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Abstract

Consider a preordered metric space $(X, d, \preceq)$. Suppose that $d(x, y) \leq d(x', y')$ if $x' \preceq x \preceq y \preceq y'$. We say that a self-map T on X is asymptotically contractive if $d(T^i x, T^i y) \to 0$ for all $x, y \in X$. We show that an order-preserving self-map T on X has a globally stable fixed point if and only if T is asymptotically contractive and there exist $x, x^* \in X$ such that $T^i x \preceq x^*$ for all $i \in \mathbb{N}$ and $x^* \preceq Tx^*$. We establish this and other fixed point results for more general spaces where d consists of a collection of distance measures. We apply our results to order-preserving nonlinear Markov operators on the space of probability distribution functions on $\mathbb{R}$.

Keywords: fixed point; order-preserving self-map; contraction; nonlinear Markov operator; global stability

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1 Introduction

The majority of fixed point theorems require a space that is complete in some sense. Fixed point theorems based on the metric approach such as the celebrated Banach contraction principle and its numerous extensions commonly assume a complete metric space (see, e.g., [5]). Results based on the order-theoretic approach such as Tarski's fixed point theorem and the Knaster-Tarski fixed point theorem typically require a complete lattice or a chain-complete partially ordered space (see, e.g., [1]). These two approaches are combined in the growing literature on fixed point theory for partially ordered complete metric spaces (e.g., [2, 3, 6, 8, 11, 12, 13, 14, 16]), where completeness still plays an indispensable role.

However, there are various situations in which it is fairly easy to construct a good candidate for a fixed point even if the underlying space may not be complete. For example, consider a self-map on a space of real valued functions on some set. Then an increasing sequence of functions majorized by a common function converges pointwise to some function in the same space. If this pointwise limit turns out to be a good candidate for a fixed point, then there is no need to verify that the entire space is complete or chain-complete.

In this paper we develop simple fixed point results for order-preserving self-maps on a space equipped with a transitive binary relation and a collection of distance measures. Most of our results assume existence of a good candidate for a fixed point instead of completeness. Some of our results use the condition that the self-map in question is asymptotically contractive, which means in our terminology that two distinct points are mapped arbitrarily close to each other after sufficiently many iterations. In the case of Markov operators induced by Markov chains, this property is an implication of the order-theoretic mixing condition introduced in [9], which is a natural property of various stochastic processes (see [9, 10]). We show that asymptotic contractiveness is not only a useful condition for showing existence of a fixed point, but also a necessary condition for existence of a globally stable fixed point.

In practice, a candidate for a fixed point must be constructed or must be shown to exist. If the underlying space is a complete metric space, then the limit of a certain Cauchy sequence serves as a good candidate. This classical approach is still common in the recent literature on fixed points of order-preserving self-maps on partially ordered complete metric spaces (e.g., [2, 3, 6, 8, 12, 14, 16]). For comparison purposes, we establish a fixed point result for such spaces as a consequence of our general results.

To illustrate how a candidate fixed point can be constructed in practice, we consider nonlinear Markov operators on the space of probability distribution functions on $\mathbb{R}$. We provide a simple sufficient condition for existence of a globally stable fixed point.

2 Definitions

Let X be a set. A binary relation $\preceq \subset X \times X$ on X is called *transitive* if for any $x, y, z \in X$,

$$x \preceq y \preceq z \quad \Rightarrow \quad x \preceq z, \tag{2.1}$$

reflexive if

$$\forall x \in X, \qquad x \preceq x, \tag{2.2}$$

and *antisymmetric* if for any $x, y \in X$,

$$x \preceq y \text{ and } y \preceq x \quad \Rightarrow \quad x = y. \tag{2.3}$$

A binary relation is called a *preorder* if it is transitive and reflexive. A preorder $\preceq$ is called a *partial order* if it is antisymmetric.

Let A be a set. Let $\Phi(A)$ be the set of functions $\phi : A \to \mathbb{R}_+$. Let $\phi, \psi \in \Phi(A)$. We write $\phi = 0$ if $\phi(a) = 0$ for each $a \in A$, and $\phi \leq \psi$ if $\phi(a) \leq \psi(a)$ for each $a \in A$. The expressions $\phi + \psi$ and $\max\{\phi + \psi\}$ are defined respectively by

$$\forall a \in A, \quad (\phi + \psi)(a) = \phi(a) + \psi(a), \tag{2.4}$$

$$\forall a \in A, \quad (\max\{\phi + \psi\})(a) = \max\{\phi(a) + \psi(a)\}. \tag{2.5}$$

For $\{\phi_i\}_{i \in \mathbb{N}} \subset \Phi(A)$, we write $\phi_i \to 0$ if $\phi_i(a) \to 0$ as $i \uparrow \infty$ for each $a \in A$.

Let $d : X \times X \times A \to \mathbb{R}_+$; the dependence of d on $(x, y, a) \in X \times X \times A$ is expressed by $d(x, y)(a)$. We treat the expression $d(x, y)$ as a function from A to $\mathbb{R}_+$; more precisely, $d(x, y)$ is the function $\phi \in \Phi(A)$ given by $\phi(a) = d(x, y)(a)$ for all $a \in A$. Under the conventions described in the previous paragraph, for any $x, y, x', y' \in X$ and $\{x_i\}_{i \in \mathbb{N}}, \{y_i\}_{i \in \mathbb{N}} \subset X$, we have the following relations:

$$d(x, y) = 0 \quad \Longleftrightarrow \quad \forall a \in A, d(x, y)(a) = 0, \tag{2.6}$$

$$d(x, y) \leq d(x', y') \quad \Longleftrightarrow \quad \forall a \in A, d(x, y)(a) \leq d(x', y')(a), \tag{2.7}$$

$$d(x_i, y_i) \to 0 \quad \Longleftrightarrow \quad \forall a \in A, d(x_i, y_i)(a) \to 0. \tag{2.8}$$

The expressions $d(x,y) + d(x',y')$ and $\max\{d(x,y), d(x',y')\}$ are defined as in (2.4) and (2.5).

We say that d is *identifying* if for any $x, y \in X$,

$$d(x,y) = 0 \quad \Rightarrow \quad x = y, \tag{2.9}$$

reflexive if

$$\forall x \in X, \quad d(x,x) = 0, \tag{2.10}$$

and *symmetric* if

$$\forall x, y \in X, \quad d(x,y) = d(y,x). \tag{2.11}$$

We say that d satisfies the *triangle inequality* if

$$\forall x, y, z \in X, \quad d(x,z) \leq d(x,y) + d(y,z). \tag{2.12}$$

We say that d is *one-dimensional* if $d(x,y)(a)$ does not depend on a for any $x, y \in X$. If d is one-dimensional, then we treat d as a function from $X \times X$ to $\mathbb{R}_+$. If d is one-dimensional, identifying, reflexive, symmetric, and satisfies the triangle inequality, then d is called a *metric*.

In what follows, the set X is assumed to be equipped with a binary relation $\preceq$ and a function $d : X \times X \times A \to \mathbb{R}_+$. Even though $\preceq$ is merely a binary relation, we regard it as a type of order.

We say that a sequence $\{x_i\}_{i \in \mathbb{N}}$ is *increasing* if $x_i \preceq x_{i+1}$ for all $i \in \mathbb{N}$. We say that a function $f : D \to \mathbb{R}$ with $D \subset \mathbb{R}$ is *increasing* if $f(x) \leq f(y)$ for any $x, y \in D$ with $x \leq y$.

We say that d is *regular* if for any $x, y, z \in X$ with $x \preceq y \preceq z$, we have

$$\max\{d(x,y), d(y,z)\} \leq d(x,z). \tag{2.13}$$

Inequality (2.13) means that if $x \preceq y$, then $d(x,y)$ increases as x "decreases" or y "increases."

Example 2.1. Let $X = \mathbb{R}$. Let $\preceq$ be the usual partial order on $\mathbb{R}$. For $x, y \in X$, define $d(x,y) = |x - y|$. Then d is one-dimensional, a metric, and regular.

Example 2.2. Let X be the set of functions on $\mathbb{R}$. Let $A = \mathbb{R}$. For $f, g \in X$, write $f \preceq g$ if $f \leq g$. Then $\preceq$ is a partial order. For $f, g \in X$ and $a \in A$, define $d(f,g)(a) = |f(a) - g(a)|$. Then d is not one-dimensional, but d is identifying, reflexive, symmetric, regular, and satisfies the triangle inequality.

Example 2.3. Let $(S, \mathscr{S})$ be a measurable space. Let X be the set of finite measures on S. For $\mu, \nu \in X$, write $\mu \preceq \nu$ if $\mu(B) \leq \nu(B)$ for each $B \in \mathscr{S}$. Then $\preceq$ is a partial order. Let A be the set of bounded measurable functions from S to $\mathbb{R}$. For $\mu, \nu \in X$ and $f \in A$, define $d(\mu, \nu)(f) = |\int f d\mu - \int f d\nu|$. Then d is not one-dimensional, but d is identifying, reflexive, symmetric, regular, and satisfies the triangle inequality.

Example 2.4. Let $X = \mathbb{R}^2$. For $x, y \in X$, write $x \preceq y$ if $\|x\| \leq \|y\|$, where $\|\cdot\|$ is the Euclidian norm. Then $\preceq$ is a preorder, but it is not a partial order since it fails to be antisymmetric. For $x, y \in X$, let $d(x, y) = \|x - y\|$. Then d is a metric, but not regular. For example, $(1/2, 0) \preceq (0, 1) \preceq (1, 0)$, but $d((0, 1), (1, 0)) = \sqrt{2} > d((1/2), 0), (1, 0)) = 1/2$.

Example 2.5. Let $X = \mathbb{R}^2$. For $x, y \in X$, write $x \preceq y$ if $x \leq y$ componentwise. Define d as in Example 2.4. Then d is a metric and regular.

Example 2.6. Let $X = \mathbb{R}^2$. For $x, y \in X$, write $x \preceq y$ if $x_1 < y_1$ or if $x_1 = y_1$ and $x_2 \leq y_2$, where $x = (x_1, x_2)$, etc. This binary relation $\preceq$ is a lexicographic order, which is a partial order. Define d as in Example 2.4. Then d is a metric, but not regular. For example, $(0, 0) \preceq (1, 100) \preceq (2, 0)$, but $d((0, 0), (1, 100)) > 100 > d((0, 0), (2, 0)) = 2$.

A self-map $T : X \to X$ is called *order-preserving* if for any $x, y \in X$,

$$x \preceq y \quad \Rightarrow \quad Tx \preceq Ty. \tag{2.14}$$

A *fixed point* of T is an element $x \in X$ such that $Tx = x$. We say that a fixed point x^* of T is *globally stable* if

$$\forall x \in X, \quad d(T^i x, x^*) \to 0. \tag{2.15}$$

Note that if x^* is a globally stable fixed point of T, then T has no other fixed point as long as d is identifying. To see this, note that if T has another fixed point x, then for any $i \in \mathbb{N}$, we have $d(x, x^*) = d(T^i x, x^*) \to 0$; thus $x = x^*$.

We say that $T : X \to X$ is *asymptotically contractive* if

$$\forall x, y \in X, \quad d(T^i x, T^i y) \to 0. \tag{2.16}$$

The term "asymptotically contractive" has been used in different meanings in the literature (e.g., [4, 15]). Our usage of the term can be justified by noting that (2.16) is an asymptotic property as well as an implication of well-known contraction properties; see (4.8) and (4.9).

3 Fixed Point Results

Let X and A be sets. Let $\preceq$ be a binary relation on X. Let $T : X \to X$. Let $d : X \times X \times A \to \mathbb{R}_+$. In this section we maintain the following assumptions:

Assumption 3.1. T is order-preserving.

Assumption 3.2. $\preceq$ is transitive.

Assumption 3.3. d is identifying.

Assumption 3.4. d is regular.

The following theorem is the most fundamental of our fixed point results.

Theorem 3.1. *Suppose that there exist $x, x^* \in X$ such that*

$$d(T^i x, T^i x^*) \to 0, \tag{3.1}$$

$$\forall i \in \mathbb{N}, \quad T^i x \preceq x^*, \tag{3.2}$$

$$x^* \preceq T x^*. \tag{3.3}$$

Then x^* is a fixed point of T.

Proof. Since T is order-preserving, (3.3) implies that

$$x^* \preceq T x^* \preceq T^2 x^* \preceq T^3 x^* \preceq \cdots . \tag{3.4}$$

This together with (3.2) implies that

$$\forall i \in \mathbb{N}, \quad T^i x \preceq x^* \preceq T^i x^*. \tag{3.5}$$

Thus by regularity of d, for any $i \in \mathbb{N}$ we have

$$d(x^*, T x^*) \le d(x^*, T^i x^*) \tag{3.6}$$

$$\le d(T^i x, T^i x^*) \to 0, \tag{3.7}$$

where the convergence holds by (3.1). It follows that $d(x^*, T x^*) = 0$; thus x^* is a fixed point of T since d is identifying. $\square$

The above proof generalizes the fixed point argument used in [10]. Under additional assumptions, conditions (3.1)–(3.3) are also necessary for existence of a fixed point.

Theorem 3.2. *Suppose that $\preceq$ is reflexive. Suppose further that d is reflexive. Then T has a fixed point if and only if there exist $x, x^* \in X$ satisfying (3.1)–(3.3).*

Proof. The "if" part follows from Theorem 3.1. For the "only if" part, let x^* be a fixed point of T. Then since $\preceq$ and d are reflexive, (3.1)–(3.3) trivially hold with $x = x^*$. $\square$

Let us now consider global stability of a fixed point. We start with a simple consequence of asymptotic contractiveness.

Lemma 3.1. *Suppose that T is asymptotically contractive and has a fixed point x^*. Then x^* is globally stable.*

Proof. To see that x^* is unique, let x be another fixed point. Then by (2.16) with $y = x^*$, we have

$$d(x, x^*) = d(T^i x, T^i x^*) \to 0. \tag{3.8}$$

Thus $x = x^*$.

For global stability, let $x \in X$ be arbitrary. Again by (2.16) with $y = x^*$ we obtain (2.15). Hence x^* is globally stable. $\square$

Theorem 3.3. *Suppose that T is asymptotically contractive. Suppose further that there exist $x, x^* \in X$ satisfying (3.2) and (3.3). Then x^* is a globally stable fixed point of T.*

Proof. Since T is asymptotically contractive, x and x^* satisfy (3.1). Thus by Theorem 3.1, x^* is a fixed point of T. Global stability follows from Lemma 3.1. $\square$

Theorem 3.4. *Suppose that $\preceq$ is reflexive. Suppose further that d is symmetric and satisfies the triangle inequality. Then T has a globally stable fixed point if and only if T is asymptotically contractive and there exist $x, x^* \in X$ satisfying (3.2) and (3.3).*

Proof. The "if" part follows from Theorem 3.3. For the "only if" part, suppose that T has a globally stable fixed point x^*. Then for any $x, y \in X$, by the triangle inequality, symmetry of d, and global stability of x^*,

$$d(T^i x, T^i y) \leq d(T^i x, x^*) + d(x^*, T^i y) \tag{3.9}$$
$$= d(T^i x, x^*) + d(T^i y, x^*) \to 0. \tag{3.10}$$

Thus (2.16) holds; i.e., T is asymptotically contractive. By reflexivity of $\preceq$, (3.2) and (3.3) hold with $x = x^*$. $\square$

4 The Case of a Complete Metric Space

In this section, in addition to Assumptions 3.1–3.4, we maintain the following assumptions.

Assumption 4.1. (X, d) is a complete metric space.

Assumption 4.2. For any increasing sequence $\{x_i\}_{i\in\mathbb{N}} \subset X$ converging to some $x \in X$, we have $x_i \preceq x$ for all $i \in \mathbb{N}$.

Assumption 4.3. For any increasing sequence $\{x_i\}_{i\in\mathbb{N}} \subset X$ converging to some $x \in X$, if there exists $y \in X$ such that $x_i \preceq y$ for all $i \in \mathbb{N}$, then $x \preceq y$.

Assumptions 4.2 and 4.3 hold if $\preceq$ is closed (i.e., a closed subset of $X \times X$). To see this, let $\{x_i\}_{i\in\mathbb{N}}$ be an increasing sequence converging to some $x \in X$. Then given any $i \in \mathbb{N}$, we have $x_i \preceq x_j$ for all $j \geq i$; thus letting $j \uparrow \infty$, we obtain $x_i \preceq x$. Furthermore, if there exists $y \in X$ such that $x_i \preceq y$ for all $i \in \mathbb{N}$, then letting $i \uparrow \infty$ yields $x \preceq y$.

Assumption 4.2 is standard in the recent literature on fixed point theory for partially ordered metric spaces (e.g., [2, 3, 6, 8, 12, 16]). Our approach differs in that it also utilizes Assumption 4.3.

Theorem 4.1. *Suppose that for any $y, z \in X$, we have*

$$y \preceq z \quad \Rightarrow \quad d(T^i y, T^i z) \to 0. \tag{4.1}$$

Suppose further that there exist $x, \overline{x} \in X$ such that

$$x \preceq Tx. \tag{4.2}$$

$$\forall i \in \mathbb{N}, \quad T^i x \preceq \overline{x}. \tag{4.3}$$

Then T has a fixed point.

Proof. For $i \in \mathbb{N}$, let $x_i = T^i x$. It follows from (4.2) that $\{x_i\}_{i\in\mathbb{N}}$ is increasing. We show that $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy. To this end, let $\epsilon > 0$. By (4.1)–(4.3) there exists $N \in \mathbb{N}$ such that $d(T^N x, T^N \overline{x}) < \epsilon$. Let $i, j \geq N$ with $i \leq j$. Let $m = j - N$. Since $x_N \preceq x_i \preceq x_j$, by regularity of d we have

$$\begin{aligned} d(x_i, x_j) &\leq d(x_N, x_j) && (4.4)\\ &= d(T^N x, T^j x) = d(T^N x, T^N T^m x) && (4.5)\\ &\leq d(T^N x, T^N \overline{x}) < \epsilon, && (4.6) \end{aligned}$$

where the first inequality in (4.6) holds by (4.3) and regularity of d. Since $i, j \geq N$ are arbitrary, it follows that $\{x_i\}$ is Cauchy.

Now, since $\{x_i\}$ is Cauchy and X is complete, it converges to some $x^* \in X$. By (4.2) and Assumption 4.2, we have

$$\forall i \in \mathbb{N}, \quad x \preceq T^i x \preceq x^*. \tag{4.7}$$

Thus (3.2) holds. Condition (3.1) follows from (4.7) and (4.1) with $y = x$ and $z = x^*$. From (4.7) we have $T^{i+1}x \preceq Tx^*$ for all $i \in \mathbb{N}$. Thus by Assumption 4.3, $x^* \preceq Tx^*$. Hence (3.3) holds. It follows by Theorem 3.1 that x^* is a fixed point of T. □

A simple sufficient condition for (4.1) is that for some $k \in [0, 1)$,

$$y \preceq z \quad \Rightarrow \quad d(Ty, Tz) \leq kd(y, z). \tag{4.8}$$

This condition is used in [12, Theorem 2.1]. A weaker condition is used in [2, Theorem 2.1] to establish a result that implies the following.

Corollary 4.1. *Let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\lim_{i\uparrow\infty} \psi^i(t) = 0$ for each $t > 0$. Suppose that for any $y, z \in X$, we have*

$$y \preceq z \quad \Rightarrow \quad d(Ty, Tz) \leq \psi(d(y, z)). \tag{4.9}$$

Suppose further that there exists $x \in X$ satisfying (4.2). Then T has a fixed point.

Proof. For any $i \in \mathbb{N}$ and $y, z \in X$ with $y \preceq z$, it follows from (4.9) that

$$d(T^i y, T^i z) \leq \psi(d(T^{i-1}y, T^{i-1}z)) \leq \cdots \leq \psi^i(d(y, z)) \to 0. \tag{4.10}$$

Thus (4.1) holds. Let $\{x_i\}_{i\in\mathbb{N}}$ be as in the proof of Theorem 4.1. It is shown in [2] that $\{x_i\}_{i\in\mathbb{N}}$ is Cauchy, so that it converges to some $x^* \in X$. By Assumption 4.2, we have $T^i x \preceq x^*$ for all $i \in \mathbb{N}$. Thus (4.3) holds with $\overline{x} = x^*$. Now the conclusion follows by Theorem 4.1. □

The core part of the proof of [2, Theorem 2.1] is to show that $\{T^i x\}$ is Cauchy, which can in fact be done without Assumptions 3.4 and 4.3. Hence the corresponding part of [2, Theorem 2.1] is not directly comparable to Theorem 4.1. The same remark applies to [12, Theorem 2.1]. In [2, 12], instead of Assumptions 3.4 and 4.3, the recursive structure of (4.8) or (4.9) is utilized to show that $\{T^i x\}$ is Cauchy and that its limit is a fixed point. See, e.g., [2, 3, 6, 8, 12, 14] for extensions.

5 Nonlinear Markov Operators

In this section we consider the case in which T is a self-map on the space of probability distribution functions on $\mathbb{R}$. Such a map is often called a nonlinear Markov operator; linear Markov operators are often associated with Markov chains. Since our approach does not require linearity, we allow T to be nonlinear. The analysis of this section can be extended to Markov chains on considerably more general spaces than $\mathbb{R}$ along the lines of [7, 9, 10].

Let F be the set of probability distribution functions on $\mathbb{R}$; i.e., each $f \in F$ is an increasing and right-continuous function from $\mathbb{R}$ to $[0, 1]$ such that

$$\lim_{x \downarrow -\infty} f(x) = 0, \tag{5.1}$$

$$\lim_{x \uparrow \infty} f(x) = 1. \tag{5.2}$$

We define the binary relation $\preceq$ on F by

$$f \preceq g \quad \Longleftrightarrow \quad \forall x \in \mathbb{R}, f(x) \geq g(x). \tag{5.3}$$

Note that $\preceq$ is a partial order. This partial order is known as "stochastic dominance." We also write $f \geq g$ if $f(x) \geq g(x)$ for all $x \in \mathbb{R}$. Hence $f \preceq g$ if and only if $f \geq g$.

In what follows we take as given an order-preserving self-map $T : F \to F$. Let $A = \mathbb{R}$. For $f, g \in F$ and $a \in A$, define

$$d(f, g)(a) = |f(a) - g(a)|. \tag{5.4}$$

It is easy to see that Assumptions 3.2–3.4 hold under(5.3) and (5.4), and that d is symmetric and satisfies the triangle inequality.

It is shown in [9, Theorem 3.1] that T is asymptotically contractive if it is the linear Markov operator on F associated with an "order mixing" Markov chain. Informally, a Markov chain is order mixing if given any two independent versions $\{X_t\}$ and $\{Y_t\}$ of the same chain with different initial conditions, we have $X_t \leq Y_t$ at least once with probability one. This is a natural property of various stochastic processes; see [9, 10].

The following result is a restatement of Theorem 3.4.

Theorem 5.1. *T has a globally stable fixed point if and only if T is asymptotically contractive and there exist $f, f^* \in F$ such that*

$$\forall i \in \mathbb{N}, \quad T^i f \preceq f^*, \tag{5.5}$$
$$f^* \preceq T f^*. \tag{5.6}$$

The next result provides a sufficient condition for the existence of $f, f^* \in F$ satisfying (5.5) and (5.6).

Theorem 5.2. *Suppose that T is asymptotically contractive. Suppose further that there exist $f, \overline{f} \in F$ such that*

$$f \preceq Tf \tag{5.7}$$
$$\forall i \in \mathbb{N}, \quad T^i f \preceq \overline{f}. \tag{5.8}$$

Then T has a globally stable fixed point f^*.

Proof. Let
$$f^* = \inf_{i\in\mathbb{N}}(T^i f), \tag{5.9}$$
where the infimum is taken pointwise. By construction, f^* satisfies (5.5). We verify that $f^* \in F$, and that (5.6) holds.

To see that $f^* \in F$, note that since each f_i is increasing, so is f^*. From (5.7)–(5.9) it follows that $\overline{f} \leq f^* \leq f$. Thus $f^*(x) \in [0, 1]$ for all $x \in \mathbb{R}$; furthermore, $\lim_{x\downarrow-\infty} f^*(x) = 0$ and $\lim_{x\uparrow\infty} f^*(x) = 1$. That f^* is right continuous or, equivalently, upper semicontinuous (since f^* is increasing) follows from the fact that the pointwise infimum of a family of upper semicontinuous functions is upper semicontinuous (see [1, p. 43]).

It remains to verify (5.6). Since $f^* \leq T^i f$ for all $i \in \mathbb{N}$, we have $Tf^* \leq T^{i+1}$ for all $i \in \mathbb{N}$. Since this inequality holds pointwise, taking the infimum of the right-hand side over $i \in \mathbb{N}$ and recalling that $\{T^i f\}$ is increasing with respect to $\preceq$ (i.e., decreasing with respect to $\leq$), we obtain $Tf^* \leq f^*$; hence $f^* \preceq Tf^*$. □

One way to ensure the existence of $\overline{f}$ satisfying (5.8) is by assuming that $\{T^i f\}$ is "tight" (with $\{T^i f\}$ viewed as a sequence of probability measures). In this case, the sequence $\{T^i f\}$ has a weak limit, which can be used as an upper bound on the sequence. This is the approach taken in [10].

Although (5.7) and (5.8) imply that $\{T^i f\}$ is tight, Theorem 5.2 does not follow from [10, Theorem 3.1, Lemma 6.5]. First of all, T is nonlinear here.

Second, even if T is linear, there may be no Markov chain that induces T. Third, asymptotic contractiveness is weaker than order mixing. Forth, T is not assumed to be "bounded in probability" here.

If one assumes that $T\overline{f} \preceq \overline{f}$ in addition to (5.7) and (5.8), then T maps $[f, \overline{f}]$ into itself, where $[f, \overline{f}]$ is the set of functions $\tilde{f} : \mathbb{R} \to [0, 1]$ such that $f \preceq \tilde{f} \preceq \overline{f}$. In this case, the existence of a fixed point can be shown by applying the Knaster-Tarski fixed point theorem [1, p. 16] to the restriction of T to $[f, \overline{f}]$. However, since we do not assume that $T\overline{f} \preceq \overline{f}$ here, T need not be a self-map on $[f, \overline{f}]$. Thus Theorem 5.2 does not follow from the Knaster-Tarski fixed point theorem.

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