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Takashi KAMIHIGASHI

Masayuki YAO

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Research Institute for Economics and Business Administration

Kobe University

2-1 Rokkodai, Nada, Kobe 657-8501 JAPAN

Deterministic Dynamic Programming in Discrete Time: A Monotone Convergence Principle*

Takashi Kamihigashi† Masayuki Yao‡

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Abstract

We consider infinite-horizon deterministic dynamic programming problems in discrete time. We show that the value function is always a fixed point of a modified version of the Bellman operator. We also show that value iteration monotonically converges to the value function if the initial function is dominated by the value function, is mapped upward by the modified Bellman operator, and satisfies a transversality-like condition. These results require no assumption except for the general framework of infinite-horizon deterministic dynamic programming.

Keywords: Dynamic programming, Bellman operator, fixed point, value iteration.

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†RIEB, Kobe University, Rokkodai, Nada, Kobe 657-8501 Japan.
Email: tkamihig@rieb.kobe-u.ac.jp.

‡Graduate School of Economics, Keio University, 2-15-45 Mita, Minato, Tokyo 108-8345, Japan.
Email: myao@gs.econ.keio.ac.jp

1 Introduction

Infinite-horizon dynamic programming in discrete time is one of the most fundamental tools in economics. Of particular importance in dynamic programming is the convergence of the value iteration algorithm to the (true) value function. While this convergence is fairly easy to establish for models with bounded returns [1], unbounded returns are common in practice, especially in economic models. Accordingly, various results on the convergence of value iteration have been established for such models under numerous technical—especially topological—assumptions [2–9].

Recently, an order-theoretic approach that does not require topology was developed and applied to deterministic dynamic programming [10–12]. This approach can be viewed as an extension of the earlier order-theoretic approach of [13]. One of the results based on the new approach is the following [10, Theorem 2.2]: value iteration monotonically converges to the value function if the initial function is dominated by the value function, is mapped upward by the Bellman operator, and satisfies a transversality-like condition.

This result requires only two assumptions in addition to the general framework of infinite-horizon deterministic dynamic programming. First, the constraint correspondence is nonempty-valued. Second, the value function never equals $+\infty$. The second assumption ensures that the Bellman operator is well defined for any function dominated by the value function, but can be nontrivial to verify since the value function is a priori unknown.

In this paper we establish a more general result that does not require even the above assumptions. We call this result a *monotone convergence principle* since it requires no assumption except for the general framework itself. To show this principle, we follow the approach of [14] in modifying the Bellman operator in such a way that it is well defined for any function. We show that the value function is a fixed point of this modified Bellman operator. The monotone convergence principle is that value iteration monotonically converges to the value function if the initial function is dominated by the value function, is mapped upward by the modified Bellman operator, and satisfies the same transversality-like condition as in the result of [10, Theorem 2.2].

2 Dynamic Programming

Our setup closely follows those of [10, 14]. Let X be a set, and let Γ be a correspondence from X to X. Let D be the graph of Γ:

$$D = \{(x, y) \in X \times X : y \in \Gamma(x)\}. \tag{1}$$

Let $u : D \to [-\infty, \infty)$. A sequence $\{x_t\}_{t=0}^{\infty}$ in X is called a *feasible path* if $x_{t+1} \in \Gamma(x_t)$ for all $t \in \mathbb{Z}_+$. A sequence $\{x_t\}_{t=1}^{\infty}$ in X is called a *feasible path from* x_0 if the sequence $\{x_t\}_{t=0}^{\infty}$ is feasible. Let Π and $\Pi(x_0)$ denote the set of feasible paths and that of feasible paths from x_0, respectively:

$$\Pi = \{\{x_t\}_{t=0}^{\infty} \in X^{\infty} : \forall t \in \mathbb{Z}_+, x_{t+1} \in \Gamma(x_t)\}, \tag{2}$$
$$\Pi(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X^{\infty} : \{x_t\}_{t=0}^{\infty} \in \Pi\}, \qquad x_0 \in X. \tag{3}$$

Throughout the paper, we follow the convention that

$$\sup \emptyset = -\infty. \tag{4}$$

Let $\beta \geq 0$. The *value function* $v^* : X \to [-\infty, \infty]$ is defined by

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0)} \mathop{\mathrm{L}}_{T \uparrow \infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}), \qquad x_0 \in X, \tag{5}$$

where $\mathrm{L} \in \{\underline{\lim}, \overline{\lim}\}$ with $\underline{\lim} = \liminf$ and $\overline{\lim} = \limsup$. Though L can be $\underline{\lim}$ or $\overline{\lim}$, its definition is fixed for the rest of the paper. Since $u(x, y) < \infty$ for all $(x, y) \in D$, the right-hand side of (5) is well defined for any feasible path. This together with (4) means that v^* is always well defined.

Let W be the set of functions from X to $[-\infty, \infty]$. Let $V = \{v \in W : \forall x \in X, v(x) < \infty\}$. The *Bellman operator* B on V is defined by

$$(Bv)(x) = \sup_{y \in \Gamma(x)} \{u(x, y) + \beta v(y)\}, \quad x \in X. \tag{6}$$

Although Bv is well defined for any function $v \in V$, it may not be well defined for all functions in W. This is because the right-hand side of (6) is not well defined if $u(x, y) = -\infty$ and $v(y) = \infty$ for some $(x, y) \in D$. This problem and its consequences are discussed in [14].

Following [14] we avoid the above problem by slightly modifying the right-hand side of (6). For this purpose, we define

$$\check{\Gamma}(x) = \{y \in \Gamma(x) \colon u(x,y) > -\infty\}, \quad x \in X, \tag{7}$$

$$\check{\Pi} = \left\{ \{x_t\}_{t=0}^{\infty} \in \Pi : \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}) > -\infty \right\}, \tag{8}$$

$$\check{\Pi}(x_0) = \{\{x_t\}_{t=1}^{\infty} \in \Pi(x_0) : \{x_t\}_{t=0}^{T} \in \check{\Pi}\}, \qquad x_0 \in X. \tag{9}$$

Recalling (4) we see that

$$\forall x_0 \in X, \quad v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)} \mathop{\mathrm{L}}_{T\uparrow\infty} \sum_{t=0}^{T} \beta^t u(x_t, x_{t+1}). \tag{10}$$

We define the *modified Bellman operator* $\check{B}$ on W by

$$(\check{B}v)(x) = \sup_{y \in \check{\Gamma}(x)} \{u(x,y) + \beta v(y)\}, \quad x \in X. \tag{11}$$

The right-hand side above is well defined for any $v \in W$ and $x \in X$ since for any $y \in \check{\Gamma}(x)$ we have $u(x,y) \in (-\infty, \infty)$, which implies that the sum $u(x,y) + \beta v(y)$ is well defined even if $v(y) = -\infty$ or $+\infty$. The following result shows that $\check{B}$ is an extension of B to W.

Lemma 2.1. *For any $v \in V$ we have $\check{B}v = Bv$.*

Proof. Let $v \in V$ and $x \in X$. We claim that

$$\forall y \in \Gamma(x) \setminus \check{\Gamma}(x), \quad u(x,y) + \beta v(y) = -\infty. \tag{12}$$

To see this, let $y \in \Gamma(x) \setminus \check{\Gamma}(x)$. Then $u(x,y) = -\infty$. Since $v \in V$, we have $v(y) < -\infty$. Hence $u(x,y) + \beta v(y) = -\infty$; thus (12) follows.

To simplify notation, let $g(x,y) = u(x,y) + \beta v(y)$ for $y \in \Gamma(x)$. We have

$$\begin{aligned}
(Bv)(x) &= \max \left\{ \sup_{y \in \check{\Gamma}(x)} g(x,y), \sup_{y \in \Gamma(x) \setminus \check{\Gamma}(x)} g(x,y) \right\} && (13) \\
&= \max \left\{ \sup_{y \in \check{\Gamma}(x)} g(x,y), -\infty \right\} && (14) \\
&= \sup_{y \in \check{\Gamma}(x)} g(x,y) = (\check{B}v)(x), && (15)
\end{aligned}$$

where (14) uses (12). Since x was arbitrary, it follows that $Bv = \check{B}v$. $\square$

A function $v \in W$ satisfying $\check{B}v = v$ is called a *fixed point* of $\check{B}$. A fixed point of B is defined similarly. We have the following result.

Theorem 2.1. *Any fixed point of B in V is a fixed point of $\check{B}$. Furthermore, v^* is a fixed point of $\check{B}$.*

Proof. See Section 4. □

The first statement above is immediate from Lemma 2.1. The second statement uses the argument of [14, Theorem 1]. Since β is only required to be nonnegative, Theorem 2.1 applies to undiscounted problems of the type studied by [15, 16].

3 Monotone Convergence Principle

We define the partial order $\leq$ on W as follows:

$$v \leq w \quad \Leftrightarrow \quad \forall x \in X,\ v(x) \leq w(x). \tag{16}$$

It is easy to see that $\check{B}$ is order-preserving in the sense that for any $v, w \in W$,

$$v \leq w \quad \Rightarrow \quad \check{B}v \leq \check{B}w. \tag{17}$$

Given a sequence $\{v_n\}_{n\in\mathbb{N}}$ in W and a function $v \in W$, we say that $\{v_n\}_{n\in\mathbb{N}}$ *monotonically converges to v pointwise* if $\{v_n(x)\}_{n\in\mathbb{N}}$ is an increasing (i.e., nondecreasing) sequence with $\lim_{n\uparrow\infty} v_n(x) = v(x) \in [-\infty, \infty]$ for every $x \in X$. We are ready to sate what we call a *monotone convergence principle*:

Theorem 3.1. *Let $\underline{v} \in W$ satisfy*

$$\underline{v} \leq v^*, \tag{18}$$

$$\underline{v} \leq \check{B}\underline{v}. \tag{19}$$

Then the sequence $\{\check{B}^n\underline{v}\}_{n\in\mathbb{N}}$ monotonically converges to a fixed point $\underline{v}^*$ of $\check{B}$ pointwise. Furthermore, if

$$\forall \{x_t\}_{t=0}^{\infty} \in \check{\Pi}, \quad \varliminf_{t\uparrow\infty} \beta^t \underline{v}(x_t) \geq 0, \tag{20}$$

then $\underline{v}^* = v^*$; i.e., $\{\check{B}^n\underline{v}\}_{n\in\mathbb{N}}$ monotonically converges to v^* pointwise.

Proof. See Section 4. □

The results of [10, Theorem 2.2] and [12, Theorems 2, 3] easily follow from the above result; see [10, 12] for discussion of other related results in the literature.

In Section 4 we prove Theorem 3.1 by extending the proof of [12, Theorem 3]. Unlike the latter proof, we directly show the first conclusion of Theorem 3.1 without using Kleene's fixed point theorem. It is worth emphasizing that Theorem 3.1 requires no additional assumption; thus it can be regarded as a principle in deterministic dynamic programming.

4 Proofs of Theorems 2.1 and 3.1

4.1 Preliminary Result

In this subsection we state an elementary result shown in [14]. Recall from (4) that $\sup A$ is well defined for any $A \subset \overline{\mathbb{R}}$. We emphasize that none of the sets in the following result is required to be nonempty.

Lemma 4.1. *Let Y and Z be sets. Let $\Omega \subset Y \times Z$, and let $f : \Omega \to \overline{\mathbb{R}}$. For $y \in Y$ and $z \in Z$, define*

$$\Omega_y = \{z \in Z : (y, z) \in \Omega\}, \tag{21}$$

$$\Omega_z = \{y \in Y : (y, z) \in \Omega\}. \tag{22}$$

Then

$$\sup_{(y,z)\in\Omega} f(y, z) = \sup_{y\in Y} \sup_{z\in\Omega_y} f(y, z) = \sup_{z\in Z} \sup_{y\in\Omega_z} f(y, z). \tag{23}$$

Proof. See [14, Lemma 1]. □

4.2 Proof of Theorem 2.1

Let $v \in V$ be a fixed point of B. Then $v = Bv = \check{B}v$ by Lemma 2.1. Hence v is a fixed point of $\check{B}$.

To show that v^* is a fixed point of $\check{B}$, let $x_0 \in X$. Note that $\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)$ if and only if

$$u(x_0, x_1) > -\infty, \quad \operatorname*{L}_{T\uparrow\infty} \sum_{t=1}^{T} \beta^t u(x_t, x_{t+1}) > -\infty. \tag{24}$$

Therefore

$$\check{\Pi}(x_0) = \{\{x_t\}_{t=1}^{\infty} \in X \times X \times \cdots : x_1 \in \check{\Gamma}(x_0), \{x_t\}_{t=2}^{\infty} \in \check{\Pi}(x_1)\}. \tag{25}$$

We apply Lemma 4.1 with $y = x_1$, $z = \{x_t\}_{t=2}^{\infty}$, $\Omega = \check{\Pi}(x_0), Y = \check{\Gamma}(x_0)$, $Z = X \times X \times \cdots$, and $\Omega_y = \check{\Pi}(x_1)$. Note from (10) that

$$v^*(x_0) = \sup_{\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)} \left\{ u(x_0, x_1) + \operatorname*{L}_{T \uparrow \infty} \sum_{t=1}^{T} \beta^t u(x_t, x_{t+1}) \right\} \tag{26}$$

$$= \sup_{x_1 \in \check{\Gamma}(x_0)} \sup_{\{x_t\}_{t=2}^{\infty} \in \check{\Pi}(x_1)} \left\{ u(x_0, x_1) + \operatorname*{L}_{T \uparrow \infty} \sum_{t=1}^{T} \beta^t u(x_t, x_{t+1}) \right\} \tag{27}$$

$$= \sup_{x_1 \in \check{\Gamma}(x_0)} \left\{ u(x_0, x_1) + \sup_{\{x_t\}_{t=2}^{\infty} \in \check{\Pi}(x_1)} \operatorname*{L}_{T \uparrow \infty} \sum_{t=1}^{T} \beta^t u(x_t, x_{t+1}) \right\} \tag{28}$$

$$= \sup_{x_1 \in \check{\Gamma}(x_0)} \{u(x_0, x_1) + \beta v^*(x_1)\} = (\check{B} v^*)(x_0), \tag{29}$$

where (27) uses Lemma 4.1 and (25). Since x_0 was arbitrary, it follows that $\check{B} v^* = v^*$.

4.3 Proof of Theorem 3.1

We first prove two lemmas.

Lemma 4.2. *Suppose that there exists $\underline{v} \in W$ satisfying (18) and (19). Define $\underline{v}^* = \sup_{n \in \mathbb{N}} (\check{B}^n \underline{v})$, where the supremum is taken pointwise. Then $\{\check{B}^n \underline{v}\}_{n \in \mathbb{N}}$ monotonically converges to $\underline{v}^*$ pointwise. Furthermore, $\underline{v}^*$ is a fixed point of $\check{B}$.*

Proof. For $n \in \mathbb{N}$, let $v_n = \check{B}^n \underline{v}$. It follows from (17) and (19) that $\{v_n\}_{n \in \mathbb{N}}$ is an increasing sequence. Hence $\{v_n\}_{n \in \mathbb{N}}$ monotonically converges to $\underline{v}^*$ pointwise. To see that $\underline{v}^*$ is a fixed point of $\check{B}$, fix $x \in X$. Note that

$$\underline{v}^*(x) = \sup_{n \in \mathbb{N}} (\check{B} v_n)(x) = \sup_{n \in \mathbb{N}} \sup_{y \in \check{\Gamma}(x)} \{u(x, y) + \beta v_n(y)\} \tag{30}$$

$$= \sup_{y \in \check{\Gamma}(x)} \sup_{n \in \mathbb{N}} \{u(x, y) + \beta v_n(y)\} \tag{31}$$

$$= \sup_{y \in \check{\Gamma}(x)} \{u(x, y) + \beta \underline{v}^*(y)\} = (\check{B} \underline{v}^*)(x), \tag{32}$$

where (31) uses Lemma 4.1. Since $x \in X$ was arbitrary, it follows that $\underline{v}^* = \check{B}\underline{v}^*$; i.e., $\underline{v}^*$ is a fixed point of $\check{B}$. $\square$

Lemma 4.3. *Let $\underline{v} \in W$ satisfy (20). Let $v \in W$ be a fixed point of $\check{B}$ with $\underline{v} \leq v$. Then $v^* \leq v$.*

Proof. Let $v \in W$ be a fixed point of $\check{B}$ with $\underline{v} \leq v$. Let $x_0 \in X$. If $\check{\Pi}(x_0) = \emptyset$, then $v^*(x_0) = -\infty \leq v(x_0)$. For the rest of the proof, suppose that $\check{\Pi}(x_0) \neq \emptyset$. Let $\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)$. Then $x_{t+1} \in \check{\Gamma}(x_t)$ for all $t \in \mathbb{Z}_+$. We have

$$
\begin{aligned}
v(x_0) &= \sup_{y \in \check{\Gamma}(x_0)} \{u(x_0, y) + \beta v(y)\} && (33)\\
&\geq u(x_0, x_1) + \beta v(x_1) && (34)\\
&\geq u(x_0, x_1) + \beta u(x_1, x_2) + \beta^2 v(x_2) && (35)\\
&\ \ \vdots && (36)\\
&\geq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T v(x_T) && (37)\\
&\geq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) + \beta^T \underline{v}(x_T). && (38)
\end{aligned}
$$

Let $\delta > 0$. By (20) we have $\beta^T \underline{v}(x_T) \geq -\delta$ for sufficiently large T. For such T we have

$$v(x_0) \geq \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) - \delta. \tag{39}$$

Hence we have

$$v(x_0) \geq \mathop{\mathrm{L}}_{T \uparrow \infty} \sum_{t=0}^{T-1} \beta^t u(x_t, x_{t+1}) - \delta. \tag{40}$$

Since this holds for any $\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)$, applying $\sup_{\{x_t\}_{t=1}^{\infty} \in \check{\Pi}(x_0)}$ to the right-hand side of (40) and recalling (10), we have $v(x_0) \geq v^*(x_0) - \delta$. Since δ was arbitrary, we obtain $v(x_0) \geq v^*(x_0)$. Since this is true for any $x_0 \in X$, we have $v \geq v^*$. $\square$

To complete the proof of Theorem 3.1, let $\underline{v} \in W$ satisfy (18) and (19). Then by Lemma 4.2, $\{\check{B}^n \underline{v}\}_{n\in\mathbb{N}}$ monotonically converges to $\underline{v}^*$ pointwise, and $\underline{v}^*$ is a fixed point of $\check{B}$. Assume (20). We have $\underline{v}^* \leq v^*$ since $\underline{v} \leq v^*$ by (18), $\check{B}$ is order-preserving, and v^* is a fixed point of $\check{B}$ by Theorem 2.1. We also have $\underline{v}^* \geq v^*$ by Lemma 4.3. Hence $\underline{v}^* = v^*$.

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