))],\ x \in (0, \bar{x}] \text{ given.}$$

Note that given $f(x) \leq f(\bar{x})$, $c(t)$ and thus $\dot{x}(t)$ are uniformly bounded. Then, by d'Albis, Gourdel, and Le Van's (2008, Theorem 1), an optimal path to (3.17) exists, and V^A is well defined.

We then define the descending value function $V^D : [0, \bar{x}] \mapsto \mathbb{R} \cup \{-\infty\}$. Let

$$\bar{c} \equiv \sup_{\rho \in [\rho_0, \rho_I], x \in (0, \bar{x}], t \in [0, \infty)} c^*(t; x, \rho) < \infty, \tag{3.18}$$

where $c^*(t; x, \rho)$ is the ρ-optimal consumption path from x. The finiteness of $\bar{c}$ is verified as follows. For all $\rho > 0$, any optimal capital path from $x \in (0, \bar{x}]$ stays in $(0, \bar{x}]$. Thus

$$\sup_{x \in (0, \bar{x}], t \in [0, \infty)} c^*(t; x, \rho) = \sup_{x \in (0, \bar{x}]} c^*(0; x; \rho),$$

for all $\rho > 0$. Take ρ' such that $\rho' > \rho_I$. $c^*(t; x, \rho')$ is an optimal and descending path, and by Lemma 3.5, $c^*(0; x; \rho) \leq c^*(0; x; \rho')$ holds for all $\rho \geq \rho_0$ and for all $x \in (0, \bar{x}]$. (Note that if the ρ-optimal path from x

is ascending, then $c^*(0;x;\rho) < f(x) < c^*(0;x;\rho').$) Therefore,

$$\bar{c} = \sup_{\rho \in [\rho_0, \rho_I], x \in (0,\bar{x}]} c^*(0;x;\rho) \leq \sup_{x \in (0.\bar{x}]} c^*(0;x;\rho') = \sup_{t \in [0,\infty)} c^*(t;\bar{x},\rho').$$

Given $c^*(t;\bar{x},\rho')$ is a solution of the $x - c$ system, it is continuous in t. Also it converges to 0. From these, $\sup_{t \in [0,\infty)} c^*(t;\bar{x},\rho') < \infty$ and thus $\bar{c} < \infty$ hold.

Using $\bar{c}$, we define the descending value function V^D by:

$$V^D(x) \equiv \max \int_0^\infty u(c(t))e^{-\rho t}dt \tag{3.19}$$
$$\text{subject to } \dot{x}(t) = f(x(t)) - c(t),\ c(t) \in [f(x(t)), \bar{c}],\ x \in (0, \bar{x}] \text{ given.}$$

As $\dot{x}(t)$ are uniformly bounded, by d'Albis, Gourdel, and Le Van's (2008, Theorem 1), an optimal path to (3.19) exists and V^D is well defined. Note that by the monotonicity of an optimal capital path, it holds that for $x \in [0, \bar{x}]$,

$$V^*(x) = \max\{V^A(x), V^D(x)\}.$$

In what follows, we show that these value functions are continuous in the discount rate. Let us express them as the functions of ρ: e.g., $V^*(x,\rho)$.

Lemma 3.8 *For each $x \in (0, \bar{x}]$, the three value functions, $V^*(x,\rho), V^A(x,\rho), V^D(x,\rho)$, are continuous in $\rho \in [\rho_0, \rho_I]$.*

Proof. We modify the original problem (2.1) by adding the constraint $c(t) \leq \bar{c}$, where $\bar{c}$ is defined by (3.18). Obviously, this modification does not affect an optimal path. We then standardize the utility function as $\bar{u}(c) \equiv u(c) - u(\bar{c})$. By doing so, the values of these value functions become nonpositive. Fix $x > 0$. We first consider the optimal value function $\bar{V}^*$, where the "bar" means that it is standardized. If

$\rho_1 < \rho_2$,

$$\begin{aligned}\bar{V}^*(x,\rho_2) &= \int_0^\infty \bar{u}(c^*(t,\rho_2))e^{-\rho_2 t}dt \geq \int_0^\infty \bar{u}(c^*(t,\rho_1))e^{-\rho_2 t}dt \\ &\geq \int_0^\infty \bar{u}(c^*(t,\rho_1))e^{-\rho_1 t}dt = \bar{V}^*(x,\rho_1) \\ &\geq \int_0^\infty \bar{u}(c^*(t,\rho_2))e^{-\rho_1 t}dt, \end{aligned} \tag{3.20}$$

where the inequalities in the first and third lines follow from the optimality of the ρ_i-consumption paths $c^*(t,\rho_i)$ for $i = 2, 1$, respectively, and the inequality in the second line follows from the nonpositivity of $\bar{u}(c)$, which results in $\bar{V}^*(x,\rho_2) \geq \bar{V}^*(x,\rho_1)$. Select an arbitrarily $\rho' \in (0,\rho_I)$. For $\rho \in (\rho',\rho_I]$, using the inequalities in (3.20), we have

$$\begin{aligned}\lim_{\rho\searrow\rho'} |\bar{V}^*(x,\rho) - \bar{V}^*(x,\rho')| &= \lim_{\rho\searrow\rho'} \left[\int_0^\infty \bar{u}(c^*(t,\rho))e^{-\rho t}dt - \int_0^\infty \bar{u}(c^*(t,\rho'))e^{-\rho' t}dt\right] \\ &\leq \lim_{\rho\searrow\rho'} \left[\int_0^\infty \bar{u}(c^*(t,\rho))e^{-\rho t}dt - \int_0^\infty \bar{u}(c^*(t,\rho))e^{-\rho' t}dt\right] \qquad (3.21) \\ &= \lim_{\rho\searrow\rho'} \int_0^\infty \bar{u}(c^*(t,\rho))(e^{-\rho t} - e^{-\rho' t})dt = 0. \qquad (3.22)\end{aligned}$$

Similarly, for $\rho \in (0,\rho')$,

$$\begin{aligned}\lim_{\rho\nearrow\rho'} |\bar{V}^*(x,\rho) - \bar{V}^*(x,\rho')| &= \lim_{\rho\nearrow\rho} \left[\int_0^\infty \bar{u}(c^*(t,\rho'))e^{-\rho' t}dt - \int_0^\infty \bar{u}(c^*(t,\rho))e^{-\rho t}dt\right] \\ &\leq \lim_{\rho\nearrow\rho} \left[\int_0^\infty \bar{u}(c^*(t,\rho'))e^{-\rho' t}dt - \int_0^\infty \bar{u}(c^*(t,\rho'))e^{-\rho t}dt\right] \qquad (3.23) \\ &= \lim_{\rho\nearrow\rho} \int_0^\infty \bar{u}(c^*(t,\rho'))(e^{-\rho' t} - e^{-\rho t})dt = 0. \qquad (3.24)\end{aligned}$$

These results verify that $\bar{V}^*(x,\rho)$ is continuous in ρ'. As we have derived this result by using only the negativity of $\bar{u}$ and the optimality of the consumption paths, the same argument is applicable to the ascending and descending value functions, which concludes these value functions are also continuous in $\rho \in [\rho_0,\rho_I]$. ■

4 Critical capital stock

In this section, we investigate the properties of the critical capital stock, which we first define.

Definition 4.1 *The critical capital stock x^C is a positive capital stock such that every optimal capital path from $x < x^C$ converges to* 0 *and every optimal capital path from $x > x^C$ converges to the upper stationary capital x^s.*

4.1 The existence and monotonicity

Theorem 4.1 *(Existence 1) Assume $\rho > \rho_0$. (i) The critical capital stock $x^C(\rho)$ exists if and only if the upper stationary capital $x^s(\rho)$ is an optimal steady state. (ii) x^C is unique and lies in $(0, x^s(\rho)]$.*

Proof. (i) If part: By the monotonicity of the optimal capital paths (Proposition 3.2), there are intervals Y and Z on the state space $[0, \infty)$ such that every optimal capital path from $y \in Y$ converges to 0, and every optimal capital path from $z \in Z$ converges to x^s. Given $x^s(\rho)$ is an optimal steady state, $x^s(\rho) \in Z$, and thus Z is nonempty. By Proposition 3.3 (iv), $\rho > \rho_0$ implies $\sup Y > 0$. Given an optimal capital path is monotonic, we have $\sup Y \leq \inf Z$. But $\sup Y < \inf Z$ implies that any $x \in (\sup Y, \inf Z)$ is an optimal steady state and must satisfy $f'(x) = \rho$, which contradicts Assumption 2 (b). Therefore there exists a critical capital stock $x^C(\rho) = \sup Y = \inf Z$. Only if part: If $x^s(\rho)$ is not an optimal steady state, every nonconstant optimal path is a descending path and converges to 0 by Proposition 3.2. Thus $x^C(\rho)$ does not exist. (ii) If x^C exists, $x^s(\rho)$ is an optimal steady state. Then, as shown in the above proof for the sufficiency,

$$0 < \sup Y = x^C = \inf Z \leq x^s(\rho)$$

holds. That is, $x^C(\rho)$ is unique and lies in $(0, x^s(\rho)]$. ■

The following theorem shows that, except for a special case that the lower stationary capital is optimal steady state, there are two optimal paths from the critical capital stock.

Theorem 4.2 *(Relation with the lower stationary capital) (i) If x_s is an optimal steady state, then $x^C = x_s$. (ii) If x_s is not an optimal steady state, then there are two optimal paths from x^C: One converges to*

x^s *and the other converges to* 0.

Proof. (i) We prove by contradiction. Assume $x^C < x_s$. Then there is the ascending and optimal path $(x^{A*}(t), c^{A*}(t))$ from $x^{A*}(0) = x_s$. The initial consumption $c^{A*}(0)$ should satisfy $f(x_s) > c^{A*}(0)$. The total utility $V^*(x_s)$ satisfies

$$V^*(x_s) = H^*(x_s, u'(c^{A*}(0))/\rho > H^*(x_s, u'(f(c_s))/\rho = u((f(c_s))/\rho,$$

where the first equality follows from Lemma 3.3 and the inequality follows from Lemma 3.2. But as x_s is an optimal steady state, $V^*(x_s) = u((f(c_s))/\rho$ must hold, which is a contradiction. By a parallel argument, we can exclude the case of $x^C > x_s$. (ii) Given x^C is not an optimal steady state, there is an optimal path from x^C that is not constant. Assume that it is an ascending path and denote it by $(x^{A*}(t), c^{A*}(t))$ with $x^{A*}(0) = x^C$. As $c^{A*}(0) < f(x^C)$, we can extend the path for $t \in [-\varepsilon, 0)$ where ε is a small positive number. By the definition of x^C, there is an optimal and descending path from $x^{A*}(-\varepsilon)$. We denote it $(x^{D*}(t), c^{D*}(t))$ with $x^{D*}(0) = x^{A*}(-\varepsilon)$. Given $V^*(x^{D*}(0)) = V^D(x^{D*}(0)) > V^A(x^{D*}(0))$, by Lemmas 3.3 and 3.4,

$$H^*(x^{D*}(0), u'(c^{D*}(0))) > H^*(x^{A*}(-\varepsilon), u'(c^{A*}(-\varepsilon)))$$

holds. As $\varepsilon \to 0$, $x^{D*}(0) = x^{A*}(-\varepsilon) \to x^C$ and we have

$$H^*(x^C, u'(c^{D*}(0))) \geq H^*(x^C, u'(c^{A*}(0))).$$

Given an optimal path from x^C is ascending, we have

$$H^*(x^C, u'(c^{D*}(0))) = H^*(x^C, u'(c^{A*}(0))).$$

Thus, by Lemmas 3.3 and 3.4, the descending path is also an optimal path from x^C. Obviously, we obtain the same result if we assume that there is an optimal and descending path from x^C. ■

We show the monotonicity result.

Theorem 4.3 *(Monotonicity with respect to the discount rate) Let ρ_1 and ρ_2 satisfy $\rho_0 < \rho_1 < \rho_2 < \tilde{\rho}$, where $\tilde{\rho}$ is the upper discount rate defined in Corollary 3.1. Then*

$$x^C(\rho_1) < x^C(\rho_2)$$

Proof. We consider two cases: (i) $x_s(\rho_1)$ is an optimal steady state. (ii) Otherwise. (i) Note first $x^C(\rho_1) = x_s(\rho_1)$ by Theorem 4.2. Then suppose $x^C(\rho_1) \geq x^C(\rho_2)$ to derive a contradiction. Then $x^C(\rho_2) \neq x_s(\rho_2)$, which implies that there is ρ_2-ascending path from $x^C(\rho_2)$. By Lemma 3.6, then there is the ρ_1-ascending path from $x^C(\rho_2)$. However, as $x^C(\rho_1) = x_s(\rho_1)$, there is no ρ_1-ascending path from $x \leq x^C(\rho_1)$, which is a contradiction. (ii) In this case, by Theorem 4.2, we have the ρ_1-optimal ascending and descending paths $\left(x_1^{A*}(t), c_1^{A*}(t)\right)$, $\left(x_1^{D*}(t), c_1^{D*}(t)\right)$ from $x_1^{A*}(0) = x_1^{D*}(0) = x^C(\rho_1)$. Then by Lemma 3.6, there is the ρ_2-descending path $\left(x_2^D(t), c_2^D(t)\right)$ from $x_2^D(0) = x^C(\rho_1)$. If the ρ_2-ascending path $\left(x_2^A(t), c_2^A(t)\right)$ from $x^C(\rho_1)$ does not exist, then $x^C(\rho_1) < x^C(\rho_2)$. Thus, we proceed the proof assuming that it exists. By Lemma 3.5,

$$c_1^{A*}(0) < c_2^A(0) < f(x^C(\rho_1)) < c_1^{D*}(0) < c_2^D(0).$$

Then, by Lemma 3.2, we have

$$\begin{aligned} H^*(x^C(\rho_1), u'(c_2^A(0))) &< H^*(x^C(\rho_1), u'(c_1^{A*}(0))) \\ &= H^*(x^C(\rho_1), u'(c_1^{D*}(0))) < H^*(x^C(\rho_1), u'(c_2^D(0))), \end{aligned}$$

where the equality follows from the fact that at x^C, both $\left(x_1^{A*}(t), c_1^{A*}(t)\right)$ and $\left(x_1^{D*}(t), c_1^{D*}(t)\right)$ are optimal and the value of the maximized Hamiltonian is proportional to the total utility by Lemmas 3.3 and 3.4. However, by the same Lemmas, $H^*(x^C(\rho_1), u'(c_2^A(0))) < H^*(x^C(\rho_1), u'(c_2^D(0)))$ implies that the descending path $\left(x_2^D(t), c_2^D(t)\right)$ dominates the ascending path $\left(x_2^A(t), c_2^A(t)\right)$. From this, $x^C(\rho_1) < x^C(\rho_2)$ follows. ■

Applying this monotonicity theorem, we can complete the existence result of the optimality of the upper steady state, and this, in turn, together with Theorem 4.1, extends the existence result of the critical capital stock.

Proposition 4.1 *(The existence of an optimal upper steady state) There is the boundary value of the discount rate $\tilde{\rho} \in [\hat{\rho}, \rho_I]$ such that $x^s(\rho)$ is an optimal steady state if and only if $\rho \leq \tilde{\rho}$.*

Proof. If we prove that $x^s(\tilde{\rho})$ is an optimal steady state, then the statement follows from Corollary 3.1. We prove it by contradiction. Suppose that $x^s(\tilde{\rho})$ is not an optimal steady state. Then, by Proposition 3.2, an optimal capital path from $x^s(\tilde{\rho})$ converges to 0. Thus,

$$V^A(x^s(\tilde{\rho}), \tilde{\rho}) < V^D(x^s(\tilde{\rho}), \tilde{\rho}) = V^*(x^s(\tilde{\rho}), \tilde{\rho}). \tag{4.1}$$

On the other hand, as shown below, for all $\rho \in (\rho_0, \tilde{\rho})$, $x^C(\rho) < x^s(\tilde{\rho}) < x^s(\rho)$ holds. Then, an optimal path from $x^s(\tilde{\rho})$ is ascending when $\rho \in (\rho_0, \tilde{\rho})$, and we have:

$$V^D(x^s(\tilde{\rho}), \rho) < V^A(x^s(\tilde{\rho}), \rho) = V^*(x^s(\tilde{\rho}), \rho). \tag{4.2}$$

These value functions are continuous in ρ by Lemma 3.8. By taking a limit $\rho \nearrow \tilde{\rho}$, we have $V^D(x^s(\tilde{\rho}), \rho) \leq V^A(x^s(\tilde{\rho}), \rho)$, which contradicts (4.1). To complete the proof, we show that $x^C(\rho) < x^s(\tilde{\rho})$ for all $\rho \in (\rho_0, \tilde{\rho})$. Suppose otherwise, i.e., $x^C(\rho') \geq x^s(\tilde{\rho})$ for some $\rho' \in (\rho_0, \tilde{\rho})$. Then, by Corollary 3.1 and as $x^s(\rho)$ is continuous and strictly decreasing, there is $\rho'' \in (\rho', \tilde{\rho})$ such that $x^s(\rho'')$ is an optimal steady state and satisfies $x^s(\rho'') = x^C(\rho') < x^C(\rho'')$, where the inequality follows from Theorem 4.3. But this contradicts Theorem 4.1 (ii). ■

Theorem 4.4 *(Existence 2) x^C exists if and only if $\rho \in (\rho_0, \tilde{\rho}]$, where $\tilde{\rho}$ is defined in Proposition 4.1.*

Proof. It follows from Proposition 3.3 (i), Theorem 4.1 and Proposition 4.1. ■

4.2 Continuity and location

In this subsection, we first show the continuity of the critical capital stock in the discount rate and then some results on the location of the critical capital stock.

Theorem 4.5 *(Continuity) $x^C(\rho)$ is continuous on $(\rho_0, \tilde{\rho}]$, where $\tilde{\rho}$ is defined in Proposition 4.1.*

Proof. We prove by contradiction. Suppose that there is $\rho' \in (\rho_0, \tilde{\rho}]$ at which $x^C(\rho)$ is discontinuous. By Theorem 4.3, this is the case that (a) $\lim_{\rho \nearrow \rho'} x^C(\rho) < x^C(\rho')$ and/or (b) $\lim_{\rho \searrow \rho'} x^C(\rho) > x^C(\rho')$. Suppose that (a) occurs. Let $z \in (\lim_{\rho \nearrow \rho'} x^C(\rho), x^C(\rho'))$. Given $z < x^C(\rho')$,

$$V^A(z, \rho') < V^D(z, \rho') = V^*(z, \rho'). \tag{4.3}$$

Similarly, as $z > x^C(\rho)$ for $\rho < \rho'$, $V^A(z, \rho) > V^D(z, \rho)$ holds. Given these functions are continuous in ρ by Lemma 3.8, we have $V^A(z, \rho') \geq V^D(z, \rho')$ at the limit $\rho \nearrow \rho'$. However, this contradicts (4.3). Thus, case (a) is ruled out. By a parallel argument, case (b) is also ruled out. ■

$\breve{x}$ and $\underline{x}$ below are defined in (3.4) and (3.5), respectively. See also Figure 1.

Theorem 4.6 *(Location of the critical capital stock) Assume $\rho \in (\rho_0, \tilde{\rho}]$. (i) If $\rho \in (\rho_0, \hat{\rho}]$, then $x^C(\rho) \leq \breve{x}(\rho)$. (ii) If $\rho \in (\hat{\rho}, \tilde{\rho})$, then $x^C(\rho) \geq \underline{x}(\rho)$. (iii) $\lim_{\rho \searrow \rho_0} x^C(\rho) = 0$. (iv) $x^C(\tilde{\rho}) = x^s(\tilde{\rho})$.*

Proof. If $\rho \in (\rho_0, \hat{\rho}]$, $\breve{x}$ exists. Given $\tilde{\rho} \leq \rho_I$, if $\rho \in (\hat{\rho}, \tilde{\rho})$, $\underline{x}$ exists. Therefore, (i) and (ii) follow from Proposition 3.3 (ii) and (iii) and Theorem 4.1. (iii) Given $x^C(\rho) \leq \breve{x}(\rho)$ (Theorem 4.2 (i)) and $\lim_{\rho \searrow \rho_0} \breve{x}(\rho) = 0$, $\lim_{\rho \searrow \rho_0} x^C(\rho) = 0$ holds. (iv) By Theorem 4.1 (ii), $x^C(\tilde{\rho}) \leq x^s(\tilde{\rho})$. Suppose $x^C(\tilde{\rho}) < x^s(\tilde{\rho})$ to derive a contradiction. Take $y \in (x^C(\tilde{\rho}), x^s(\tilde{\rho}))$. Then, for $\rho < \tilde{\rho}$, we have $V^D(y, \rho) < V^A(y, \rho) = V^*(y, \rho)$. By the continuity of these functions in ρ, we have at the limit of $\rho \nearrow \tilde{\rho}$,

$$V^D(y, \tilde{\rho}) \leq V^A(y, \tilde{\rho}) \tag{4.4}$$

On the other hand, for $\rho > \tilde{\rho}$, as $x^s(\rho)$ is not an optimal steady state, every optimal capital path converges to 0. Thus, for $\rho > \tilde{\rho}$, $V^A(y,\rho) < V^D(y,\rho) = V^*(y,\rho)$ holds. Take the limit of $\rho \searrow \tilde{\rho}$, and we have

$$V^A(y,\tilde{\rho}) \leq V^D(y,\tilde{\rho}). \tag{4.5}$$

From (4.4) and (4.5), we have $V^A(y,\tilde{\rho}) = V^D(y,\tilde{\rho})$. This is the case that any $y \in (\tilde{x}^C, x^s(\tilde{\rho}))$ is a critical capital stock, which contradicts the uniqueness of the critical capital stock (Theorem 4.1 (ii)). ■

The critical capital stock as a function of the discount rate is strictly increasing and continuous. It increases from 0 to $x^s(\tilde{\rho})$ as the discount rate increases. Therefore, at some discount rate in $(\rho_0, \tilde{\rho}]$, the critical capital stock coincides with the lower stationary capital: $x^C = x_s$. However, this does not necessarily imply that x_s is an optimal steady state, as shown in the following proposition.[8]

Proposition 4.2 *x_s is not an optimal steady state if*

$$\rho^2 < 4f''(x_s)c_s/\sigma\left(c_s\right). \tag{4.6}$$

Proof. We prove by contradiction. Suppose (4.6) holds, but x_s is an optimal steady state. By Theorem 4.2 (i), $x^C = x_s$. Then, for any $x \in (x_s, x^s)$, an optimal path $(x^*(t), c^*(t))$ from x is ascending. From the eigenvalues (2.10), the inequality in (4.6) implies that (x_s, c_s) is the spiral source. Thus, there is t' such that $x^*(t') = x_s$ and $c^*(t') < f(x_s)$. However, this implies that x_s is not an optimal steady state by Lemmas 3.2 and 3.3, which is a contradiction. ■

Remark: Askenazy and Le Van (1999, Proposition 10) derive another condition under which x_s is not an optimal steady state:

$$\rho^2 < f''(x_s)c_s/\sigma(c_s). \tag{4.7}$$

Given

$$4f''(x_s)c_s/\sigma(c_s) > f''(x_s)c_s/\sigma(c_s),$$

[8] On the other hand, there is an example that the lower stationary capital can be an optimal steady state. See Akao, Kamihigashi and Nishimura (2015).

if (4.7) holds, then (4.6) also holds. In this sense, our sufficient condition is weaker than (4.7).

Figure 3 illustrates numerical examples with the production function:

$$f(x) = 10^{-3} \ln(x+1) + \frac{x^2}{4(x^2+1)},$$

for which $x_I = 0.5768$, $\hat{x} = 0.9985$, $\rho_0 = 10^{-3}$, $\hat{\rho} = 0.1257$, and $\rho_I = 0.1630$. The utility function is a constant intertemporal elasticity of substitution (CIES) type: $u(c) = \left(c^{1-\sigma} - 1\right)/(1-\sigma)$, $\sigma > 0$. Panel (a) depicts the production function. Panels (b-1) and (b-2) are the case when σ is 0.7. (b-1) depicts the phase diagram when the discount rate coincides with the maximum average productivity ($\rho = \hat{\rho}$). The two vertical lines show the nullclines of $\dot{c} = 0$ that locate at the lower and upper stationary capitals. In this case, by Theorem 4.4, the critical capital stock exists. In the panel, the critical capital stock is near to or slightly greater than the lower stationary capital. (b-2) is the case when the discount rate coincides with the maximum marginal productivity ($\rho = \rho_I$). The vertical line is at the inflection point x_I, where the two nullclines of $\dot{c} = 0$ merge. The phase diagram shows that with, or near to, this discount rate, the critical capital stock merges with the optimal steady state, which is suggested by Theorem 4.6 (iv). Panels (c-1)–(c-3) are the case that the elasticity of the marginal utility is smaller: $\sigma = 0.3$. With respect to the discount rate, (c-1) corresponds to (b-1), and (c-3) corresponds to (b-2). The difference from Panel (b) is that when $\rho = \rho_I$, there is no longer the critical capital stock. The heteroclinic bifurcation occurs with a lower discount rate, which is shown in Panel (c-2).

<Figure 3>

4.3 The upper bound of the discount rate

Theorem 4.4 shows that there is the upper bound of discount rate $\tilde{\rho} \in [\hat{\rho}, \rho_I]$ for the existence of the critical capital stock. At $\tilde{\rho}$, the critical capital stock merges to the upper stationary capital that is an optimal steady state. For $\rho > \tilde{\rho}$, there is no longer the critical capital stock or the optimal interior steady state, although the upper stationary capital may exist. Figure 3 indicates that the level of the upper bound depends on the curvature of the utility function. Intuitively, the higher the elasticity of the marginal utility, the more attractive a flat consumption path and the upper stationary capital may tend to keep being an optimal steady state against the increase in the discount rate. We show that this intuition is true. Specifically, we show that depending on the curvature of the utility function, the critical capital stock, as well as the optimal steady state, can survive even at a discount rate almost as high as ρ_I, or they can disappear even at a slightly greater discount rate than $\hat{\rho}$. To this end, this subsection assumes the CIES utility function:

Assumption 3:

$$u(c) = \begin{cases} c^{1-\sigma}/(1-\sigma) & \text{if } \sigma \neq 1 \\ \ln c & \text{if } \sigma = 1 \end{cases}. \tag{4.8}$$

We first show that if σ is sufficiently large, for any $\rho < \rho_I$, $x^s(\rho)$ is an optimal steady state, and thus the critical capital stock exists. Fix $\rho \in (\hat{\rho}, \rho_I)$. Consider the following piecewise linear production function $\tilde{f}(x)$:

$$\tilde{f}(x) = \begin{cases} \alpha x & \text{if } 0 \leq x < \underline{x} \\ \rho x - (\rho - \alpha)\underline{x} & \text{if } x \geq \underline{x} \end{cases}, \tag{4.9}$$

where $\underline{x}$ is defined in (3.5) and α is given by

$$\alpha = f(\underline{x})/\underline{x}. \tag{4.10}$$

Note that

$$0 < \alpha = \frac{f(\underline{x})}{\underline{x}} < \frac{f(\hat{x})}{\hat{x}} = \hat{\rho} < \rho. \tag{4.11}$$

As shown in Figure 4, $\tilde{f}(x) \geq f(x)$ with equality only if $x \in \{0, \underline{x}, x^s(\rho)\}$. From this inequality, if $x^s(\rho)$ is an optimal steady state to the problem:

$$\max_{c \geq 0} \int_0^\infty u(c)e^{-rt}dt \text{ subject to } \dot{x} = \tilde{f}(x) - c,\ x \geq 0,\ x(0) \text{ given}, \tag{4.12}$$

then $x^s(\rho)$ is also an optimal steady state to the problem (2.1).

<Figure 4>

Lemma 4.1 *Problem (4.12) has the following closed-form solution for the optimal consumption policy:*

$$\tilde{C}(x) = \begin{cases} \beta x & \text{if } 0 \le x \le \underline{x} \\ \beta \underline{x} & \text{if } \underline{x} < x \le \tilde{x}^C \\ \tilde{f}(x) & \text{if } \tilde{x}^C \le x \end{cases}, \tag{4.13}$$

where $\beta = (1/\sigma)(\rho - \alpha) + \alpha$ *and* $\tilde{x}^C$ *is given by:*

$$\tilde{x}^C = \left(1 + \frac{\rho - \alpha}{\sigma\rho}\right)\underline{x}. \tag{4.14}$$

Proof. We verify the optimality of $\tilde{C}(x)$ by using the Hamilton–Jacobi–Bellman equation. Denote by $T(x)$ the time to reach $\underline{x}$ from $x \in (\underline{x}, \tilde{x}^C)$. $T(x)$ satisfies:

$$e^{-\rho T(x)} = \frac{\rho x - (\rho + \beta - \alpha)\underline{x}}{\rho \underline{x} - (\rho + \beta - \alpha)\underline{x}}. \tag{4.15}$$

The value function $\tilde{V}(x)$ associated with the policy function (4.13) is given by:

$$\tilde{V}(x) = \begin{cases} \begin{cases} [\beta^{-\sigma}/(1-\sigma)]x^{1-\sigma} \text{ if } \sigma \neq 1 \\ \rho^{-1}(\ln \rho x + \rho^{-1}(\alpha - \beta)) \text{ if } \sigma = 1 \end{cases} & \text{for } 0 < x \le \underline{x} \\ \int_0^{T(x)} u(\beta\underline{x})e^{-\rho t}dt + \tilde{V}(\underline{x})e^{-\rho T(x)} & \text{for } \underline{x} < x \le \tilde{x}^C \\ u(\tilde{f}(x))/\rho & \text{for } \tilde{x}^C \le x \end{cases} \tag{4.16}$$

With $\tilde{V}(x)$ and $\tilde{C}(x)$,

$$\begin{aligned} \rho\tilde{V}(x) &= u(\tilde{C}(x)) + \tilde{V}'(x)[\tilde{f}(x) - \tilde{C}(x)] \\ &\ge u(c) + \tilde{V}'(x)[\tilde{f}(x) - c] \text{ for all } c \ge 0 \end{aligned} \tag{4.17}$$

holds for each $x > 0$. Let $(\tilde{x}(t), \tilde{c}(t))$ be a feasible path induced by the policy function (4.13). We compare this path with a candidate of optimal path $(x(t), c(t))$ from the same initial capital stock $x(0) = \tilde{x}(0)$. By Proposition 3.1, $(x(t), c(t))$ can be chosen in the class of the $x - c$ paths. (4.17) leads to:

$$\int_0^\infty u(\tilde{c}(t))e^{-\rho t}dt - \int_0^\infty u(c(t))e^{-\rho t}dt \geq \lim_{t\to\infty} e^{-\rho t}(\tilde{V}(x(t)) - \tilde{V}(\tilde{x}(t))). \tag{4.18}$$

Thus, we prove optimality if we can show that the right-hand side of (4.18) is nonnegative. This obviously holds if the utility function is bounded from below or either $x(t)$ or $\tilde{x}(t)$ does not converge to 0. In other words, we need to check the case that $\sigma \geq 1$ and either of $x(t)$ or $\tilde{x}(t)$ at least converges to 0. As the Jacobian matrix of the $x - c$ system at the origin $(0, 0)$ is

$$J = \begin{bmatrix} \alpha & -1 \\ 0 & (\alpha - \rho)/\sigma \end{bmatrix}, \tag{4.19}$$

the origin is saddle point by (4.11). The stable eigenvector of (4.19) is given by $(1, \beta)$. Assume that $(x(t), c(t))$ converges to the origin. Then, with a sufficiently small initial value $x(0)$,

$$x(t) = x(0)\exp\left[\left(\frac{\alpha - \rho}{\sigma}\right)t\right] + o(x(t)),$$

where $\lim_{x\to 0} o(x) = 0$. Then we have:

$$\lim_{t\to\infty} e^{-\rho t}\tilde{V}(x(t)) = \begin{cases} \lim_{t\to\infty} \frac{\beta^{-\sigma}x(t)^{1-\sigma}e^{-\rho t}}{1-\sigma} = \lim_{t\to\infty} \frac{\beta^{-\sigma}x(0)^{1-\sigma}e^{-\beta t}}{1-\sigma} = 0 \text{ for } \sigma \neq 1 \\ \lim_{t\to\infty} e^{-\rho t}\left[\left(\ln \rho x(0)e^{(\alpha-\beta)t}\right)/\rho + (\alpha - \beta)/\rho^2\right] = 0 \text{ for } \sigma = 1 \end{cases}. \tag{4.20}$$

Therefore, if an $x - c$ path $(x(t), c(t))$ converges to the origin, then $\lim_{t\to\infty} e^{-\rho t}\tilde{V}(x(t)) = 0$. Given $(\tilde{x}(t), \tilde{c}(t))$ is also an $x - c$ path, if it converges to the origin, we have $\lim_{t\to\infty} e^{-rt}\tilde{V}(\tilde{x}(t)) = 0$. Therefore the right-hand side of (4.18) is 0, and the proof completes. ■

Define

$$\tilde{\sigma} \equiv \frac{-\gamma(x^s)}{x_s - \underline{x}}\rho^{-1}, \quad \rho \in (\hat{\rho}, \rho_I) \tag{4.21}$$

where $\gamma(x)$ is the gain function defined in (3.1).

Proposition 4.3 *For any $\rho \in (\hat{\rho}, \rho_I)$, $x^s(\rho)$ is an optimal steady state if $\sigma \geq \tilde{\sigma}$.*

Proof. Fix $\rho \in (\hat{\rho}, \rho_I)$. Given

$$\gamma(x^s) = \gamma(\underline{x}) = \left(\frac{f(\underline{x})}{\underline{x}} - \rho \right) \underline{x} = (\alpha - \rho)\, \underline{x},$$

$$x^s = \left(1 + \frac{\rho - \alpha}{\tilde{\sigma}\rho} \right) \underline{x} \tag{4.22}$$

holds. Then for all $\sigma \geq \tilde{\sigma}$, $\tilde{x}^C \leq x^s$ by (4.14). Therefore, x^s is an optimal steady state to the problem (4.12), which implies that x^s is also an optimal steady state to the problem (2.1) and thus $\rho \leq \tilde{\rho}$. ■

Remark: $\tilde{\sigma}$ is strictly increasing in ρ with $\tilde{\sigma}(\hat{\rho}) = 0$ and $\lim_{\rho \nearrow \rho_I} \tilde{\sigma}(\rho) = \infty$.

Second, we show that for any $\rho > \hat{\rho}$, we have $\rho > \tilde{\rho}$ if σ is sufficiently small. This implies that the critical capital stock, together with the higher interior optimal steady state, may disappear, even when the discount rate is slightly greater than $\hat{\rho}$.

For the proof, we prepare the following lemma. This lemma shows that if the utility function is linear, then for any $\rho > \hat{\rho}$, the upper stationary capital x^s is not an optimal steady state. Let c^M satisfy $c^M > f(x^s)$. Let $x^M(t)$ be the capital path from x^s induced with the most rapid approach policy: $c(t) = c^M$ if $x(t) > 0$ and $c(t) = 0$ if $x(t) = 0$. Define the value function associated with the most rapid approach path by:

$$V_L(c^M) \equiv \int_0^{T^*(c^M)} c^M e^{-\rho t} dt = \int_0^{T^*(c^M)} \gamma(x^M(t)) e^{-\rho t} dt + x^s, \tag{4.23}$$

where $T^*(c^M)$ is the first time when $x^M(t) = 0$.

Lemma 4.2 *Let $\rho > \hat{\rho}$. Consider a linear utility version of the problem (2.1) with the maximum consumption $c^M > f(x^s)$:*

$$\max_{c(t)} \int_0^\infty c(t) e^{-\rho t} dt \text{ subject to } \dot{x}(t) = f(x(t)) - c(t),\ c(t) \in [0, c^M],\ x(t) \geq 0,\ x(0) = x^s \text{ given.}$$

When c^M is sufficiently large, it holds that

$$V_L(c^M) \equiv \int_0^{T^*(c^M)} c^M e^{-\rho t} dt > \int_0^{\infty} f(x^s) e^{-\rho t} dt. \tag{4.24}$$

Proof. Fix $\rho > \hat{\rho}$. We have:

$$V_L(c^M) - \int_0^{\infty} f(x^s) e^{-\rho t} dt = \int_0^{T^*(c^M)} \gamma(x^M(t)) e^{-\rho t} dt - \int_0^{\infty} \gamma(x^s) e^{-\rho t} dt. \tag{4.25}$$

It is easily verified that $T^*(c^M)$ and $V_L(c^M)$ are continuous. Given $\rho > \hat{\rho}$,

$$0 > \gamma(x^M(t)) \geq \gamma(x_s) \text{ for all } t \geq 0. \tag{4.26}$$

where x_s is the lower stationary capital. Also,

$$\int_0^{T^*(c^M)} \gamma(x_s) e^{-\rho t} dt \to 0 \text{ as } c^M \to \infty, \tag{4.27}$$

given $\lim_{c^M \to \infty} T^*(c^M) = 0$. (4.26) and (4.27) imply:

$$\int_0^{T^*(c^M)} \gamma(x^M(t)) e^{-\rho t} dt \to 0 \text{ as } c^M \to \infty. \tag{4.28}$$

On the other hand, as $\rho > \hat{\rho}$, $\gamma(x^s) < 0$. Therefore, with sufficient large c^M, (4.25) is positive. That is, (4.24) holds. ∎

Proposition 4.4 *For any $\rho > \hat{\rho}$, there is $\sigma^M(\rho) < 1$ such that $x^s(\rho)$ is not an optimal steady state if $\sigma \leq \sigma^M(\rho)$.*

Proof. Fix $\rho > \hat{\rho}$. Choose c^M so that (4.24) in Lemma 4.2 holds. Let $\delta > 0$ such that:

$$2\delta = \int_0^{T^*(c^M)} c^M e^{-\rho t} dt - \frac{f(x^s)}{\rho}$$

Also, choose a sufficiently small $\sigma^M > 0$ such that

$$\left|\int_0^{T^*(c^M)} c^M e^{-\rho t} dt - \int_0^{T^*(c^M)} u(c^M) e^{-\rho t} dt\right| = \left|\left(\int_0^{T^*(c^M)} \gamma(x^M(t)) e^{-\rho t} dt + x^s\right) - \int_0^{T^*(c^M)} \frac{\left(c^M\right)^{1-\sigma^M}}{1-\sigma^M} e^{-\rho t} dt\right| < \delta,$$

and

$$\left|\int_0^{\infty} f(x^s) e^{-\rho t} dt - \int_0^{\infty} u(f(x^s)) e^{-\rho t} dt\right| = \left|\frac{f(x^s)}{\rho} - \frac{f(x^s)^{1-\sigma^M}}{\rho(1-\sigma^M)}\right| < \delta.$$

Then,

$$\int_0^{T^*(c^M)} u(c^M) e^{-\rho t} dt - \int_0^{\infty} u(f(x^s)) e^{-\rho t} dt > \left[\left(\int_0^{T^*(c^M)} \gamma(x^M(t)) e^{-\rho t} dt + x^s\right) - \delta\right] - \left[\frac{f(x^s)}{\rho} + \delta\right] = 0. \tag{4.29}$$

Therefore, x^s is not an optimal steady state when the elasticity is σ^M. Obviously, this conclusion holds with any elasticity σ such that $\sigma < \sigma^M$, because with the elasticity, (4.24) holds. ■

5 Concluding remarks

Nonconvexity is ubiquitous in the real world. We always have the opportunity to take a path converging to a lower steady state by the reason that it is optimal as a certain criterion. However, it may be undesirable from other viewpoints such as sustainability and intergenerational equity. A theoretical inquiry on the critical capital stock should provide the basic knowledge needed to avoid such an undesirable path, as well as to understand why such a path has been experienced in history.

Appendix

Proof of Proposition 3.1

If an optimal path $x^*(t)$ satisfies $x^*(t) > 0$ for all $t \geq 0$, then by Michel (1982, Theorem), the costate variable $q^*(t) = u'\left[c^*(t)\right]$ exists for all $t \geq 0$, which implies $c^*(t) > 0$ by $\lim_{c \searrow 0} u'(c) = \infty$ in Assumption 1. Thus, we only have to prove that there is no finite extinction time $T^* = \inf\{t > 0 | x^*(t) = 0\}$. Given it is trivial in the case of $u(0) = -\infty$, we consider the case that $u(c)$ is bounded from below and assume $u(0) = 0$. Suppose that the finite extinction time $T^* < \infty$ exists. Then, there is a tuple $(c^*(t), x^*(t), T^*)$ that is a solution to a free final time problem with the constraint $x(T) \geq 0$:

$$\max_{c(t)\geq 0, T} \int_0^T u\left(c(t)\right) e^{-\rho t} dt$$

$$\text{subject to } \dot{x}(t) = f\left(x(t)\right) - c(t),\ x(T) = 0,\ x(0) = x > 0 \text{ given, } T > 0 \text{ free.}$$

By Seierstad and Sydsæter (1987, Chapter 2, p. 143, Theorem 11), at the extinction time T^*, there exists $c^*(T^*) > 0$ such that $q^*(T^*) = u'\left[c^*(T^*)\right]$ and the Hamiltonian at $t = T^*$ satisfies:

$$H(x^*(T^*), q^*(T^*)) = u(c^*(T^*)) + u'\left[c^*(T^*)\right](-c^*(T^*)) = 0. \tag{.1}$$

Given u is strictly concave and $u(0) = 0$,

$$\begin{aligned} 0 &< \left[u(c^*(T^*)) - u(0)\right] - u'(c^*(T^*))\left[c^*(T^*) - 0\right] \\ &= u(c^*(T^*)) - u'(c^*(T^*))c^*(T^*), \end{aligned}$$

which contradicts (.1). Therefore, $x^*(t) > 0$ for all $t \geq 0$.

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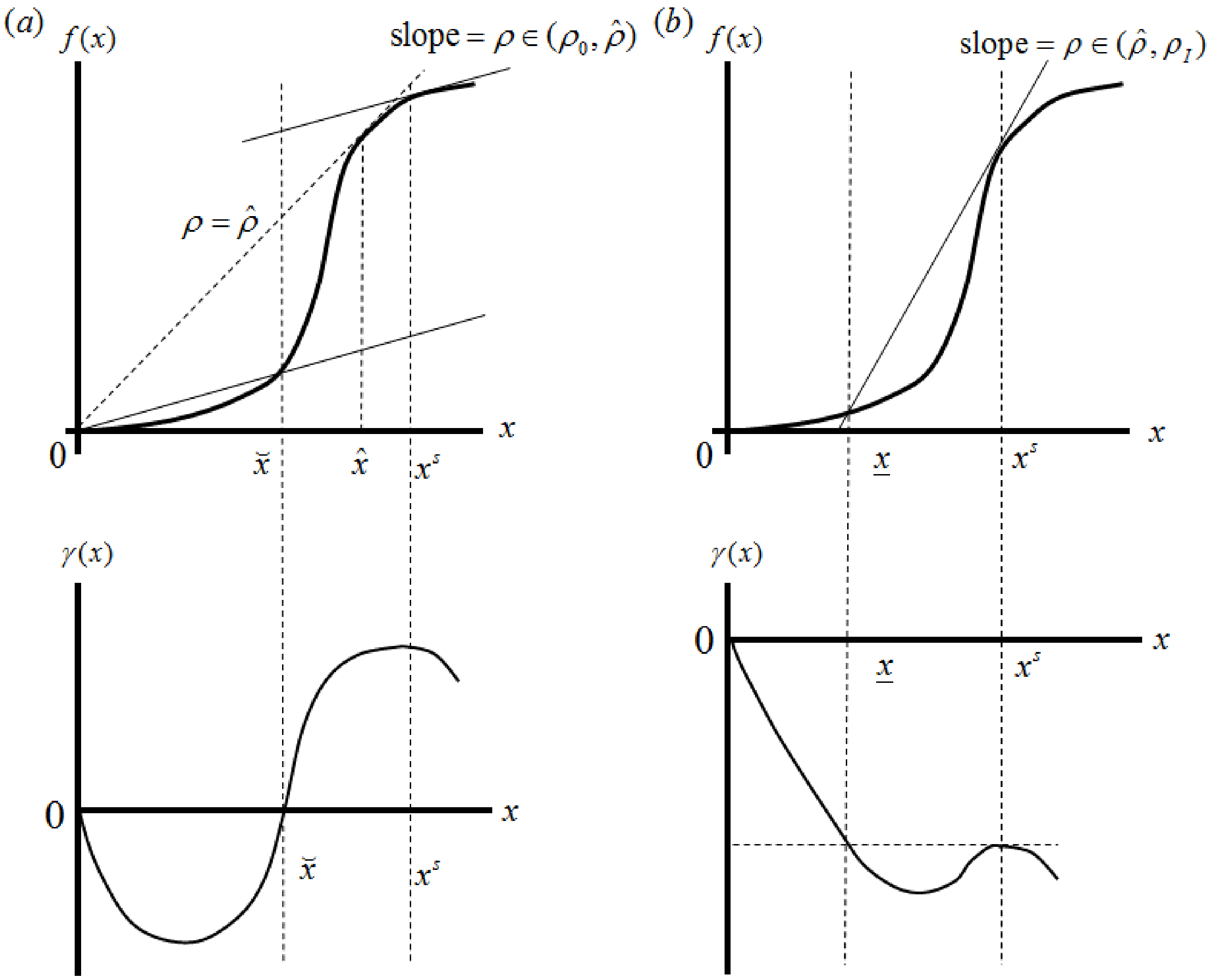

Figure 1. Production function and gain function

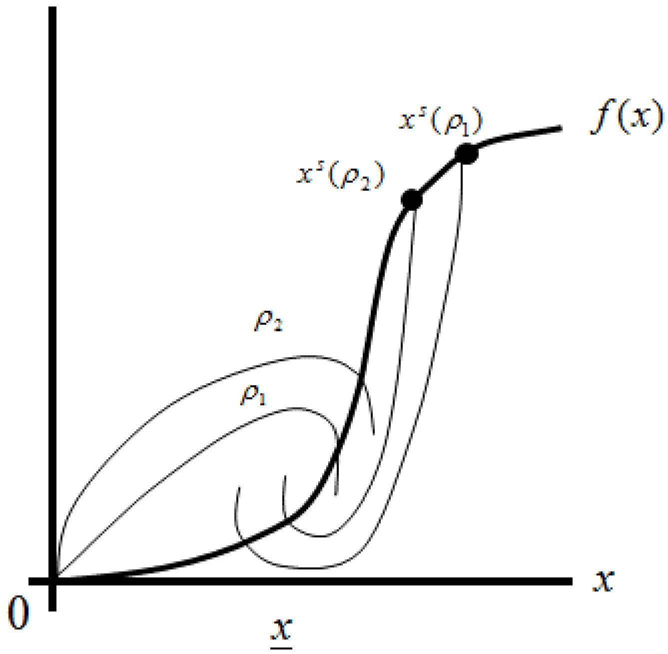

Figure 2. Ascending and descending paths when $\rho = \rho_1, \rho_2$ $(\rho_1 < \rho_2)$

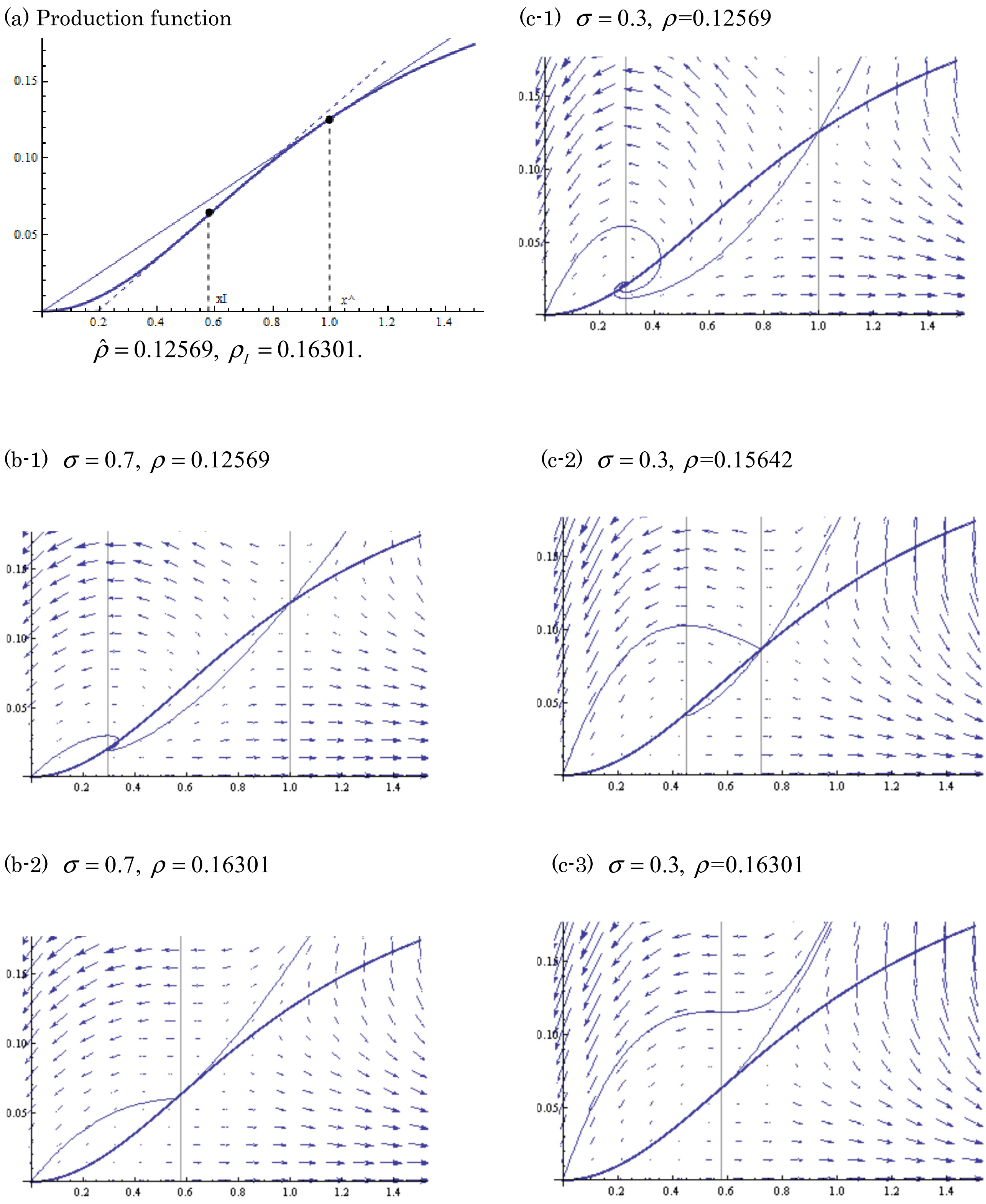

Figure 3. Numerical simulations.

Figure 3

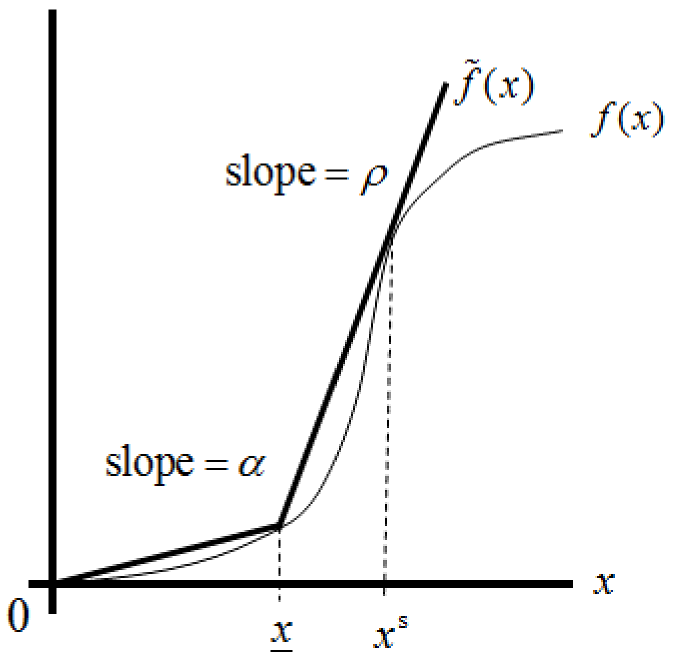

Figure 4. Piecewise linear production function