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An Order-Theoretic Mixing Condition for Monotone Markov Chains*

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Abstract

We discuss stability of discrete-time Markov chains satisfying monotonicity and an order-theoretic mixing condition that can be seen as an alternative to irreducibility. A chain satisfying these conditions has at most one stationary distribution. Moreover, if there is a stationary distribution, then the chain is stable in an order-theoretic sense.

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1 Introduction

In this paper we consider an order-theoretic mixing condition for Markov chains called "order mixing." This condition was introduced in Kamihigashi and Stachurski (2009) and shown to imply stability of a stationary distribution under certain conditions.[1] Informally, a Markov chain is order mixing if, given any two independent versions $\{X_t\}_{t\in\mathbb{Z}_+}$ and $\{X'_t\}_{t\in\mathbb{Z}_+}$ of the same process with different initial conditions, we have

$$\mathbb{P}\left(\cup_{t\in\mathbb{Z}_+}\{X_t \leq X'_t\}\right) = 1. \tag{1}$$

We show that if the chain is monotone in a standard sense as well as order mixing, then there is at most one stationary distribution. Moreover, if there is a stationary distribution, then the chain is stable in an order-theoretic sense. These results are particularly useful when the chain fails to be irreducible or cannot easily be verified to be irreducible.

Bhattacharya and Lee (1988, 1997) and others (e.g., Bhattacharya and Majumdar, 2007; Bhattacharya et al., 2010) have used an alternative order-theoretic mixing condition called "splitting" to study stability of Markov chains in the absence of irreducibility. As we show in section 4, order mixing is weaker than the splitting condition.[2]

There is also a natural connection between order mixing and the classical stability theory for Markov chains initiated by Doeblin (1938). The latter studies chains such that, given any two independent versions $\{X_t\}$ and $\{X'_t\}$ of the same process, we have

$$\mathbb{P}\left(\cup_{t\in\mathbb{Z}_+}\{X_t = X'_t\}\right) = 1. \tag{2}$$

This "classical coupling" condition implies that there is at most one stationary distribu-

[1] See Stachurski (2009) for a special case of this result.

[2] See Hopenhayn and Prescott (1992) for a mixing condition for compact state spaces that is similar to the splitting condition.

tion, and, if there is one, then the chain is stable in terms of convergence in total variation. Conditions (1) and (2) thus have similar implications, but the latter implies a stronger stability property. On the other hand, the order mixing condition (1) is considerably weaker than (2), and is applicable even when convergence in total variation fails. Since existence of a "successful coupling" often implies convergence in total variation (see Thorisson, 2000), order mixing can be viewed as an alternative to such coupling methods.

In section 4 we present an example that is not irreducible and does not satisfy the splitting condition. Standard coupling methods are not applicable to this example either. However, it is easily shown to be order mixing.

The rest of the paper is organized as follows. In section 2 we review basic definitions concerning Makov chains and formalize the concept of order mixing. In section 3 we present the main results of the paper. In section 4 we discuss the splitting condition and the example mentioned above.

2 Preliminaries

We begin with basic definitions concerning Markov chains, starting with chains on arbitrary measurable spaces, and then moving to monotone chains on preordered spaces.

2.1 Standard Definitions

Let $(\mathcal{X}, \mathscr{X})$ be a measurable space, let $\mathcal{P}(\mathcal{X})$ be the probability measures on $(\mathcal{X}, \mathscr{X})$, and let $b\mathcal{X}$ be the bounded measurable functions from $\mathcal{X}$ to $\mathbb{R}$. A *stochastic kernel* on $\mathcal{X}$ is a function $Q\colon \mathcal{X} \times \mathscr{X} \to [0,1]$ such that $Q(x, \cdot) \in \mathcal{P}(\mathcal{X})$ for each $x \in \mathcal{X}$, and $Q(\cdot, B)$ is measurable for each $B \in \mathscr{X}$.

Each stochastic process we consider is assumed to be an independent draw from a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$; this is to avoid defining a new probability measure for each stochastic process introduced. Given a stochastic kernel Q on $\mathcal{X}$, a discrete-time, $\mathcal{X}$-valued stochastic process $\{X_t\}_{t\in\mathbb{Z}_+}$ is called *Markov-*(Q,μ) if X_0 has distribution $\mu \in \mathcal{P}(\mathcal{X})$, and $Q(x,\cdot)$ is the conditional distribution of X_{t+1} given $X_t = x$ (more formally, $\mathbb{P}[X_{t+1} \in B \mid X_0,\ldots,X_t] = Q(X_t, B)$ almost surely for all $B \in \mathscr{X}$). If μ is the probability measure $\delta_x \in \mathcal{P}(\mathcal{X})$ concentrated on $x \in \mathcal{X}$, we call $\{X_t\}$ *Markov-*(Q,x) rather than Markov-(Q,δ_x). We say that $\{X_t\}$ is *Markov-*Q if $\{X_t\}$ is Markov-(Q,μ) for some $\mu \in \mathcal{P}(\mathcal{X})$.

Given $h \in b\mathcal{X}$, we define the right Markov operator by

$$h \mapsto Qh, \qquad (Qh)(x) = \int h(y)Q(x,dy). \tag{3}$$

The t-th iterate has the property $(Q^t h)(x) = \mathbb{E}h(X_t)$ for any Markov-(Q,x) chain $\{X_t\}$. Given $\mu \in \mathcal{P}(\mathcal{X})$, we define the left Markov operator by

$$\mu \mapsto \mu Q, \qquad (\mu Q)(B) = \int Q(x,B)\mu(dx), \tag{4}$$

with the property $(\mu Q^t)(B) = \mathbb{P}\{X_t \in B\}$ for any Markov-(Q,μ) chain $\{X_t\}$. A distribution $\mu^* \in \mathcal{P}(\mathcal{X})$ is called *stationary* if $\mu^* Q = \mu^*$.

To simplify notation, for $\mu \in \mathcal{P}(\mathcal{X})$ and $h \in b\mathcal{X}$, we define

$$\langle \mu, h\rangle = \int h(x)\mu(dx).$$

It is easy to see that, for any Markov-(Q,μ) chain $\{X_t\}$, we have

$$\langle \mu Q^t, h\rangle = \langle \mu, Q^t h\rangle = \mathbb{E}h(X_t) \quad \text{for all } t \in \mathbb{N}. \tag{5}$$

For Markov-(Q,μ) chain $\{X_t\}$ and $C \in \mathscr{X}$, we let $L_Q(\mu, C)$ represent the probability that $\{X_t\}$ ever visits C. That is,

$$L_Q(\mu, C) = \mathbb{P}\left(\cup_{t \in \mathbb{Z}_+} \{X_t \in C\}\right). \tag{6}$$

Since all Markov-(Q, μ) chains induce the same probability distribution on the product space $(\mathcal{X}^\infty, \mathscr{X}^\infty)$, the value of $L_Q(\mu, C)$ does not depend on the particular Markov-(Q, μ) chain chosen. If $\mu = \delta_x$ for some $x \in \mathcal{X}$, then we write $L_Q(x, C)$ rather than $L_Q(\delta_x, C)$. A standard conditioning argument shows that

$$L_Q(\mu, C) = \int L_Q(x, C)\mu(dx). \tag{7}$$

2.2 Monotonicity

Next we specialize to monotone Markov chains on preordered spaces. Our main interest is on well-behaved subsets of $\mathbb{R}^n$ with the usual partial order, but we want to emphasize that topology plays no role in the analysis. For this reason, we take our state space $\mathcal{E}$ to be a preordered space equipped with preorder $\leq$.[3] Letting $I_b = \{x \in \mathcal{E} : b \leq x\}$ and $D_b = \{x \in \mathcal{E} : x \leq b\}$, we define $\mathscr{E}$ as the sigma-algebra generated by the family of sets $\{I_b : b \in \mathcal{E}\} \cup \{D_b : b \in \mathcal{E}\}$; this mimics the construction of the Borel sigma-algebra on $\mathbb{R}^n$. The graph of $\leq$ is denoted $\mathbb{G}$. That is,

$$\mathbb{G} = \{(x, x') \in \mathcal{E} \times \mathcal{E} : x \leq x'\}.$$

We assume that the graph $\mathbb{G}$ of $\leq$ is measurable in the product space $\mathcal{E} \times \mathcal{E}$ (i.e., $\mathbb{G} \in \mathscr{E} \otimes \mathscr{E}$). This is satisfied if, for example, $\mathcal{E} = \mathbb{R}^n$ and $\leq$ is the usual partial order.

A function $h : \mathcal{E} \to \mathbb{R}$ is called *increasing* if $h(x) \leq h(y)$ whenever $x \leq y$. A set $C \subset \mathcal{E}$ is

[3] A binary relation is called a *preorder* if it is reflexive and transitive.

called *increasing* if its indicator function $\mathbb{1}_C$ is increasing. The set of all increasing bounded measurable functions from $\mathcal{E}$ to $\mathbb{R}$ is denoted by $ib\mathcal{E}$. A stochastic kernel Q on $\mathcal{E}$ is called *increasing* if $Qh \in ib\mathcal{E}$ for any $h \in ib\mathcal{E}$ (and hence $Q^n h \in ib\mathcal{E}$ for any $n \in \mathbb{N}$).

Lemma 2.1. *If $\mu, \nu \in \mathcal{P}(\mathcal{E})$ satisfy $\langle \mu, h \rangle = \langle \nu, h \rangle$ for all $h \in ib\mathcal{E}$, then $\mu = \nu$.*

Proof. Let $\mu, \nu \in \mathcal{P}(\mathcal{E})$ with $\langle \mu, h \rangle = \langle \nu, h \rangle$ for all $h \in ib\mathcal{E}$. Let $\mathscr{E}^i$ be the increasing sets in $\mathscr{E}$. Note that μ and ν agree on $\mathscr{E}^i$ since $\mathbb{1}_C \in ib\mathcal{E}$ for any $C \in \mathscr{E}^i$. The family $\mathscr{E}^i$ is closed under finite intersections; thus it is a π-system . Furthermore, $\mathscr{E}^i$ generates $\mathscr{E}$ since, given $b \in \mathcal{E}$, any sigma-algebra containing $\mathscr{E}^i$ contains both I_b and D_b, the latter because $\mathcal{E} \setminus D_b$ is increasing. It follows by Billingley (1995, p. 42) that μ and ν agree on $\mathscr{E}$. $\square$

Given $\mathcal{E}$-valued, independent Markov-Q chains $\{X_t\}_{t \in \mathbb{Z}_+}$ and $\{X'_t\}_{t \in \mathbb{Z}_+}$, the $\mathcal{E} \times \mathcal{E}$-valued process $\{(X_t, X'_t)\}_{t \in \mathbb{Z}_+}$ is also Markov. In particular, if we define the product kernel $Q \times Q$ on $\mathcal{E} \times \mathcal{E}$ by

$$(Q \times Q)((x, x'), A) = \int \int \mathbb{1}_A(y, y') Q(x, dy) Q(x', dy') \tag{8}$$

for $(x, x') \in \mathcal{E} \times \mathcal{E}$ and $A \in \mathscr{E} \otimes \mathscr{E}$, then $Q \times Q$ is a stochastic kernel on $\mathcal{E} \times \mathcal{E}$, and $\{(X_t, X'_t)\}_{t \in \mathbb{Z}_+}$ is Markov-$Q \times Q$.

Using the product kernel $Q \times Q$ on $\mathcal{E} \times \mathcal{E}$ in (6), we now formally define order mixing: we say that Q (or any Markov-Q chain) is *order mixing* if

$$L_{Q \times Q}((x, x'), \mathbb{G}) = 1 \quad \text{for all } (x, x') \in \mathcal{E} \times \mathcal{E}. \tag{9}$$

This makes precise the idea that independent Markov-Q chains $\{X_t\}$ and $\{X'_t\}$ starting at x and x' attain $X_t \leq X'_t$ eventually with probability one. If Q is order mixing, then (7) and (9) imply that

$$L_{Q \times Q}((\mu, \nu), \mathbb{G}) = 1 \quad \text{for all } \mu, \nu \in \mathcal{P}(\mathcal{E}). \tag{10}$$

Remark 2.1. It is easy to verify that if Q^m is order mixing for some $m \in \mathbb{N}$, then Q itself is order mixing.

3 Stability

Let $(\mathcal{E}, \mathscr{E})$ be as in the previous section. Let Q be a stochastic kernel on $\mathcal{E}$.

Theorem 3.1. *If Q is increasing and order mixing, then for any $\mu, \nu \in \mathcal{P}(\mathcal{E})$, we have*

$$\lim_{t \to \infty} \{ \langle \mu Q^t, h \rangle - \langle \nu Q^t, h \rangle \} = 0 \quad \textit{for all } h \in ib\mathcal{E}. \tag{11}$$

Before proving the theorem, we state the most important implications:

Corollary 3.1. *If the conditions of theorem 3.1 hold, then Q has at most one stationary distribution. Moreover, if a stationary distribution μ^* exists, then for any $\mu \in \mathcal{P}(\mathcal{E})$, we have*

$$\lim_{t \to \infty} \langle \mu Q^t, h \rangle = \langle \mu^*, h \rangle \quad \textit{for all } h \in ib\mathcal{E}. \tag{12}$$

Proof. First, if both μ and ν are stationary, then by (11) we have $\langle \mu, h \rangle = \langle \nu, h \rangle$ for all $h \in ib\mathcal{E}$. Equality of μ and ν now follows by lemma 2.1. Regarding the second claim, suppose that Q has a (necessarily unique) stationary distribution μ^*. Let $\mu \in \mathcal{P}(\mathcal{E})$, and let $h \in ib\mathcal{E}$. Applying (11) again we obtain $\langle \mu Q^t, h \rangle \to \langle \mu^*, h \rangle$. Since μ and h were arbitrary, we obtain (12). □

While the stability criterion (12) is not traditional, it implies that μQ^t converges weakly to μ^* if $\mathcal{E} = \mathbb{R}^n$ with the standard partial order. Indeed, in this case, (12) implies that $\lim_{t \to \infty} (\mu Q^t)(D_b) = \mu^*(D_b)$ for *all* $b \in \mathbb{R}^n$ and, *a fortiori*, for all continuity points b of the distribution function of μ^*; i.e., μQ^t converges weakly to μ^* (e.g., Billingsley, 1995, p. 378).

Proof of theorem 3.1. It is sufficient to prove that, for any $\mu, \nu \in \mathcal{P}(\mathcal{E})$, we have

$$\limsup_{t\to\infty}\{\langle \mu Q^t, h\rangle - \langle \nu Q^t, h\rangle\} \leq 0 \quad \text{for all } h \in ib\mathcal{E}. \tag{13}$$

Indeed, if this is the case, then, reversing the roles of μ and ν, we also have

$$\limsup_{t\to\infty}\{\langle \nu Q^t, h\rangle - \langle \mu Q^t, h\rangle\} \leq 0 \quad \text{for all } h \in ib\mathcal{E},$$

or, equivalently, $\liminf_{t\to\infty}\{\langle \mu Q^t, h\rangle - \langle \nu Q^t, h\rangle\} \geq 0$. This and (13) yield (11).

Let $\mu, \nu \in \mathcal{P}(\mathcal{E})$. To verify (13), fix $h \in ib\mathcal{E}$. Adding a sufficiently large constant to h (which does not affect the inequality in (13)), we can assume without loss of generality that $h \geq 0$. Let $\{(X_t, X'_t)\}$ be Markov-$(Q \times Q, \mu \times \nu)$. Let $\tau = \inf\{t \in \mathbb{Z}_+ : X_t \leq X'_t\}$, where $\inf \emptyset = \infty$. Since Q is order mixing, we have $\mathbb{P}\{\tau < \infty\} = 1$ by (10). Note from (5) that, for given $t \in \mathbb{N}$,

$$\langle \nu Q^t, h\rangle = \mathbb{E} h(X'_t) \geq \mathbb{E}\mathbb{1}\{\tau \leq t\} h(X'_t) = \mathbb{E}[\mathbb{E}[\mathbb{1}\{\tau \leq t\} h(X'_t) | \mathscr{F}_\tau]], \tag{14}$$

where $\mathscr{F}_\tau$ is the pre-τ sigma-algebra.[4] The inner expectation on the rightmost side of (14) can be written (almost surely) as

$$\mathbb{1}\{\tau \leq t\}\mathbb{E}[h(X'_t)|\mathscr{F}_\tau] = \mathbb{1}\{\tau \leq t\}(Q^{t-\tau}h)(X'_\tau), \tag{15}$$

where the equality is due to the strong Markov property (e.g., Meyn and Tweedie, 2009).[5] Since Q is increasing, $Q^{t-\tau}h \in ib\mathcal{E}$ for any $t \geq \tau$ (with $Q^0 h = h$). Moreover, by the

[4] If $\mathscr{F}_t$ is the sigma-algebra generated by $(X_0, X'_0), \dots, (X_t, X'_t)$, then $\mathscr{F}_\tau$ is all sets $B \in \mathscr{F}$ such that $\{\tau = t\} \cap B \in \mathscr{F}_t$ for all $t \in \mathbb{Z}_+$. See Meyn and Tweedie (2009, p. 66).

[5] Equation (15) can be seen in more detail as follows: by the strong Markov property and Fubini's theorem (e.g., Billingsley, 1995), provided that $\tau \leq t$, we have $\mathbb{E}[h(X'_t)|\mathscr{F}_\tau] = \int h(y')(Q \times Q)^{t-\tau}((X_\tau, X'_\tau), d(y, y')) = \int \left[\int h(y') Q^{t-\tau}(X'_\tau, dy')\right] Q^{t-\tau}(X_\tau, dy) = \int h(y') Q^{t-\tau}(X'_\tau, dy') = (Q^{t-\tau}h)(X'_\tau)$.

definition of τ we have $X_\tau \leq X'_\tau$, so $(Q^{t-\tau}h)(X'_\tau) \geq (Q^{t-\tau}h)(X_\tau)$. Combining this with (14) and (15), we have

$$\langle \nu Q^t, h\rangle \geq \mathbb{E}[\mathbb{1}\{\tau \leq t\}(Q^{t-\tau}h)(X_\tau)].$$

Applying the arguments from (14) and (15) to $\{X_t\}$ instead of $\{X'_t\}$, we have

$$\mathbb{E}\mathbb{1}\{\tau \leq t\}h(X_t) = \mathbb{E}[\mathbb{1}\{\tau \leq t\}(Q^{t-\tau}h)(X_\tau)].$$

$$\therefore \quad \langle \nu Q^t, h\rangle \geq \mathbb{E}\mathbb{1}\{\tau \leq t\}h(X_t) = \mathbb{E}h(X_t) - \mathbb{E}\mathbb{1}\{\tau \geq t+1\}h(X_t).$$

Recalling $\mathbb{E}h(X_t) = \langle \mu Q^t, h\rangle$ and setting $M = \sup_{x\in\mathcal{E}} h(x)$, we now have

$$\langle \mu Q^t, h\rangle - \langle \nu Q^t, h\rangle \leq \mathbb{P}\{\tau \geq t+1\}M.$$

The right-hand side converges to zero because $\mathbb{P}\{\tau < \infty\} = 1$, and we obtain (13). $\square$

4 Discussions

4.1 The Splitting Condition

Bhattacharya and Lee (1988, 1997) consider stability of monotone Markov chains in the absence of irreducibility.[6] Their environment consists of a sequence of IID monotone (increasing or decreasing) maps $\{\alpha_t\}$ from $\mathcal{E}$ to itself, and a process $\{X_t\}$ generated by

$$X_t = \alpha_t X_{t-1} = \alpha_t \circ \cdots \circ \alpha_1(x),$$

[6] See Bhattacharya and Majumdar (2007) and Bhattacharya et al. (2010) for extensions of their work.

where $x \in \mathcal{E}$ is the initial condition. The corresponding kernel is $Q(x, B) = \mathbb{P}\{\alpha_1(x) \in B\}$. They consider a "splitting condition," which requires the existence of a $c \in \mathcal{E}$ and $m \in \mathbb{N}$ such that

(a) $\mathbb{P}\{\alpha_m \circ \cdots \circ \alpha_1(x) \leq c, \ \forall x \in \mathcal{E}\} > 0$, and

(b) $\mathbb{P}\{\alpha_m \circ \cdots \circ \alpha_1(x) \geq c, \ \forall x \in \mathcal{E}\} > 0$.

As we now show, this condition implies order mixing. Without loss of generality, suppose that (a) and (b) hold with $m = 1$; if $m > 1$, a small modification of the following argument shows that Q^m is order mixing, and if Q^m is order mixing then Q is order mixing by remark 2.1. Let $\{\alpha_t\}$ and $\{\alpha'_t\}$ be independent versions of the IID map sequence, let $X_t = \alpha_t(X_{t-1})$ and $X'_t = \alpha'_t(X'_{t-1})$ with initial conditions $X_0 = x$ and $X'_0 = x'$ respectively, and let $\tau = \inf\{t \in \mathbb{Z}_+ : X_t \leq X'_t\}$. We need to show that $\mathbb{P}\{\tau < \infty\} = 1$.

To see this, let $A_t = \{\alpha_t(x) \leq c, \ \forall x\}$, $A'_t = \{\alpha'_t(x) \geq c, \ \forall x\}$, and $B_t = A_t \cap A'_t$. Let $\epsilon = \mathbb{P}(B_t) = \mathbb{P}(A_t)\mathbb{P}(A'_t)$, which does not depend on t since $\{\alpha_t\}$ and $\{\alpha'_t\}$ are IID. By the splitting condition (with $m = 1$), $\epsilon > 0$. Moreover, we have $B_t \subset \{X_t \leq X'_t\}$, and hence $\cup_t B_t \subset \{\tau < \infty\}$, or, equivalently, $\{\tau = \infty\} \subset \cap_t(\Omega \setminus B_t)$. Since the events B_t are independent and $\mathbb{P}(B_t) = \epsilon$, we then have

$$0 \leq \mathbb{P}\{\tau = \infty\} \leq \mathbb{P} \cap_t (\Omega \setminus B_t) = \lim_{T \to \infty} (1 - \epsilon)^T = 0.$$

Hence $\mathbb{P}\{\tau < \infty\} = 1$, and order mixing is established.

Two remarks are in order. First, the above argument does not require the IID maps α_t to be monotone, so that the splitting condition implies order mixing whenever both conditions are well defined. Second, under additional assumptions that may not be restrictive in applications, the splitting condition ensures that there exists a unique stationary distribution μ^* and that μQ^t converges to μ^* uniformly and exponentially in a suitable metric for all initial distributions μ (see also Bhattacharya et al., 2010). In other words, the splitting condition alone implies a stronger version of Corollary 3.1. Hence, order mixing can be an alternative to the splitting condition only if the existence of a stationary distribution is known (or can be shown) and the rate of convergence is not an issue.

4.2 AR(1) Processes on $\mathbb{R}_+$

Consider the AR(1) process on $\mathcal{E} = \mathbb{R}_+$ given by

$$X_{t+1} = \theta X_t + U_{t+1}, \tag{16}$$

where $\theta \in (0, 1)$ and $\{U_t\}$ is a nonnegative IID process. Since the right-hand side of (16) is increasing in X_t, the associated stochastic kernel is increasing. The splitting condition does not hold here, since, given any $c \in \mathbb{R}_+$ and $m \in \mathbb{N}$, we have $\mathbb{P}\{X_m < c\} \to 0$ as $X_0 \uparrow \infty$.[7]

For the rest of the discussion, we assume that

$$\theta = 1/3, \qquad U_t = \begin{cases} 2/3 & \text{with probability } 1/2, \\ 0 & \text{with probability } 1/2. \end{cases} \tag{17}$$

In this case, the chain is not irreducible. To see this, note that the stationary distribution of the chain is the distribution of $\sum_{i \in \mathbb{Z}_+} 3^{-i} U_i$. Thus the support of the stationary distribution is the Cantor set, which we denote by Λ:

$$\Lambda = \left\{ \sum_{i \in \mathbb{N}} \frac{a_i}{3^i} : a_i \in \{0, 2\} \text{ for all } i \in \mathbb{N} \right\}. \tag{18}$$

[7] We should mention that the process here is an "average contraction," for which various stability results are available (e.g., Bhattacharya and Majumdar, 2007, sec. 3.7). Here we use this simple model to illustrate an order mixing chain that can easily be shown to satisfy neither irreducibility nor standard coupling conditions.

It is easy to see that if $X_t \in [0,1]$, then $\mathbb{P}[X_{t+1} \in \Lambda \,|\, X_t] > 0$ if and only if $X_t \in \Lambda$. Therefore, if $X_0 \in [0,1] \setminus \Lambda$, then $\mathbb{P}\{X_t \in \Lambda\} = 0$ all $t \in \mathbb{Z}_+$. Since any irreducibility measure must be supported on Λ (see Meyn and Tweedie, 2009, p. 232), it follows that the chain cannot be irreducible.

To consider coupling methods, let $\{X'_t\}$ be an independent version of (16) under (17). Suppose that X_0 is rational and X'_0 is irrational. Then, for all $t, t' \in \mathbb{Z}_+$, the support of X_t is a set of rationals, and that of $X'_{t'}$ is a set of irrationals. Thus the supports of X_t and $X'_{t'}$ are always disjoint, so there is no successful distributional shift-coupling, which implies that there is neither successful exact coupling nor successful distributional coupling (see Thorisson (2000) for definitions).

While none of the above methods works here, it is not difficult to show that the chain is order mixing. To see this, let $\{X_t\}$ and $\{X'_t\}$ be independent versions of (16) with the corresponding IID processes $\{U_t\}$ and $\{U'_t\}$. Let $V_t = X_t - X'_t$ and $W_t = U_{t+1} - U'_{t+1}$ for all $t \in \mathbb{Z}_+$. Then $\{V_t\}$ is an AR(1) process on $\mathbb{R}$: $V_{t+1} = (1/3)V_t + W_{t+1}$. It is easy to see that[8]

$$\mathbb{P}\left(\cup_{t\in\mathbb{Z}_+}\{V_t \le 0\}\right) = 1. \tag{19}$$

Since $V_t = X_t - X'_t$ by definition, it follows that $\{X_t\}$ is order mixing.

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[8]For example, (19) can be shown directly by estimating $\mathbb{P}(\cap_{t\in\mathbb{Z}_+}\{V_t \ge 0\})$. Alternatively, it can be shown by considering the random walk $\hat{V}_{t+1} = \hat{V}_t + W_{t+1}$ with $\hat{V}_0 = V_0$. Since $\mathbb{E}W_t = 0$, we have $\liminf_{t\to\infty} \hat{V}_t = -\infty$ by the Chung-Fuchs theorem (e.g., Chow and Teicher, 1997, p. 154). Since $V_t \le \hat{V}_t$ as long as $V_i \ge 0$ for all $i \le t$, we obtain (19).

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