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**Abstract**

This paper establishes a simple no-bubble theorem that applies to a wide range of deterministic models with infinitely lived consumers. Our model assumes only a sequential budget constraint and strictly monotone preferences. In this general setup, we show that asset bubbles are impossible if a consumer can reduce his asset holdings permanently. This is a substantial generalization of the result of Kocherlakota (1992) on asset bubbles and short-sales constraints.

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# 1 Introduction

Since the global financial crisis of 2007-2008, there has been a surge of interest in rational asset pricing bubbles, or simply "asset bubbles." Numerous economic mechanisms that give rise to asset bubbles are still being proposed, and the implications of asset bubbles on various economic issues are actively discussed in the current literature; we refer the reader to Miao (2014) for a short survey on recent developments.

In constructing potential models of asset bubbles, it is important to understand conditions for existence or nonexistence of asset bubbles. For example, a sufficient condition for nonexistence of asset bubbles is useful not only in avoiding specifications that do not lead to asset bubbles, but also in understanding economic mechanisms that give rise to asset bubbles.

In this paper, we establish a simple "no-bubble theorem" that can be used to rule out asset bubbles in a wide range of deterministic models with infinitely lived consumers. More specifically, we consider a consumer facing a sequential budget constraint and having strictly monotone preferences. We show that asset bubbles are impossible if the consumer can reduce his asset holdings permanently starting from an arbitrary period.

A result of this nature was first shown by Kocherlakota (1992) using a transversality condition. Our result is a substantial generalization of his result. Another closely related result is a no-bubble result obtained by Kamihigashi (2001) again using a transversality condition. This result directly applies to various deterministic representative-agent models in continuous time in which the agent has strictly monotone, differentiable instantaneous utility functions. A discrete-time version of this result can be shown using the results of Kamihigashi (2003, 2005),[1] but we establish considerably more general results in this paper.

Our no-bubble theorem is related to other results in the literature (e.g., Santos and Woodford, 1997; Montrucchio, 2004). We discuss these and other existing results in some detail after stating our theorem.

The rest of the paper is organized as follows. In Section 2, we present the model along with necessary assumptions. In Section 3, we formally define asset bubbles. In Section 4, we present some examples satisfying our assumptions. In Section 5, we state and prove our no-bubble theorem. In

[1]See Kamihigashi (2008a, 2008b) for results on asset bubbles consistent with transversality conditions.

Section 6, we show some corollaries of our theorem and discuss some closely related results in the literature. In Section 7, we provide some concluding comments.

## 2 The Model

Time is discrete and denoted by $t \in \mathbb{Z}_+$. We assume that there are at least one consumption good (used as the numeraire) and at least one asset that pays a dividend of $d_t$ units of the consumption good in each period $t \in \mathbb{Z}_+$. Let $p_t$ be the price of the asset in period $t \in \mathbb{Z}_+$. Consider an infinitely lived consumer who faces the following constraints:

$$c_t + p_t s_t = (p_t + d_t)s_{t-1} + y_t,\ c_t \geq 0, \quad \forall t \in \mathbb{Z}_+, \tag{2.1}$$
$$s \in \mathcal{S}(s_{-1}, p, d, y), \tag{2.2}$$

where $c_t$ is consumption in period $t$, $y_t \in \mathbb{R}$ is (net) income in period $t$, $s_t$ is asset holdings at the end of period $t$ with $s_{-1}$ historically given, and $\mathcal{S}(s_{-1}, p, d, y)$ is a set of sequences in $\mathbb{R}$ with $p = \{p_t\}_{t=0}^{\infty}, d = \{d_t\}_{t=0}^{\infty}$, and $y = \{y_t\}_{t=0}^{\infty}$. We present several examples of (2.2) in Section 4.

There can be any number of consumption goods and any number of assets; one can interpret $y_t$ to include all the corresponding variables implicitly. In addition, there can be any number of heterogeneous or homogeneous consumers. These numbers are unimportant here since we focus on the implications of the optimal behavior of only one consumer on the price sequence of only one asset. The results shown below apply to a wide range of models in which there are at least one consumption good, one asset, and one consumer.

Let $\mathcal{C}$ be the set of nonnegative sequences $\{c_t\}_{t=0}^{\infty}$ in $\mathbb{R}$. For any $c \in \mathcal{C}$, we let $\{c_t\}_{t=0}^{\infty}$ denote the sequence representation of $c$, and vice versa. In other words, we use $c$ and $\{c_t\}_{t=0}^{\infty}$ interchangeably; likewise, we use $s$ and $\{s_t\}_{t=0}^{\infty}$ interchangeably. We define the inequalities $<$ and $\leq$ on $\mathcal{C}$ as follows:

$$c \leq c' \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_t,\ c_t \leq c'_t, \tag{2.3}$$
$$c < c' \quad \Leftrightarrow \quad c \leq c' \text{ and } \exists t \in \mathbb{Z}_+, c_t < c'_t. \tag{2.4}$$

The consumer's preferences are represented by a binary relation $\prec$ on $\mathcal{C}$. For any $c, c' \in \mathcal{C}$, the consumer strictly prefers $c'$ to $c$ if and only if $c \prec c'$. Throughout the paper, we maintain the following assumption.

**Assumption 2.1.** $d_t \geq 0$ and $p_t \geq 0$ for all $t \in \mathbb{Z}_+$.

This assumption is standard but proves to be useful. Throughout the paper, we maintain the following assumption unless otherwise specified.

**Assumption 2.2.** $p_t > 0$ for each $t \in \mathbb{Z}_+$.

Although this assumption is required for some of the variables introduced below to be well-defined (see, e.g., (3.1) and (3.2)), we emphasize that it is not always assumed. In particular, if the asset is intrinsically useless, i.e., $d_t = 0$ for all $t \in \mathbb{Z}_+$, then it is more than natural to consider the possibility that $p_t = 0$ for all $t \in \mathbb{Z}_+$. In fact, one of our results deals with such a case without assuming Assumption 2.2; see Proposition 6.1.

We say that a pair of sequences $c = \{c_t\}_{t=0}^{\infty}$ and $s = \{s_t\}_{t=0}^{\infty}$ in $\mathbb{R}$ is a *plan*; a plan $(c, s)$ is *feasible* if it satisfies (2.1) and (2.2); and a feasible plan $(c^*, s^*)$ is *optimal* if there exists no feasible plan $(c, s)$ such that $c^* \prec c$.

Most of our results take an optimal plan $(c^*, s^*)$ as given and require the following assumption.

**Assumption 2.3.** For any $c \in C$ with $c^* < c$, we have $c^* \prec c$.

This assumption is satisfied if $\prec$ is strictly monotone in the sense that for any $c, c' \in \mathcal{C}$ with $c < c'$, we have $c \prec c'$. Although this latter requirement is also reasonable, there are important cases in which it is not satisfied; see Section 4. We present some examples of preferences satisfying Assumption 2.3 in Section 4.

# 3 Asset bubbles

Although we do not consider an explicit maximization problem, we can define asset bubbles using the budget constraint in (2.1). For this purpose, we introduce some notation. For $t \in \mathbb{N}$, let

$$R_t = \frac{p_t + d_t}{p_{t-1}}, \tag{3.1}$$

which is the rate of return on the asset, or the implicit interest rate. For $t \in \mathbb{Z}_+$ and $i \in \mathbb{N}$, we define

$$q_t^i = \prod_{j=t+1}^{t+i} \frac{1}{R_j}, \tag{3.2}$$

which can be interpreted as the period $t$ price of one unit of consumption in period $t+i$. We also define $q_t^0 = 1$ for all $t \in \mathbb{Z}_+$. Note that for all $t, i, n \in \mathbb{Z}_+$, we have

$$q_t^i q_{t+i}^n = q_t^{i+n}. \tag{3.3}$$

Let $t \in \mathbb{Z}_+$. Note from (3.1) and (3.2) that $p_t = q_t^1(p_{t+1} + d_{t+1})$. By repeated application of this equation and (3.3), we have

$$\begin{align}
p_t &= q_t^1 d_{t+1} + q_t^1 p_{t+1} \tag{3.4}\\
&= q_t^1 d_{t+1} + q_t^2 d_{t+2} + q_t^2 p_{t+2} \tag{3.5}\\
&\vdots \tag{3.6}\\
&= \sum_{i=1}^{n} q_t^i d_{t+i} + q_t^n p_{t+n} \quad (\forall n \in \mathbb{N}) \tag{3.7}\\
&= \sum_{i=1}^{\infty} q_t^i d_{t+i} + \lim_{n\to\infty} q_t^n p_{t+n}, \tag{3.8}
\end{align}$$

where both the infinite sum and the limit exist, as we now argue. Note that the finite sum in (3.7) is increasing in $n \in \mathbb{N}$ (as a consequence of Assumptions 2.1 and 2.2). Since the left-hand side of (3.7) equals $p_t$ for all $n \in \mathbb{N}$, we have

$$\forall n \in \mathbb{Z}_+, \quad q_t^n p_{t+n} \geq q_t^{n+1} p_{t+n+1}. \tag{3.9}$$

It follows that both the infinite sum and the limit in (3.8) exist in $\mathbb{R}_+$.

Using (3.8), we decompose $p_t$ into two components:

$$p_t = f_t + b_t, \tag{3.10}$$

where $f_t$ is called the *fundamental value* of the asset and $b_t$ is called the *bubble*, which are defined, respectively, as follows:

$$\begin{align}
f_t &= \sum_{i=1}^{\infty} q_t^i d_{t+i}, \tag{3.11}\\
b_t &= \lim_{n\uparrow\infty} q_t^n p_{t+n}. \tag{3.12}
\end{align}$$

Note from (3.12) and (3.3) that

$$b_0 = \lim_{i\uparrow\infty} q_0^i p_i = \lim_{i\uparrow\infty : i\geq t} q_0^t q_t^{i-t} p_i = q_0^t \lim_{n\uparrow\infty} q_t^n p_{t+n} = q_0^t b_t. \tag{3.13}$$

Therefore (under Assumption 2.2)

$$b_0 = 0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ b_t = 0. \tag{3.14}$$

This together with (3.10) implies that

$$p_0 = f_0 \quad \Leftrightarrow \quad \forall t \in \mathbb{Z}_+,\ p_t = f_t. \tag{3.15}$$

# 4 Examples

In this section, we present several examples of (2.2). We also discuss some examples of preferences that satisfy Assumptions 2.3. Many of these examples are used in Section 6.

## 4.1 Constraints on Asset Holdings

The simplest example of (2.2) would be the following:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq 0. \tag{4.1}$$

This constraint is often used in representative-agent models; see, e.g., Lucas (1978) and Kamihigashi (1998).

Kocherlakota (1992) uses a more general version of (4.1):

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \sigma, \tag{4.2}$$

where $\sigma \in \mathbb{R}$. If $\sigma < 0$, then the above constraint is a short-sales constraint.

The following constraint is even more general:

$$\forall t \in \mathbb{Z}_+, \quad s_t \geq \underline{s}_t, \tag{4.3}$$

where $\underline{s}_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Note that (4.2) is a special case of (4.3) with $\underline{s}_t = \sigma$ for all $t \in \mathbb{Z}_+$.

So far we have only considered inequality constraints on $s_t$, but other types of constraints are also covered by (2.2). For example, the right-hand side of the budget constraint in (2.1) is the consumer's wealth at the beginning of period $t$; thus it may be reasonable to require it to be nonnegative:

$$\forall t \in \mathbb{N}, \quad (p_t + d_t)s_{t-1} + y_t \geq 0. \tag{4.4}$$

This is clearly an example of (2.2), and in fact a special case of (4.3) with

$$\forall t \in \mathbb{Z}_+, \quad \underline{s}_t = -y_{t+1}/(p_{t+1} + d_{t+1}). \tag{4.5}$$

In addition to (4.2), Kocherlakota (1992) considers the following "wealth constraint":

$$\forall t \in \mathbb{Z}_+, \quad p_t s_t + \sum_{i=1}^{\infty} q_t^i y_{t+i} \geq 0, \tag{4.6}$$

which is another example of (2.2). The left-hand side above is the period-$t$ value of the consumer's current asset holdings and future income stream. Note that (4.6) is in fact a special case of (4.3) with

$$\forall t \in \mathbb{Z}_+, \quad \underline{s}_t = -\sum_{i=1}^{\infty} q_t^i y_{t+i}/p_t. \tag{4.7}$$

See, e.g., Wright (1987) and Huang and Werner (2000) for equivalence relations between different budget constraints.

## 4.2 Preferences

**Example 4.1.** A typical objective function in a consumer's maximization problem takes the form

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \tag{4.8}$$

where $\beta \in (0,1)$ and $u : \mathbb{R}_+ \to [-\infty, \infty)$ is a strictly increasing function. Suppose further that $u$ is bounded, and define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad \sum_{t=0}^{\infty} \beta^t u(c_t) < \sum_{t=0}^{\infty} \beta^t u(c'_t). \tag{4.9}$$

Then $\prec$ clearly satisfies Assumption 2.3.

If $u$ is unbounded, i.e., if $u(0) = -\infty$, then the above definition of $\prec$ may not satisfy Assumption 2.3. In particular, given $c^*, c \in \mathcal{C}$ with $c^* < c$, Assumption 2.3 does not hold if $c^*_t = c_t = 0$ for some $t \in \mathbb{Z}_+$. In this case,

$$\sum_{t=0}^{\infty} \beta^t u(c^*_t) = \sum_{t=0}^{\infty} \beta^t u(c_t) = -\infty, \tag{4.10}$$

provided that both sums are well-defined. The next example considers an optimality criterion that handles this and other problems.

**Example 4.2.** For $t \in \mathbb{Z}_+$, let $u_t : \mathbb{R}_+ \to [-\infty, \infty)$ be a strictly increasing function. In this case, the infinite sum $\sum_{t=0}^{\infty} u_t(c_t)$ may not be well-defined. Even if it is always well-defined, it may not be strictly increasing, as discussed above. To deal with these problems, consider the binary relation $\prec$ defined by

$$c \prec c' \quad \Leftrightarrow \quad \varliminf_{T \uparrow \infty} \sum_{t=0}^{T} [u_t(c_t) - u(c_t')] < 0, \tag{4.11}$$

where we follow the convention that $u_t(-\infty) - u_t(-\infty) = 0$; see Dana and Le Van (2006) for related optimality criteria. It is easy to see that $\prec$ here satisfies Assumption 2.3.

Continuing with this example, suppose that (2.2) is given by (4.1). Suppose further that each $u_t$ is differentiable on $\mathbb{R}_{++}$, and that there exists an optimal plan $(c^*, s^*)$ such that

$$\forall t \in \mathbb{Z}_+, \quad c_t^* > 0,\ s_t^* = 1. \tag{4.12}$$

Then the standard Euler equation holds for all $t \in \mathbb{Z}_+$:

$$u_t'(c_t^*) p_t = u_{t+1}'(c_{t+1}^*)(p_{t+1} + d_{t+1}). \tag{4.13}$$

This together with (3.2) implies that

$$\forall t \in \mathbb{Z}_+, \forall i \in \mathbb{N}, \quad q_t^i = \frac{u_{t+i}'(c_{t+i}^*)}{u_t'(c_t^*)}. \tag{4.14}$$

In this case, the fundamental value $f_t$ takes the familiar form

$$f_t = \sum_{i=1}^{\infty} \frac{u_{t+i}'(c_{t+i}^*)}{u_t'(c_t^*)} d_{t+i}. \tag{4.15}$$

**Example 4.3.** Let $v : \mathcal{C} \to \mathbb{R}$ be a strictly increasing function. Define the binary relation $\prec$ by

$$c \prec c' \quad \Leftrightarrow \quad v(c_0, c_1, c_2, \dots) < v(c_0', c_1', c_2', \dots). \tag{4.16}$$

Note that (4.16) satisfies Assumption 2.3 without any additional condition on $v$. For example, $v$ can be a recursive utility function.

# 5 The No-Bubble Theorem

## 5.1 The Statement

We are ready to state our no-bubble theorem.

**Theorem 5.1.** *Let Assumptions 2.1 and 2.2 hold. Let $(c^*, s^*)$ be an optimal plan satisfying Assumption 2.3. Suppose that there exist $\epsilon > 0$ and $\tau \in \mathbb{Z}_+$ such that*

$$\forall t \geq \tau,\ s_t \geq s_t^* - \epsilon \quad \Rightarrow \quad s \in \mathcal{S}(s_{-1}, p, d, y). \tag{5.1}$$

*Then $b_0 = 0$.*

It seems remarkable that asset bubbles can be ruled out under such simple conditions. In particular, no explicit utility function is assumed, and the only requirement on the binary relation $\prec$ is Assumption 2.3, which is a weak form of strict monotonicity.

There are related results in the literature that also do not require an explicit utility function. In particular, Santos and Woodford (1997) consider a stochastic economy with incomplete markets and heterogeneous consumers where each consumer's preferences are represented by a strictly monotone partial order. They (Theorems 3.1 and 3.3) rule out asset bubbles (more precisely, show the existence of state prices such that no bubble arises) by assuming the finiteness of the value of the aggregate endowment stream or by imposing a sufficient degree of impatience on consumers' preferences.[2] Theorem 5.1 follows neither approach, and uses (5.1) instead.[3]

There are other related results in the literature. We discuss them after proving the theorem.

## 5.2 Proof of Theorem 5.1

Suppose by way of contradiction that

$$b_0 = \lim_{n\to\infty} q_0^n p_n > 0. \tag{5.2}$$

[2]See Werner (2014) for an extension of their results.

[3]See Kamihigashi (1998, Proposition 3.1) for another related result that assumes only strict monotonicity of preferences. However, this result does not directly deals with asset bubbles; thus discussing it in detail is beyond the scope of this paper.

This together with (3.9) implies that

$$\forall n \in \mathbb{Z}_+, \quad q_0^n p_n \geq b_0. \tag{5.3}$$

Note from (3.8) and (5.2) that $\sum_{n=1}^{\infty} q_0^n d_n < p_0$. From (5.3) and this, we have

$$\sum_{n=1}^{\infty} \frac{d_n}{p_n} \leq \sum_{n=1}^{\infty} \frac{q_0^n d_n}{b_0} < \frac{p_0}{b_0}. \tag{5.4}$$

Let $\epsilon > 0$ and $\tau \in \mathbb{Z}_+$ be as given by (5.1). Let $\delta \in (0, \epsilon)$. We construct an alternative plan $(c^\delta, s^\delta)$ as follows:

$$c_t^\delta = \begin{cases} c_t^* & \text{if } t \neq \tau, \\ c_\tau^* + p_\tau \delta & \text{if } t = \tau, \end{cases} \tag{5.5}$$

$$s_t^\delta = \begin{cases} s_t^* & \text{if } t \leq \tau - 1, \\ s_\tau^* - \delta & \text{if } t = \tau, \\ (p_t + d_t) s_{t-1}^\delta + y_t - c_t^* & \text{if } \tau \geq \tau + 1. \end{cases} \tag{5.6}$$

It suffices to show that $(c^\delta, s^\delta)$ is feasible for $\delta > 0$ sufficiently small; for then, we have $c^* \prec c^\delta$ by (5.5) and Assumption 2.3, contradicting the optimality of $(c^*, s^*)$.

For the rest of the proof, we only consider variables in periods $t \geq \tau$; thus for simplicity we assume without loss of generality that $\tau = 0$. For $t \geq \tau = 0$, define

$$\delta_t = s_t^* - s_t^\delta. \tag{5.7}$$

Note from (2.1) and (5.7) that $p_t \delta_t = (p_t + d_t)\delta_{t-1}$ for all $t \in \mathbb{N}$; thus

$$\delta_t = \delta_{t-1} \frac{p_t + d_t}{p_t} = \delta_{t-2} \frac{p_{t-1} + d_{t-1}}{p_{t-1}} \frac{p_t + d_t}{p_t} = \cdots \tag{5.8}$$

$$= \delta \prod_{i=1}^{t} \frac{p_i + d_i}{p_i} \leq \delta \prod_{i=1}^{\infty} \frac{p_i + d_i}{p_i}, \tag{5.9}$$

where the equality in (5.9) holds since $\delta_0 = \delta$ by (5.6), and the inequality holds since $d_t \geq 0$ for all $t \in \mathbb{Z}_+$ by Assumption 2.1.

Since $(c^\delta, s^\delta)$ satisfies (2.1) by construction, to show that $(c^\delta, s^\delta)$ is feasible, it suffices to verify that $\delta_t \le \epsilon$ for all $t \in \mathbb{Z}_+$; for then, we have $s \in \mathcal{S}(y_{-1}, p, d, y)$ by (5.1) and (5.7). To this end, note that

$$\ln\left(\prod_{i=1}^\infty \frac{p_i + d_i}{p_i}\right) = \sum_{i=1}^\infty \ln\left(\frac{p_i + d_i}{p_i}\right) \tag{5.10}$$

$$= \sum_{i=1}^\infty \ln\left(1 + \frac{d_i}{p_i}\right) \tag{5.11}$$

$$\le \sum_{i=1}^\infty \frac{d_i}{p_i} \le \frac{p_0}{b_0}, \tag{5.12}$$

where the first inequality holds by concavity of $\ln(\cdot)$, and the last inequality uses (5.4). It follows that

$$\prod_{i=1}^\infty \frac{p_i + d_i}{p_i} < \infty. \tag{5.13}$$

Using this and recalling (5.8) and (5.9), we can choose $\delta > 0$ small enough that $\delta_t \le \epsilon$ for all $t \in \mathbb{Z}_+$, as desired.

# 6 Corollaries and Related Results

In this section, we show various corollaries of Theorem 5.1 partly to discuss related results in the literature. We also consider the case of fiat money. Let us start with a simple consequence of Theorem 5.1:

**Corollary 6.1.** *Let Assumptions 2.1 and 2.2 hold. Let $(c^*, s^*)$ be an optimal plan satisfying Assumption 2.3. Suppose that (2.2) is given by (4.3) with $\underline{s}_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Suppose that*

$$\varliminf_{t\uparrow\infty}(s_t^* - \underline{s}_t) > 0. \tag{6.1}$$

*Then $b_0 = 0$.*

*Proof.* Let $\epsilon \in (0, \varliminf_{t\uparrow\infty}(s_t^* - \underline{s}_t))$. Then there exists $\tau \in \mathbb{Z}_+$ such that $s_t^* - \underline{s}_t \ge \epsilon$, or $s_t^* - \epsilon \ge \underline{s}_t$, for all $t \ge \tau$. Thus (5.1) holds. Hence $b_0 = 0$ by Theorem 5.1. □

The following is immediate from the above result.

**Corollary 6.2.** *Let Assumptions 2.1 and 2.2 hold. Let $(c^*, s^*)$ be an optimal plan satisfying Assumption 2.3. Suppose that (2.2) is given by (4.3) with $\underline{s}_t \in \mathbb{R}$ for all $t \in \mathbb{Z}_+$. Suppose that $b_0 > 0$. Then*

$$\varliminf_{t\uparrow\infty}(s_t^* - \underline{s}_t) = 0. \tag{6.2}$$

The above result is a substantial generalization of Kocherlakota's (1992) result on asset bubbles and short-sales constraints:

**Corollary 6.3** (Kocherlakota, 1992, Proposition 3)**.** *Let Assumptions 2.1 and 2.2 hold. Suppose that the consumer's preferences are represented by (4.8) with $\beta \in (0,1)$ and $u : \mathbb{R}_+ \to [-\infty, \infty)$. Suppose that $u$ is $C^1$ on $\mathbb{R}_{++}$, strictly increasing, concave, and bounded above or below by zero. Let $(c^*, s^*)$ be an optimal plan such that*

$$\forall t \in \mathbb{Z}_+, \quad c_t^* > 0, \tag{6.3}$$

$$\left|\sum_{t=0}^{\infty} \beta^t u(c_t^*)\right| < \infty. \tag{6.4}$$

*Suppose that (2.2) is given by (4.2) for some $\sigma \in \mathbb{R}$. Suppose that $b_0 > 0$. Then*

$$\varliminf_{t\uparrow\infty} s_t^* = \sigma. \tag{6.5}$$

*Proof.* It is easy to see that the hypothesis of Corollary 6.2 holds with $\underline{s}_t = \sigma$ for all $t \in \mathbb{Z}_+$. Thus we have (6.2) with $\underline{s}_t = \sigma$ for all $t \in \mathbb{Z}_+$, which implies (6.5). □

In view of Corollaries 6.2 and 6.3, one can see that most of the assumptions on the consumer's preferences in the latter result are unnecessary.

Now, recall that the proof of Theorem 5.1 ((5.2)–(5.4)) shows that if $b_0 > 0$, then

$$\sum_{i=1}^{\infty} \frac{d_i}{p_i} < \infty. \tag{6.6}$$

Since $d_0/p_0 < \infty$ (under Assumption 2.2), we have the following implication:

$$b_0 > 0 \quad \Rightarrow \quad \sum_{i=0}^{\infty} \frac{d_i}{p_i} < \infty. \tag{6.7}$$

The contrapositive of this result is shown by Montrucchio (2004) for a fairly general stochastic model:[4]

**Corollary 6.4** (Montrucchio, 2004, Theorem 2)**.** *Let Assumptions 2.1 and 2.2 hold. Suppose that*

$$\sum_{i=0}^{\infty} \frac{d_i}{p_i} = \infty. \tag{6.8}$$

*Then* $b_0 = 0$.

Montrucchio (2004) uses a martingale argument to show the above result for a stochastic model with arbitrary state prices satisfying a no-arbitrage condition.

The following result considers the case of fiat money.

**Proposition 6.1.** *Let Assumption 2.1 hold. Let* $(c^*, s^*)$ *be an optimal plan satisfying Assumption 2.3. Suppose that there exist* $\epsilon > 0$ *and* $\tau \in \mathbb{Z}_+$ *satisfying (5.1). Suppose further that*

$$\forall t \geq \tau, \quad d_t = 0. \tag{6.9}$$

*Then*

$$\forall t \geq \tau, \quad p_t = 0. \tag{6.10}$$

*Proof.* Let $\epsilon > 0$ and $\tau \in \mathbb{Z}_+$ be as in (5.1). Without loss of generality, we assume that $\tau = 0$. Note from (6.9) and (3.11) that

$$\forall t \in \mathbb{Z}_+, \quad f_t = 0. \tag{6.11}$$

Suppose by way of contraction that $p_t > 0$ for some $t \geq \tau = 0$. Without loss of generality, we assume that $t = 0$; i.e., $p_0 > 0$.

[4] A similar result is shown by Bosi et al. (2014) for "bubbles" on physical capital.

First, suppose that $p_t > 0$ for all $t \in \mathbb{Z}_+$. Then Assumption 2.2 holds; thus by Theorem 5.1 we have $b_0 = 0$. Hence by (6.11), we obtain $p_0 = 0$, a contradiction.

Next, suppose that $p_t = 0$ for some $t \in \mathbb{N}$. Let $T$ be the first $T \in \mathbb{Z}_+$ with

$$p_T > 0, \quad p_{T+1} = 0. \tag{6.12}$$

We construct an alternative plan $(c, s)$ as follows:

$$c_t = \begin{cases} c_t^* & \text{if } t \neq T, \\ c_T^* + p_T \epsilon & \text{if } t = T, \end{cases} \tag{6.13}$$

$$s_t = \begin{cases} s_t^* & \text{if } t \leq T - 1, \\ s_T^* - \epsilon & \text{if } t \geq T. \end{cases} \tag{6.14}$$

It is easy to see from (5.1), (2.1), and (6.12) that $(c, s)$ is feasible. But we have $c^* \prec c$ by (6.13) and Assumption 2.3, contradicting the optimality of $(c^*, s^*)$. □

Finally, we present two results that apply to representative-agent models.

**Corollary 6.5.** *Let Assumptions 2.1 and 2.2 hold. Suppose that (2.2) is given by (4.1). Let $(c^*, s^*)$ be an optimal plan satisfying Assumption 2.3 and*

$$\forall t \in \mathbb{Z}_+, \quad s_t^* = 1. \tag{6.15}$$

*Then $b_0 = 0$.*

*Proof.* Note that (6.15) and (4.1) imply (5.1) with $\epsilon = 1$ and $\tau = 0$. Thus Theorem 5.1 applies. □

The following result is immediate from the above result.

**Corollary 6.6.** *Let Assumptions 2.1 and 2.2 hold. Assume the setup of Example 4.2 (up to (4.15)). Then*

$$p_0 = \sum_{i=1}^{\infty} \frac{u_i'(c_t^*)}{u_0'(c_0^*)} d_t. \tag{6.16}$$

A similar result is shown in Kamihigashi (2001, Section 4.2.1) for a continuous-time model with a nonlinear constraint. It is known that a stochastic version of Corollary 6.6 requires additional assumptions; see Kamihigashi (1998) and Montrucchio and Privileggi (2001).

# 7 Concluding Comments

In this paper, we established a simple no-bubble theorem for deterministic infinite-horizon models with sequential budget constraints and strictly monotone preferences. Essentially the only condition required for the nonexistence of asset bubbles is that a consumer can reduce his asset holdings permanently starting from an arbitrary period. This is a substantial generalization of the result of Kocherlakota (1992) on asset bubbles and short-sales constraints. Our result can be used in any context where his result is used.

Since our analysis does not involve explicit optimization, it can easily be extended to a stochastic model to a certain degree. In fact, many of our arguments can be repeated by assuming that all sequences are sample paths of stochastic processes.[5] However, care must be taken when one considers any action taken by a consumer, who does not have access to future information. In addition, the definition of a bubble is not unambiguous when markets are incomplete; see, e.g., Santos and Woodford (1997) and Montrucchio (2004). We leave a stochastic extension of our no-bubble theorem for future research.

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[5]See Kamihigashi (2011) for sample path properties of certain bubble processes.

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